

Mackey functors

Throughout this presentation of Mackey functors, the letter R denotes a commutative and associative ring with unit, and G denotes a finite group.

1. Three equivalent definitions

There are (at least) three equivalent definitions of Mackey functors for G over R . Each of them uses a different object associated to G .

1.1. Using the poset of subgroups of G . This definition of Mackey functors uses the poset of subgroups of G . It goes back to Green ([2])

Definition 1.1.1 : *A Mackey functor M for G over R consists of the following data :*

- *An R -module $M(H)$, for any subgroup H of G .*
- *Maps of R -modules*

$$t_H^K : M(H) \rightarrow M(K) \quad r_H^K : M(K) \rightarrow M(H) \quad c_{x,H} : M(H) \rightarrow M({}^x H) \quad ,$$

whenever $H \subseteq K$ are subgroups of G , and $x \in G$.

These maps are subject to the following conditions :

- *(transitivity) If $H \subseteq K \subseteq L$ are subgroups of G , then*

$$t_K^L t_H^K = t_H^L \quad r_H^K r_K^L = r_H^L \quad .$$

If x and y are elements of G , and if H is a subgroup of G , then

$$c_{y, {}^x H} c_{x,H} = c_{yx,H} \quad .$$

- *(compatibility) If $H \subseteq K$ are subgroups of G , and if $x \in G$, then*

$$c_{x,K} t_H^K = t_{{}^x H}^{{}^x K} c_{x,H} \quad c_{x,H} r_H^K = r_{{}^x H}^{{}^x K} c_{x,K} \quad .$$

- *(triviality) If H is a subgroup of G and x is an element of H , then*

$$t_H^H = r_H^H = c_{x,H} = Id_{M(H)} \quad .$$

- (Mackey axiom) If $H \subseteq K \supseteq L$ are subgroups of G , then

$$r_H^K t_L^K = \sum_{x \in [H \backslash K / L]} t_{H \cap^x L}^H c_{x, H^x \cap L} r_{H^x \cap L}^L, \quad ,$$

where $[H \backslash K / L]$ is a set of representatives of the double cosets modulo H and L in K .

If M and M' are Mackey functors for G over R , a morphism of Mackey functors $f : M \rightarrow M'$ is a collection of morphisms $f_H : M(H) \rightarrow M'(H)$, where H is a subgroup of G , such that for any subgroups $H \subseteq K$ of G and any $x \in G$

$$f_K t_H^K = t_H^K f_H \quad f_H r_H^K = r_H^K f_K \quad f_{xH} c_{x,H} = c_{x,H} f_H \quad .$$

If $f : M \rightarrow M'$ and $f' : M' \rightarrow M''$ are morphisms of Mackey functors, the composition $f'f$ is the morphism of Mackey functors defined by $(f'f)_H = f'_H f_H$, for $H \subseteq G$. The Mackey functors for G over R , with these morphisms and this composition of morphism, form a category, denoted by $\text{Mack}_R(G)$.

It follows in particular from these definitions that if H is a subgroup of G and M is a Mackey functor for G over R , then $M(H)$ is an $R\overline{N}_G(H)$ -module, where $\overline{N}_G(H)$ denotes the group $N_G(H)/H$.

1.2. Using the category of G -sets. The second definition of Mackey functors for G over R is due to Dress ([1]). It uses the category $G\text{-set}$ of finite G -sets : its objects are the finite sets with a left G -action, its morphisms are the G -equivariant maps, and composition of morphisms is composition of maps.

Definition 1.2.1 : A Mackey functor $\tilde{M}$ for G over R is a bivariate functor from $G\text{-set}$ to the category $RG\text{-Mod}$ of left RG -modules, i.e. a pair of functors $(\tilde{M}_*, \tilde{M}^*)$ from $G\text{-set}$ to $RG\text{-Mod}$, with $\tilde{M}_*$ covariant, and $\tilde{M}^*$ contravariant, which coincide on objects (i.e. $\tilde{M}_*(X) = \tilde{M}^*(X)$ for any finite G -set X , and this common value is denoted by $\tilde{M}(X)$). This bivariate functor is subject to the two following additional conditions :

- (additivity) If X and Y are finite G -sets, and if i_X and i_Y are the respective inclusion maps from X and Y to their disjoint union $X \sqcup Y$, then the maps

$$\tilde{M}(X) \oplus \tilde{M}(Y) \xrightarrow{(\tilde{M}_*(i_X), \tilde{M}_*(i_Y))} \tilde{M}(X \sqcup Y) \xrightarrow{\begin{pmatrix} \tilde{M}^*(i_X) \\ \tilde{M}^*(i_Y) \end{pmatrix}} \tilde{M}(X) \oplus \tilde{M}(Y)$$

are mutual inverse isomorphisms.

- (cartesian squares) If

$$\begin{array}{ccc} X & \xrightarrow{a} & Y \\ b \downarrow & & \downarrow c \\ Z & \xrightarrow{d} & T \end{array}$$

is a cartesian (i.e. pull-back) square of finite G -sets, then

$$\tilde{M}_*(b)\tilde{M}^*(a) = \tilde{M}^*(d)\tilde{M}_*(c) \quad .$$

A morphism of Mackey functors is a natural transformation of bivariant functors, and composition of morphisms is composition of natural transformations. The category of Mackey functors for this definition will be denoted by $\widetilde{\text{Mack}}_R(G)$.

1.3. Using the Mackey algebra. The following definition is due to Thévenaz and Webb ([4]) :

Definition 1.3.1 : The Mackey algebra $\mu_R(G)$ of G over R is the associative R -algebra defined by generators $\hat{t}_H^K$, $\hat{r}_H^K$, and $\hat{c}_{x,H}$, for subgroups $H \subseteq K$ of G and $x \in G$, subject to the following relations :

- (transitivity) If $H \subseteq K \subseteq L$ are subgroups of G , then

$$\hat{t}_K^L \hat{t}_H^K = \hat{t}_H^L \quad \hat{r}_H^K \hat{r}_K^L = \hat{r}_H^L \quad .$$

If x and y are elements of G , and if H is a subgroup of G , then

$$\hat{c}_{y,xH} \hat{c}_{x,H} = \hat{c}_{yx,H} \quad .$$

- (compatibility) If $H \subseteq K$ are subgroups of G , and if $x \in G$, then

$$\hat{c}_{x,K} \hat{t}_H^K = \hat{t}_{xH}^x \hat{c}_{x,H} \quad \hat{c}_{x,H} \hat{r}_H^K = \hat{r}_{xH}^x \hat{c}_{x,K} \quad .$$

- (triviality) If H is a subgroup of G and x is an element of H , then

$$\hat{t}_H^H = \hat{r}_H^H = \hat{c}_{x,H} \quad .$$

- (Mackey axiom) If $H \subseteq K \supseteq L$ are subgroups of G , then

$$\hat{r}_H^K \hat{t}_L^K = \sum_{x \in [H \backslash K / L]} \hat{t}_{H \cap xL}^H \hat{c}_{x, H^x \cap L} \hat{r}_{H^x \cap L}^L \quad ,$$

where $[H \backslash K / L]$ is a set of representatives of the double cosets modulo H and L in K .

- (vanishing) All products of generators, different from those appearing in the previous four relations, are zero.
- (identity) The sum of the elements $\hat{t}_H^H$, over all subgroups H of G , is equal to the identity element of $\mu_R(G)$.

Definition 1.3.2 : A Mackey functors $\hat{M}$ for G over R is a $\mu_R(G)$ -module, and a morphism of Mackey functors is a morphism of $\mu_R(G)$ -modules.

Remark 1.3.3 : The Mackey algebra $\mu_R(G)$ has a natural anti-automorphism σ_G , defined by R -linearity from

$$t_H^K \mapsto r_H^K \quad r_H^K \mapsto t_H^K \quad c_{x,H} \mapsto c_{x^{-1},H} \quad ,$$

for $H \subseteq K \subseteq G$ and $x \in G$. In particular, any left $\mu_R(G)$ -module can be viewed as a right $\mu_R(G)$ -module via this anti-automorphism.

1.4. Equivalence of the definitions.

- [1 $\rightarrow$ 2] If M is a Mackey functor in the first sense, and X is a finite G -set, define

$$\tilde{M}(X) = \left(\bigoplus_{x \in X} M(G_x) \right)^G \quad ,$$

where the group G permutes the components of the direct sum via its action on X , and via the maps $c_{g,G_x} : M(G_x) \rightarrow M(G_{gx})$, for $g \in G$ and $x \in X$.

If $f : X \rightarrow Y$ is a map of G -sets, and if $u = (u_x)_{x \in X}$ is an element of $\tilde{M}(X)$, define the element $M_*(f)(u)$ of $\tilde{M}(Y)$ by

$$\forall y \in Y, \quad M_*(f)(u)_y = \sum_{x \in G_y \backslash f^{-1}(y)} t_{G_x}^{G_y} u_x \quad .$$

Conversely, if $v = (v_y)_{y \in Y}$ is an element of $\tilde{M}(Y)$, define the element $M^*(f)(v)$ of $\tilde{M}(X)$ by

$$\forall x \in X, \quad (M^*(f)(v))_x = r_{G_x}^{G_{f(x)}} v_{f(x)} \quad .$$

Then $\tilde{M}$ is a Mackey functor in the second sense, and the correspondence $M \mapsto \tilde{M}$ is an equivalence of category from $\mathbf{Mack}_R(G)$ to $\widetilde{\mathbf{Mack}}_R(G)$.

- [2 $\rightarrow$ 1] If $\tilde{M}$ is a Mackey functor in the second sense, and if H is a subgroup of G , set

$$M(H) = \tilde{M}(G/H) \quad .$$

If $H \subseteq K$ are subgroups of G , let $p_H^K : G/H \rightarrow G/K$ denote the natural projection map. Define then

$$t_H^K = \tilde{M}_*(p_H^K) \quad r_H^K = \tilde{M}^*(p_H^K) \quad .$$

If $x \in G$, and if H is a subgroup of G , let $\gamma_{x,H}$ denote the map of G -sets from G/H to $G/^xH$ defined by $\gamma_{x,H}(gH) = gx^{-1}xH$. Then set

$$c_{x,H} = \tilde{M}_*(\gamma_{x,H}) \quad .$$

Then M is a Mackey functor in the first sense, and the correspondence $\tilde{M} \mapsto M$ extends to an equivalence of categories from $\widetilde{\mathbf{Mack}}_R(G)$ to $\mathbf{Mack}_R(G)$, which is a quasi-inverse to the equivalence $1 \rightarrow 2$.

- $[1 \rightarrow 3]$ If M is a Mackey functor in the first sense, set

$$\hat{M} = \bigoplus_{H \subseteq G} M(H) \quad .$$

Let the generator $\hat{t}_H^K$ of $\mu_R(G)$ act on $\hat{M}$ by sending the component $M(L)$, for $L \neq H$, to 0, and the component $M(H)$ to $M(K)$ via the map t_H^K . Similarly, let $\hat{r}_H^K$ send $M(L)$ to 0 if $L \neq K$, and to $M(H)$ via the map r_H^K otherwise. Finally, if $x \in G$, let $\hat{c}_{x,H}$ send the component $M(L)$ to 0 if $L \neq H$, and to $M(^xH)$ via the map $c_{x,H}$ otherwise.

Then $\hat{M}$ is a $\mu_R(G)$ -module, and the correspondence $M \mapsto \hat{M}$ extends to an equivalence of categories from $\mathbf{Mack}_R(G)$ to $\mu_R(G)\text{-Mod}$.

- $[3 \rightarrow 1]$ Let $\hat{M}$ be a $\mu_R(G)$ -module. If H is a subgroup of G , define

$$M(H) = \hat{t}_H^H \hat{M} \quad .$$

The relations on the generators of $\mu_R(G)$ imply that if $H \subseteq K$ and $x \in G$, then

$$\hat{t}_H^K M(H) \subseteq M(K) \quad \hat{r}_H^K M(K) \subseteq M(H) \quad \hat{c}_{x,H} M(H) \subseteq M(^xH) \quad ,$$

and this defines respectively a map $t_H^K : M(H) \rightarrow M(K)$, a map $r_H^K : M(K) \rightarrow M(H)$, and a map $c_{x,H} : M(H) \rightarrow M(^xH)$.

Now M is a Mackey functor in the first sense, and the correspondence $\hat{M} \mapsto M$ extends to an equivalence of categories from $\mu_R(G)\text{-Mod}$ to $\mathbf{Mack}_R(G)$, which is a quasi-inverse to the equivalence $1 \rightarrow 3$.

Notation 1.4.1 : *In view of these equivalence, the three categories $\mathbf{Mack}_R(G)$, $\widetilde{\mathbf{Mack}}_R(G)$ and $\mu_R(G)\text{-Mod}$ will sometimes be identified, and a Mackey functor will be considered as an object of any of them. This will lead in particular to equalities such as $M(H) = M(G/H)$ for a Mackey functor M and a subgroup H of G .*

2. Examples

2.1. Representation groups. The various representation groups attached to subgroups of G can generally be viewed as Mackey functors :

- Let k be a field. If H is a subgroup of G , denote by $R_k(G)$ the Grothendieck group of the category of finitely generated kG -modules, for relations given by short exact sequences.

If $H \subseteq K$ are subgroups of G , the restriction of modules induces a map $r_H^K = \text{Res}_H^K : R_k(K) \rightarrow R_k(H)$, whereas induction of modules induces a map $t_H^K = \text{Ind}_H^K : R_k(H) \rightarrow R_k(K)$. If $x \in G$, there is an obvious conjugation map $c_{x,H} : R_k(H) \rightarrow R_k({}^xH)$. With this notation R_k becomes a Mackey functor for G over $\mathbb{Z}$.

One can check easily that if X is a finite G -set, then $R_k(X)$ is the Grothendieck group of G -equivariant k -vector bundles over X .

Similarly, one can consider the Grothendieck group $P_k(G)$ of the category of finitely generated *projective* kG -modules, for relations given by direct sum decomposition. It is also a Mackey functor for the previous operations of restriction, induction, and conjugation.

- If H is a subgroup of G , denote by $B(H)$ the Burnside group of H , i.e. the Grothendieck group of the category of finite H -sets, for relations given by disjoint unions decomposition. Then if $H \subseteq K$ are subgroups of G , the restriction of K -sets gives a map $r_H^K = \text{Res}_H^K : B(K) \rightarrow B(H)$. Similarly, induction of H -sets (defined by $\text{Ind}_H^K X = K \times_H X$) induces a map $B(H) \rightarrow B(K)$. If $x \in G$, there is again an obvious conjugation map $c_{x,H} : B(H) \rightarrow B({}^xH)$, and B is a Mackey functor for G over $\mathbb{Z}$.

One can show that for any finite G -set X , the group $B(X)$ is the Grothendieck group of the category of finite G -sets over X , for relations given by disjoint union decomposition.

2.2. (Co)homology functors. Let M be an RG -module, and l be a non-negative integer. If K is a subgroup of G , define $M(K)$ as the cohomology group $H^l(K, \text{Res}_K^G M)$ (or $H_l(K, \text{Res}_K^G M)$). Then the operations of transfer, restriction, and conjugation, define a structure of Mackey functor on M . If X is a finite G -set, then $\tilde{M}(X) \cong H^l(G, RX \otimes_R M)$ (or $H_l(G, RX \otimes_R M)$). Similarly, the Tate (co)homology groups have a natural structure of Mackey functor.

These Mackey functors have the following additional property : the composition $t_H^K r_H^K$ is equal to the multiplication by $|K : H|$, for any subgroups $H \subseteq K$ of G . Mackey functors with this property are called *cohomological*.

3. Fixed points, fixed quotients, and simple Mackey functors

Since the category of Mackey functors for G over R is equivalent to the category of modules over an R -algebra, it is an abelian category. One can try to classify and describe its simple and projective objects.

3.1. Fixed points and fixed quotients. Let V be an RG -module. The zero-th cohomology functor $H^0(-, V)$ is denoted by FP_V , and it is called the *fixed points functor* associated to V . Similarly, the zero-th homology functor $H_0(-, V)$ is denoted by FQ_V , and it is called the *fixed quotient functor* associated to V . If H is a subgroup of G

$$FP_V(H) = V^H \quad FQ_V(H) = V_H \quad .$$

If $H \subseteq K$ are subgroups of G , then the maps t_H^K and r_H^K for FP_V are the relative trace maps and restriction maps, i.e.

$$\forall v \in V^H, \quad t_H^K(v) = \sum_{x \in K/H} xv \quad \forall w \in V^K, \quad r_H^K(v) = v \quad .$$

If $g \in G$, then the conjugation map $c_{g,H}$ is given by

$$\forall v \in V^H, \quad c_{g,H}(v) = gv \quad .$$

Similarly, the maps t_H^K and r_H^K for FQ_V are given by

$$\forall v \in V_H, \quad t_H^K(v) = v \quad \forall w \in V_K, \quad r_H^K(w) = \sum_{x \in H \backslash K} xw \quad ,$$

and the map $c_{g,H}$ is given by $c_{g,H}(v) = gv$, for $v \in V_H$.

If X is a finite G -set, one can show that

$$FP_V(X) = \text{Hom}_{RG}(RX, V) \quad FQ_V(X) = RX \otimes_{RG} V \quad ,$$

where RX is the permutation module associated to X , i.e. the free R -module with basis X .

If $f : X \rightarrow Y$ is a map of finite G -sets, let $Rf : RX \rightarrow RY$ and ${}^tRf : RY \rightarrow RX$ be the R -linear maps defined by

$$\forall x \in X, \quad (Rf)(x) = f(x) \quad \forall y \in Y, \quad ({}^tRf)(y) = \sum_{x \in f^{-1}(y)} x \quad .$$

Then the map $(FP_V)_*(f)$ is given by composition with tRf , and the map $(FP_V)^*(f)$ is given by composition with Rf . Similarly, the map $(FQ_V)_*(f)$ is the map $Rf \otimes_{RG} Id_V$, and $(FQ_V)^*(f) = {}^tRf \otimes_{RG} Id_V$.

The main property of these constructions is the following :

Proposition 3.1.1 : *The correspondence $V \mapsto FP_V$ (resp. $V \mapsto FQ_V$) is a functor from $RG\text{-Mod}$ to $\mathbf{Mack}_R(G)$, which is right adjoint (resp. left adjoint) to the evaluation functor $M \mapsto M(1)$.*

Notation 3.1.2 : *Let H be a subgroup of G , and let V be an $R\overline{N}_G(H)$ -module. The composite bivariant functor*

$$X \mapsto X^H \mapsto FP_V(X^H) = \text{Hom}_{R\overline{N}_G(H)}(R(X^H), V)$$

from $G\text{-set}$ to $R\text{-Mod}$ is denoted by $FP_{H,V}$. If X is a finite $G\text{-set}$, set

$$S_{H,V}(X) = \text{Tr}_1^{\overline{N}_G(H)}\left(\text{Hom}_R(R(X^H), V)\right) \subset FP_{H,V}(X) \quad ,$$

where $\text{Tr}_1^{\overline{N}_G(H)}$ is the relative trace map.

One can show that $FP_{H,V}$ is a Mackey functor, and that $S_{H,V}$ is a sub-Mackey functor of $FP_{H,V}$.

Definition 3.1.3 : *Let M be a Mackey functor for G over R . A minimal subgroup for M is a subgroup H of G , such that $M(H) \neq 0$, but $M(K) = 0$ for every proper subgroup K of H .*

Proposition 3.1.4 : *(Thévenaz-Webb [3])*

1. *Let H be a subgroup of G , and V be a simple $R\overline{N}_G(H)$ -module. Then $S_{H,V}$ is a simple Mackey functor for G over R . Moreover H is a minimal subgroup for $S_{H,V}$, and $S_{H,V}(H) \cong V$.*
2. *Let S be a simple Mackey functor for G over R . Let H be a minimal subgroup for S , and set $V = S(H)$. Then V is a simple $R\overline{N}_G(H)$ -module, and S is isomorphic to $S_{H,V}$.*
3. *If K is a subgroup of G and W is a simple $R\overline{N}_G(K)$ -module, the simple functors $S_{H,V}$ and $S_{K,W}$ are isomorphic if and only if the pairs (H, V) and (K, W) are conjugate under G .*

4. The Dress construction and projective Mackey functors

4.1. The Dress construction. Let M be a Mackey functor for G over R , and let X be a fixed finite G -set. The functor $\pi_X : Y \mapsto Y \times X$ is an endofunctor of the category $G\text{-set}$, which commutes to disjoint unions, and preserves cartesian squares. If M is a Mackey functor for G over R , it follows that the functor $M \circ \pi_X$ is also a Mackey functor for G over R , which is denoted by M_X , and called the *Dress construction* associated to M and X .

The construction $M \mapsto M_X$ is an endofunctor of the category $\mathbf{Mack}_R(G)$. One can show that this functor is self-adjoint.

4.2. The Burnside functor. Let RB be the Burnside functor over R , defined by composing the Burnside functor B from $G\text{-set}$ to $\mathbb{Z}\text{-Mod}$ with the functor $A \mapsto R \otimes_{\mathbb{Z}} A$ from $\mathbb{Z}\text{-Mod}$ to $R\text{-Mod}$. The main properties of RB are summarized in the following proposition :

Proposition 4.2.1 : 1. *Let M be a Mackey functor for G over R . Then*

$$\mathrm{Hom}_{\mathbf{Mack}_R(G)}(RB, M) \cong M(G) \quad .$$

2. *More generally, if X is a finite G -set, then*

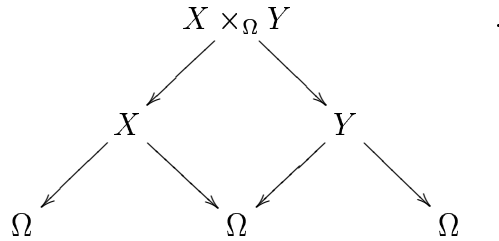
$$\mathrm{Hom}_{\mathbf{Mack}_R(G)}(RB_X, M) \cong M(X) \quad .$$

3. *The functor B_X is a projective Mackey functor, and any Mackey functor is isomorphic to a quotient of a direct sum of such Mackey functors.*

4. *Let $\Omega = \bigsqcup_{H \subseteq G} G/H$. Then RB_Ω is a progenerator in $\mathbf{Mack}_R(G)$, and there is an isomorphism of R -algebras*

$$\mathrm{End}_{\mathbf{Mack}_R(G)}(RB_\Omega) \cong \mu_R(G) \quad .$$

5. *There is an isomorphism $\mathrm{End}_{\mathbf{Mack}_R(G)}(RB_\Omega) \cong RB(\Omega \times \Omega)$, and the corresponding algebra structure on $RB(\Omega \times \Omega)$ is obtained by linearity from the pull-back product $(X, Y) \mapsto X \times_\Omega Y$ in the following diagram*



With this identification $\mu_R(G) \cong RB(\Omega \times \Omega)$, the anti-automorphism σ_G of $\mu_R(G)$ sends the G -set $(X, (f, g))$ over $\Omega \times \Omega$ to $(X, (g, f))$.

References

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