

Green functors

Recall that the letter R denotes a commutative and associative ring with unit, and G denotes a finite group.

1. Two equivalent definitions

There are (at least) two equivalent definitions of Green functors for G over R .

1.1. Using the poset of subgroups of G . This definition of Green functors uses the poset of subgroups of G . It goes back to Green ([7])

Definition 1.1.1 : *A Green functor A for G over R is a Mackey functor for G over R , together with an R -algebra structure on $A(H)$, for each subgroup H of G . The Mackey structure and the algebras structures have to be compatible in the following sense :*

- *If $H \subseteq K$ are subgroups of G , and if $x \in G$, the maps r_H^K and $c_{x,H}$ are maps of R -algebras.*
- *(Frobenius relations) If $H \subseteq K$ are subgroups of G , if $a \in A(H)$ and $b \in A(K)$, then*

$$b(t_H^K a) = t_H^K \left((r_H^K b) a \right) \quad (t_H^K a) b = t_H^K \left(a (r_H^K b) \right) \quad .$$

A morphism of Green functors $f : A \rightarrow B$ is a morphism of Mackey functors such that for each subgroup H of G , the map $f_H : A(H) \rightarrow B(H)$ is a map of R -algebras.

Remark 1.1.2 : The R -algebras considered here are always supposed unital, and the maps of R -algebras must preserve identity elements.

1.2. Using the category of G -sets. The following definition of Green functors ([3] Section 2.2) is analogous to the Dress definition of Mackey functors.

Definition 1.2.1 : A Green functor A for G over R is a Mackey functor for G over R endowed for any G -sets X and Y with bilinear maps

$$A(X) \times A(Y) \rightarrow A(X \times Y)$$

denoted by $(a, b) \mapsto a \times b$ which are bifunctorial, associative, and unitary, in the following sense:

- (Bifunctoriality) If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are morphisms of G -sets, then the squares

$$\begin{array}{ccc} A(X) \times A(Y) & \xrightarrow{\times} & A(X \times Y) \\ A_*(f) \times A_*(g) \downarrow & & \downarrow A_*(f \times g) \\ A(X') \times A(Y') & \xrightarrow{\times} & A(X' \times Y') \end{array}$$

$$\begin{array}{ccc} A(X) \times A(Y) & \xrightarrow{\times} & A(X \times Y) \\ A^*(f) \times A^*(g) \uparrow & & \uparrow A^*(f \times g) \\ A(X') \times A(Y') & \xrightarrow{\times} & A(X' \times Y') \end{array}$$

are commutative.

- (Associativity) If X , Y and Z are G -sets, then the square

$$\begin{array}{ccc} A(X) \times A(Y) \times A(Z) & \xrightarrow{Id_{A(X)} \times (\times)} & A(X) \times A(Y \times Z) \\ (\times) \times Id_{A(Z)} \downarrow & & \downarrow \times \\ A(X \times Y) \times A(Z) & \xrightarrow{\times} & A(X \times Y \times Z) \end{array}$$

is commutative, up to identifications $(X \times Y) \times Z \simeq X \times Y \times Z \simeq X \times (Y \times Z)$.

- (Unitarity) If $\bullet$ denotes the G -set with one element, there exists an element $\varepsilon_A \in A(\bullet)$ such that for any G -set X and for any $a \in A(X)$

$$A_*(p_X)(a \times \varepsilon_A) = a = A_*(q_X)(\varepsilon_A \times a)$$

denoting by p_X (resp. q_X) the (bijective) projection from $X \times \bullet$ (resp. from $\bullet \times X$) to X .

If A and B are Green functors for the group G , a morphism of Green functors $f : A \rightarrow B$ is a morphism of Mackey functors such that for any G -sets X and Y , the square

$$\begin{array}{ccc} A(X) \times A(Y) & \xrightarrow{\times} & A(X \times Y) \\ f_X \times f_Y \downarrow & & \downarrow f_{X \times Y} \\ B(X) \times B(Y) & \xrightarrow[\times]{} & B(X \times Y) \end{array}$$

is commutative. The composition of morphisms of Green functors is the composition of morphisms of Mackey functors. The category of Green functors for G over R is denoted by $\mathbf{Green}_R(G)$.

Remark 1.2.2 : The most concise way to express this definition (Street [9]) uses the monoidal structures on $G\text{-set}$ and $R\text{-Mod}$, given respectively by direct product of G -sets and tensor product of R -modules. More precisely, let $C : G\text{-set} \times G\text{-set} \rightarrow G\text{-set}$ denote the direct product functor, and $T : R\text{-Mod} \times R\text{-Mod} \rightarrow R\text{-Mod}$ denote the tensor product functor. A Green functor for G over R is just a Mackey functor for G over R , viewed as a bivariant functor from $G\text{-set}$ to $RG\text{-Mod}$, which is monoidal, i.e. endowed with a natural transformation $T \circ A \rightarrow A \circ C$ of bivariant functors, which has to be compatible with the various isomorphisms corresponding to the associativity and unit objects of the monoidal structures.

1.3. Equivalence of the definitions.

- [1 $\rightarrow$ 2] If A is a Green functor for the first definition, recall that the corresponding Mackey functor in the sense of Dress is defined for a finite G -set X by

$$A(X) = \left(\bigoplus_{x \in X} A(G_x) \right)^G .$$

If Y is another finite G -set, define a product map $A(X) \times A(Y) \rightarrow A(X \times Y)$ in the following way : if $u = (u_x)_{x \in X} \in A(X)$ and $v = (v_y)_{y \in Y} \in A(Y)$, then the product $u \times v$ is defined by

$$(u \times v)_{x,y} = r_{G_{x,y}}^{G_x}(u_x) r_{G_{x,y}}^{G_y}(v_y) \quad ,$$

for $x \in X$ and $y \in Y$, where $G_{x,y} = G_x \cap G_y$. The identity element ε of A is the identity element of the algebra $A(G) = A(\bullet)$.

- [2 $\rightarrow$ 1] If A is a Green functor in the second sense, then recall that $A(H)$ is defined by $A(H) = A(G/H)$ for a subgroup H of G . Then the product $A(H) \times A(H) \rightarrow A(H)$ is defined by

$$a.b = A^*(\delta_{G/H})(a \times b) \quad ,$$

where $\delta_{G/H}$ is the diagonal inclusion $G/H \rightarrow (G/H) \times (G/H)$. The identity element ε_H of $A(H)$ is equal to $r_H^G \varepsilon$, where ε is the identity element of A .

2. Examples

2.1. Representations rings. The representation groups associated to subgroups of G have generally a natural ring structure, for which they can be viewed as Green functors :

- If k is a field, then the tensor product of RH -modules (over k) induces a ring structure on $R_k(H)$, for $H \subseteq G$. The identity element is the image of the trivial module k . One can check that R_k is a Green functor for G over $\mathbb{Z}$.
- If H is a subgroup of G , then the direct product of H -sets gives a ring structure on the Burnside group $B(H)$. Its identity element is the image of the trivial H -set $\bullet$. This endows the Burnside functor B with a Green functor structure.

Recall that if X is a finite G -set, then $B(X)$ is the Grothendieck group of the category of finite G -sets over X . If Y is another finite G -set, then the product $B(X) \times B(Y) \rightarrow B(X \times Y)$ is induced by the obvious product sending a G -set U over X and a G -set V over Y to their direct product $U \times V$ over $X \times Y$.

More generally, the functor RB is a Green functor for G over R . It is an initial object in the category $\text{Green}_R(G)$.

2.2. Cohomology rings. Let K be a subgroup of G . The cup product in cohomology defines a ring structure on

$$H^\oplus(K, R) = \bigoplus_{l=0}^{\infty} H^l(K, R) \quad ,$$

for which $H^\oplus(-, R)$ becomes a Green functor. The subfunctor $H^0(-, R)$ is also a Green functor, usually denoted by FP_R . More generally, if A is a G -algebra over R , the functor FP_A is a Green functor for G over R .

3. Modules over a Green functor

Green functors can be viewed as generalized R -algebras : a Green functor for the trivial group over R is nothing but an R -algebra. Similarly, there are (at least) two equivalent definitions of the notion of module over a Green functor :

3.1. Using the poset of subgroups of G .

Definition 3.1.1 : *Let A be a Green functor for G over R . A module M over A (or an A -module) is a Mackey functor for G over R , together with a structure of $A(H)$ -module on $M(H)$, for each subgroup H of G . The Mackey structure and the module structures have to be compatible in the following sense :*

- *If $H \subseteq K$ are subgroups of G , and if $x \in G$, then*

$$\forall a \in A(K), \forall m \in M(K), r_H^K(am) = r_H^K(a)r_H^K(m) \quad ,$$

$$\forall a \in A(H), \forall m \in M(H), c_{x,H}(am) = c_{x,H}(a)c_{x,H}(m) \quad .$$

- *(Frobenius relations) If $H \subseteq K$ are subgroups of G , then*

$$\forall a \in A(K), \forall m \in M(H), a(t_H^K m) = t_H^K \left((r_H^K a) m \right) \quad ,$$

$$\forall a \in A(H), \forall m \in M(K), (t_H^K a) m = t_H^K \left(a(r_H^K m) \right) \quad .$$

A morphism of A -modules Green functors $f : M \rightarrow N$ is a morphism of Mackey functors such that for each subgroup H of G , the map $f_H : M(H) \rightarrow N(H)$ is a map of $A(H)$ -modules.

3.2. Using the category of G -sets.

Definition 3.2.1 : *A module M over the Green functor A for G over R is a Mackey functor for G over R , endowed for any G -sets X and Y with bilinear maps*

$$A(X) \times M(Y) \rightarrow M(X \times Y)$$

denoted by $(a, m) \mapsto a \times m$ which are bifunctorial, associative, and unitary, in the following sense:

- (Bifunctoriality) If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are morphisms of G -sets, then the squares

$$\begin{array}{ccc} A(X) \times M(Y) & \xrightarrow{\times} & M(X \times Y) \\ A_*(f) \times M_*(g) \downarrow & & \downarrow M_*(f \times g) \\ A(X') \times M(Y') & \xrightarrow[\times]{} & M(X' \times Y') \end{array}$$

$$\begin{array}{ccc} A(X) \times M(Y) & \xrightarrow{\times} & M(X \times Y) \\ A^*(f) \times M^*(g) \uparrow & & \uparrow M^*(f \times g) \\ A(X') \times M(Y') & \xrightarrow[\times]{} & M(X' \times Y') \end{array}$$

are commutative.

- (Associativity) If X , Y and Z are G -sets, then the square

$$\begin{array}{ccc} A(X) \times A(Y) \times M(Z) & \xrightarrow{Id_{A(X)} \times (\times)} & A(X) \times M(Y \times Z) \\ (\times) \times Id_{A(Z)} \downarrow & & \downarrow \times \\ A(X \times Y) \times M(Z) & \xrightarrow[\times]{} & M(X \times Y \times Z) \end{array}$$

is commutative, up to identifications $(X \times Y) \times Z \simeq X \times Y \times Z \simeq X \times (Y \times Z)$.

- (Unitarity) For any G -set X and for any $m \in M(X)$

$$A_*(q_X)(\varepsilon_A \times m) = m$$

denoting by q_X the (bijective) projection from $\bullet \times X$ to X .

A morphism of A -modules $f : M \rightarrow N$ is a morphism of Mackey functors such that for any G -sets X and Y , the square

$$\begin{array}{ccc} A(X) \times M(Y) & \xrightarrow{\times} & M(X \times Y) \\ Id \times f_Y \downarrow & & \downarrow f_{X \times Y} \\ A(X) \times N(Y) & \xrightarrow[\times]{} & N(X \times Y) \end{array}$$

is commutative. The composition of morphisms of A -modules is the composition of morphisms of Mackey functors. The category of A -modules is denoted by $A\text{-Mod}$.

3.3. Examples.

- Let V be an RG -module. Then the functor FP_V is a module over the Green functor FP_R . More generally, if $l \in \mathbb{N}$, the cohomology functor $H^l(-, V)$ and the homology functor $H_l(-, V)$ are FP_R -modules.

Conversely, it has been observed by Puig that the FP_R -modules can be characterized among the Mackey functor over R as the *cohomological* Mackey functors (recall that it means that $t_H^K r_H^K$ is equal to the multiplication by the index $|K : H|$, for any subgroups $H \subseteq K$ of G).

- Let M and N be Mackey functors for G over R . If X is a finite G -set, define

$$\mathcal{H}(M, N)(X) = \text{Hom}_{\text{Mack}_R(G)}(M, N_X) \quad .$$

This definition can be extended to give a natural Mackey functor structure on $\mathcal{H}(M, N)$. In the case $M = N$, there is similarly a natural Green functor structure on $\mathcal{H}(M, M)$. If A is any Green functor for G over R , it is equivalent to give M a structure of A -module, or to give a morphism of Green functors $A \rightarrow \mathcal{H}(M, M)$ ([3] 2.1.2).

Since the Green functor RB is an initial object in the category $\text{Green}_R(G)$, it follows that the category of Mackey functors for G over R is equivalent to the category of RB -modules.

- The previous example shows that the category $\text{Mack}_R(G)$ admits an *internal Hom construction*. There is also an internal tensor product in $\text{Mack}_R(G)$: if M and N are Mackey functors for G over R , this construction gives another Mackey functor $M \hat{\otimes} N$ for G over R . The value of $M \hat{\otimes} N$ at the finite G -set X can be defined as

$$(M \hat{\otimes} N)(X) = \hat{M} \otimes_{\mu_R(G)} \widehat{N_X} \quad ,$$

where $\hat{M}$ is the $\mu_R(G)$ -module associated to M , viewed as a right $\mu_R(G)$ -module using the anti-automorphism of the Mackey algebra.

This construction is functorial in M and N , and there is an adjunction

$$\text{Hom}_{\text{Mack}_R(G)}(M \hat{\otimes} N, P) \cong \text{Hom}_{\text{Mack}_R(G)}\left(N, \mathcal{H}(M, P)\right) \quad .$$

- If A is a Green functor for G over R , the opposite Green functor A^{op} is the Mackey functor A , equipped with the opposite product $\times^{op}$, defined for finite G -sets X and Y by

$$\forall a \in A(X), \forall b \in A(Y), \quad a \times^{op} b = A_* \left(\begin{smallmatrix} y, x \\ \downarrow \\ x, y \end{smallmatrix} \right) (b \times a) \quad ,$$

where $\begin{pmatrix} y, x \\ \downarrow \\ x, y \end{pmatrix}$ is the map from $Y \times X$ to $X \times Y$ sending (y, x) to (x, y) . The identity element of A^{op} is the identity element of A .

There is an obvious notion of right module over a Green functor, and the category of right modules over A is equivalent to the category of left modules over A^{op} .

If A and B are Green functors for G over R , there is also an obvious notion of (A, B) -bimodule, or A -module- B . Moreover the tensor product $A \hat{\otimes} B^{op}$ has a natural structure of Green functor, with canonical Green functor homomorphisms $A \rightarrow A \hat{\otimes} B^{op}$ and $B^{op} \rightarrow A \hat{\otimes} B^{op}$, and the category of A -modules- B is equivalent to the category of $A \hat{\otimes} B^{op}$ -modules.

- Let A be a Green functor. Then A is an A -module- A . If M is an A -module, and if X is a finite G -set, there is a natural structure of A -module on the Dress construction M_X . This gives an endofunctor of the category $A\text{-Mod}$, which is self adjoint. Moreover, if M is an A -module, then

$$\text{Hom}_{A\text{-Mod}}(A_X, M) \cong M(X) \quad .$$

In particular the modules A_X are projective A -modules, and the module A_Ω is a progenerator of the category $A\text{-Mod}$.

4. Associated categories and algebras

4.1. The category associated to a Green functor. Let A be a Green functor for G over R . Let $\mathcal{C}_A$ denote the following category :

- The objects of $\mathcal{C}_A$ are the finite G -sets.
- If X and Y are finite G -sets, then

$$\text{Hom}_{\mathcal{C}_A}(X, Y) = A(Y \times X) \quad .$$

- If X, Y , and Z are finite G -sets, then the composition of the morphisms $f \in A(Y \times X)$ and $g \in A(Z \times Y)$ in $\mathcal{C}_A$ is the element $g \circ f$ of $A(Z \times X)$ defined by

$$g \circ f = A_* \begin{pmatrix} z, y, x \\ \downarrow \\ z, x \end{pmatrix} A^* \begin{pmatrix} z, y, x \\ \downarrow \\ z, y, y, x \end{pmatrix} (g \times f) \quad ,$$

where $\begin{pmatrix} z, y, x \\ \downarrow \\ z, x \end{pmatrix}$ is the map from $Z \times Y \times X$ to $Z \times X$ sending (z, y, x) to (z, x) , and $\begin{pmatrix} z, y, x \\ \downarrow \\ z, y, y, x \end{pmatrix}$ is the map from $Z \times Y \times X$ to $Z \times Y \times Y \times X$ sending (z, y, x) to (z, y, y, x) .

- The identity morphism of the finite G -set X is the element

$$A_* \left(\begin{smallmatrix} x \\ \downarrow \\ x, x \end{smallmatrix} \right) A^* \left(\begin{smallmatrix} x \\ \downarrow \\ \bullet \end{smallmatrix} \right) (\varepsilon)$$

of $A(X \times X)$, where $\left(\begin{smallmatrix} x \\ \downarrow \\ x, x \end{smallmatrix} \right)$ is the diagonal inclusion from X to $X \times X$, and $\left(\begin{smallmatrix} x \\ \downarrow \\ \bullet \end{smallmatrix} \right)$ is the unique map from X to the trivial G -set $\bullet$.

Proposition 4.1.1 : [[3] 3.3.5] *The category of A -modules is equivalent to the category of R -linear functors from $\mathcal{C}_A$ to $R\text{-Mod}$.*

4.2. The algebra associated to a Green functor. Since A_Ω is a pro-generator of the category $A\text{-Mod}$, it follows that $A\text{-Mod}$ is equivalent to the category of modules over the algebra

$$\mu(A) = \text{End}_{A\text{-Mod}}(A_\Omega) \quad ,$$

which is also isomorphic to the algebra $\text{End}_{\mathcal{C}_A}(\Omega) = A(\Omega \times \Omega)$. The algebra $\mu(A)$ can also be defined by generators and relations using the following result :

Proposition 4.2.1 : *Let A be a Green functor for the group G . Then $\mu(A)$ is isomorphic to the R -algebra defined by the following generators and relations :*

- *The generators are:*
 - The elements t_K^H and r_K^H , for $K \subseteq H \subseteq G$.
 - The elements $c_{x,H}$ for $x \in G$ and $H \subseteq G$.
 - The elements $\lambda_{K,a}$ for $K \subseteq G$ and $a \in A(K)$.
- *The relations are :*
 - The relations of the Mackey algebra for r_K^H , t_K^H , and $c_{x,H}$, i.e.

$$t_K^H t_L^K = t_L^H, \quad r_L^K r_K^H = r_L^H \quad \forall L \subseteq K \subseteq H$$

$$c_{y,xH} c_{x,H} = c_{yx,H} \quad \forall x, y, H$$

$$t_H^H = r_H^H = c_{h,H} \quad \forall h \in H$$

$$c_{x,H} t_K^H = t_{xK}^H c_{x,K}, \quad c_{x,K} r_K^H = r_{xK}^H c_{x,H} \quad \forall x, K, H$$

$$\sum_H t_H^H = \sum_H r_H^H = 1$$

$$r_K^H t_L^H = \sum_{x \in K \setminus H/L} t_{K \cap xL}^K c_{x, K \cap xL} r_{K^x \cap L}^L \quad \forall K \subseteq H \supseteq L$$

the other products of r_H^K , t_H^K and $c_{g,H}$ being zero.

– The additional following relations :

$$\lambda_{K,a} + \lambda_{K,a'} = \lambda_{K,a+a'}, \quad \lambda_{K,a} \lambda_{K,a'} = \lambda_{K,aa'} \quad \forall a, a' \in A(K), \forall K \subseteq G$$

$$\lambda_{K,z\varepsilon_K} = z t_K^K \quad \forall K \subseteq G, \quad z \in R$$

$$r_K^H \lambda_{H,a} = \lambda_{K,r_K^H(a)} r_K^H \quad \forall a \in A(H), \forall K \subseteq H \subseteq G$$

$$\lambda_{H,a} t_K^H = t_K^H \lambda_{K,r_K^H(a)} \quad \forall a \in A(H), \forall K \subseteq H \subseteq G$$

$$t_K^H \lambda_{K,a} r_K^H = \lambda_{H,t_K^H(a)} \quad \forall a \in A(K), \forall K \subseteq H \subseteq G$$

$$\lambda_{xH, c_{x,H}(a)} c_{x,H} = c_{x,H} \lambda_{H,a} \quad \forall x \in G, \forall a \in A(H), \forall H \subseteq G$$

5. Simple modules and simple Green functors

5.1. Simple modules. The classification and description of simple Mackey functors can be generalized to an arbitrary Green functor, in the following form ([3] Chapter 11) :

Notation 5.1.1 : Let A be a Green functor for G over R . If H is a subgroup of G , denote by $\bar{A}(H)$ the Brauer quotient of A in H , defined by

$$\bar{A}(H) = A(H) / \sum_{K \subset H} t_K^H A(K) \quad ,$$

and denote by $a \mapsto \bar{a}$ the projection map from $A(H)$ to $\bar{A}(H)$. This map induces an R -algebra structure on $\bar{A}(H)$, together with a natural action of the group $\bar{N}_G(H)$. Denote by $\hat{A}(H)$ the semi-direct product $\bar{A}(H) \otimes R \bar{N}_G(H)$.

Let V be a simple $\hat{A}(H)$ -module. If X is a finite G -set, then $R(X^H)$ is a $\bar{N}_G(H)$ -module. Set

$$S_{H,V}(X) \simeq Tr_1^{\bar{N}_G(H)} \left(\text{Hom}_R(R(X^H), V) \right)$$

If $f : X \rightarrow Y$ is a morphism of finite G -sets, then for $\alpha \in S_{H,V}(X)$ and $y \in Y^H$, set

$$S_{H,V*}(f)(\alpha)(y) = \sum_{\substack{x \in X^H \\ f(x)=y}} \alpha(x)$$

If $g : Y \rightarrow X$ is a morphism of finite G -sets, then for $\alpha \in S_{H,V}(X)$ and $y \in Y^H$, set

$$S_{H,V}^*(g)(\alpha)(y) = \alpha g(y)$$

Finally if $a \in A(X)$ and $f \in S_{H,V}(Y)$, then define a morphism $a \times f$ from $R((X \times Y)^H) = R(X^H \times Y^H)$ to V by

$$(a \times f)(x, y) = \left(\overline{A^*(m_x)(a)} \otimes 1 \right) \cdot f(y)$$

where m_x is the morphism of G -sets from G/H to X defined by $m_x(uH) = ux$.

One can show that these definitions give an A -module structure on $S_{H,V}$. Moreover :

Proposition 5.1.2 : *Let A be a Green functor for the group G .*

1. *If S is a simple A -module, and H is a minimal subgroup for S , then $V = S(H)$ is a simple $\hat{A}(H)$ -module, and S is isomorphic to $S_{H,V}$.*
2. *Conversely, if H is a subgroup of G , and V is a simple $\hat{A}(H)$ -module, then $S_{H,V}$ is a simple A -module, the group H is minimal for $S_{H,V}$, and moreover $S_{H,V}(H) \simeq V$.*
3. *Let H and K be subgroups of G . If V is a simple $\hat{A}(H)$ -module, and if W is a simple $\hat{A}(K)$ -module, then the modules $S_{H,V}$ and $S_{K,W}$ are isomorphic if and only if the pairs (H, V) and (K, W) are conjugate under G .*

5.2. Simple Green functors. A simple Green functor is a Green functor A for G over R , such that A is simple as A -module- A (or $A \hat{\otimes} A^{op}$ -module). This is equivalent to requiring that A has no non-trivial *functorial two-sided ideal*, in the sense of Thévenaz ([10]). The simple Green functors can be described using the following result (for details, see [3] 11.5) :

Proposition 5.2.1 : [Thévenaz [10] Theorem 12.11]

1. *Let A be a simple Green functor for G . Then there exists a subgroup M of G , a normal subgroup H of M , and a simple algebra S on which M/H acts projectively, such that*

$$A \simeq \text{Ind}_M^G \text{Inf}_{M/H}^M FP_S \quad .$$

The triple (M, H, S) is unique up to conjugation by G (and up to isomorphism of M/H -algebras for S).

2. *Conversely, if $H \trianglelefteq M$ are subgroups of G , if S is a simple algebra on which M/H acts projectively, then $\text{Ind}_M^G \text{Inf}_{M/H}^M FP_S$ is a simple Green functor.*

6. An example of related construction

6.1. Crossed G -monoids.

Definition 6.1.1 : A G -monoid is a monoid endowed with a left G -action by monoid automorphisms. A G -monoid which is a group is called a G -group. A morphism of G -monoids is a G -equivariant monoid homomorphism.

Example 6.1.2 : Denote by G^c the set G on which the group G acts by conjugation. Then the multiplication map $G^c \times G^c \rightarrow G^c$ endows G^c with a structure of G -group. More generally, if N is a normal subgroup of G , then G acts on N by conjugation, and N is a G -group for this action.

Definition 6.1.3 : A crossed G -monoid is a pair (Γ, φ) , where Γ is a G -monoid, and $\varphi : \Gamma \rightarrow G^c$ is a morphism of G -monoids. A morphism $\theta : (\Gamma, \varphi) \rightarrow (\Gamma', \varphi')$ of crossed G -monoids is a morphism of G -monoids $\theta : \Gamma \rightarrow \Gamma'$ such that $\varphi' \circ \theta = \varphi$.

6.2. Associated Green functors.

Proposition 6.2.1 : [[4],[1]] Let (Γ, φ) be a crossed G -monoid. If A is a Green functor for G over R , let A_Γ denote the Mackey functor obtained by the Dress construction from the G -set Γ . If X and Y are finite G -set, define a product map $\times_\Gamma : A_\Gamma(X) \otimes_R A_\Gamma(Y) \rightarrow A_\Gamma(X \times Y)$ by

$$\forall a \in A_\Gamma(X), \forall b \in A_\Gamma(Y), a \otimes b \mapsto a \times_\Gamma b = A_* \left(\begin{array}{c} x, \gamma, y, \gamma' \\ \downarrow \\ x, \varphi(\gamma)y, \gamma\gamma' \end{array} \right) (a \times b) \quad ,$$

where $\left(\begin{array}{c} x, \gamma, y, \gamma' \\ \downarrow \\ x, \varphi(\gamma)y, \gamma\gamma' \end{array} \right)$ is the map from $X \times \Gamma \times Y \times \Gamma$ to $X \times Y \times \Gamma$ sending (x, γ, y, γ') to $(x, \varphi(\gamma)y, \gamma\gamma')$. Let moreover ε_{A_Γ} denote the element $A_* \left(\begin{array}{c} \bullet \\ \downarrow \\ 1_\Gamma \end{array} \right) (\varepsilon_A)$ of $A(\Gamma) \cong A_\Gamma(\bullet)$, where $\left(\begin{array}{c} \bullet \\ \downarrow \\ 1_\Gamma \end{array} \right)$ is the map sending the unique element of $\bullet$ to the identity element of Γ .

Then A_Γ is a Green functor for G over R , and the correspondence $A \mapsto A_\Gamma$ is an endo-functor of the category $\mathbf{Green}_R(G)$.

In particular, the evaluation $A_\Gamma(\bullet) \cong A(\Gamma)$ of A_Γ at the trivial G -set is an R -algebra. One can express the product formula for this algebra in the decomposition $A(\Gamma) \cong \left(\bigoplus_{\gamma \in \Gamma} A(G_\gamma) \right)^G$:

Proposition 6.2.2 : *Let a and b be elements of $A(\Gamma)$. Then for $\gamma \in \Gamma$, the γ component of $a \times_{\Gamma} b$ is given by*

$$(a \times_{\Gamma} b)_{\gamma} = \sum_{\substack{(\alpha, \beta) \in G_{\gamma} \backslash \Gamma \times \Gamma \\ \alpha\beta = \gamma}} t_{G_{\alpha, \beta}}^{G_{\gamma}} (r_{G_{\alpha, \beta}}^{G_{\alpha}} a_{\alpha} \cdot r_{G_{\alpha, \beta}}^{G_{\beta}} b_{\beta}) \quad .$$

6.3. Examples.

- Let B denote the Burnside Green functor, and let $\Gamma = G^c$. Then the ring $B(\Gamma) = B(G^c)$ is called the *crossed Burnside ring*. It is the Grothendieck ring of the monoidal category of *crossed G -sets*, i.e. G -sets over G^c . This ring has been studied by Yoshida (see also [2])
- Let A denote the cohomology Green functor $H^{\oplus}(-, R)$, and let $\Gamma = G^c$. Then one can show that the algebra $A(\Gamma)$ is isomorphic to the Hochschild cohomology algebra of the group algebra RG . In this case, the above product formula has been conjectured by Cibils ([5]) and Cibils and Solotar ([6]), and proved by Siegel and Witherspoon ([8]).
- (Cibils) Let A denote the Grothendieck ring of the category of finitely generated RG -modules, for relations given by direct sum decompositions. If $\Gamma = G^c$, then the ring $A(\Gamma)$ is isomorphic to the Grothendieck ring of Hopf bimodules for the Hopf algebra RG .

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