Prospects in Topology

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IN HONOR OF
WILLIAM BROWDER

edited by

Frank Quinn

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PROSPECTS IN TOPOLOGY

FRANK QUINN, EDITOR

Proceedings of a conference in honor of William Browder
Princeton, March 1994

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Algebraic K-Theory of Local Number Fields: The Unramified Case

M. Bökstedt
I. Madsen

1 Introduction

For any unital ring A the cyclotomic trace of [BHM] gives a map, in the category of spectra,

$$\mathrm{trc}\colon K(A) \to \mathrm{TC}(A)$$

from Quillen's K-theory spectrum to the topological cyclic homology spectrum. The latter spectrum was introduced in [BHM], but we refer the reader to [HM2], for a detailed account of the construction.

If A is a finite dimensional algebra over the Witt vectors $W(k)$ of a perfect field k of characteristic $p \neq 0$, then the cyclotomic trace becomes a homotopy equivalence in non-negative degrees after completion at p, or equivalently,

$$\mathrm{trc}_*\colon \pi_i(K(A);\mathbb{Z}_p) \cong \pi_i(\mathrm{TC}(A);\mathbb{Z}_p), \quad i \geq 0 \tag{1.1}$$

by one of the main results from [HM2], based in part on a relative theorem due to R. McCarthy.

In this paper we evaluate the homotopy type of $\mathrm{TC}(A)$ for certain discrete valuation rings, namely those which have characteristic zero, finite residue fields, and are absolutely unramified (A/pA is the residue field). These rings can be displayed as the Witt-vectors of finite fields with p^s elements, or equivalently as the ring of integers in local number fields which are unramified extensions of $\mathbb{Q}_p$, the field of p-adic numbers, cf. [Ser].

Let K denote the topological periodic K-theory spectrum whose $2n$'th space is $BU \times \mathbb{Z}$, and let $hF(\Psi^g)$ be the homotopy fiber of $\Psi^g - 1$,

$$hF(\Psi^g) \to K \to K,$$

i.e. the "homotopy fixed point spectrum" of the Adams operation Ψ^g, $g \in \mathbb{Z}$. This is a very well understood spectrum. In particular its homotopy groups are given by

$$\pi_i hF(\Psi^g) = \begin{cases} \mathbb{Z}/(g^n - 1) & \text{if } i = 2n-1 \\ \mathbb{Z} & \text{if } i = 0, -1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Given any spectrum T we let $T[n,\infty)$ denote its "$n-1$-connected" cover; there is a map

$$f_n\colon T[n,\infty) \to T$$

with $\pi_i(f_n)$ an isomorphism for $i \geq n$ and $\pi_n T[n,\infty) = 0$ for $i < n$. Quillen proved in [Q1] that $hF(\Psi^g)$ is intimately related to the algebraic K-theory of the finite field $\mathbb{F}_g$ when $g = l^a$ is a prime power:

$$\begin{aligned} K(\mathbb{F}_g)^\wedge_p &\simeq hF(\Psi^g)^\wedge_p[0,\infty), \\ K(\mathbb{F}_g)^\wedge_l &\simeq H\mathbb{Z}_l \quad p \neq l. \end{aligned} \tag{1.3}$$

Here $(-)^\wedge_p$ indicates p-adic completion and $H\mathbb{Z}_l$ denotes the Eilenberg-Maclane spectrum with $\pi_0 H\mathbb{Z}_l = \mathbb{Z}_l$, the l-adic integers, and $\pi_i H\mathbb{Z}_l = 0$ for $i \neq 0$.

Let us now fix a prime p and write $A_s = W(\mathbb{F}_{p^s})$ for the Witt-vectors of the finite field with p^s elements. Our main result can be stated as follows.

Choose an integer g which generates the units $(\mathbb{Z}/p^2)^\times$ in the ring $\mathbb{Z}/p^2$ (or equivalently a topological generator of $\mathbb{Z}_p^\times$).

Theorem. *For an odd prime p, there is a homotopy equivalence of p-adic spectra,*

$$\begin{aligned} K(A_s)^\wedge_p \simeq hF(\Psi^g)^\wedge_p[0,\infty] \vee \Sigma\big(hF(\Psi^g)^\wedge_p[0,\infty)\big) \\ \vee \Sigma(K^\wedge_p[2,\infty)) \vee \bigvee^{s-1} \Sigma^{-1}(K^\wedge_p[2,\infty)), \end{aligned}$$

where $\Sigma(-)$ denotes the suspension of the listed spectrum.

It is customary in topology to write $J_p = hF(\Psi^g)^\wedge_p$ and we shall adopt this notation in the rest of the paper. Since p is fixed throughout, we shall often drop the subscript p on J, $J = J_p$.

For readers which are not so familiar with the notion of spectra, let us rewrite the theorem above on the level of spaces as

$$(BGL(A_s)^+ \times \mathbb{Z})^\wedge_p \simeq J[0,\infty) \times B(J[0,\infty)) \times SU^\wedge_p \times \prod^{s-1} U^\wedge_p \tag{1.4}$$

where $(\ )^+$ is Quillen's plus construction, and $B(-)$ denotes the classifying space, $\Omega BX \simeq X$, and where SU resp. U are the infinite special unitary group, resp. unitary group. In particular, (1.4) implies the following results about the higher K-groups of A_s: for $i \geq 0$,

$$K_i(A_s;\mathbb{Z}_p) \cong \Big(\pi_i(J) \oplus \pi_i(BJ) \oplus \pi_i(SU) \oplus \bigoplus^{s-1} \pi_i(U)\Big) \otimes \mathbb{Z}_p. \tag{1.5}$$

Concretely, $\pi_0(J) = \mathbb{Z}_p$, $\pi_{2n}(J) = 0$ for $n > 0$ and

$$\pi_{2n-1}(J) = \begin{cases} \mathbb{Z}/p^{v_p(n+1)} & \text{if } n \equiv 0 \pmod{p-1} \\ 0 & \text{otherwise} \end{cases}$$

where $v_p(-)$ denotes the p-adic valuation, and

$$\pi_i(U)=\begin{cases}\mathbb{Z} & \text{for } i \text{ odd}\\ 0 & \text{for } i \text{ even}\end{cases}$$

while $\pi_i(SU)=\pi_i(U)$ for $i\neq 0$ and $\pi_1(SU)=0$. Finally, $\pi_i(BJ)=\pi_{i-1}(J)$. The left-hand side of (1.5) are the higher K-groups with p-adic coefficients, i.e.

$$K_i(A_s;\mathbb{Z}_p)=\underset{\leftarrow}{\lim}\,\pi_i\big(BGL(A_s)^+\times\mathbb{Z};\mathbb{Z}/p^n\big).$$

The first two terms of (1.5) are p-torsion groups except for $i=0,1$. Rationally,

$$\dim\big(K_{2i-1}(A_s;\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\big)=s \quad \text{for } i\geq 2. \tag{1.6}$$

This is in agreement with an old result of Wagoner [W], since one knows from [P] that $K_i(A_s;\mathbb{Z}_p)=\underset{\leftarrow}{\lim}K_i(A_s/p^k;\mathbb{Z}_p)$.

Suppose l is a prime different from p. Then

$$K_i(A_s;\mathbb{Z}_l)\simeq K_i(\mathbb{F}_{p^s};\mathbb{Z}_l)=K_i(\mathbb{F}_{p^s})\otimes\mathbb{Z}_l \tag{1.7}$$

by a result of Suslin, cf. [Sus], Theorem 2.8, and these groups we listed in (1.2), (1.3). Hence, in total, one now knows all higher K-groups of A_s with finite or adic coefficients. What is missing is the knowledge of the uniquely divisible part of $K_i(A_s)$; this is an entirely different story.

Let F_s be the quotient field. By localization, [Q2], there is a cofibration of spectra

$$K(A_s)\to K(F_s)\to \Sigma K(\mathbb{F}_{p^s})$$

where the cofiber is known by (1.3). Moreover, the homotopy exact sequence breaks up into short exact sequences

$$0\to K_i(A_s;\mathbb{Z}_p)\to K_i(F_s;\mathbb{Z}_p)\to K_{i-1}(\mathbb{F}_{p^s})\otimes\mathbb{Z}_l\to 0$$

cf. [Sou], so the above also calculates $K_i(F_s;\mathbb{Z}_l)$ for all l.

In the theorem we excluded $p=2$. The expected result for $p=2$, currently evolving, is similar but not identical to the case of odd primes. We refer the reader to sect. 6 below, where we also discuss the relations of our results to étale K-theory, and a possible attack on the ramified case.

In the rest of the paper p will denote an odd prime unless otherwise specified.

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This paper is intimately related to [BM], and consequently there are many cross references to that paper. We have strived, however to make the presentation relatively selfcontained. Finally it is a pleasure to acknowledge valuable conversations with L. Hesselholt; he pointed out a mistake in our initial formulation of the results.

2 Topological cyclic homology

Let A be a unital ring and let $T(A)$ denote the spectrum whose 0'th space is the topological Hochschild homology, $\mathrm{THH}(A)$. It is a connected equivariant S^1-spectrum, and moreover, a cyclotomic spectrum in the notion of [HM2]. In particular, there are maps

$$F,R\colon T(A)^{C_n}\to T(A)^{C_m}$$

of the fixed sets under the cyclic groups of order n and m, provided m divides n. The topological cyclic homology is defined to be

$$\mathrm{TC}(A)=\underset{\overleftarrow{F,R}}{\mathrm{holim}}\,T(A)^{C_n}$$

cf. [HM2], sect. 3. We are in this paper only interested in its p-adic completion, where by [HM2], Theorem 3.1 one can simplify the construction by only using fixed sets under the groups $C_{p^n}\subset S^1$ of order p^n. Let us define

$$\mathrm{TF}(A)=\underset{\overleftarrow{F}}{\mathrm{holim}}\,T(A)^{C_{p^n}},\quad \mathrm{TR}(A)=\underset{\overleftarrow{R}}{\mathrm{holim}}\,T(A)^{C_{p^n}}. \tag{2.1}$$

We note that

$$\pi_0T(A)^{C_{p^n}}=W_{n+1}(A),$$

the Witt vectors of length $n+1$, and that π_0F, π_0R are the Frobenius and restriction maps of Witt vectors by [HM2], Theorem 2.3. One defines

$$\mathrm{TC}(A,p)=\mathrm{TF}(A)^{hR}=\mathrm{TR}(A)^{hF}. \tag{2.2}$$

Here hR and hF indicates homotopy fixed sets, i.e.

$$\mathrm{TF}(A)^{hR}\longrightarrow\mathrm{TF}(A)\xrightarrow{R-1}\mathrm{TF}(A)$$

is a cofibration of spectra. One has

$$\mathrm{TC}(A)^{\wedge}_p\simeq\mathrm{TC}(A,p)^{\wedge}_p. \tag{2.3}$$

Let Γ be a discrete group, and $\Lambda B\Gamma$ the free loop space of its classifying space. Then

$$T(A[\Gamma])\simeq T(A)\wedge\Lambda B\Gamma_+$$

(in the category of spectra), and the maps R and F can be calculated as follows (on the level of prespectra)

$$R_{A[\Gamma]} = R \wedge \Delta_p^{-1}, \quad F_{A[G]} = F \wedge \mathrm{incl} \tag{2.4}$$

as maps from $T(A)^{C_{p^n}} \wedge \Lambda B\Gamma_+^{C_{p^n}}$ to $T(A)^{C_{p^{n-1}}} \wedge \Lambda B\Gamma_+^{C_{p^{n-1}}}$. Here $\Delta_p \colon \Lambda B\Gamma^{C_{p^{n-1}}} \to \Lambda B\Gamma^{C_{p^n}}$ is the homeomorphism which maps a loop $\lambda(z)$ into the loop $\lambda(z^p)$. This follows from [HM2], Theorem 6.1 upon identifying the realization of the cyclic nerve $N^{\mathrm{cy}}(\Gamma)$ with $\Lambda B\Gamma$, cf. [BHM], Proposition 2.6.

Suppose now that Γ is a finite group whose order is prime to p. Then

$$(\Lambda B\Gamma)_p^\wedge \simeq \pi_0 \Lambda B\Gamma = \Gamma/\mathrm{conjugacy} \tag{2.5}$$

and Δ_p becomes $\Delta_p(\gamma) = \gamma^p$, [BHM], lemma 7.11. We decompose $\Gamma/\mathrm{conjugacy}$ into its orbits under Δ_p, say

$$\Gamma/\mathrm{conjugacy} = \coprod \Gamma_+^\alpha.$$

Since the C_{p^n}-action comes from a circle action

$$T(A[\Gamma])^{C_{p^n}} = \bigvee_\alpha T(A)^{C_{p^n}} \wedge \Gamma_+^\alpha,$$

and hence

$$\begin{aligned} \mathrm{TF}(A[\Gamma]) &= \bigvee \mathrm{TF}(A) \wedge \Gamma_+^\alpha \\ \mathrm{TC}(A[\Gamma], p) &= \bigvee (\mathrm{TF}(A) \wedge \Gamma_+^\alpha)^{hR}. \end{aligned}$$

Suppose Γ^α has cardinality $s(\alpha)$. Then

$$(\mathrm{TF}(A) \wedge \Gamma_+^\alpha)^{hR^{s(\alpha)}} = (\mathrm{TF}(A)^{hR^{s(\alpha)}} \wedge \Gamma_+^\alpha).$$

The rest of the R-action amounts to an action of the cyclic group $C_{s(\alpha)}$, which acts freely on Γ^α, by definition. Thus

$$\mathrm{TC}(A, p) = \bigvee (\mathrm{TF}(A)^{hR^{s(\alpha)}} \wedge \Gamma_+^\alpha)^{hC_{s(\alpha)}} = \bigvee \mathrm{TF}(A)^{hR^{s(\alpha)}}. \tag{2.6}$$

We shall use (2.6) for $\Gamma = C_f$ with $(p, f) = 1$ and $A = \mathbb{Z}_p$. In this case the splitting (2.6) is analogous to the decomposition

$$\mathbb{Z}_p[C_f] = \prod_{d|f} \mathbb{Z}_p \otimes \mathbb{Z}[\zeta_d] = \prod_{d|f} \prod^{r(d)} \mathbb{Z}_p[\zeta_d] \tag{2.7}$$

with $r(d)$ the index of the local Galois group $G_d = \{p^i \mid i \in \mathbb{Z}\}$ in the global Galois group $\mathbb{Z}/d^\times$ of $\mathbb{Q}(\zeta_d)/\mathbb{Q}$. Recall from the introduction the notation $A_s =$

$W(\mathbb{F}_{p^s})$, the ring of integers in the unramified extension of $\mathbb{Q}_p$ of degree s. Then $A_s = \mathbb{Z}_p[\zeta_f]$ for suitable f (For given f, s is the minimal number with $p^s \equiv 1 \pmod f$).

Theorem 2.8. *There is a cofibration*

$$\mathrm{TC}(A_s)_p^\wedge \longrightarrow \mathrm{TF}(\mathbb{Z}_p)_p^\wedge \xrightarrow{R^s - 1} \mathrm{TF}(\mathbb{Z}_p)_p^\wedge.$$

Proof. Pick f so that $A_s = \mathbb{Z}_p[\zeta_f]$. We claim that $\mathrm{TF}(\mathbb{Z}_p \otimes \mathbb{Z}[\zeta_f])_p^\wedge$ is homotopy equivalent to the cofiber of

$$\bigvee \mathrm{TF}(\mathbb{Z}_p[C_d])_p^\wedge \to \mathrm{TF}(\mathbb{Z}_p[C_f])_p^\wedge \tag{$*$}$$

where the wedge runs over all divisors d of f with $d < f$. This follows for example from general induction theory: $\pi_*(\mathrm{TF}(\mathbb{Z}_p C_f); \mathbb{Z}_p)$ is a Mackey functor, hence a module over the Burnside ring $A(C_f) \otimes \mathbb{Z}_p$. The localization at the prime ideal $q(C_f) \subset A(C_f) \otimes \mathbb{Z}_p$, given by the C_f-sets X with $\mathrm{card}(X^{C_f}) \equiv 0 \pmod p$, is on the one hand equal to $\pi_*(\mathrm{TF}(\mathbb{Z}_p \otimes \mathbb{Z}[\zeta_f]); \mathbb{Z}_p)$ by [HaM], Proposition 6.17 and on the other hand

$$\begin{aligned} &\pi_*(\mathrm{TF}(\mathbb{Z}_p[C_f]); \mathbb{Z}_p)_{q(C_f)} - p) \\ &\quad = \ker\bigl(\mathrm{Res} \colon \pi_*(\mathrm{TF}(\mathbb{Z}_p C_f); \mathbb{Z}_p) \to \bigoplus \pi_*(\mathrm{TF}(\mathbb{Z}_p C_d); \mathbb{Z}_p)\bigr) \\ &\quad = \mathrm{coker}\bigl(\mathrm{Ind} \colon \bigoplus \pi_*(\mathrm{TF}(\mathbb{Z}_p C_d); \mathbb{Z}_p) \to \pi_*(\mathrm{TF}(\mathbb{Z}_p C_f); \mathbb{Z}_d)\bigr) \end{aligned}$$

The cofiber of $(*)$ is $\mathrm{TF}(\mathbb{Z}_p)_p^\wedge \wedge (\mathbb{Z}/f)^\times$ and thus

$$\mathrm{TC}(\mathbb{Z}_p \otimes \mathbb{Z}[\zeta_f])_p^\wedge = \bigvee^{r(f)} (\mathrm{TF}(\mathbb{Z}_p)^{hR^s})_p^\wedge.$$

Since $\mathbb{Z}_p \otimes \mathbb{Z}[\zeta_f] = \prod^{r(f)} \mathbb{Z}_p[\zeta_f]$ the result follows. □

More generally, let $E \subset F$ be an unramified extension of local fields of degree s, and let $B \subset A$ be the corresponding extension of the integers in E and F. Then

$$\mathrm{TC}(A)_p^\wedge \to \mathrm{TF}(B)_p^\wedge \xrightarrow{R^s - 1} \mathrm{TF}(B)_p^\wedge \tag{2.9}$$

is a cofibration where p is the residue characteristic.

The general scheme for calculation, presented in the next three sections when $A = A_s$ and $B = \mathbb{Z}_p$, is to calculate $\pi_*(\mathrm{TF}(B); \mathbb{Z}_p)$ by comparing fixed sets to homotopy fixed sets. More precisely there is a cofibration diagram (cf. [HM2], Proposition 4.1)

$$\begin{tikzcd} T(B)_{hC_{p^n}} \arrow[r, "N^h"] & T(B)^{hC_{p^n}} \arrow[r, "R^h"] & \hat{\mathbb{H}}(C_{p^n}, T(B)) \\ T(B)_{hC_{p^n}} \arrow[u] \arrow[r, "N"] & T(B)^{C_{p^n}} \arrow[u, "\Gamma_n"'] \arrow[r, "R"] & T(B)^{C_{p^{n-1}}} \arrow[u, "\hat{\Gamma}_n"'] \end{tikzcd} \tag{2.10}$$

for each n. The upper sequence is the norm cofibration, valid for any equivariant spectrum. The lower sequence is the fundamental cofibration of cyclotomic spectra. One may take limits over n and observe that

$$\begin{aligned} \underset{\leftarrow}{\operatorname{holim}}\, T(B)^{hC_{p^n}} &\simeq_p T(B)^{hS^1} \\ \underset{\leftarrow}{\operatorname{holim}}\, T(B)_{hC_{p^n}} &\simeq_p \Sigma T(B)_{hS^1} \\ \underset{\leftarrow}{\operatorname{holim}}\, \hat{\mathbb{H}}(C_{p^n}; T(B)) &\simeq_p \hat{\mathbb{H}}(S^1, T(B)) \end{aligned} \tag{2.11}$$

where the notation is as follows: for any compact Lie group G, and any G-spectrum T

$$T_{hG} = T \wedge_G EG_+, \quad T^{hG} = \operatorname{Map}(EG_+, T),$$
$$\hat{\mathbb{H}}(G, T) = (\widetilde{EG} \wedge \operatorname{Map}(EG_+, T))^G$$

with EG the free contractible G-space, $\widetilde{EG} = \Sigma(EG)$; the smash products are taken in the category of G-spectra. The first equivalence in (2.11) is an easy consequence of the fact that the homotopy direct limit of BC_{p^n} is equivalent to BS^1 after p-adic completion. The second equivalence is proved in [BHM], lemma 5.15 when $T(B)$ has finite p-type, but is true in general. The third equivalence follows from the two first and the norm cofibration displayed in the upper line of (2.12).

The homotopy limit of diagram (2.10) then becomes the p-completion of

$$\begin{tikzcd} \Sigma T(B)_{hS^1} \arrow[r, "N^h"] & T(B)^{hS^1} \arrow[r, "R^h"] & \hat{\mathbb{H}}(S^1, T(B)) \\ \Sigma T(B)_{hS^1} \arrow[u] \arrow[r, "N"] & \mathrm{TF}(B) \arrow[u, "\Gamma"] \arrow[r, "R"] & \mathrm{TF}(B) \arrow[u, "\hat{\Gamma}"] \end{tikzcd} \tag{2.12}$$

Finally one has the following conjecture similar to the affirmed Segal conjecture for the sphere spectrum:

Conjecture 2.13. *If B is the ring of integers in a local field with residue field of characteristic p then the p-completion of Γ and $\hat{\Gamma}$ in (2.12) defines homotopy equivalences onto the connected covers $T(B)^{hS^1}[0,\infty)$ and $\hat{\mathbb{H}}(S^1, T(B))[0,\infty)$.*

By a result of S. Tsalidis, explained in the next section, the conjecture follows from the statement that

$$\hat{\Gamma}_1 \colon T(B) \to \hat{\mathbb{H}}(C_p, T(B))[0,\infty) \tag{2.14}$$

be a p-adic homotopy equivalence; (2.14) in turn was proved in [BM] for $B = \mathbb{Z}_p$, $p > 2$.

3 Spectral sequences

In [BM] the skeleton spectral sequences ([GM], [BM], sect. 2) with abutments $\pi_*(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p)$ and $\pi_*(\hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p)); \mathbb{F}_p)$ were studied for odd primes p. The spectral sequences are of homology type, and

$$E^2_{p,q}(C_{p^n}) = H^{-p}(C_{p^n}; \pi_q(T(\mathbb{Z}_p); \mathbb{F}_p)) \Rightarrow \pi_{q-p}(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p)$$
$$\hat{E}^2_{p,q}(C_{p^n}) = \hat{H}^{-p}(C_{p^n}; \pi_q(T(\mathbb{Z}_p); \mathbb{F}_p)) \Rightarrow \pi_{q-p}(\hat{\mathbb{H}}(C_{p^n}, T(\mathbb{Z}_p)); \mathbb{F}_p)$$

where $\hat{H}$ denotes Tate cohomology. The E^2-terms are

$$E^2_{*,*}(C_{p^n}) = E\{u_n\} \otimes S\{t\} \otimes E\{e\} \otimes S\{f\}$$
$$\hat{E}^2_{*,*}(C_{p^n}) = E\{u_n\} \otimes S\{t, t^{-1}\} \otimes E\{e\} \otimes S\{f\}$$

with $E\{-\}$ and $S\{-\}$ denoting the exterior and symmetric algebras, respectively. The bi-degrees of the listed generators are

$$\deg u_n = (-1, 0), \ \deg t = (-2, 0), \ \deg e = (0, 2p-1), \ \deg f = (0, 2p).$$

Both spectral sequences are spectral sequences of algebras, that is, the differentials d^r are derivations. It was conjectured in [BM], (4.3) that the non-zero differentials are generated from the following statements:

$$\begin{array}{ll} \text{(i)} & d^{2\mathfrak{p}(k+1)}(t^{p^k}) = \lambda_k t^{p^k + \mathfrak{p}(k+1)} e f^{\mathfrak{p}(k)} \quad \text{for } k < n, \text{ with } \lambda_k \in \mathbb{F}_p^\times \\ \text{(ii)} & d^{2\mathfrak{p}(n)+1}(u_n) = \mu_n t^{\mathfrak{p}(n)+1} f^{\mathfrak{p}(n-1)+1} \quad \text{with } \mu_n \in \mathbb{F}_p^\times \\ \text{(iii)} & d^r(tf) = 0, \ d^r(te) = 0, \ d^r(1) = 0 \text{ for all } r \\ & \text{where } \mathfrak{p}(k) = p(p^k - 1)/(p-1) \text{ with } p(0) = 1. \end{array} \tag{3.1}$$

Since the differentials are derivations, (i), (iii) imply that

$$d^{2\mathfrak{p}(k)}(t^{p^k i}) = i\lambda_{k-1} t^{p^k i + \mathfrak{p}(k+1)} e f^{\mathfrak{p}(k)}$$

for all i ($i \in \mathbb{Z}$ for $\hat{E}^{2\mathfrak{p}(k)}_{*,*}(n)$), and it is equivalent to give the differential structure for $E^r_{*,*}(C_{p^n})$ and $\hat{E}^r_{*,*}(C_{p^n})$.

In [BM], sect. 5 we showed

Lemma 3.2 ([BM]). *Conjecture 3.1 is true for $n = 1$, and 3.1 (iii) is true for all n.*

We shall not repeat the argument here. It is based upon a comparison with the corresponding spectral sequences for the sphere spectrum,

$$\begin{aligned} (S^0)^{hS^1} &\to T(\mathbb{Z}_p)^{hS^1} \to T(\mathbb{Z}_p)^{hC_{p^n}} \\ \hat{\mathbb{H}}(S^1, S^0) &\to \hat{\mathbb{H}}(S^1, T(\mathbb{Z}_p)) \to \hat{\mathbb{H}}(C_{p^n}, T(\mathbb{Z}_p)), \end{aligned} \tag{3.3}$$

the known structure in low degrees of the spectral sequence for $(S^0)^{hS^1}$, and upon the fact that S^0 is a so called split spectrum so that

$$(S^0)^{hS^1} \simeq \mathrm{Map}(BS^1_+, S^0).$$

The obvious map from S^0 to $\mathrm{Map}(BS^1_+, S^0)$ composed with the first map in (3.3) has the property that it takes the non-trivial elements

$$v_1 \in \pi_{2p-2}(S^0; \mathbb{F}_p), \quad \alpha_1 = \beta_1(v_1) \in \pi_{2p-3}(S^0; \mathbb{F}_p)$$

into tf and te, respectively.

For $n > 1$, conjecture 3.1 was derived from the assertion that there exists a factorization

$$\begin{tikzcd}
(S^0)^{hS^1} \arrow[rr] \arrow[dr] & & T(\mathbb{Z}_p)^{hS^1} \\
& J[0,\infty)^{hS^1} \arrow[ur] &
\end{tikzcd}$$

which we could not prove. The differentials in 3.1 (i), (ii) correspond to the differentials for $J[0,\infty)^{hS^1}$ upon substituting te, tf for α_1, v_1.

Lemma 3.4 ([BM]). *The map*

$$\hat{\Gamma}_1 \colon T(\mathbb{Z}_p)^\wedge_p \to \hat{\mathbb{H}}(C_p, T(\mathbb{Z}_p))[0,\infty)$$

is a homotopy equivalence.

Given (3.1) it is routine to evaluate the E^∞-terms, and observe that

$$\pi_i(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p) \cong \pi_i(\hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p)); \mathbb{F}_p)$$

for $i \geq 0$. In fact we proved in [BM], that the following are homotopy equivalences

$$T(\mathbb{Z}_p)^{hC_{p^n}}[0,\infty) \underset{\simeq}{\xleftarrow{\Gamma_n}} T(\mathbb{Z}_p)^{C_{p^n}} \underset{\simeq}{\xrightarrow{\hat{\Gamma}_{n+1}}} \hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p))[0,\infty)$$

In his 1994 thesis [T], S. Tsalidis proved by independent methods

Theorem 3.5 (Tsalidis). *If* $\Gamma_1 \colon T(\mathbb{Z}_p)^{C_p} \to T(\mathbb{Z}_p)^{hC_p}[0,\infty)$ *is a homotopy equivalence then so are*

$$\begin{aligned} \Gamma_n &: T(\mathbb{Z}_p)^{C_{p^n}} \to T(\mathbb{Z}_p)^{hC_{p^n}}[0,\infty) \\ \hat{\Gamma}_{n+1} &: T(\mathbb{Z}_p)^{C_{p^n}} \to \hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p))[0,\infty) \end{aligned}$$

for all n.

Tsalidis' proof is modelled upon the corresponding statement for the sphere spectrum—known as (part of) the Segal conjecture. It does not however follow from the existing literature on the Segal conjecture, e.g. [C], [AHJM], [CMP], since $T(\mathbb{Z}_p)$ is not a split C_{p^n}-spectrum, but Tsalidis overcomes this difficulty. The start of the induction is provided by lemma 3.4 upon using diagram (2.10).

In [BM], (4.13) it was remarked (without proof) that the differentials listed in (3.1) above were the first non-zero differentials. More precisely:

Proposition 3.6. *Suppose the differentials (3.1), (i) and (3.1), (ii) are true for* $E^r_{*,*}(C_{p^n})$ *and* $\hat{E}^r_{*,*}(C_{p^n})$, *then the differentials for* $E^r_{*,*}(C_{p^{n+1}})$ *and* $\hat{E}^r_{*,*}(C_{p^{n+1}})$ *are as listed, except that* λ_n *and* μ_{n+1} *might be zero. If* λ_n *is non-zero, so is* μ_{n+1}

Based on this proposition Tsalidis proves that Theorem 3.5 implies (3.1) for all n. We owe a proof of Proposition 3.6; it did not seem that important at the time of writing [BM] as we had no way of proving that λ_n and μ_{n+1} were non-zero. See also remark 3.11 below.

The inductive proof is based upon comparison with the spectral sequences for $T(\mathbb{Z}_p)^{hS^1}$ and $\hat{\mathbb{H}}(S^1, T(\mathbb{Z}_p))$, and the obvious restriction maps

$$F \colon \hat{\mathbb{H}}(S^1, T(\mathbb{Z}_p)) \to \hat{\mathbb{H}}(C_{p^n}, T(\mathbb{Z}_p)).$$

The skeleton spectral sequence for $\pi_*(\hat{\mathbb{H}}(S^1, T(\mathbb{Z}_p)); \mathbb{F}_p)$ has E^2-term

$$\hat{E}^2_{*,*}(S^1) = S\{t, t^{-1}\} \otimes E\{e\} \otimes S\{f\}.$$

It is an upper half plane convergent spectral sequence, cf. [HM1]. We shall also make use of the transfer map

$$V_n \colon \hat{\mathbb{H}}(C_{p^n}, T(\mathbb{Z}_p)) \to \hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p))$$

which works for all equivariant spectra, and in particular for the S^1/C_p-spectrum $\hat{\mathbb{H}}(C_p, T(\mathbb{Z}_p))$, cf. [HM2], Proposition 2.2.

Proof of Proposition 3.6. Suppose inductively that 3.1 (i) and 3.1 (ii) hold for $\hat{E}^r_{*,*}(C_{p^n})$. The restriction map F induces a map of spectral sequences, which on the E^2-term is the obvious injection. The terms te and tf in $\hat{E}^2_{*,*}(S^1)$ are permanent cycles by lemma 3.2. It follows from the injectivity of F on $\hat{E}^2_{*,*} = \hat{E}^{2p}_{*,*}$ that $d^{2p}(t) = t^{p+1}e$ in $E^2_{*,*}(S^1)$, and hence that $d^{2p}(t^{i+j}f^j) = it^{i+p+j}ef^j$ for $i \in \mathbb{Z}$ and $j \geq 0$. Thus

$$\hat{E}^{2p+1}_{*,*}(S^1) = \mathbb{F}_p\{t^{pi+j}f^j \mid i \in \mathbb{Z},\ j \geq 0\} \oplus \mathbb{F}_p\{t^{pi+j}ef^j \mid i \in \mathbb{Z},\ j \geq 0\}$$

where $\mathbb{F}_p\{\cdots\}$ is the $\mathbb{F}_p$ vector space generated by the listed elements. Similarly we calculate $\hat{E}^{2p+1}_{*,*}(C_{p^n})$. Since $d^{2p}(u_n) = 0$ we get

$$\hat{E}^{2p+1}_{*,*}(C_{p^n}) = E\{u_n\} \otimes \hat{E}^{2p+1}_{*,*}(S^1)$$

and F is injective on $\hat{E}^{2p+1}$. By our induction hypothesis, $\hat{E}^{2p+1}_{*,*}(C_{p^n}) = \hat{E}^{2\mathfrak{p}(2)}_{*,*}(C_{p^n})$ when $n > 1$, and

$$d^{2\mathfrak{p}(2)}(t^{pi+j}f^j) = it^{pi+j+\mathfrak{p}(2)}ef^{j+\mathfrak{p}(1)}$$

which makes it possible to calculate $\hat{E}^{2\mathfrak{p}(2)+1}_{*,*}(S^1)$ and to check it injects into $\hat{E}^{2\mathfrak{p}(2)+1}_{*,*}(C_{p^n})$ provided $n \geq 2$. In this fashion we obtain

$$\begin{aligned}\hat{E}^{2\mathfrak{p}(n)+1}_{*,*}(S^1) &= \mathbb{F}_p\{t^{p^n i+j}f^j \mid i \in \mathbb{Z},\ j \geq 0\} \oplus \mathbb{F}_p\{t^{p^n i+j}ef^j \mid i \in \mathbb{Z},\ j \geq 0\} \oplus \\ &\bigoplus_{k=1}^{n-1}\mathbb{F}_p\{t^{p^k i+j}ef^j \mid i \in \mathbb{Z},\ v_p(i) < n-k,\ \mathfrak{p}(k-1) \leq j < \mathfrak{p}(k)\} \qquad (3.7)\end{aligned}$$

where $v_p(i)$ denotes the p-adic valuation and $\hat{E}^{2\mathfrak{p}(n)+1}_{*,*}(C_{p^n}) = \hat{E}^{2\mathfrak{p}(n)+1}_{*,*}(S^1) \otimes E\{u_n\}$, so that F is still an injection on $\hat{E}^{2\mathfrak{p}(n)+1}_{*,*}$.

We next show that $\hat{E}^{2\mathfrak{p}(n)+1}_{*,*}(S^1) = \hat{E}^{2\mathfrak{p}(n+1)}_{*,*}(S^1)$. Let us look in total degree $2p^n - 1$. We have the elements $z(k-1)$, $k = 1, \ldots, n-1$, in the last $n-1$ summands of (3.7),

$$z(k-1) = t^{p^k(1-p^{n-k})+\mathfrak{p}(k-1)}ef^{\mathfrak{p}(k-1)},$$

and in addition the elements

$$z(n-1,l) = t^{p^{n+1}l+\mathfrak{p}(n-1)}ef^{\mathfrak{p}(n-1)+p^n l},\ l \geq 0$$

in the second summand of (3.7). There are no further elements in degree $2p^n - 1$ in $\hat{E}^{2\mathfrak{p}(n)+1}_{*,*}(S^1)$. One knows in advance that all elements of (3.7) involving e are infinite cycles. Indeed we can compare with the spectral sequence converging to p-integral homotopy with E^2-term

$$\hat{H}^{-p}(S^1; \pi_* T(\mathbb{Z}_p)) = S\{t, t^{-1}\} \otimes \pi_* T(\mathbb{Z}_p)$$

and abutment $\pi_*\hat{\mathbb{H}}(S^1, T(\mathbb{Z}_p))$. The homotopy groups of $T(\mathbb{Z}_p)$ are concentrated in odd degrees, and in degree zero,

$$\pi_{2n-1}T(\mathbb{Z}_p) = \mathbb{Z}/n \otimes \mathbb{Z}_p$$

and the reduction map

$$\pi_{2n-1}T(\mathbb{Z}_p) \to \pi_{2n-1}(T(\mathbb{Z}_p); \mathbb{F}_p)$$

is surjective, [B]. Thus in positive fiber degrees the p-integral spectral sequence is concentrated in odd total degrees, and the reduction map from the integral spectral sequence to the mod p spectral sequence is injective on the E^2-level with image consisting of the elements of the form $t^i ef^j$, which must therefore be permanent cycles.

In particular the elements $z(k-1)$, $z(n-1,l)$ are all infinite cycles in the mod p spectral sequence $\hat{E}^r_{*,*}(S^1)$. Since d^r raises fiber degrees by $r-1$ and since the spectral sequence lies in the upper half plane, $z(k-1)$, $k = 1, \ldots, n-1$ and $z(n-1)$ cannot be d^r-boundaries when $r \geq 2\mathfrak{p}(n)+1$. However $z(n-1,1) = t^{p^{n+1}+\mathfrak{p}(n-1)}ef^{p^n+\mathfrak{p}(n-1)}$ could be a boundary, in that

$$d^{2\mathfrak{p}(n+1)}t^{-p^n} = \lambda_n t^{p^{n+1}+\mathfrak{p}(n-1)}ef^{p^n+\mathfrak{p}(n-1)}, \quad \lambda_n \in \mathbb{F}_p.$$

Since $p^n + \mathfrak{p}(n-1) = \mathfrak{p}(n)$ and $p^{n+1} + \mathfrak{p}(n-1) = \mathfrak{p}(n+1) - p^n$ this is equivalent to the statement

$$d^{2\mathfrak{p}(n+1)}t^{p^n} = \lambda_n t^{p^n+\mathfrak{p}(n+1)}ef^{\mathfrak{p}(n)}, \quad \lambda_n \in \mathbb{F}_p \qquad (3.8)$$

in $\hat{E}^{2\mathfrak{p}(n+1)}_{*,*}(S^1)$. Note also that the argument implies that $\hat{E}^{2\mathfrak{p}(n)+1}_{*,*}(S^1) = \hat{E}^{2\mathfrak{p}(n+1)}_{*,*}(S^1)$ as postulated.

Consider now the maps

$$F_n \colon \hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p)) \to \hat{\mathbb{H}}(C_{p^n}, T(\mathbb{Z}_p))$$
$$V_n \colon \hat{\mathbb{H}}(C_{p^n}, T(\mathbb{Z}_p)) \to \hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p))$$

where F_n is the inclusion of fixed sets and V_n is the transfer. On the level of homotopy (and in the spectral sequences) V_n is F_n-multiplicative in the sense that

$$V_n(xF_n(y)) = V_n(x)y \qquad (3.9)$$

for $x \in \hat{E}^r_{*,*}(C_{p^n})$, $y \in \hat{E}^r_{*,*}(C_{p^{n+1}})$, cf. [HM2], (2.2). On $\hat{E}^2_{*,*}(C_{p^n})$, $V_n(u_n) = u_{n+1}$. This is just the usual statement that the transfer or induction from $\hat{H}^1(C_{p^n}; \mathbb{F}_p)$ to $\hat{H}^1(C_{p^{n+1}}; \mathbb{F}_p)$ is an isomorphism. Since the elements $t^i ef^j$ and $t^i f^j$ are in the image of F_n, it follows from (3.9) that the elements

$$u_{n+1}t^i ef^j, u_{n+1}t^i f^j \in \hat{E}^2_{*,*}(C_{p^{n+1}})$$

are in the image of V_n, and inductively that they survive to $\hat{E}^r_{*,*}(C_{p^{n+1}})$ if they survive to $\hat{E}^r_{*,*}(C_{p^n})$. We conclude that

$$\hat{E}^{2\mathfrak{p}(n)+1}_{*,*}(C_{p^{n+1}}) \cong \hat{E}^{2\mathfrak{p}(n)+1}_{*,*}(S^1) \otimes E\{u_{n+1}\} \cong \hat{E}^{2\mathfrak{p}(n)+1}_{*,*}(C_{p^n})$$

with the last isomorphism induced by F_n. In total degree $2p^{n+1} - 2$ and fiber degree larger than $2\mathfrak{p}(n+1)$ there are the elements

$$t^{p^{n+1}l+\mathfrak{p}(n)+1}f^{p^n l+\mathfrak{p}(n)+1}, \quad u_{n+1}et^{p^{n+1}l+\mathfrak{p}(n)}f^{p^n l+\mathfrak{p}(n)}.$$

Using $F \colon \hat{E}^r_{*,*}(S^1) \to \hat{E}^r_{*,*}(C_{p^{n+1}})$ and (3.8) we conclude that $\hat{E}^{2\mathfrak{p}(n)+1}_{*,*}(C_{p^{n+1}}) = \hat{E}^{2\mathfrak{p}(n+1)}_{*,*}(C_{p^{n+1}})$, and that $d^{2\mathfrak{p}(n+1)}$ is determined by

$$\begin{aligned} d^{2\mathfrak{p}(n+1)}(t^{-p^n}) &= \lambda_n t^{\mathfrak{p}(n)}ef^{\mathfrak{p}(n)} \\ d^{2\mathfrak{p}(n+1)}(u_{n+1}t^{-p^{n+1}}) &= \nu_{n+1}u_{n+1}et^{\mathfrak{p}(n)}f^{\mathfrak{p}(n)} \qquad (3.10)\end{aligned}$$

with $\lambda_n, \nu_{n+1} \in \mathbb{F}_p$. This proves the first part of Proposition 3.6. To prove the second part we must show that $\nu_{n+1} = 0$. Here is an argument based on Tsalidis' Theorem 3.5.

With the induction hypothesis we can evaluate the spectral sequence

$$E^r_{*,*}(C_{p^n}) \Rightarrow \pi_*(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p).$$

The result we need (cf. [BM], (4.4)) is that $E^{2\mathfrak{p}(n)+2}_{i,j}(C_{p^n}) = E^{\infty}_{i,j}(C_{p^n})$ for $j - i \geq 0$ is concentrated in the strip $-2\mathfrak{p}(n) \leq p \leq 0$, and that

$$0 \neq (tf)^{\mathfrak{p}(n)} \in E^{2\mathfrak{p}(n)+2}_{*,*}(C_{p^n})$$

but $(tf)^{\mathfrak{p}(n)+1} = 0$. As mentioned above, in the paragraph following lemma 3.2, the unit $S^0 \to T(\mathbb{Z}_p)^{hC_{p^n}}$ maps $v_1 \in \pi_{2p-2}(S^0; \mathbb{F}_p)$ into tf (modulo higher filtration, of course), so $v_1^{\mathfrak{p}(n)} \neq 0$ in $\pi_*(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p)$. More importantly, $v_1^{\mathfrak{p}(n)+1} = 0$. This is true modulo filtration by the vanishing of $(tf)^{\mathfrak{p}(n)+1}$ in $E^{2\mathfrak{p}(n)+2}_{*,*}(C_{p^n})$. Moreover, as $E^{2\mathfrak{p}(n)+2}_{i,j}(C_{p^n}) = 0$ for $i < -2\mathfrak{p}(n)$ when $j - i = 2(p-1)\mathfrak{p}(n) \geq 0$, the vanishing of $v_1^{\mathfrak{p}(n)+1}$ in $\pi_*(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p)$ follows.

By Theorem 3.5 we can conclude that $v_1^{\mathfrak{p}(n)+1} = 0$ in $\pi_*(\hat{H}(C_{p^{n+1}}; T(\mathbb{Z}_p)); \mathbb{F}_p)$. Suppose now in (3.10) that $\nu_{n+1} \neq 0$. Then $\hat{E}^{2\mathfrak{p}(n+1)+1}_{2p^{n+1}-1,0} = 0$ and consequently

$$d^{2\mathfrak{p}(n+1)+1}\colon \hat{E}^{2\mathfrak{p}(n+1)+1}_{2p^{n+1}-1,0} \to \hat{E}^{2\mathfrak{p}(n+1)+1}_{-2\mathfrak{p}(n)-2,2\mathfrak{p}(n+1)}$$

vanishes. The range contains $(tf)^{\mathfrak{p}(n)+1}$ which must then survive to $\hat{E}^{\infty}$ contradicting the vanishing of $v_1^{\mathfrak{p}(n)+1}$. We conclude that $\nu_{n+1} = 0$, so that the first possible differential on $u_{n+1}t^{-p^{n+1}}$ is

$$d^{2\mathfrak{p}(n+1)+1}(u_{n+1}t^{-p^{n+1}}) = \mu_{n+1}(tf)^{\mathfrak{p}(n)+1}.$$

Since $t^{p^{n+1}i}$ is a $d^{2\mathfrak{p}(n+1)+1}$-cycle in $\hat{E}^r_{*,*}(S^1)$ it is a $d^{2\mathfrak{p}(n+1)+1}$-cycle in $\hat{E}^r_{*,*}(C_{p^{n+1}})$ and the $d^{2\mathfrak{p}(n+1)+1}$ differential on $u_n = (u_n t^{-p^{n+1}})t^{p^{n+1}}$ is as claimed. □

Remark 3.11. We must admit that in [BM] we overlooked the possibility that $d^{2\mathfrak{p}(n+1)}(u_{n+1}) \neq 0$ corresponding to $\nu_{n+1} \neq 0$ in the argument above.

Theorem 3.12 (Tsalidis). *The differentials in the spectral sequences*

$$E^r_{*,*}(C_{p^{n+1}}) \Rightarrow \pi_*(T(\mathbb{Z}_p)^{hC_{p^{n+1}}}; \mathbb{F}_p)$$
$$\hat{E}^r_{*,*}(C_{p^{n+1}}) \Rightarrow \pi_*(\hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p)); \mathbb{F}_p)$$

are given by (3.1).

Proof. The case $n = 0$ and 3.1 (iii) is covered in lemma 3.2. By Proposition 3.6 it suffices to establish that $\lambda_n \neq 0$ and $\mu_{n+1} \neq 0$. Tsalidis proceeds by induction. Given the statement for $n - 1$, the argument of Proposition 3.6 or [BM], (4.4) shows that

$$\dim_{\mathbb{F}_p} \pi_{2p^n-1}(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p) = n$$

(generated by $ef^{p^{n-1}-1}, t^p ef^{p^{n-1}}, t^{\mathfrak{p}(2)} ef^{p^{n-1}+\mathfrak{p}(1)}, \ldots, t^{\mathfrak{p}(n-1)} ef^{p^{n-1}+\mathfrak{p}(n-2)}$). By Theorem 3.5, $\dim_{\mathbb{F}_p} \pi_{2p^n-1}(\hat{\mathbb{H}}(C_{p^n}, T(\mathbb{Z}_p)); \mathbb{F}_p) = n$. Given Proposition 3.6,

$$\begin{aligned} \hat{E}^{2\mathfrak{p}(n+1)}_{*,*}(C_{p^{n+1}}) &= \mathbb{F}_p\{u^{\epsilon}_{n+1} t^{p^n i+j} f^j \mid i \in \mathbb{Z},\ j \geq 0,\ \epsilon = 0,1\} \\ &\oplus \mathbb{F}_p\{u^{\epsilon}_{n+1} t^{p^n i+j} ef^j \mid i \in \mathbb{Z},\ j \geq 0,\ \epsilon = 0,1\} \oplus \\ &\bigoplus_{k=1}^{n-1} \mathbb{F}_p\{u^{\epsilon}_{n+1} t^{p^k i+j} ef^j \mid v_p(i) < n-k,\ \mathfrak{p}(k-1) \leq j < \mathfrak{p}(k),\ \epsilon = 0,1\}. \end{aligned}$$

In total degree $2p^n - 1$ we have the classes

$$y(k) = t^{\mathfrak{p}(k)-p^n} ef^{\mathfrak{p}(k-1)}, \quad k = 1, \ldots, n$$

which must all survive to $\hat{E}^{\infty}_{*,*}(C_{p^{n+1}})$, since they are infinite cycles and have fiber degree less that $2\mathfrak{p}(n+1) - 1$. The class

$$y(n+1) = t^{\mathfrak{p}(n+1)-p^n} ef^{\mathfrak{p}(n)} \in \hat{E}^{2\mathfrak{p}(n+1)}_{*,*}$$

has precisely fiber degree $2\mathfrak{p}(n+1) - 1$. It is an infinite cycle (cf. the proof of Proposition 3.6), and

$$d^{2\mathfrak{p}(n+1)}(t^{-p^n}) = \lambda_n y(n+1).$$

If $\lambda_n = 0$ then $y(n+1)$ will survive to $\hat{E}^{\infty}_{*,*}(C_{p^{n+1}})$ and would violate that $\dim_{\mathbb{F}_p} \hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p)) = n$. Tsalidis' argument for $d^{2\mathfrak{p}(n+1)+1}(u_{n+1}) \neq 0$ is a similar counting argument, but in fact our proof of Proposition 3.6 already shows this. □

Corollary 3.13. *The spectral sequences $E^r_{*,*}(S^1)$ and $\hat{E}^r_{*,*}(S^1)$ converging to $\pi_*(T(\mathbb{Z}_p)^{hS^1}; \mathbb{F}_p)$ and $\pi_*(\hat{\mathbb{H}}(S^1, T(\mathbb{Z}_p)); \mathbb{F}_p)$ has E^{∞}-terms*

$$E^{\infty}_{*,*}(S^1) = E\{e\} \otimes S\{tf\} \oplus \prod_{k=0}^{\infty} \mathbb{F}_p\{t^i ef^j \mid \mathfrak{p}(k) \leq i < \mathfrak{p}(k+1),\ v_p(i-j) \geq k\}$$

$$\hat{E}^{\infty}_{*,*}(S^1) = E\{e\} \otimes S\{tf\} \oplus \prod_{k=0}^{\infty} \mathbb{F}_p\{t^{p^{k+1}i+j} ef^j \mid i \in \mathbb{Z},\ \mathfrak{p}(k) \leq j < \mathfrak{p}(k+1)\}.$$

Proof. The restriction homomorphisms induce isomorphisms

$$\pi_*(T(\mathbb{Z}_p)^{hS^1}; \mathbb{F}_p) \xrightarrow{\cong} \varprojlim \pi_*(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p)$$
$$\pi_*(\hat{\mathbb{H}}(S^1, T(\mathbb{Z}_p)); \mathbb{F}_p) \xrightarrow{\cong} \varprojlim \pi_*(\hat{\mathbb{H}}(C_{p^n}, T(\mathbb{Z}_p)); \mathbb{F}_p)$$

and using the above theorem we find that the differentials in the spectral sequences are determined from (3.1), (i) and (3.1), (iii). A routine calculation calculates the E^∞-terms. □

In even degrees we have only the powers of tf, and the unit

$$S^0 \to T(\mathbb{Z}_p)^{hS^1} \overset{R^h}{\to} \hat{\mathbb{H}}(S^1, T(\mathbb{Z}_p))$$

maps the periodicity element $v_1 \in \pi_{2p-2}(S^0; \mathbb{F}_p)$ into tf. In odd degrees $2r-1$, one can conveniently list the generators as follows. Write

$$(r-p)/(p-1) = a_0 + a_1 p + a_2 p^2 + \cdots \tag{3.14}$$

with $0 \le a_0 < p$ and $0 < a_i \le p$ for $i > 0$. We call it the p-expansion of $(r-p)/(p-1)$. Define

$$\begin{aligned} x_{2r-1}(k) &= t^{a_0+\cdots+a_k p^k} ef^{(r-p+a_0+\cdots+a_k p^k)/p} \in E^\infty_{*,*}(S^1) \\ \hat{x}_{2r-1}(k) &= t^{p-r+a_0 p+\cdots+a_k p^{k+1}} ef^{a_0+a_1 p+\cdots+a_k p^k} \in \hat{E}^\infty_{*,*}(S^1). \end{aligned} \tag{3.15}$$

Then in total degree $2r-1$,

$$\begin{aligned} E^\infty \pi_{2r-1}(T(\mathbb{Z}_p)^{hS^1}; \mathbb{F}_p) &= \prod_{k=1}^{\infty} \mathbb{F}_p\{x_{2r-1}(k)\} \\ E^\infty \pi_{2r-1}(\hat{\mathbb{H}}(S^1, T(\mathbb{Z}_p)); \mathbb{F}_p) &= \prod_{k=1}^{\infty} \mathbb{F}_p\{\hat{x}_{2r-1}(k)\}. \end{aligned} \tag{3.16}$$

In (3.14) we let $l = l(r)$ be the minimal number so that $a_k = a_{k+1}$ for $k > l$, i.e.

$$(r-p)/(p-i) = \sum_{k=0}^{l} a_k p^k + a_\infty p^{l+1}/(1-p) \in \mathbb{Z}_p \tag{3.17}$$

with $a_l \ne a_\infty$.

Remark 3.18. The ordinary p-adic expansion

$$r/(p-1) = \sum_{i=0}^{\infty} b_i p^i, \quad 0 \le b_i < p-1$$

is related to the p-expansion of (3.14) by $a_0 = b_0$ and $a_i = b_i + 1$ for $i \ge 1$. If we write $r/(p-1) = x + y/(p-1)$ with x an integer and $y = a_\infty - 1$ we see by performing the subtraction (in base p) $r/(p-1) - y/(p-1)$ that $r > 0$ precisely when $a_l > a_\infty$.

One can use (3.15) to evaluate

$$E^\infty(R^h)\colon E^\infty \pi_{2r-1}(T(\mathbb{Z}_p)^{hS^1}; \mathbb{F}_p) \to E^\infty \pi_{2r-1}(\hat{\mathbb{H}}(S^1, T(\mathbb{Z}_p)); \mathbb{F}_p).$$

Indeed, one just checks incidence relations $x_{2r-1}(i) = \hat{x}_{2r-1}(j)$. The result is (cf. [BM], sect. 7): The classes $x_{2r-1}(i)$ with $E^\infty(R^h)(x_{2r-1}(i)) \ne 0$ are as follows:

$$\begin{array}{rl} \text{(i)} & E^\infty(R^h)(x_{2r-1}(k)) = \hat{x}_{2r-1}(k-1) \quad \text{for } k > l \\ \text{(ii)} & E^\infty(R^h)(x_{2r-1}(l-1)) = \hat{x}_{2r-1}(l-1) \quad \text{if } r \equiv 1 \pmod{p-1},\ r > 1 \\ \text{(iii)} & E^\infty(R^h)(x_{2r-1}(\kappa)) = \hat{x}_{2r-1}(l-1) \quad \text{if } r = p + p^{\kappa+\nu+2} + (p-1)a \\ & \qquad \text{with } \mathfrak{p}(\kappa) \le a < \mathfrak{p}(\kappa+1),\ \nu > 0 \text{ and } \kappa \ge 0. \end{array} \tag{3.19}$$

In exceptional case (ii), $(r-p)/(p-1) = \sum_{n=0}^{l-1} a_k p^k$, $a_l = p$ and $a_\infty = p-1$. In the exceptional case (iii), $r \equiv 2 \pmod{p-1}$, and (3.14) takes the form:

$$\begin{aligned} (r-p)/(p-1) = \quad & \sum_{k=0}^{\kappa} a_k p^k + p \cdot p^{\kappa+1} + (p-1) \cdot p^{\kappa+2} + \cdots \\ & +(p-1)p^{\kappa+\nu+1} + (p-2)p^{\kappa+\nu+2} + \cdots . \end{aligned}$$

So $l = \kappa + \nu + 1$ and $a = \sum_{k=0}^{\kappa} a_k p^k$.

We close this section by listing the E^∞-terms of the spectral sequences converging to $\pi^*(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_n)$ and $\pi_*(\hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p)); \mathbb{F}_p)$ in positive total degrees. The calculations are routine, given Theorem 3.12; in fact most of the calculations were done in the course of proving Proposition 3.6. We only need the result in odd positive degrees. With the notation of (3.15) we have for $r > 0$:

$$\begin{aligned} E^\infty \pi_{2r-1}(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p) &= \begin{cases} \mathbb{F}_p\{x_{2r-1}(0), \ldots, x_{2r-1}(n-1)\}, & n \ge v_p(r) \\ \mathbb{F}_p\{x_{2r-1}(0), \ldots, x_{2r-1}(n)\}, & n < v_p(r) \end{cases} \\ E^\infty \pi_{2r-1}(\hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p)); \mathbb{F}_p) &= \begin{cases} \mathbb{F}_p\{\hat{x}_{2r-1}(0), \ldots, \hat{x}_{2r-1}(n-1)\}, & n \ge v_p(r) \\ \mathbb{F}_p\{\hat{x}_{2r-1}(0), \ldots, \hat{x}_{2r-1}(n)\}, & n < v_p(r). \end{cases} \end{aligned} \tag{3.20}$$

Moreover, the restriction maps

$$\begin{aligned} F\colon E^\infty \pi_{2r-1}(T(\mathbb{Z}_p)^{hS^1}; \mathbb{F}_p) &\to E^\infty \pi_{2r-1}(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p) \\ F\colon E^\infty \pi_{2r-1}(\hat{\mathbb{H}}(S^1, T(\mathbb{Z}_p)); \mathbb{F}_p) &\to E^\infty \pi_{2r-1}(\hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p)); \mathbb{F}_p) \end{aligned} \tag{3.21}$$

are the obvious surjections which annihilates the classes $x_{2r-1}(i)$ and $\hat{x}_{2r-1}(i)$ not present in the target. Indeed, F preserve filtrations, and the classes from (3.15) not present on the right hand side in (3.21) lie in filtration degrees where the target is zero.

4 The map $R\colon \mathrm{TF}(\mathbb{Z}_p) \to \mathrm{TF}(\mathbb{Z}_p)$

In this section we shall use the diagram

$$\begin{tikzcd}
T(\mathbb{Z}_p)^{hS^1} \arrow[r, "R^h"] & \hat{\mathbb{H}}(S^1, T(\mathbb{Z}_p)) \\
\mathrm{TF}(\mathbb{Z}_p) \arrow[u, "\Gamma"] \arrow[r, "R"] & \mathrm{TF}(\mathbb{Z}_p) \arrow[u, "\hat{\Gamma}"']
\end{tikzcd} \tag{4.1}$$

and the calculations of sect. 3 to study the endomorphism induced by R on mod p homotopy. It follows from Theorem 3.5 that Γ and $\hat{\Gamma}$ define homotopy equivalences, onto the connective covers $T(\mathbb{Z}_p)^{hS^1}[0,\infty)$ and $\hat{\mathbb{H}}(S^1, T(\mathbb{Z}_p))[0,\infty)$, respectively. Our main result is the following theorem where $\mathrm{TF}_*(\mathbb{Z}_p; \mathbb{F}_p) = \pi_*(\mathrm{TF}(\mathbb{Z}_p); \mathbb{F}_p)$. With the notation of (3.17) we have

Theorem 4.2. (i) *There are classes $\xi_{2r-1}(i) \in \mathrm{TF}_{2r-1}(\mathbb{Z}_p; \mathbb{F}_p)$ so that $\mathrm{TF}_{2r-1}(\mathbb{Z}_p; \mathbb{F}_p) = \prod_{i=0}^{\infty} \mathbb{F}_p\{\xi_{2r-1}(i)\}$ and such that*

$$\begin{aligned}
&R(\xi_{2r-1}(i)) = \xi_{2r-1}(i-1) \quad \textit{for } i > l, \textit{ with } l = l(r) \textit{ defined in (3.17)}\\
&R(\xi_{2r-1}(l-1)) = \xi_{2r-1}(l-1) \quad \textit{if } r > 1 \textit{ and } r \equiv 1 \pmod{p-1}\\
&R(\xi_{2r-1}(\kappa)) = \xi_{2r-1}(l-1) \quad \textit{if } r = p + p^{\kappa+\nu+1} + (p-1)a\\
&\qquad \textit{with } \mathfrak{p}(\kappa) \le a < \mathfrak{p}(\kappa+1) \textit{ and } \nu > 0\\
&R(\xi_{2r-1}(i)) = 0 \quad \textit{in all other cases.}
\end{aligned}$$

(ii) $\mathrm{TF}_{2r}(\mathbb{Z}_p; \mathbb{F}_p) = \mathbb{F}_p\{v_1^{r/p-1}\}$ *if $r \equiv 0 \pmod{p-1}$ and is zero otherwise, and R is the identity in even degrees.*

The proof of this result will occupy the main part of the section, but first let us derive the structure of the homotopy fiber $hF(R^s)$ in

$$hF(R^s) \xrightarrow{S} \mathrm{TF}(\mathbb{Z}_p) \xrightarrow{R^s-1} \mathrm{TF}(\mathbb{Z}_p).$$

This gives the exact sequences of $\mathbb{F}_p$ vector spaces

$$\begin{gathered}
0 \to \mathrm{TF}_{2r}(\mathbb{Z}_p; \mathbb{F}_p) \xrightarrow{\partial_*} \pi_{2r-1}(hF(R^s); \mathbb{F}_p) \to \ker(R^s-1)_{2r-1} \to 0\\
0 \to \mathrm{cok}(R^s-1)_{2r-1} \xrightarrow{\partial_*} \pi_{2r-2}(hF(R^s); \mathbb{F}_p) \to \mathrm{TF}_{2r-2}(\mathbb{Z}_p; \mathbb{F}_p) \to 0
\end{gathered} \tag{4.3}$$

upon taking into considerations that $R^s = 1$ on $\mathrm{TF}_{2*}(\mathbb{Z}_p; \mathbb{F}_p)$. Since $R(\xi_{2r-1}(l)) = 0$ we have the exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \prod_{i=l}^{\infty} \mathbb{F}_p\{\xi_{2r-1}(i)\} \arrow[r] \arrow[d, "R^s-1"] & \mathrm{TF}_{2r-1}(\mathbb{Z}_p; \mathbb{F}_p) \arrow[r] \arrow[d, "R^s-1"] & \prod_{i=0}^{l-1} \mathbb{F}_p\{\xi_{2r-1}(i)\} \arrow[r] \arrow[d, "R^s-1"] & 0 \\
0 \arrow[r] & \prod_{i=l}^{\infty} \mathbb{F}_p\{\xi_{2r-1}(i)\} \arrow[r] & \mathrm{TF}_{2r-1}(\mathbb{Z}_p; \mathbb{F}_p) \arrow[r] & \prod_{i=0}^{l-1} \mathbb{F}_p\{\xi_{2r-1}(i)\} \arrow[r] & 0
\end{tikzcd}$$

The right-hand vertical arrow has kernel and cokernel $\mathbb{F}_p\{\xi_{2r-1}(l-1)\}$ when $r \equiv 1 \pmod{p-1}$ and $r > 1$. (If $r = 1$ then $l = 0$ and $\prod_{i=0}^{l-1} = 0$), and is otherwise an isomorphism by Theorem 4.2 (i). The left hand vertical map is surjective, and its kernel is an s-dimensional vector space with generators

$$\zeta_{2r-1}^{(k)} = \varprojlim\{\xi_{2r-1}(i) \mid i \equiv k \pmod{s},\ i \ge l\} = \sum_{i \equiv k \pmod{s}} \xi_{2r-1}(i), \quad k = 1, \dots, s. \tag{4.4}$$

Indeed, $R^s\xi_{2r-1}(i) = \xi_{2r-1}(i-s)$ for $i \ge l+s$ and $R^s\xi_{2r-1}(i) = 0$ if $i < l+s$. We have proved

Corollary 4.5. $\pi_i(hF(R^s); \mathbb{F}_p) = 0$ *for $i < -1$ and*

$$\text{(i)} \qquad \pi_{2r-2}(hF(R^s); \mathbb{F}_p) = \begin{cases} \mathbb{F}_p \oplus \mathbb{F}_p & \textit{if } r \equiv 1 \pmod{p-1},\ r > 1\\ \mathbb{F}_p & \textit{if } r = 1\\ 0 & \textit{otherwise} \end{cases}$$

$$\text{(ii)} \qquad \pi_{2r-1}(hF(R^s); \mathbb{F}_p) = \begin{cases} \mathbb{F}_p^{\oplus(s+1)} & \textit{if } r \equiv 1 \pmod{p-1},\ r > 1\\ \mathbb{F}_p^{\oplus s} & \textit{if } r = 1\\ \mathbb{F}_p^{\oplus(s+1)} & \textit{if } r \equiv 0 \pmod{p-1},\ r > 0\\ \mathbb{F}_p^{\oplus s} & \textit{if } r = 0\\ \mathbb{F}_p^{\oplus s} & \textit{otherwise, } r > 0. \end{cases}$$

In short there is an (abstract) isomorphism

$$\pi_*(hF(R^s); \mathbb{F}_p) \cong \pi_{*-1}(J; \mathbb{F}_p) \oplus \pi_*(J; \mathbb{F}_p) \oplus \pi_*(SU \times U^{(s-1)}; \mathbb{F}_p)$$

for $* \ge 0$, where J is the connective image of J space.

We are now ready to begin the proof of Theorem 4.2. Let us first introduce a filtration on the homotopy groups of $T(\mathbb{Z}_p)^{hS^1}$, $\hat{\mathbb{H}}(S^1, T(\mathbb{Z}_p))$ and $\mathrm{TF}(\mathbb{Z}_p)$ by setting

$$\begin{aligned}
W_*^{n+1} &= \ker\big(\pi_*(T(\mathbb{Z}_p)^{hS^1}; \mathbb{F}_p) \to \pi_*(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p)\big)\\
\hat{W}_*^{n+1} &= \ker\big(\pi_*(\hat{\mathbb{H}}(S^1, T(\mathbb{Z}_p)); \mathbb{F}_p) \to \pi_*(\hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p)); \mathbb{F}_p)\big)\\
V_*^{n+1} &= \ker\big(\pi_*(\mathrm{TF}(\mathbb{Z}_p); \mathbb{F}_p) \to \pi_*(T(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p)\big)
\end{aligned} \tag{4.6}$$

where the maps are induced from the restriction maps F which includes S^1 fixed sets into the C_{p^n} or $C_{p^{n+1}}$ fixed set in the first two cases, and the restriction map $\varprojlim T(\mathbb{Z}_p)^{C_{p^n}} \to T(\mathbb{Z}_p)^{C_{p^n}}$ in the last case. By convention, W_*^0, $\hat{W}_*^0$ and V_*^0 are the entire homotopy group in each of the three cases.

Since there are p-adic homotopy equivalences (cf. (2.11)):

$$\begin{gathered}
T(\mathbb{Z}_p)^{hS^1} \to \operatorname{holim}_{\leftarrow} T(\mathbb{Z}_p)^{hC_{p^n}}\\
\hat{\mathbb{H}}(S^1, T(\mathbb{Z}_p)) \to \operatorname{holim}_{\leftarrow} \hat{\mathbb{H}}(C_{p^{n+1}}; T(\mathbb{Z}_p))
\end{gathered}$$

we get from Theorem 3.5 *filtered* isomorphisms

$$\pi_*(\hat{\mathbb{H}}(S^1,T(\mathbb{Z}_p));\mathbb{F}_p) \xleftarrow{\hat{\Gamma}} \pi_*(\mathrm{TF}(\mathbb{Z}_p);\mathbb{F}_p) \xrightarrow{\Gamma} \pi_*(T(\mathbb{Z}_p)^{hS^1};\mathbb{F}_p). \tag{4.7}$$

The isomorphic filtration quotients in odd degrees

$$\hat{W}^k_{2r-1}/\hat{W}^{k+1}_{2r-1} \cong V^k_{2r-1}/V^{k+1}_{2r-1} \cong W^k_{2r-1}/W^{k+1}_{2r-1} \tag{4.8}$$

are easily calculated from (3.12) and (3.21) to be

$$W^k_{2r-1}/W^{k+1}_{2r-1} = \begin{cases} \mathbb{F}_p & \text{if } k \neq v_p(r) \\ 0 & \text{if } k = v_p(r) \end{cases} \tag{4.9}$$

and the $\mathbb{F}_p$ is generated by $x_{2r-1}(k-1)$ if $k > v_p(r)$ and by $x_{2r-1}(k)$ if $k < v_p(r)$.

We point out that the filtrations W^k_{2r-1} and $\hat{W}^k_{2r-1}$ (for $k \neq v_p(r)$) simply amounts to a renumbering of the skeleton filtrations associated to the spectral sequences which converges to $\pi_{2r-1}(T(\mathbb{Z}_p)^{hS^1};\mathbb{F}_p)$ and $\pi_{2r-1}(\hat{\mathbb{H}}(S^1;T(\mathbb{Z}_p));\mathbb{F}_p)$. The precise renumbering however depends on r or more precisely on the p-expansion of $(r-1)/(p-1)$ in (3.14). Let $E^0\mathrm{TF}_{2r-1}(\mathbb{Z}_p;\mathbb{F}_p)$ denote the associated graded vector space in the V-filtration. Then (4.1), (4.8), (4.9) and (3.19) calculates

$$E^0R\colon E^0\mathrm{TF}_{2r-1}(\mathbb{Z}_p;\mathbb{F}_p) \to E^0\mathrm{TF}_{2r-1}(\mathbb{Z}_p;\mathbb{F}_p)$$

to be

$$E^0R(x_{2r-1}(k)) = \begin{cases} x_{2r-1}(k-1) & \text{if } k > l \\ x_{2r-1}(k) & \text{if } k = l-1 \text{ and } r \equiv 1 \pmod{p-1},\ r > 1 \\ x_{2r-1}(l-1) & \text{if } k = \kappa \text{ and } r = p + p^{\kappa+\nu+1} + a(p-1) \\ & \qquad \text{with } \mathfrak{p}(\kappa) \leq a < \mathfrak{p}(\kappa+1),\ \nu > 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

Theorem 4.2 simply asserts that we can lift this formula to $\mathrm{TF}_{2r-1}(\mathbb{Z}_p;\mathbb{F}_p)$ for suitable choices of classes $\xi_{2r-1}(k) \in \mathrm{TF}_{2r-1}(\mathbb{Z}_p;\mathbb{F}_p)$ lifting $x_{2r-1}(k)$.

Proof of Theorem 4.2. There are two main points: one needs to lift classes in the kernel of E^0R to classes in the kernel of R, and when $r \equiv 1 \pmod{p-1}$, $r > 1$ one needs to exhibit $\xi_{2r-1}(l-1)$ which is invariant under R. If we can solve these two points then the rest follows algebraically.

The first problem of lifting elements in the kernel of E^0R to elements in the kernel of R was treated in [BM], Theorem 2.15. It is based on the pleasant fact that the structure of the norm cofibration in homotopy

$$\Sigma T(\mathbb{Z}_p)_{hS^1} \xrightarrow{N^h} T(\mathbb{Z}_p)^{hS^1} \xrightarrow{R^h} \hat{\mathbb{H}}(S^1,T(\mathbb{Z}_p))$$

is intimately related to the structure of the spectral sequence $E^r_{*,*}(S^1)$ which converges to $\hat{\mathbb{H}}(S^1,T(\mathbb{Z}_p))$. Indeed, the usual spectral skeleton spectral sequence for $T(\mathbb{Z}_p)_{hS^1}$ is contained in $\hat{E}^r_{*,*}(S^1)$:

$$E^2_{i,j}(T(\mathbb{Z}_p)_{hS^1}) = \hat{E}^2_{i+2,j}(S^1), \quad i \geq 0.$$

The differentials in $\hat{E}^r_{i,j}(S^1)$ inside the range $i \geq 2$ corresponds to differentials in $E^r_{*,*}(T(\mathbb{Z}_p)_{hS^1})$. The differentials

$$d^r\colon \hat{E}^r_{i,j}(S^1) \to \hat{E}_{i-r,j+r-1}(S^1)$$

with $i \geq 2$ and $i \leq r$ on the other hand detects the norm map

$$N^h\colon \pi_*(\Sigma T(\mathbb{Z}_p)_{hS^1};\mathbb{F}_p) \to \pi_*(\hat{\mathbb{H}}(S^1,T(\mathbb{Z}_p));\mathbb{F}_p).$$

See sect. 2 of [BM] for more details. In our case the classes $x_{2r-1}(k) \in E^\infty_{*,*}(S^1)$ which are annihilated by $E^\infty R^h$ are the classes which are killed by differentials in $\hat{E}^r_{*,*}(S^1)$. Concretely, let us write

$$(r-p)/(p-1) = a + p^s b/(p-1), \quad a = \sum_{i=0}^{k} a_i p^i$$

with $b = 0$ or $v_p(b) = 0$. From remark 3.18 and a minor computation we see that $k \leq l(r)$ and $r > 0$ implies that $b > 0$. It is easy to see that $b = 0$ if and only if $x_{2r-1}(k)$ belongs to case 3.19 (ii), and that $x_{2r-1}(k)$ belongs to case 3.19 (iii) if and only if $b = 1$ and $s \geq k+3$. Thus $E^0R(x_{2r-1}(k)) = 0$ implies that either $b \geq 2$ or $b = 1$ and $s \leq k+2$. In both these cases $a + p^{s-1}b \geq \mathfrak{p}(s-1)$ and hence $(r-p+a)/p \geq \mathfrak{p}(s-1)$. But this means that there exists a class

$$u = t^{a-\mathfrak{p}(s)} f^{(r-p+a)/p-\mathfrak{p}(s-1)} \in E^2(\hat{\mathbb{H}}(S^1,T(\mathbb{Z}_p));\mathbb{F}_p)$$

and using that $v_p(b) = 0$ we have

$$d^{2\mathfrak{p}(s)}u = \lambda x_{2r-1}(k), \quad \lambda \in \mathbb{F}_p^\times.$$

In this situation there is a representative for $x_{2r-1}(k)$ which lies in the image of N^h namely N^h applied to a class in $\pi_*(\Sigma T(\mathbb{Z}_p)^{hS^1};\mathbb{F}_p)$ represented by u. Since $R^h \circ N^h$ is trivial, this ensures a representative $y_{2r-1}(k)$ for $x_{2r-1}(k)$ which is annihilated by R^h, and we can use $\xi_{2r-1}(k) = \Gamma^{-1}(y_{2r-1}(k)) \in \mathrm{TF}_{2r-1}(\mathbb{Z}_p;\mathbb{F}_p)$. This represents $x_{2r-1}(k)$ and lies in the kernel of R.

The second problem to find a representative for $x_{2r-1}(l-1)$ invariant under R was solved in sect. 7, and in the appendix, of [BM]. It is based on the diagram of spectra

$$\begin{tikzcd}
K(\mathbb{Z}_p) \arrow[r, "\mathrm{Trc}"] \arrow[d, "\mathrm{tr}"] & \mathrm{TC}(\mathbb{Z}_p) \arrow[r, "S"] \arrow[d, "S"] & T(\mathbb{Z}_p)^{hS^1} \arrow[d, "\mathrm{id}"] \\
T(\mathbb{Z}_p) & \mathrm{TF}(\mathbb{Z}_p) \arrow[l, "F"'] \arrow[r, "\Gamma"] & T(\mathbb{Z}_p)^{hS^1}
\end{tikzcd}$$

and the fact that in dimension $2p-1$,

$$\mathrm{tr}\colon K_{2p-1}(\mathbb{Z}_p;\mathbb{F}_p) \to \pi_{2p-1}(T(\mathbb{Z}_p);\mathbb{F}_p)$$

is surjective. Let $\mathrm{tr}(e_K) = e$. Since $K_*(\mathbb{Z}_p;\mathbb{F}_p)$ is a $\pi_*(S^0;\mathbb{F}_p)$ module, we have the element $e_K v_1^{(r-1)/(p-1)} \in K_{2r-1}(\mathbb{Z}_p;\mathbb{F}_p)$ and it maps to

$$e(tf)^{(r-p)/(p-1)} \in E^\infty \pi_{2r-1}(T(\mathbb{Z}_p)^{hS^1};\mathbb{F}_p).$$

But this is precisely $x_{2r-1}(l-1)$. Thus we set

$$\xi_{2r-1}(l-1) = S \circ \mathrm{Trc}\big(e_k v_1^{(r-p)/(p-1)}\big).$$

Since $(R - \mathrm{id})S$ is null homotopic, $\xi_{2r-1}(l-1)$ is invariant under R.

Finally, since $E^0R\colon V_{2r-1}^{k+1}/V_{2r-1}^{k} \to V_{2r-1}^{k}/V_{2r-1}^{k-1}$ is an isomorphism when $k \geq l$ we can always find representatives $\xi_{2r-1}(k)$ for $k > l$ with the wanted property, and also a representative $\xi_{2r-1}(\kappa)$ with $R\xi_{2r-1}(\kappa) = \xi_{2r-1}(l-1)$ in the exceptional case $r \equiv 2 \pmod{p-1}$. □

5 The homotopy types

In this section we examine the homotopy type of the fiber $hF(R^s)$ in the cofibration

$$hF(R^s) \longrightarrow \mathrm{TF}(\mathbb{Z}_p) \xrightarrow{R^s-1} \mathrm{TF}(\mathbb{Z}_p).$$

Every spectrum is a module over the sphere spectrum, so $v_1 \in \pi_{2p-2}(S^0;\mathbb{F}_p)$ acts on the mod p homotopy groups, and in particular on $\pi_*(hF(R^s);\mathbb{F}_p)$. We shall examine this action and begin by the action on

$$E^0\pi_*(\mathrm{TF}(\mathbb{Z}_p);\mathbb{F}_p) = \prod_{k=0}^{\infty} V_*^k/V_*^{k+1}$$

where $V_*^0 \supseteq V_*^1 \supseteq V_*^2 \supseteq \cdots$ is the filtration introduced in (4.6). We recall from (4.8) and (4.9) that $V_{2r-1}^k/V_{2r-1}^{k+1}$ is zero for $k = v_p(r)$ and otherwise is a single copy of $\mathbb{F}_p$, generated by $x_{2r-1}(k-1)$ resp. $x_{2r-1}(k)$ when $k > v_p(r)$ resp. $k < v_p(r)$.

Let $(a_0, a_1, \ldots)$ be the coefficients which appears in the p-expansion (3.14) of $(r-p)/(p-1)$.

Lemma 5.1. *In* $E^0\pi_{2r-1}(\mathrm{TF}(\mathbb{Z}_p);\mathbb{F}_p)$,

$$v_1 \cdot x_{2r-1}(k) = \begin{cases} x_{2s-1}(k) & \textit{if } (a_0, a_1, \ldots, a_k) < (p-1, p, \ldots, p) \\ x_{2s-1}(k+1) & \textit{if } (a_0, a_1, \ldots, a_k, a_{k+1}) = (p-1, p, \ldots, p, p) \\ 0 & \textit{otherwise} \end{cases}$$

in the left lexicographic ordering; $s = r + p - 1$.

Proof. We apply Γ and calculate instead in $E^0\pi_*(T(\mathbb{Z}_p)^{hS^1};\mathbb{F}_p) \cong E^\infty_{*,*}(S^1)$ where by (3.15),

$$x_{2r-1}(k) = t^{a(k)} e f^{1/p(r-p+a(k))}, \quad a(k) = a_0 + a_1 p + \cdots + a_k p^k$$

and where $v_1 \cdot x_{2r-1}(k) = tf x_{2r-1}(k)$. This is non-zero if $t^{a(k)+1} e f^{1/p(r-1+a(k)+1)} \neq 0$, i.e. if $a(k)+1$ appears as a partial sum in the p-expansion of $(s-1)/p-1$. This easily gives the claimed result. □

Example 5.2. Consider $r = p + p^{n+2}$. The p-expansion of $(r-p)/(p-1)$ has coefficients $(0, p, (p-1), \ldots, (p-1), (p-2), (p-2), \ldots)$ with n entries $(p-1)$, and $x_{2r-1}(0) = ef^{p^{n+1}}$. It follows that

$$v_1^{\mathfrak{p}(n+2)-1} x_{2r-1}(0) \neq 0 \text{ and } v_1^{\mathfrak{p}(n+2)} x_{2r-1}(0) = 0$$

(in $E^0\pi_*(\mathrm{TF}(\mathbb{Z}_p);\mathbb{F}_p)$). From (the graded version of) Theorem 4.2,

$$R(x_{2r-1}(0)) = x_{2r-1}(l-1) = v_1^{p^{n+1}} e f^{p^n}$$

and hence

$$R(v_1^a x_{2r-1}(0)) = v_1^{p^{n+1}+a} e f^{p^n} \tag{$*$}$$

which is non-zero for $a < \mathfrak{p}(n)$. Now, in the notation of (4.2),

$$v_1^a x_{2r-1}(0) = x_{2t-1}(\kappa), \quad t = p + p^{n+2} + a(p-1),$$

and $(*)$ is precisely the (exceptional) formula $Rx_{2t-1}(\kappa) = x_{2t-1}(l-1)$ in (3.19) (iii).

Proposition 5.3. *For every* $s \geq 1$ *and* $i \geq 0$,

$$v_1\colon \pi_i(hF(R^s);\mathbb{F}_p) \to \pi_{i+2p-2}(hF(R^s);\mathbb{F}_p)$$

is injective. It is an isomorphism for $i \neq 1$.

Proof. This follows from the exact sequences (4.3) and lemma 5.1. Indeed, the action of v_1 in $\mathrm{TF}_{2r}(\mathbb{Z}_p;\mathbb{F}_p)$ is an isomorphism because $\mathrm{TF}_{2*}(\mathbb{Z}_p;\mathbb{F}_p) = S\{v_1\}$, so we must check how v_1 acts on $\ker(R^s-1)_{2r-1}$ and on $\mathrm{cok}(R^s-1)_{2r-1}$. We have already calculated these two modules in the course of proving corollary 4.5, e.g.

$$\ker(R^s-1)_{2r-1} = \begin{cases} \mathbb{F}_p^{\oplus(s+1)} & \text{if } r \equiv 1 \pmod{p-1},\ r > 1 \\ \mathbb{F}_p^{\oplus s} & \text{otherwise.} \end{cases}$$

The generators are $\xi_{2r-1}(l-1) = e_K \cdot v_1^{(r-p)/(p-1)}$ with $e_K \in K_{2p-1}(\mathbb{Z}_p;\mathbb{F}_p)$ and

$$\zeta_{2r-1}^{(k)} = \varprojlim\{\xi_{2r-1}(i) \mid i \equiv k \pmod{s},\ i \geq l\}.$$

We must check that $v_1\zeta_{2r-1}^{(k)} = \zeta_{2r+2p-3}^{(k)}$. This is a consequence of lemma 5.1 as follows: In the p-expansion

$$(r-p)/(p-1) = \sum_{k=0}^{l} a_k p^k + a_\infty p^{l+1} + a_\infty p^{l+2} + \cdots$$

the constant terms $a_\infty \le p-1$ by remark 3.18. Thus

$$(a_0, \dots, a_i) < (p-1, p, \dots, p, p) \text{ when } i \ge l+1$$

and hence that $v_1 \cdot x_{2r-1}(i) = x_{2r+2p-3}(i)$. Since $\xi_{2r-1}(l+1)$ corresponds to $x_{2r-1}(l+1)$ modulo higher filtration, it follows that

$$v_1\xi_{2r-1}(i) = \xi_{2r+2p-3}(i) \in E^0\pi_*(\mathrm{TF}(\mathbb{Z}_p); \mathbb{F})$$

for large i. Thus $v_1\zeta^{(k)}_{2r-1} = \zeta^{(k)}_{2r+2p-3}$ in $E^0\pi_*(\mathrm{TF}(\mathbb{Z}_p); \mathbb{F}_p)$ and v_1 maps the s-dimensional vector space spanned by the $\zeta^{(k)}_{2r-1}$ isomorphically to the s-dimensional vector space spanned by the $\zeta^{(k)}_{2r+2p-3}$. Hence

$$v_1 \colon \ker(R^s - 1)_{2r-1} \to \ker(R^s - 1)_{2r+2p-3}$$

is an isomorphism.

For the cokernel,

$$\mathrm{cok}(R^s - 1)_{2r-1} = \begin{cases} \mathbb{F}_p & \text{if } r \equiv 1 \ (\operatorname{mod} p-1),\ r > 1 \\ 0 & \text{otherwise} \end{cases}$$

the generator of $\mathbb{F}_p$ is $e_K v_1^{r-p/p-1}$ and these generators are visibly stable under multiplication by v_1. □

Before we continue it is in order to remind the reader about L_1-localization (= localization at p-local topological K-theory, p a fixed odd prime), cf. [Bou], [Rav]. The L_1-localization of the sphere spectrum fits into a cofibration

$$L_1(S^0) \to J \to \Sigma^{-1}H\mathbb{Q},$$

where J is the p-local periodic J spectrum as in sect. 1. Moreover for any spectrum E,

$$L_1(E) \simeq E \wedge L_1(S^0).$$

In particular, if E has finite type then

$$L_1(E)^\wedge_p \simeq E \wedge J^\wedge_p.$$

Let E/p be the cofiber of $p \colon E \to E$, so that S^0/p is the Moore spectrum. The element $v_1 \in \pi_{2p-2}(S^0/p) = \pi_{2p-2}(S^0; \mathbb{F}_p)$ factors to define a map $v_1 \colon \Sigma^{2p-2}(S^0/p) \to S^0/p$, [A1] and we let $S^p/p[1/v_1]$ be the homotopy colimit of the desuspended iterates of v_1. It is a fundamental fact of homotopy theory that

$$S^0/p[1/v_1] \simeq L_1(S^0)/p,$$

and hence more generally that $E/p[1/v_1] = L_1(E)/p$, [Mi]. Thus multiplication by v_1 defines an isomorphism on $\pi_*(L_1(E); \mathbb{F}_p)$, and moreover:

Lemma 5.4. *Let E be any spectrum (of finite type) such that $v_1 \colon \pi_i(E; \mathbb{F}_p) \to \pi_{i+2p-2}(E; \mathbb{F}_p)$ is an isomorphism for $i \ge N$. Then the connected covers $E[N,\infty)$ and $L_1(E)[N,\infty)$ agree after completion at p.* □

In particular, this applies to $hF(R^s)$ to give

$$hF(R^s)[2,\infty) \simeq_p L_1(hF(R^s))[2,\infty). \tag{5.5}$$

Let $bu = K[2,\infty)$ be the spectrum with 0'th space BU, and spectrum structure induced from Bott periodicity. Theorem 2.8 and the following proves the theorem of the introduction.

Theorem 5.6. *There is a p-adic homotopy equivalence*

$$hF(R^s)[0,\infty) \simeq_p J[0,\infty) \vee \Sigma J[0,\infty) \vee \Sigma bu \vee \bigvee^{s-1} \Sigma^{-1}bu,$$

in agreement with the abstract isomorphism of homotopy groups from corollary 4.5.

Proof. Let us first comment on the case $s = 1$ which was treated in [BM], Theorem 9.17, where

$$hF(R) \simeq \mathrm{TC}(\mathbb{Z}_p) \simeq K(\mathbb{Z}_p).$$

Denote by

$$f \colon S^0 \to \mathrm{TC}(\mathbb{Z}_p), \quad g \colon \Sigma(S^0) \to \mathrm{TC}(\mathbb{Z}_p)$$

the unit map and the map which represents $1 - p \in \pi_1 K(\mathbb{Z}_p) = \mathbb{Z}_p^\times$. If $S \colon hF(R) \to \mathrm{TF}(\mathbb{Z}_p)$ is the inclusion of the fiber, then keeping the notation of (4.4)

$$(Sf)_*(v_1) = v_1 \in \mathrm{TF}_{2p-2}(\mathbb{Z}_p; \mathbb{F}_p)$$
$$(Sg)_*(\Sigma v_1) = v_1\zeta_1^{(1)} \in \mathrm{TF}_{2p-1}(\mathbb{Z}_p; \mathbb{F}_p).$$

Moreover, $v_1\zeta_1^{(1)} = \zeta^{(1)}_{2p-1}$ since this is true modulo filtration and since the elements lie in the R-invariant part of $\mathrm{TF}_{2p-1}(\mathbb{Z}_p; \mathbb{F}_p)$, generated by e and $\zeta^{(1)}_{2p-1}$.

We claim that the mod p Bockstein operator is non-trivial on $f_*(v_1)$ and $g_*(v_1)$. This is proved in [BM], lemma 9.8. We shall not repeat the proof here, but just mention that it uses comparison with topological cyclic homology in the Waldhausen setting where TC splits off the spectrum $S^0 \vee \Sigma(S^0)$.

In the notation of the previous section, and with $\alpha_1 = \beta_1 v_1$

$$(Sf)_*(\alpha_1) = \zeta^{(1)}_{2p-1}, \quad g_*(\Sigma\alpha_1) = \partial_*(e)$$

cf. (4.3), (4.4). It is a consequence of (5.5) that $f \vee g$ factors as

$$\begin{tikzcd}
S^0 \vee \Sigma S^0 \arrow[r, "f \vee g"] \arrow[d] & hF(R) \arrow[d, "\mathrm{id}"] \\
J[0,\infty) \vee \Sigma J[0,\infty) \arrow[r, "\bar f \vee \bar g"] & hF(R)
\end{tikzcd}$$

Now $\pi_*(J[0,\infty);\mathbb{F}_p) = E\{\alpha_1\} \otimes S\{v_1\}$ and since $v_1^k \cdot \zeta_{2p-1}^{(0)} = \zeta_{2p+2(p-1)k-1} \neq 0$ and $v_1^k e \neq 0$, $\pi_*(\bar{f};\mathbb{F}_p)$ and $\pi_*(\bar{g};\mathbb{F}_p)$ are both injective. (The higher Bockstein structure: $\beta_n(v_1^{p^n}) = v_1^{p^n-1}\alpha_1$ shows that the same is the case on $\mathbb{Z}_p$-integral homotopy groups).

We can use the diagram

$$\begin{tikzcd}
hF(R) \arrow[r] \arrow[d, "i"] & \mathrm{TF}(\mathbb{Z}_p) \arrow[r, "R-1"] \arrow[d, "\mathrm{id}"] & \mathrm{TF}(\mathbb{Z}_p) \arrow[d, "(R^s-1)/(R-1)"] \\
hF(R^s) \arrow[r] & \mathrm{TF}(\mathbb{Z}_p) \arrow[r, "R^s-1"] & \mathrm{TF}(\mathbb{Z}_p)
\end{tikzcd}$$

to conclude that $i\circ(\bar{f}\vee\bar{g})$ is also injective on homotopy groups; indeed $i_*(\zeta_{2r-1}^{(1)}) = \sum_{k=1}^{s}\zeta_{2r-1}^{(k)}$ by the definition of the classes.

We now look in the cofiber

$$J[0,\infty) \vee \Sigma J[0,\infty) \to hF(R^s) \to \overline{hF}(R^s).$$

The mod p homotopy groups of $\overline{hF}(R^s)$ are concentrated in odd degrees, and by corollary 4.5 abstractly given by

$$\pi_*(\overline{hF}(R^s);\mathbb{F}_p) \cong \pi_*(\Sigma bu \vee \bigvee^{s-1} \Sigma^{-1}bu;\mathbb{F}_p).$$

Moreover,

$$v_1\colon \pi_{2r-1}(\overline{hF}(R^s);\mathbb{F}_p) \to \pi_{2r+2p-3}(\overline{hF}(R^s);\mathbb{F}_p)$$

is an isomorphism for $r \neq 1$, and injective for $r = 1$. It follows from [Rog1] that

$$\overline{hF}(R^s) \simeq_p \Sigma bu \vee \bigvee^{s-1} \Sigma^{-1}bu$$

giving the cofibration

$$J[0,\infty) \vee \Sigma J[0,\infty) \to hF(R^s) \to \Sigma bu \vee \bigvee^{s-1} \Sigma^{-1}bu. \qquad (*)$$

By [MST], [Rog1] or [A2], this cofibration is necessarily split:

$$[bu, J[0,\infty)_p^{\wedge}] = [bu, \Sigma^2 J[0,\infty)_p^{\wedge}] = 0,$$

and the homomorphisms

$$[bu, \Sigma J[0,\infty)_p^{\wedge}] \to \mathrm{Hom}(\pi_*(bu_p^{\wedge}), \pi^*(\Sigma J[0,\infty)_p^{\wedge}))$$
$$[bu, \Sigma^3 J[0,\infty)_p^{\wedge}] \to \mathrm{Hom}(\pi_*(bu_p^{\wedge}), \pi_*(\Sigma^3 J[0,\infty)_p^{\wedge}))$$

are injective. Since the exact homotopy sequence of $(*)$ is short exact, $(*)$ must split. □

6 Concluding remarks

One would like of course to extend the results of this paper to cover the discrete valuation rings which are ramified. Let us discuss the most obvious case, namely the ring $\mathbb{Z}_p[\zeta_p]$. There are at least two ways to attack the calculation of $\mathrm{TC}(\mathbb{Z}_p[\zeta_p])$.

One could follow the general scheme explained at the end of sect. 2, and begin by calculating $T(\mathbb{Z}_p[\zeta_p])$, then its homotopy fixed sets and Tate theory for C_{p^n} and S^1, and verify (2.14). This would give $\pi_*(\mathrm{TF}(\mathbb{Z}_p[\zeta_p]);\mathbb{F}_p)$ and a calculation of π_*R would then lead to $\mathrm{TC}_*(\mathbb{Z}_p[\zeta_p])$.

An alternative approach, which we find promising, is to compare $\mathbb{Z}_p[\zeta_p]$ with $\mathbb{Z}_p[C_p]$ and use the general decomposition of $T(\mathbb{Z}_p[G])$. This leads to the following cofibration, cf. [M]:

$$\mathrm{TC}(\mathbb{Z}_p) \wedge BC_{p+} \to \mathrm{TC}(\mathbb{Z}_p[C_p]) \to \bigvee^{p-1} \Sigma(T(\mathbb{Z}_p) \wedge_{S_1} \rho^*(BC_p)_+) \qquad (6.1)$$

where $\rho^*(BC_p)$ denotes BC_p with a free S^1-action (i.e. $\rho^*(ES^1/C_p)$ with $\rho\colon S^1 \to S^1/C_p$ the p'th root isomorphism). One knows the homotopy groups of the left-hand term by our main theorem and standard facts from algebraic topology, and it is not so hard to work out entirely the homotopy groups of the right-hand wedge terms.

The real difficulty is in the comparison of the group ring with its maximal order, i.e. the calculation of the relative term in

$$\mathrm{TC}(\mathbb{Z}_p[C_p] \to \mathcal{M}_p) \to \mathrm{TC}(\mathbb{Z}_p[C_p]) \to \mathrm{TC}(\mathcal{M}_p)$$

where $\mathcal{M}_p = \mathbb{Z}_p \times \mathbb{Z}_p[\zeta_p]$.

Let us compare with the localization theorem in algebraic K-theory which imply the cofibrations

$$K(\mathbb{Z}_p[C_p] \to \mathbb{Q}_p[C_p]) \to K(\mathbb{Z}_p[C_p]) \to K(\mathbb{Q}_p[C_p])$$
$$K(\mathbb{F}_p) \times K(\mathbb{F}_p) \to K(\mathcal{M}_p) \to K(\mathbb{Q}_p[C_p])$$

since the relative K-theory of the integers in a local field is equivalent to the K-theory of the residue field. Moreover, as $K(\mathbb{F}_p)_p^{\wedge} \simeq H\mathbb{Z}_p$ the equality of TC and K for orders give

$$\mathrm{TC}(\mathbb{Z}_p[C_p] \to \mathcal{M}_p)_p^{\wedge} \to K(\mathbb{Z}_p[C_p] \to \mathbb{Q}_p[C_p])_p^{\wedge} \to H\mathbb{Z}_p \vee H\mathbb{Z}_p. \qquad (6.2)$$

It is a standard conjecture in algebraic K-theory that $K(A)_p^{\wedge}$ is essentially equal to the completion of its L_1-localization for integers in (local) fields, i.e. that

$$K(A)_p^{\wedge} \to L_1K(A)_p^{\wedge} \qquad (6.3)$$

is injective on homotopy, and induces a homotopy equivalence of 1-connected covers: $K(A)_p^{\wedge}[2,\infty) \simeq L_1K(A)_p^{\wedge}[2,\infty)$. This statement is sometimes called the (local) Lichtenbaum-Quillen conjecture.

Let us compare with (6.1). The right-hand term in (6.1) has trivial L_1-localization after completion since this is the case for $T(\mathbb{Z}_p)$, and since localization commutes with smash products. The localization of the left-hand term in (6.1) can be determined from our main result to be

$$(J \vee \Sigma J \vee \bigvee^{p} \Sigma K) \wedge BC_{p+}.$$

Using these results one may work out that

$$\mathrm{TC}(\mathbb{Z}_p[C_p])^\wedge_p \to L_1\mathrm{TC}(\mathbb{Z}_p[C_p])^\wedge_p$$

is not injective on homotopy groups, not even in high dimensions. Thus $K(\mathbb{Z}_p[C_p])^\wedge_p$ is not "L_1-local". This however in itself does not contradict (6.3). Indeed one expects $K(\mathbb{Z}_p[C_p] \to \mathcal{M}_p)^\wedge_p$ to be related to $K(\mathbb{F}_p[C_p])^\wedge_p$ which has trivial L_1-localization; it seems a good working hypothesis that

$$L_1(K(\mathbb{Z}_p[C_p]) \to \mathcal{M}_p)^\wedge_p = 0$$

and (6.3) would then be equivalent to the statement that

$$\mathrm{TC}(\mathbb{Z}_p[C_p] \to \mathcal{M}_p)^\wedge_p[2,\infty) \to \mathrm{TC}(\mathbb{Z}_p[C_p])^\wedge_p[2,\infty) \to L_1\mathrm{TC}(\mathbb{Z}_2[C_p])^\wedge_p[2,\infty) \qquad (6.4)$$

be a cofibration. We plan to return to these questions in a future paper.

For any local number field F, that is finite extension F of $\mathbb{Q}_p$, Dwyer and Mitchell has in [DH], Theorem 13.3 evaluated $L_1K(F)^\wedge_p$. With our notions the result is

$$L_1K(F)^\wedge_p \simeq hF(\Psi^q)^\wedge_p \vee \Sigma hF(\Psi^q)^\wedge_p \vee \bigvee^{r} \Sigma K^\wedge_p, \quad r = |F : \mathbb{Q}_p| \qquad (6.5)$$

where q is a topological generator of a certain subgroup $\Gamma_F \subseteq \mathbb{Z}_p^\times$. To describe Γ_F, let $F_0 = F[\zeta_p]$ have degree d_F over F and let a_F be the maximal a such that F_0 contains p^a'th roots of one. Then $\Gamma_F \subseteq \mathbb{Z}_p^\times$ is the subgroup generated by $1 + p^{a_F}\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$ and the d_F'th roots of one. If F is unramified over $\mathbb{Q}_p$ then $d_F = p - 1$ and $a_F = 1$, so (6.5) is in agreement with our theorem since $L_1K(F)^\wedge_p \simeq L_1K(A)^\wedge_p$, where A is the integers in F, and since $L_1K[n,\infty)^\wedge_p \simeq K^\wedge_p$ for all n. Given (6.1) it seems somewhat surprising that the result only depends on the degree and Γ_F, in particular when one, as we do, believes in (6.3). But it is not impossible as far as we can judge at present.

In the proof of (6.5), Dwyer and Mitchell uses that

$$L_1K(F)^\wedge_p[2,\infty) \simeq K^{et}(F)^\wedge_p[2,\infty) \qquad (6.6)$$

where the right-hand side is the étale K-theory of F. By definition, étale K-theory has the homotopy limit property:

$$K^{et}(F)^G \simeq K^{et}(F)^{hG} \qquad (6.7)$$

for any subgroup G of the Galois group of $F/\mathbb{Q}_p$. The same equation then holds on connective covers. Returning to the unramified case one then gets

$$K(A_s)^G[2,\infty]^\wedge_p \simeq K(A_s)^{hG}[2,\infty)^\wedge_p$$

and one wonders what the Galois action might be on the right-hand side of

$$K(A_s) \simeq_p J[0,\infty) \times \Sigma J[0,\infty) \times \Sigma(K[2,\infty)) \times \prod^{r-1} \Sigma^{-1}(K(2,\infty)).$$

The first guess that it be trivial on the J-factors cannot be true since

$$J^{hG} = \mathrm{Map}(BG_+, J)$$

is more than a single copy of J when $(p, |G|) \neq 0$.

Let us finally comment on $p = 2$ which was excluded in our theorem. Rognes has in [Rog2] calculated $\hat{\mathbb{H}}(C_2, T(\mathbb{Z}_2))$ and has verified (2.14), hence conjecture (2.13). It appears that one can use the argument of sect. 3 to evaluate the spectral sequences $\hat{E}^r_{*,*}(C_{2^n})$ and $\hat{E}^r_{*,*}(S^1)$ at 2, and that one obtains an answer very similar to the answer for odd primes. In particular multiplication by the 8-dimensional Adams periodicity operator at 2 is injective on mod 2 homotopy group. In the basic case of $K(\mathbb{Z}_2) = \mathrm{TC}(\mathbb{Z}_2)$ it seems entirely likely that there are cofibrations

$$\begin{aligned} &\Sigma J[0,\infty) \to K(\mathbb{Z}_2)^\wedge_2 \to X \\ &SU^\wedge_2 \to X \to J[0,\infty) \end{aligned} \qquad (6.8)$$

where as above J is the *complex* image of J space, and *not* its real analogue. In contrast to our results at odd primes, we do not think that both cofibrations in (6.8) are split.

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