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# THE EIGHTFOLD WAY TO BP-OPERATIONS

or

$E_*E$  AND ALL THAT

by

J.M. BOARDMAN\*

It is held that diligent adherence to the Eightfold Way will lead to Nirvana, the Great Enlightenment. We do not promise as much; we merely study left/right module/comodule structures on homology/cohomology. Our main purpose is to reinterpret Adams' work [1] on  $E_*E$ . This territory has been well worked over, but this will not prevent us from working it over some more. Throughout, the graded stable homotopy category  $S_{-h*}$  described in [4] is the convenient context. All homology and cohomology groups are taken in the reduced sense, as vanishing on a one-point space. All rings are understood to have an identity element.

In §1 we study the classical case of ordinary cohomology as initiated by Milnor [7]. In §2 we define spectra with coefficients, and in §3 and §4 develop the necessary tools for handling them. Then in §5 and §6 we present our theory of universal cohomology operations.

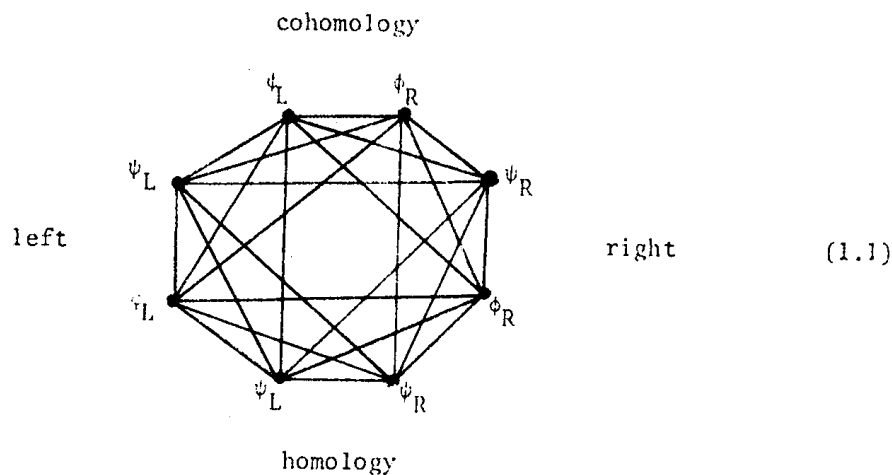
Our claim is that the machinery is more than just a clean way to set up the formal properties of cohomology operations; it is a practical way of computing and managing them. In §7 we apply the theory to the rest of Milnor's paper [7] and in §8 and §9 consider the spectra MU and BP. In §10 we extend the theory to unstable BP-operations. Lest anyone think this is purely a theoretical paper, in §11 we give the computations for desuspending the stunted projective space  $P_{16}^{26}$ , after Wilson [10].

§1. THE CLASSICAL CASE. In this section we study ordinary cohomology  $H^*X$  and homology  $H_*X$  with coefficients in the field  $F_p$  of  $p$  elements, where  $p$  is a fixed prime. By definition, the Steenrod algebra  $A^*$  of cohomology operations acts on  $H^*X$ . Milnor [7] gave  $A^*$  a Hopf algebra structure and

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introduced the dual Hopf algebra  $A_*$ . For this section we shall assume  $X$  finite for simplicity, so that  $H_*X$  and  $H^*X$  have finite total dimension and are strict duals. (In §7 we generalize to arbitrary  $X$ .)

We contemplate the mandala



The vertices represent the steps of our Eightfold Way, which are, in clockwise order from the top,

1. Right action, on cohomology,  $\phi_R: H^*X \otimes A^* \rightarrow H^*X$ ;
2. Right coaction, on cohomology,  $\psi_R: H^*X \rightarrow H^*X \otimes A_*$ ;
3. Right action, on homology,  $\phi_R: H_*X \otimes A^* \rightarrow H_*X$ ;
4. Right coaction, on homology,  $\psi_R: H_*X \rightarrow H_*X \otimes A_*$ ;
5. Left coaction, on homology,  $\psi_L: H_*X \rightarrow A_* \otimes H_*X$ ;
6. Left action, on homology,  $\phi_L: A^* \otimes H_*X \rightarrow H_*X$ ;
7. Left coaction, on cohomology,  $\psi_L: H^*X \rightarrow A_* \otimes H^*X$ ;
8. Left action, on cohomology,  $\phi_L: A^* \otimes H^*X \rightarrow H^*X$ .

The lines in the diagram represent two-headed arrows, which arise from the various functors discussed below.

We first need some notation and conventions. We recall that the tensor product  $V \otimes W$  of graded vector spaces over the field  $F_p$  is associative and has  $F_p$  as unit, by isomorphisms  $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$  and  $F_p \otimes V \cong V \cong V \otimes F_p$  that we shall suppress from our notation. It is also commutative, by the isomorphism  $T: V \otimes W \cong W \otimes V$  defined by  $T(x \otimes y) = (-)^{\deg(x)\deg(y)} y \otimes x$ . The standard sign convention applies throughout, under which one introduces a sign whenever the written order of two symbols of odd degree is reversed. For example, if  $\alpha: V \rightarrow V'$  and  $\beta: W \rightarrow W'$  are linear maps of nonzero degree, their tensor product  $\alpha \otimes \beta: V \otimes W \rightarrow V' \otimes W'$  is defined by

$$(\alpha \otimes \beta)(x \otimes y) = (-)^{\deg(\beta)\deg(x)} \alpha x \otimes \beta y.$$

The dual graded vector space  $V^*$  of  $V$  is equipped with the evaluation map  $e: V^* \otimes V \rightarrow F_p$ , which we write  $e(f \otimes x) = \langle f, x \rangle = fx$ . For each  $x \in V$ , the formula

$$\langle x, f \rangle = (-)^{\deg(x)\deg(f)} \langle f, x \rangle$$

defines a linear map  $V^* \rightarrow F_p$ , that is, an element of  $V^{**}$ . The resulting map  $V \rightarrow V^{**}$  is an isomorphism if  $V$  is finite-dimensional in each degree, and allows us to identify  $V^{**}$  with  $V$ . Similarly, the formula

$$\langle f \otimes g, x \otimes y \rangle = (-)^{\deg(g)\deg(x)} \langle f, x \rangle \langle g, y \rangle \quad (x \in V, y \in W, f \in V^*, g \in W^*)$$

defines a canonical homomorphism  $V^* \otimes W^* \rightarrow (V \otimes W)^*$  which is an isomorphism if  $V$  or  $W$  has finite total dimension, or in certain other cases.

We next introduce the functors that appear in (1.1), to prepare for Theorem 1.2. We usually start either from a left  $A^*$ -module  $V$  with action  $\phi_L: A^* \otimes V \rightarrow V$  or a left  $A_*$ -comodule  $V$  with coaction  $\psi_L: V \rightarrow A_* \otimes V$ , in which case we write  $\psi_L x = \sum_i a_i \otimes x_i$ . The definitions for right modules and comodules are similar, often with a sign adjustment, and are omitted. Throughout,  $V$  and  $W$  will be graded vector spaces of finite total dimension. We take typical elements  $x \in V$ ,  $f \in V^*$  and  $\alpha \in A^*$ .

Conjugation, C. If  $V$  is a left  $A^*$ -module with action  $\phi_L$  we define the right  $A^*$ -module  $CV$  as  $V$  with the action  $x\alpha = (-)^{\deg(x)\deg(\alpha)} (c\alpha)x$ , where we use the canonical antiautomorphism  $c$  of the Hopf algebra  $A^*$ . Equivalently,  $C\phi_L$  is the composite

$$V \otimes A^* \xrightarrow{1 \otimes c} V \otimes A^* \xrightarrow{T} A^* \otimes V \xrightarrow{\phi_L} V.$$

If  $W$  is another left  $A^*$ -module, so is  $V \otimes W$ , and  $T: C(V \otimes W) \xrightarrow{\sim} CW \otimes CV$  is an isomorphism of right  $A^*$ -modules. Similarly for right  $A^*$ -modules and left and right  $A_*$ -comodules. The functors  $C$  give the horizontal lines in (1.1).

Duality, D. This is simply an alternative notation,  $DV = V^*$ . If  $V$  is a left  $A^*$ -module with action  $\phi_L: A^* \otimes V \rightarrow V$ , then  $DV = V^*$  is the left  $A_*$ -comodule with coaction  $D\phi_L = \phi_L^*: V^* \rightarrow A_* \otimes V^*$ . (Note that the dual of  $V \otimes W$  is  $V^* \otimes W^*$  rather than  $W^* \otimes V^*$ , because Milnor and Moore [8, p.222] declared it to be so. The distinction is important precisely when  $V = W$ .) The functors  $D$  give the vertical lines in (1.1).

Partial duals, D' and D''. We do not have to dualize completely. For  $D'$  we just dualize  $A^*$  (or  $A_*$ ). If  $V$  is a left  $A_*$ -comodule, we make  $D'V = V$  a left  $A^*$ -module with action

$$(D'\psi_L)(\alpha \otimes x) = \sum_i \langle c\alpha, a_i \rangle x_i = \sum_i \langle \alpha, ca_i \rangle x_i.$$

The inclusion of  $c$  is necessary to make  $D'$  an action. Equivalently,  $D'\psi_L$  is the composite

$$A^* \otimes V \xrightarrow{c \otimes \psi_L} A^* \otimes A_* \otimes V \xrightarrow{e \otimes 1} V.$$

If  $W$  is another left  $A_*$ -comodule, so is  $V \otimes W$ , and the identity map  $D'(V \otimes W) = D'V \otimes D'W$  becomes an isomorphism of left  $A^*$ -modules. Similarly for right  $A_*$ -comodules.

If instead  $V$  is a left  $A^*$ -module, the coaction  $D'\phi_L: V \rightarrow A_* \otimes V$  that makes  $D'V = V$  a left  $A_*$ -comodule has to be defined indirectly, by

$$(\alpha \otimes 1)(D'\phi_L)x = \phi_L(c \alpha \otimes x) \quad \text{for all } \alpha \in A^*,$$

where  $\alpha \otimes 1: A_* \otimes V \rightarrow F_p \otimes V = V$ . Again, if  $W$  is another left  $A^*$ -module, the identity map  $D'(V \otimes W) = D'V \otimes D'W$  is an isomorphism of left  $A^*$ -comodules.

Alternatively, we can dualize just  $V$ . Given a left  $A^*$ -module  $V$ , we define the left  $A^*$ -module  $D''V$  to be  $V^*$  with action  $D''\phi_L$  defined by

$$\langle (D''\phi_L)(\alpha \otimes f), x \rangle = (-)^{\deg(\alpha)\deg(f)} \langle f, \phi_L(c\alpha \otimes x) \rangle.$$

Given a left  $A_*$ -comodule  $V$ , the left  $A_*$ -comodule  $D''V = V^*$  has coaction  $D''\psi_L: V^* \rightarrow A_* \otimes V^*$  defined indirectly by

$$(1 \otimes x)(D''\psi_L)f = \sum_i (-)^? \langle f, x_i \rangle c a_i,$$

where we write  $1 \otimes x: A_* \otimes V^* \rightarrow A_* \otimes F_p = A_*$  by identifying  $V$  with  $V^{**}$ .

If we dualize both we find  $D'D''V = DV$  in all variations, as expected.

Shuffles,  $S'$  and  $S''$  (for want of a better name). These are very similar to  $D'$  and  $D''$  except that conjugation  $c$  is not required, at the cost of changing sides. If  $V$  is a left  $A^*$ -module we define the right  $A^*$ -module  $S'V = V^*$  as having the action  $S'\phi_L: V^* \otimes A^* \rightarrow V^*$  given by

$$\langle (S'\phi_L)(f \otimes \alpha), x \rangle = \langle f, \phi_L(\alpha \otimes x) \rangle.$$

In simpler notation, this takes the very appealing form  $\langle f\alpha, x \rangle = \langle f, \alpha x \rangle$ ; we simply shuffle the action from  $V$  to  $V^*$ . Given a left comodule  $V$ , the right  $A_*$ -comodule  $S'V = V^*$  has coaction determined by

$$(x \otimes 1)(S'\psi_L)f = \sum_i (-)^? \langle f, x_i \rangle a_i.$$

Also, if  $V$  is a left  $A_*$ -comodule, we define the right  $A^*$ -module  $S'V = V$  as having the action given by

$$(S'\psi_L)(x \otimes \alpha) = \sum_i (-)^? \langle \alpha, a_i \rangle x_i$$

where  $\psi_L x = \sum_i a_i \otimes x_i$ . If  $V$  is a left  $A^*$ -module, the right  $A_*$ -comodule  $S'V = V$  has coaction defined indirectly by

$$(1 \otimes \alpha)(S'\phi_L)x = \phi_L(\alpha \otimes x).$$

**SUMMARY.** We start from the  $A^*$ -module  $H^*X$  and apply the various functors. There are three aspects or variances to watch, namely left/right,

$A^*$ -module/ $A_*$ -comodule, and homology/cohomology. Each may or may not be reversed by the above functors.

- C reverses left/right;
- D reverses module/comodule and homology/cohomology;
- D' reverses module/comodule;
- D'' reverses homology/cohomology;
- S' reverses left/right and module/comodule;
- S'' reverses left/right and homology/cohomology.

Yes, there is a missing functor (besides the identity), the conjugate dual CD. It reverses everything. It corresponds to the missing diagonals in (1.1).

- THEOREM 1.2. (a) The diagram (1.1) commutes;  
 (b) any of the eight structures determines all the others;  
 (c) all functors respect tensor products.

If we start from the usual left action on cohomology,  $A^* \otimes H^*X \rightarrow H^*X$ ,

- (d) the Künneth isomorphisms  $H^*(X \wedge Y) \cong H^*X \otimes H^*Y$  and  $H_*(X \wedge Y) \cong H_*X \otimes H_*Y$  are isomorphisms for all the structures;  
 (e) all four coactions are ring homomorphisms.

PROOF. There is less to this theorem than meets the eye. If we take C, D and S' as the generating functors, all the others can be expressed in terms of them. Specifically, one has to check that  $S'' = CD''$ ,  $D' = CS'$ ,  $D = D'D''$  (already done), and that the squares of C, D and S' are the identity functor. The diagram commutes because C, D and S' commute with each other. Then (c) has to be verified only for C, D and S' (or D'). Because the comultiplication on  $A^*$  is defined to make  $H^*(X \wedge Y) \cong H^*X \otimes H^*Y$  an isomorphism of left  $A^*$ -modules, (d) and (e) follow..

Since all eight structures are equivalent, one may well wonder why bother with them all. Classically,  $H^*X$  is by definition an  $A^*$ -module, and Milnor passes to the right coaction on cohomology by using  $S' = DS''$ . His major contribution is the observation that the multiplicativity of the coaction on cohomology is far more transparent than the original Cartan formula, which here takes the form that the cup product multiplication  $H^*X \otimes H^*X \rightarrow H^*X$  is a homomorphism of  $A^*$ -modules. We also find in §6 that the equivalence tends to break down when we generalize. Nevertheless, the left/right symmetry persists and one could just as well do everything using only the left structures. However, it is a historical fact that this was not done. (Perhaps the absence of conjugation in S' does make it more natural than the partial dual D' ?) As we shall see in §7, §8 and §9, Milnor and virtually everyone since has studied  $A_*$  by means of the right coaction on cohomology.

§2. SPECTRA WITH COEFFICIENTS. In this section we introduce spectra with coefficients, which are the main tool for our theory. Many of the definitions can obviously be generalized enormously; at this time, however, we shall refrain.

Let  $E$  be a commutative ring spectrum, by which we understand a spectrum  $E$  equipped with a commutative and associative multiplication map  $\mu: E \wedge E \rightarrow E$  and a unit map  $i: S \rightarrow E$ , where the necessary diagrams commute in the homotopy category. To abbreviate, we shall write  $\pi$  for  $\pi_* E$ , a commutative graded ring (with the usual sign,  $yx = (-)^{\deg(x)\deg(y)} xy$ ). Given right  $\pi$ -module  $M$ , we wish to extend  $E$  to a spectrum  $M \otimes_{\pi} E$ , which we call  $E$  with coefficients  $M$ . Since all tensor products in this section are taken over  $\pi$ , we generally write them simply as  $\otimes$ .

In fact, we shall be slightly more general. Given a spectrum  $F$ , we define a  $\pi$ -action on  $F$  exactly as in any graded additive category.

DEFINITION 2.1. A  $\pi$ -action on  $F$  consists of a map  $W(k): F \rightarrow F$  for each  $k \in \pi$ , subject to the usual identities  $W(k+k') = W(k) + W(k')$ ,  $W(1) = 1$ , and  $W(kk') = W(k) \circ W(k')$ . (It follows that  $\deg(W(k)) = \deg(k)$ .)

In other words, a homomorphism  $\pi \rightarrow \{F, F\}_*$  of graded rings. Because  $\pi$  is commutative, there is no distinction between left actions and right actions.

EXAMPLES of actions.

1.  $E$ , with action  $W(k) = \mu \circ (k \wedge 1): E = S \wedge E \rightarrow E \wedge E \rightarrow E$ .
2. If  $F$  has a  $\pi$ -action, so does  $U(F)$  for any additive functor  $U$ , namely  $U(W(k))$ ; for instance  $F \wedge Y$  and  $Y \wedge F$ .
3. The cohomology  $E^*X$  and homology  $E_*X$  have  $\pi$ -actions, naturally in  $X$ , by regarding them as functors of  $E$ . A  $\pi$ -action on a graded group is exactly a left  $\pi$ -module structure.
4.  $E$ , with action  $W(k) = \mu \circ (1 \wedge k): E = E \wedge S \rightarrow E \wedge E \rightarrow E$ . This looks like the right counterpart to the first example, but is in fact identical.
5. Any  $E$ -module  $G$ , with action  $W(k) = \phi \circ (k \wedge 1): G = S \wedge G \rightarrow E \wedge G \rightarrow G$ , where  $\phi: E \wedge G \rightarrow G$  is the  $E$ -action on  $G$ . An obvious generalization of the first example.

DEFINITION 2.2. Given a free right  $\pi$ -module  $M$  with basis elements  $m_r$  in degree  $d(r)$  and a spectrum  $F$  with  $\pi$ -action, we define the tensor product spectrum  $M \otimes_{\pi} F = \bigvee_r \Sigma^{d(r)} F$ , equipped with the injections of summands  $i_r: F \rightarrow M \otimes_{\pi} F$  of degree  $d(r)$ . In case  $F = E$ , we call this the spectrum  $E$  with coefficients  $M$ , as suggested by Lemma 2.3, below.

EXAMPLE.  $\pi \otimes_{\pi} F = F$ .

LEMMA 2.3. The homotopy groups of  $M \otimes_{\pi} F$  are given by  
 $\pi_*(M \otimes_{\pi} F) \cong M \otimes_{\pi} \pi_* F$ . In particular,  $\pi_*(M \otimes_{\pi} E) \cong M$ .

PROOF. We have  $\pi_*(M \otimes_{\pi} F) \cong \bigoplus_r \Sigma^{d(r)} \pi_* F \cong M \otimes_{\pi} \pi_* F$ .

To see that this isomorphism and others are canonical, we have to get away from the given basis of  $M$ .

LEMMA 2.4. (a) For each  $m \in M$  there is a map  $m \otimes F: F \rightarrow M \otimes_{\pi} F$ , with the properties (i)  $(m+m') \otimes F = (m \otimes F) + (m' \otimes F)$ , (ii)  $mk \otimes F = (m \otimes F) \circ W(k)$ , and (iii)  $m_r \otimes F = i_r$ .

(b) Suppose given a map  $h(m): F \rightarrow Y$  for each  $m \in M$ , satisfying the axioms (i)  $h(m) + h(m') = h(m+m')$  and (ii)  $h(mk) = h(m) \circ W(k)$ . Then there exists a unique map  $h: M \otimes_{\pi} F \rightarrow Y$  such that  $h(m \otimes F) = h(m)$  for all  $m$ .

PROOF. Write  $m$  in terms of the basis as  $m = \sum_r m_r k_r$ , where the sum is of course finite. In (a) we are forced to take  $m \otimes F = \sum_r i_r \circ W(k_r)$ , and this works. Since  $M \otimes F$  is a graded wedge of copies of  $F$  with injections  $i_r = m_r \otimes F$ , we are forced to define  $h$  for (b) by  $h \circ i_r = h(m_r)$  for all  $r$ , and then the axioms on  $h(m)$  make  $h(m \otimes F) = h(m)$  true in general.

THEOREM 2.5. The spectrum  $M \otimes_{\pi} F$  is functorial in  $M$  as well as in  $F$ , and is independent of the choice of basis of  $M$ . Given a homomorphism  $g: M \rightarrow N$  of free right  $\pi$ -modules, there is a unique map  $g \otimes F: M \otimes_{\pi} F \rightarrow N \otimes_{\pi} F$  such that  $(g \otimes F) \circ (m \otimes F) = gm \otimes F$  for all  $m \in M$ .

PROOF. Lemma 2.4(b) defines the map  $g \otimes F$ . Functoriality and independence of  $M \otimes F$  from the choice of basis follow.

CONVENTION. In forming the tensor product  $M \otimes_{\pi} N$ , we insist on  $M$  being a right  $\pi$ -module and  $N$  a left  $\pi$ -module, just as if  $\pi$  were not commutative. It becomes a left  $\pi$ -module if  $M$  is a  $\pi$ -bimodule, and a right  $\pi$ -module if  $N$  is a bimodule.

In view of the commutativity of  $\pi$ , there is no great algebraic distinction between left  $\pi$ -modules and right  $\pi$ -modules: if  $M$  is a left  $\pi$ -module, we can easily make it a right  $\pi$ -module by defining  $mk = (-)^{\deg(k)\deg(m)} km$ , and vice versa. However, it will be necessary to keep track of a large number of different and unrelated  $\pi$ -module structures, often two or more on the same group. We shall therefore declare or arrange some modules to be left modules and others to be right modules. We shall make much use of  $\pi$ -bimodules, which have a left action and a right action, usually unrelated (although we do insist on  $(km)k' = k(mk')$ ).

Extra structure on  $M$  also passes to  $M \otimes F$ .

LEMMA 2.6. Assume  $M$  is a  $\pi$ -bimodule that is free as a right  $\pi$ -module.

Then

(a) the spectrum  $M \otimes_{\pi} F$  inherits a  $\pi$ -action;

(b) if  $N$  is a free right  $\pi$ -module, we have canonical associativity,

$N \otimes_{\pi} (M \otimes_{\pi} F) \cong (N \otimes_{\pi} M) \otimes_{\pi} F$ . (In the future we shall simply write  $N \otimes_{\pi} M \otimes_{\pi} F$ .)

PROOF. We use the action  $W(k) = L(k) \otimes F$  on  $M \otimes F$ , where  $L(k): M \rightarrow M$  is the left action by  $k$  on the bimodule  $M$ . Let  $M$  have basis elements  $m_r$  in degree  $d(r)$  as right  $\pi$ -module and let  $N$  have basis elements  $n_s$  in degree  $e(s)$ . Then  $N \otimes M$  is also a free right  $\pi$ -module with basis elements  $n_s \otimes m_r$  in degree  $d(r)+e(s)$ . By construction,

$$N \otimes (M \otimes F) = \bigvee_s \sum \epsilon(s) M \otimes F = \bigvee_s \bigvee_r \sum \epsilon(s)+d(r) F$$

and

$$(N \otimes M) \otimes F = \bigvee_{r,s} \sum d(r)+e(s) F$$

are visibly isomorphic. A double application of Lemma 2.4(b) defines the canonical isomorphism  $\alpha: N \otimes (M \otimes F) \xrightarrow{\sim} (N \otimes M) \otimes F$  by

$$\alpha(n \otimes (m \otimes F)) = (n \otimes m) \otimes F: F \rightarrow (N \otimes M) \otimes F$$

for all  $m \in M$  and  $n \in N$ , and  $\alpha$  is in fact an isomorphism of decompositions.

By a right  $\pi$ -algebra  $M$  we shall understand a ring  $M$  with ring homomorphism  $\eta_R: \pi \rightarrow M$  that makes  $M$  an algebra over  $\pi$  in the ordinary sense. It is no different from a left  $\pi$ -algebra, except that we use  $\eta_R$  to make  $M$  a right  $\pi$ -module and reserve the right to endow  $M$  with a left  $\pi$ -module structure unrelated to  $\eta_R$ .

LEMMA 2.7. If  $M$  is a right  $\pi$ -algebra that is free as a right  $\pi$ -module then  $M \otimes_{\pi} E$  is canonically a ring spectrum, and is commutative if  $M$  is. Its multiplication map makes the diagram

$$\begin{array}{ccc} E \wedge E & \xrightarrow{(m \otimes E) \wedge (m' \otimes E)} & (M \otimes_{\pi} E) \wedge (M \otimes_{\pi} E) \\ \downarrow \mu & & \downarrow \mu \\ E & \xrightarrow{mm' \otimes E} & M \otimes_{\pi} E \end{array}$$

commute for all  $m, m' \in M$ , and this property characterizes it.

PROOF. By distributivity of the smash product,  $(M \otimes E) \wedge (M \otimes E)$  breaks up as a graded wedge of copies of  $E \wedge E$ , so that we can define the multiplication map on  $M \otimes E$  by requiring the diagram to commute whenever  $m$  and  $m'$  are basic elements. It follows from Lemma 2.4(a) and the structure on  $E$  that the diagram commutes generally. The unit map is simply  $(1 \otimes E) \circ i: S \rightarrow E \rightarrow M \otimes E$ , and commutativity and the unit property are clear. Associativity is easy, since  $(M \otimes E) \wedge (M \otimes E) \wedge (M \otimes E)$  similarly breaks up as a graded wedge of copies of  $E \wedge E \wedge E$ .

We give a substantial example of a tensor product spectrum. By the conventions in §3 it is appropriate to use the action on  $E$  to make  $X_*E$  a right  $\pi$ -module.

LEMMA 2.8. Suppose  $X_*E$  is a free right  $\pi$ -module. Then we have a canonical isomorphism  $X_*E \xrightarrow{\sim} X_*E \otimes_{\pi} E$ . It is an isomorphism of ring spectra

if  $X$  is a ring spectrum.

PROOF. Given  $m \in X_*E$ , that is, a map  $m: S \rightarrow X_*E$ , we construct the map

$$h(m): E = S \wedge E \xrightarrow{m \wedge 1} X_*E \wedge E \xrightarrow{1 \wedge u} X_*E.$$

Lemma 2.4(b) constructs the desired map  $h: X_*E \otimes E \rightarrow X_*E$  such that  $h(m \otimes E) = h(m)$  for all  $m$ . Since  $h$  induces the identity homomorphism  $X_*E \rightarrow X_*E$  on homotopy groups, it is an isomorphism.

If  $X$  is a ring spectrum, so is  $X_*E$  and  $X_*E$  becomes a right  $\pi$ -algebra. By Lemmas 2.4 and 2.7, we have only to check that  $h(1)$  is correct and that  $\mu_0(h(m) \wedge h(m')) = h(mm') \circ \mu_0: E_*E \rightarrow X_*E$  for all  $m, m' \in X_*E$ .

We obviously need to study the cohomology and homology theories defined by the spectrum  $M \otimes F$ .

THEOREM 2.9. (a)  $(M \otimes_\pi F)_*X \cong M \otimes_\pi (F_*X)$  for all  $X$ ;

(b)  $(M \otimes_\pi F)^*X \cong M \otimes_\pi (F^*X)$  for all finite  $X$ ;

(c) if  $F$  is highly connected and  $d(r) \rightarrow \infty$  then  $(M \otimes_\pi F)^*X \cong \hat{M} \otimes_\pi F^*X$ ,

where we complete in the sense of allowing infinite sums  $\sum_r m_r \otimes y_r$ .

REMARK. So in (a) and (b) the parentheses are redundant. From now on we shall simply write  $M \otimes_\pi F_*X$  and  $M \otimes_\pi F^*X$  whenever possible.

PROOF. For each  $m \in M$  the map  $m \otimes F: F \rightarrow M \otimes F$  induces  $F_*X \rightarrow (M \otimes F)_*X$ . Hence a canonical homomorphism  $M \otimes (F_*X) \rightarrow (M \otimes F)_*X$ , and similarly in cohomology,  $M \otimes (F^*X) \rightarrow (M \otimes F)^*X$ . These are evidently isomorphisms when  $M$  is free on one generator, whatever  $X$  is. Then (a) follows because both sides are strongly additive functors of  $M$ . The same holds for (b), provided  $X$  is finite, since

$$(M \otimes F)^*X = \{X, \bigvee_r M_r \otimes F\}^* = \bigoplus_r \{X, M_r \otimes F\}^*,$$

where we write  $M$  as a direct sum  $\bigoplus_r M_r$  of free modules of rank 1. The hypotheses in (c) ensure that we have a product

$$M \otimes F = \bigvee_r M_r \otimes F = \prod_r M_r \otimes F,$$

so that  $(M \otimes F)^*X = \prod_r M_r \otimes (F^*X)$ .

REMARK. The right side of (a) is a homology theory even if  $M$  is only a flat  $\pi$ -module. Then the Brown-Whitehead-Adams representation theorem [2] defines a spectrum  $M \otimes F$  to satisfy (a), uniquely up to isomorphism. Unfortunately, the isomorphisms are not in general well defined and there may be problems with functoriality and in making  $M \otimes E$  a ring spectrum as in Lemma 2.7, although these problems disappear when the universal coefficient theorem of §4 applies.

REMARK. A homomorphism  $g: M \rightarrow \pi$  of right  $\pi$ -modules induces a map of spectra  $g \otimes F: M \otimes F \rightarrow \pi \otimes F = F$  and therefore a homomorphism  $(g \otimes F)_*: (M \otimes F)^*X \rightarrow F^*X$ . The value of this homomorphism on a typical infinite

element  $\sum_r m_r \otimes y_r$  in (c) is not in general obvious. We get an infinite series  $\sum_r (gm_r)y_r$  in  $F^*X$  that converges to the value with respect to the usual filtration of  $F^*X$  defined by the skeletons of  $X$ . The sum of the series is well defined only if this filtration of  $F^*X$  is Hausdorff; if not, the homomorphisms  $(g \otimes F)_*$  define some extra structure on  $F^*X$ .

COROLLARY 2.10. If  $X_*E$  is free right  $\pi$ -module, then

$$\pi_*(X_*E \wedge Y) = X_*(E \wedge Y) = (X_*E)_*Y \cong X_*E \otimes_{\pi} E_*Y$$

for any spectrum  $Y$ .

PROOF. Combine Lemma 2.8 with Theorem 2.9(a).

§3. TWO-FACED ALGEBRA. Here we collect various comments on left and right modules. We work over the commutative groundring  $\pi = \pi_*E$ , where  $E$  is a commutative ring spectrum, and all tensor products are taken over  $\pi$ . As mentioned in §2, it is desirable to declare some modules to be left modules and others to be right modules.

CONVENTION (Adams). A  $\pi$ -action on  $F$  induces left  $\pi$ -module structures on  $F_*G$  and  $F^*G$ , right  $\pi$ -module structures on  $G_*F$  and  $G^*F$ , and a  $\pi$ -bimodule structure (consisting of two essentially equivalent  $\pi$ -actions) on  $\pi_*F$ .

This convention makes  $E_*X$  and  $E^*X$  left  $\pi$ -modules, as before. Further, for a space  $X$  (rather than a spectrum),  $E^*(X, \emptyset)$  becomes a commutative left  $\pi$ -algebra (compare §2), and for a commutative ring spectrum  $F$ ,  $E_*F$  becomes a commutative left  $\pi$ -algebra. This is our major supply of modules. Since we insist on using one left module and one right module to form a tensor product over  $\pi$ , a certain amount of trading appears inevitable. This we organize as follows.

DEFINITION 3.1. The formal conjugate  $cM$  of  $M$  is a copy of  $M$  having an element  $cm$  for each  $m \in M$ . If  $M$  is a left  $\pi$ -module, we make  $cM$  a right  $\pi$ -module by defining  $(cm)k = (-)^{\deg(k)\deg(m)} c(km)$ . Similarly, if  $M$  is a right  $\pi$ -module,  $cM$  becomes a left  $\pi$ -module by  $k(cm) = (-)^{\deg(k)\deg(m)} c(mk)$ . So if  $M$  is a  $\pi$ -bimodule,  $cM$  is another  $\pi$ -bimodule with the two actions in effect interchanged.

The commutativity isomorphism  $c: F_*G \xrightarrow{\sim} G_*F$  induces an isomorphism  $F_*G \xrightarrow{\sim} G_*F$ , which is also an isomorphism of rings if  $F$  and  $G$  are commutative ring spectra. It is reasonable and consistent with the Adams convention to use this isomorphism to identify the formal copy  $c(F_*G)$  of  $F_*G$  with  $G_*F$ , together with any module structures present. In particular, we have the important special case  $c(E_*E) = E_*E$ , where we identify  $cm = cm$  (the  $cm$  on the left is a formal copy of  $m$ , and the  $cm$  on the right is the image of  $m$  under the automorphism  $c = c_*$ .) Of course,  $\pi$  is itself a bimodule and we identify  $c\pi$  with  $\pi$  directly, by  $ck = k$ . Tensor products behave as

expected: there is a canonical isomorphism  $c(M \otimes N) \cong cN \otimes cM$  of groups, together with left and/or right module structures if present.

It is in defining duals that we must part company with traditional algebra. Nevertheless, all proofs are elementary and largely omitted.

DEFINITION 3.2. Given a left  $\pi$ -module  $M$ , we define the dual left  $\pi$ -module  $M^*$  to be  $\text{Hom}_\pi(M, \pi)$ . We make it a left  $\pi$ -module by defining

$$\langle kf, m \rangle = k \langle f, m \rangle = (-)^{\deg(f)\deg(k)} \langle f, km \rangle \quad (m \in M, f \in M^*, k \in \pi).$$

If  $M$  is a bimodule, we make  $M^*$  a bimodule with right  $\pi$ -module structure defined by  $\langle fk, m \rangle = (-)^{\deg(k)\deg(m)} \langle f, mk \rangle$ .

Here and elsewhere,  $\text{Hom}_\pi(, )$  will invariably denote the group of left module homomorphisms. Thus the definition of  $M^*$  is asymmetric and the two duals  $M^*$  and  $(cM)^*$  of a bimodule  $M$  are unrelated in general.

The interaction of duals and tensor products requires some care.

LEMMA 3.3. Let  $M$  be a  $\pi$ -bimodule and  $N$  a left  $\pi$ -module. Then

(a) there is a natural transformation  $\theta: M^* \otimes_\pi N^* \rightarrow (M \otimes_\pi N)^*$  of left modules (or of bimodules if  $N$  is a bimodule) defined by

$$\langle \theta(f \otimes g), m \otimes n \rangle = (-)^{\deg(g)\deg(m)} \langle f, m \langle g, n \rangle \rangle ;$$

(b) if  $L$  is another bimodule, the diagram

$$\begin{array}{ccc} L^* \otimes_\pi M^* \otimes_\pi N^* & \xrightarrow{\theta \otimes 1} & (L \otimes_\pi M)^* \otimes_\pi N^* \\ \downarrow 1 \otimes \theta & & \downarrow \theta \\ L^* \otimes_\pi (M \otimes_\pi N)^* & \xrightarrow{\theta} & (L \otimes_\pi M \otimes_\pi N)^* \end{array}$$

commutes;

(c) both composites  $M^* \xrightarrow{\sim} M^* \otimes_\pi \pi = M^* \otimes_\pi \pi^* \rightarrow (M \otimes_\pi \pi)^* \xrightarrow{\sim} M^*$  and  $M^* \xrightarrow{\sim} \pi \otimes_\pi M^* = \pi^* \otimes_\pi M^* \rightarrow (\pi \otimes_\pi M)^* \xrightarrow{\sim} M^*$  are the identity homomorphism.

PROOF. (a) In terms of diagrams,  $\theta(f \otimes g)$  is the composite

$$M \otimes N \xrightarrow{1 \otimes g} M \otimes \pi = M \xrightarrow{f} \pi.$$

Then (c) follows trivially, and in (b) both composites evaluated on  $f \otimes g \otimes h$  reduce to

$$L \otimes M \otimes N \xrightarrow{1 \otimes 1 \otimes h} L \otimes M \otimes \pi = L \otimes M \xrightarrow{1 \otimes g} L \otimes \pi = L \xrightarrow{f} \pi.$$

REMARK. The formula  $\langle f, m \rangle \langle g, n \rangle$  we used in §1 makes no sense here, because  $f \otimes g: M \otimes N \rightarrow \pi \otimes \pi$  is undefined unless  $f$  is a homomorphism of  $\pi$ -bimodules. Our formula for  $\theta(f \otimes g)$  evades such a hypothesis.

REMARK. There is no commutativity statement in Lemma 3.3.

We may use this to dualize coalgebras, in a sense.

LEMMA 3.4. Let  $R$  be a two-faced coalgebra in the sense of a  $\pi$ -bimodule  $R$  equipped with homomorphisms  $\psi: R \rightarrow R \otimes_\pi R$  and  $\epsilon: R \rightarrow \pi$  of bimodules that make the diagrams

$$\begin{array}{ccccc}
 R & \xrightarrow{\psi} & R \otimes_{\pi} R & & R \xrightarrow{\psi} R \otimes_{\pi} R & & R \xrightarrow{\psi} R \otimes_{\pi} R \\
 \downarrow \psi & & \downarrow 1 \otimes \psi & & \downarrow = & & \downarrow = \\
 R \otimes_{\pi} R & \xrightarrow{\psi \otimes 1} & R \otimes_{\pi} R \otimes_{\pi} R & & R \xrightarrow{=} R \otimes_{\pi} \pi & & R \xrightarrow{=} \pi \otimes_{\pi} R \\
 & & & & & & \downarrow \epsilon \otimes 1 \\
 & & & & & & \pi \otimes_{\pi} R
 \end{array}$$

commute. Then

(a) the dual  $R^*$  is a ring with unit ring homomorphism  $\epsilon^*: \pi \rightarrow R^*$ , and the bimodule structure on  $R^*$  is given by multiplication with  $\epsilon^*$  on either side;

(b) if the left  $\pi$ -module  $M$  is a left  $R$ -comodule with coaction  $\psi: M \rightarrow R \otimes_{\pi} M$  that is a homomorphism of left  $\pi$ -modules, then  $M^*$  becomes a left  $R^*$ -module.

PROOF. We use Lemma 3.3 to construct the multiplication

$$R^* \otimes R^* \rightarrow (R \otimes R)^* \xrightarrow{\psi^*} R^*$$

on  $R^*$  for (a) and the action

$$R^* \otimes M^* \rightarrow (R \otimes M)^* \xrightarrow{\psi^*} M^*$$

on  $M^*$  for (b). To verify the axioms we need parts (b) and (c) of Lemma 3.3.

REMARK. We need all the stated module structures on  $R$ ,  $\psi$  and  $\epsilon$  in order to form the diagrams at all.

REMARK. The ring  $R^*$  is not a  $\pi$ -algebra in the ordinary sense because the image of  $\epsilon^*: \pi \rightarrow R^*$  will not in general be central, so that the left and right  $\pi$ -module structures on  $R^*$  are in general quite different. In particular, multiplication in  $R^*$  is not  $\pi$ -bilinear. The identity element of  $R^*$  is the counit homomorphism  $\epsilon: R \rightarrow \pi$ , regarded as an element of  $R^*$ .

§4. A UNIVERSAL COEFFICIENT THEOREM. While the universal coefficient theorem 2.9 is completely satisfactory for the homology  $(M \otimes E)_* X$ , the cohomology version for  $(M \otimes E)^* X$  has some disadvantages. Unless  $X$  is finite, we had to restrict  $M$ . In this section we discuss a different kind of universal coefficient theorem in which we impose conditions on  $X$  but not on  $M$ .

An element  $y \in (M \otimes E)^* X$  is by definition a map  $y: X \rightarrow M \otimes E$ . We use it to induce a homomorphism of right  $\pi$ -modules

$$X_* E \xrightarrow{y_*} M \otimes E_* E \xrightarrow{1 \otimes \epsilon} M \otimes \pi \stackrel{\sim}{=} M \quad (4.1)$$

where we use the usual augmentation  $\epsilon: E_* E \rightarrow \pi$  defined as  $\mu_*: \pi_*(E_* E) \rightarrow \pi_* E$ . We use the formal conjugation 3.1 to produce a homomorphism of left  $\pi$ -modules.

THEOREM 4.2. (Universal coefficient theorem) For suitable  $E, X$  and  $M$  the homomorphism (4.1) induces an isomorphism  $(M \otimes_{\pi} E)^* X \stackrel{\sim}{=} \text{Hom}_{\pi}(E_* X, {}^c M)$ .

REMARK. If we take  $M = \pi$  the assertion becomes  $E^* X \stackrel{\sim}{=} \text{Hom}_{\pi}(E_* X, \pi) = (E_* X)^*$ , which is usually quite false. This gives some indication of when the result might be valid.

PROOF. The standard method is due to Atiyah. We compare the Atiyah-Hirzebruch spectral sequences

$$E_2^{S,*} = H^S(X; M) \text{ converging to } (M \otimes E)_* X$$

and

$$E_{S,*}^2 = H_S(X; \pi) \text{ converging to } E_* X.$$

We apply the functor  $\text{Hom}(-, cM)$  to the second to obtain another spectral sequence

$$E_2^{S,*} = \text{Hom}_{\pi}(H_S(X; \pi), cM) \text{ converging to } \text{Hom}_{\pi}(E_* X, cM).$$

There is a natural map from the first spectral sequence to this one, to which we apply the comparison theorem.

There are clearly difficulties with the method. The third spectral sequence will not be a spectral sequence in general unless  $M$  is an injective  $\pi$ -module or everything in the second spectral sequence is projective. There are convergence questions to settle. To apply the comparison theorem at all we need an isomorphism of  $E_2$ -terms,  $H^S(X; M) \cong \text{Hom}_{\pi}(H_S(X; \pi), cM)$ , which may or may not occur. However, the method works often enough for our purposes.

REMARK. The cases that suffice for our present purposes are:

1.  $E = H(F_p)$ , the mod  $p$  Eilenberg-MacLane spectrum, and any  $X$ ;
2.  $E = MU$ , with  $H_*(X; \mathbb{Z})$  free abelian;
3.  $E = BP$ , with  $H_*(X; \mathbb{Z}_{(p)})$  free over  $\mathbb{Z}_{(p)}$ .

In all these cases all differentials vanish and convergence is satisfactory. For further details see Adams [1, Lemma 4.2 on p.48]. The theorem is then universal to the extent that we place no restrictions on the module  $M$  (other than freeness).

§5. UNIVERSAL OPERATIONS. As before we take a commutative ring spectrum  $E$  with coefficient ring  $\pi = \pi_* E$ . All tensor products in this section are taken over  $\pi$ .

It is natural to look for operations that preserve the three kinds of elementary structure on the cohomology  $E^* X$  and homology  $E_* X$ . First, they are abelian groups; but every operation automatically preserves the additive structure, as does any natural transformation of additive functors. Second, they are left  $\pi$ -modules. Obviously the module actions themselves preserve the module structure ( $\pi$  being commutative), but in the typical situation one soon finds that they are the only such operations. (The exception is ordinary (co)homology with coefficients  $F_p$ , where the module structure is so trivial that any operation is forced to preserve it. In some ways, ordinary cohomology is a most extraordinary cohomology theory.) Third, we have multiplicative structure consisting of pairings  $E^* X \times E^* Y \rightarrow E^*(X \wedge Y)$  etc., that make  $E^*(X, \emptyset)$

an algebra when  $X$  is a space. We therefore seek merely additive operations and multiplicative operations.

Any map  $E \rightarrow E$  of spectra induces a cohomology operation on  $E^*(-)$  and a homology operation on  $E_*(-)$ . Since cohomology is by definition representable, every cohomology operation is induced by a unique map. The Brown-Whitehead-Adams representation theorem [2] gives the same result for homology, except perhaps for uniqueness. We shall therefore confuse maps and operations.

It is often extremely convenient to handle many operations at once. One map  $E \rightarrow M \otimes E$  induces a whole collection of operations, by composition with the maps  $g \otimes E: M \otimes E \rightarrow \pi \otimes E \cong E$  for the various right  $\pi$ -module homomorphisms  $g: M \rightarrow \pi$ . The main idea of this section is that a proper choice of  $M$  will give all possible operations.

DEFINITION 5.1. Let  $R$  be a free right  $\pi$ -module. We call an operation  $\psi_L: E \rightarrow R \otimes_\pi E$  a universal additive operation if given any operation  $\theta: E \rightarrow M \otimes_\pi E$  with  $M$  a free right  $\pi$ -module, there exists a unique homomorphism  $g: R \rightarrow M$  of right  $\pi$ -modules that makes the diagram

$$\begin{array}{ccc} E & \xrightarrow{\quad \quad} & R \otimes_\pi E \\ & \searrow \psi_L & \downarrow g \otimes E \\ & & M \otimes_\pi E \end{array}$$

$\theta$

commute. Similarly, we call  $\psi_L$  a universal multiplicative operation if  $R$  and  $M$  are also commutative right  $\pi$ -algebras (see Lemma 2.7),  $\psi_L$  and  $\theta$  are maps of ring spectra, and  $g$  is required to be a homomorphism of right  $\pi$ -algebras.

In particular, by taking  $M = \pi$ , we recover the general (additive or multiplicative) operation on  $E$ . Either kind of universal operation is of course unique up to isomorphism if it exists. Existence already implies much structure.

THEOREM 5.2. If  $\psi_L: E \rightarrow R \otimes_\pi E$  is a universal multiplicative operation, then  $R$  is a "two-faced Hopf algebra" with commutative multiplication.

PROOF. We mean that  $R$  is both a two-faced coalgebra (in the sense of Lemma 3.4) and a commutative ring, such that  $\psi$  and  $\epsilon$  are ring homomorphisms. Further, there are left and right unit homomorphisms (in general distinct)  $\eta_L: \pi \rightarrow R$  and  $\eta_R: \pi \rightarrow R$  that induce the left and right  $\pi$ -module structures on  $R$ .

By hypothesis  $R$  is a commutative right  $\pi$ -algebra, with right unit  $\eta_R$ . If we apply  $\psi_L$  to the sphere spectrum  $S$  we obtain the ring homomorphism  $\psi_L S: E_* S \rightarrow R \otimes E_* S \cong R$ , which will serve as  $\eta_L$ . Since  $R$  is commutative,  $R \otimes R$  is again a ring, and by the universal property we fill in homomorphisms of right  $\pi$ -algebras  $\psi: R \rightarrow R \otimes R$  and  $\epsilon: R \rightarrow \pi$  that make the diagrams

$$\begin{array}{ccc}
 \text{(a)} & \begin{array}{ccc} E & \xrightarrow{\psi_L} & R \otimes E \\ \downarrow \psi_L & & \downarrow \psi \otimes 1 \\ R \otimes E & \xrightarrow{1 \otimes \psi_L} & R \otimes R \otimes E \end{array} & \\
 \text{(b)} & \begin{array}{ccc} E & \xrightarrow{\psi_L} & R \otimes E \\ \downarrow = & & \downarrow \epsilon \otimes 1 \\ E & \xrightarrow{=} & \pi \otimes E \end{array} & (5.3)
 \end{array}$$

commute. If we apply these diagrams to  $S$ , we see that  $\psi$  and  $\epsilon$  are also homomorphisms of left  $\pi$ -modules. Further use of the universal property shows that  $\psi$  and  $\epsilon$  satisfy the two-faced coalgebra axioms of Lemma 3.4.

Construction of universal operations is also easy if we assume enough. The universal coefficient theorem 4.2 classifies the maps from  $E$  to  $M \otimes E$  by the group  $\text{Hom}_\pi(E_*E, cM)$ . This suggests a candidate.

**THEOREM 5.4** Assume that  $A = E_*E$  is a free right  $\pi$ -module. Then

$$\psi_L: E = E_*S \xrightarrow{1_*i} E_*E \xrightarrow{\sim} E_*E \otimes_\pi E = A \otimes_\pi E \text{ (using Lemma 2.8)}$$

is both the universal additive operation and the universal multiplicative operation, provided the universal coefficient theorem 4.2 holds for  $(M \otimes_\pi E)_*E$  for all  $M$ .

**PROOF.** Theorem 4.2 sets up a 1-1 correspondence between maps  $\theta: E \rightarrow M \otimes E$  of spectra and homomorphisms  $g = g(\theta): A \rightarrow M$  of right modules by

$$g = g(\theta): A = E_*E \xrightarrow{\theta_*E} M \otimes E_*E \xrightarrow{1 \otimes \epsilon} M \otimes_\pi \pi \xrightarrow{\sim} M.$$

If we define  $\psi_L: E \rightarrow A \otimes E$  by  $g(\psi_L) = 1$ , in other words,

$$A = E_*E \xrightarrow{\psi_L E} A \otimes E_*E \xrightarrow{1 \otimes \epsilon} A \otimes_\pi \pi \xrightarrow{\sim} A \quad (5.5)$$

is the identity, then naturality shows that  $\psi_L$  is the universal additive operation. But the stated map fulfills this condition. That is, the composite homomorphism

$$A = E_*E = (E_*S)_*E \xrightarrow{(1_*i)_*} (E_*E)_*E = E_*E \otimes E_*E \xrightarrow{1 \otimes \epsilon} E_*E \otimes_\pi \pi \xrightarrow{\sim} A$$

is the identity. This is easy to see once we recognize the composite  $(E_*E)_*E \rightarrow A$  as  $(1_*\nu)_*: \pi_*(E_*E_*E) \rightarrow \pi_*(E_*E)$ . Further,  $\psi_L$  also serves as the universal multiplicative operation because  $\theta_*E$  and hence  $g(\theta)$  are multiplicative whenever  $\theta$  is.

Theorem 5.2 provides all the standard structure on  $A$  for free, except for the internal conjugation antiautomorphism  $c$ . Closer examination shows that our definitions are not really very different from those of Adams. The identity element of  $A$  is clearly  $i_*i: S = S_*S \rightarrow E_*E$ . Right  $\pi$ -module structures were used all along, to build  $A \otimes E$ . We can identify  $\eta_R$  with  $i_*E: \pi = S_*E \rightarrow E_*E = A$ , since this is a homomorphism of right  $\pi$ -modules that takes 1 to 1. Our left unit homomorphism is by construction

$$\eta_L = \psi_L S: \pi = E_*S = (E_*S)_*S \xrightarrow{(1_*i)_*} (E_*E)_*S \xrightarrow{\sim} A \otimes E_*S \xrightarrow{\sim} A,$$

which we may identify with  $E_*i: E_*S \rightarrow E_*E$ ; it therefore induces the same left

$\pi$ -module structure on  $A = E_*E$  we had before.

We can similarly recognize the structure maps  $\psi$  and  $\epsilon$  in Theorem 5.2.

LEMMA 5.6. We have  $\psi = \psi_L E: A = E_*E \rightarrow A \otimes_{\pi} E_*E = A \otimes_{\pi} A$  and  
 $\epsilon = \mu_*: A = \pi_*(E_*E) \rightarrow \pi_*E = \pi$ .

PROOF. We have from (5.3a) the commutative diagram

$$\begin{array}{ccccc}
 E_*E & \xrightarrow{\psi_L E} & A \otimes E_*E & \xrightarrow{1 \otimes \epsilon} & A \otimes \pi = A \\
 \downarrow \psi_L E & & \downarrow \psi \otimes 1 & & \downarrow \psi \\
 A \otimes E_*E & \xrightarrow{1 \otimes \psi_L E} & A \otimes A \otimes E_*E & \xrightarrow{1 \otimes 1 \otimes \epsilon} & A \otimes A \otimes \pi = A \otimes A
 \end{array}$$

in which (5.5) identifies the top and bottom rows with identity homomorphisms. Hence  $\psi = \psi_L E$ . A similar diagram handles  $\epsilon$ .

The one structure that Theorem 5.2 does not define is the conjugation  $c$  in  $A$ . The standard Hopf algebra definition [8, §8] makes no sense in this context. Instead, it is defined as induced by the commutativity switch isomorphism,  $E_*E \cong E_*E$ , as in §3. Its properties are obvious.

LEMMA 5.7. The conjugation  $c$  in  $A = E_*E$  has the properties:  
 $c$  is a ring automorphism,  $c \circ c = 1$ ,  $c \circ \eta_L = \eta_R$ ,  $c \circ \eta_R = \eta_L$ ,  $\epsilon \circ c = \epsilon$ , and the  
identification  $cA = A$  gives  $c\psi = \psi: A \rightarrow A \otimes_{\pi} A$ .

PROOF. We can write the comultiplication  $\psi$  more symmetrically as

$$\pi_*(E_*E) = \pi_*(E_*S_*E) \xrightarrow{(1 \wedge i \wedge 1)_*} \pi_*(E_*E_*E) = \pi_*(E_*E) \otimes \pi_*(E_*E),$$

where the second isomorphism is induced by

$$\pi_*(E_*E) \times \pi_*(E_*E) \rightarrow \pi_*(E_*E_*E) \xrightarrow{(1 \wedge \mu \wedge 1)_*} \pi_*(E_*E_*E).$$

Then all properties are obvious.

§6. THE EIGHTFOLD WAY REVISED. In this section we assume that Theorem 5.4 applies to the commutative ring spectrum  $E$ , so that we have the two-faced Hopf algebra  $A$  and the universal operation  $\psi_L: E \rightarrow A \otimes E$ . (All tensor products are taken over  $\pi = \pi_*E$ .) We generalize (1.1) to this situation. We find that all eight structures are still present in some shape or form, although it is clear that any constructions that depend on inverting duality isomorphisms have no place here. We also drop all restrictions on  $X$ .

Then Lemma 3.4 makes the dual  $A^*$  into a ring with unit homomorphism  $\epsilon^*: \pi \rightarrow A^*$ , not central. On the other hand, the natural algebra for cohomology operations is  $E^*E = \{E, E\}_*$ , a ring under composition, containing a copy of  $\pi$  as the subring of all actions  $W(k): E \rightarrow E$ .

LEMMA 6.1. The isomorphism  $E^*E \cong A^*$  given by the universal coefficient theorem 4.2 is an isomorphism of rings and of bimodules.

PROOF. Given  $\alpha: E \rightarrow E$ , the corresponding element of  $A^*$  is the homo-

morphism

$$A = E_*E \xrightarrow{E_*\alpha} E_*E \xrightarrow{\epsilon} \pi$$

of left  $\pi$ -modules. If  $\alpha = W(k)$ , this is the homomorphism that takes  $a \in A$  to  $(-)^{\deg(a)\deg(k)} \epsilon(ak)$ . On the other hand,  $\epsilon^*k \in A^*$  is the homomorphism that takes  $a$  to  $k \cdot \epsilon(a)$ . Since these agree, the two unit homomorphisms into  $E^*E$  and  $A^*$  correspond.

Let  $\alpha$  and  $\beta$  in  $E^*E$  give rise to homomorphisms  $f$  and  $g$  in  $A^*$  respectively. The product of  $f$  and  $g$  in  $A^*$  is defined as

$$A \xrightarrow{\psi} A \otimes A \xrightarrow{1 \otimes g} A \otimes \pi = A \xrightarrow{f} \pi.$$

We have to show this agrees with

$$E_*E \xrightarrow{E_*\beta} E_*E \xrightarrow{E_*\alpha} E_*E \xrightarrow{\epsilon} \pi.$$

These are the homomorphisms induced on homotopy groups by the composite maps

$$E_*E = E_*S_*E \xrightarrow{1_*i_*1} E_*E_*E \xrightarrow{1_*1_*\beta} E_*E_*E \xrightarrow{1_*\mu} E_*E \xrightarrow{1_*\alpha} E_*E \xrightarrow{\mu} E$$

and

$$E_*E \xrightarrow{1_*\beta} E_*E \xrightarrow{1_*\alpha} E_*E \xrightarrow{\mu} E$$

respectively. These agree because  $\mu \circ (1_*\beta) \circ (i_*1) = \beta: E \rightarrow E$ . We have an isomorphism of rings. It follows that we have an isomorphism of bimodules because the bimodule structure on  $E^*E$  was defined in §2 as composition with the  $\pi$ -actions  $W(k)$  on either side.

We now list the eight structures as in §1, but in a different order.

1. The standard left action of  $A^* = E^*E$  on cohomology,  $E^*X$ ;
2. Left coaction on homology,  $\psi_{L*}: E_*X \rightarrow A \otimes E_*X$ , given by Theorem 5.4.

This is a genuine coaction according to (5.3).

3. Left coaction on cohomology,  $\psi_{L*}: E^*X \rightarrow (A \otimes E)^*X$ , given by Theorem 5.4. If  $X$  is finite, we can write  $(A \otimes E)^*X = A \otimes E^*X$  by Theorem 2.9(b) and we also have a genuine coaction. Otherwise, provided 2.9(c) holds, we can write  $A \hat{\otimes} E^*X$  instead. Although no longer strictly a coaction, (5.3) still applies and the structure is just as useful.

4. Left action of  $A^* = E^*E$  on homology,  $E_*X$ . This is one of the best-kept secrets of stable homotopy theory, even in the classical case of ordinary homology. We simply regard  $E_*X$  as a functor of  $E$  also, and then a map  $\alpha: E \rightarrow E$  induces the desired operation or natural transformation  $\alpha_*X: E_*X \rightarrow E_*X$ .

The one construction that survives intact is conjugation, provided we use the formal conjugation 3.1 consistently. This allows us to deduce the four right structures from the left structures with no extra work.

5. Right coaction on homology,  $c(E_*X) \rightarrow c(E_*X) \otimes A$ , deduced from the left

coaction by applying the canonical isomorphisms  $c(M \otimes N) \cong cM \otimes cN$  and identifying  $cA = A$ . (We include the  $c$  with  $E_*X$  merely to maintain the Adams convention; it is often ignored or omitted in practice.)

6. Right coaction on cohomology,  $c(E^*X) \rightarrow c((A \otimes E)^*X)$ , where we can often write the right side as  $c(E^*X) \hat{\otimes} A$ .

7. Right action on cohomology,  $E^*X$ , by  $cA^*$ , the opposite ring to  $A^*$ , with multiplication  $\alpha \cdot c\beta = (-)^{\deg(\alpha)\deg(\beta)} c(\beta\alpha)$ .

8. Right action on homology,  $E_*X$ , by  $cA^*$ .

One difference from the classical case is that because the dual  $A^*$  is defined asymmetrically and  $\eta_L \neq \eta_R$  in general, the conjugation  $c$  in  $A$  is not linear and does not pass to  $A^*$ . In general, there is no antiautomorphism of  $A^*$  that preserves  $\eta: \pi \rightarrow A^*$ .

EXAMPLE. Let  $\pi$  be a finite-dimensional commutative algebra over the rationals  $Q$ , concentrated in degree zero, and take  $E = H(\pi)$ , an Eilenberg-MacLane ring spectrum. In this case we have an excellent grasp of the algebra of cohomology operations: it is  $\text{End}_Q(\pi)$ , with  $\pi$  embedded as the subring of  $\pi$ -linear endomorphisms. One can show that there exists a  $\pi$ -preserving anti-automorphism of  $\text{End}_Q(\pi)$  if and only if  $\pi$  is a Poincaré duality algebra over  $Q$ .

CONJECTURE. For  $E = MU$  or  $BP$ , we conjecture that there exists no antiautomorphism of the ring  $E^*E$  that preserves  $\pi$ . We have no proof at this time, mainly for lack of interest. Note that these groups  $E^*E$  are large and the statement is algebraic; we are not restricting to continuous antiautomorphisms, whatever they might be.

We next study what is left of the various constructions that form the edges of the diagram (1.1). As already noted, conjugation  $C$  works perfectly, provided we use  $cA^*$  instead of  $A^*$  for the right actions. Because it works so well, we concentrate on the left structures.

As for duality  $D$ ,  $E^*X$  is no longer the dual of  $E_*X$  in any generality. However, the Kronecker product still gives a homomorphism  $K: E^*X \rightarrow (E_*X)^*$  of  $\pi$ -modules. Given the left coaction of  $A$  on  $E_*X$ , Lemma 3.4 constructs a left action of  $A^*$  on  $(E_*X)^*$ , which we may compare with the left action on  $E^*X$ .

LEMMA. 6.2. The Kronecker product homomorphism  $K: E^*X \rightarrow (E_*X)^*$  is a homomorphism of left  $A^*$ -modules.

PROOF. We have to show that  $f(Ku) = K(\alpha u): E_*X \rightarrow \pi$ , where  $u \in E^*X$  and  $f \in A^*$  corresponds to  $\alpha: E \rightarrow E$ . By Lemma 3.4,  $f(Ku)$  is the composite

$$E_*X \xrightarrow{\psi_L X} A \otimes E_*X \xrightarrow{1 \otimes E_*u} A \otimes E_*E \xrightarrow{1 \otimes \varepsilon} A \otimes \pi = A \xrightarrow{f} \pi.$$

We compare with

$$K(\alpha u): E_*X \xrightarrow{E_*u} E_*E \xrightarrow{E_*\alpha} E_*E \xrightarrow{\epsilon} \pi.$$

By definition,  $f = \epsilon \circ E_*\alpha$ , and the required equality follows from the diagram

$$\begin{array}{ccccc} E_*X & \xrightarrow{E_*u} & E_*E & \xrightarrow{=} & A \\ \downarrow \psi_L^X & & \downarrow \psi_L^E & & \downarrow = \\ A \otimes E_*X & \xrightarrow{1 \otimes E_*u} & A \otimes E_*E & \xrightarrow{1 \otimes \epsilon} & A \end{array}$$

which commutes by naturality and (5.5).

In favorable cases  $K$  is an isomorphism by Theorem 4.2, but in general neither structure determines the other.

However, partial duality  $D'$ , in which we dualize from  $A$  to  $A^*$ , works well by hypothesis of Theorem 5.4.

LEMMA 6.3. Given  $f \in A^*$  corresponding to  $\alpha \in E^*E$ , the action of  $f$  on  $E^*X$  defined by

$$E^*X \xrightarrow{\psi_L^X} (A \otimes E)^*X \xrightarrow{(cf \otimes 1)_*} (\pi \otimes E)^*X = E^*X$$

agrees with the standard left action of  $\alpha$  on  $E^*X$ . And similarly for homology,  $E_*X$ .

PROOF. The action of  $f$  on homology or cohomology is induced by the map of spectra

$$E = E \wedge S \xrightarrow{1 \wedge i} E \wedge E \xrightarrow{\alpha \wedge 1} E \wedge E \xrightarrow{u} E,$$

which simplifies to  $\alpha$ .

The necessity of having a homomorphism of right  $\pi$ -modules to form the tensor product forced us to introduce the conjugate  $cf$  of  $f$ . It may well be considered more natural to avoid conjugation by putting the coaction on the other side, using the shuffle  $S'$  instead of  $D'$ . Our conventions require the right module  $cE^*X$  instead, but this conjugation is purely formal and often omitted. For simplicity we assume  $X$  finite, although there is an obvious extension when Theorem 2.9(c) applies, as it does in all our applications.

COROLLARY 6.4. Given  $f \in A^*$  corresponding to  $\alpha: E \rightarrow E$ , the left action of  $\alpha$  on  $E^*X$  may be recovered from the right coaction on  $cE^*X$  as the composite

$$cE^*X \xrightarrow{\psi_R^X} cE^*X \otimes A \xrightarrow{1 \otimes f} cE^*X \otimes \pi = cE^*X,$$

and similarly for homology.

The other partial duality  $D''$  is far less useful. For actions, in the classical case it involved conjugation in  $A^*$ , which is not available here, quite apart from the lack of duality between  $E^*X$  and  $E_*X$ . For coactions, the following result expresses some relation between the coactions on  $E^*X$  and  $E_*X$ , but its significance is obscure.

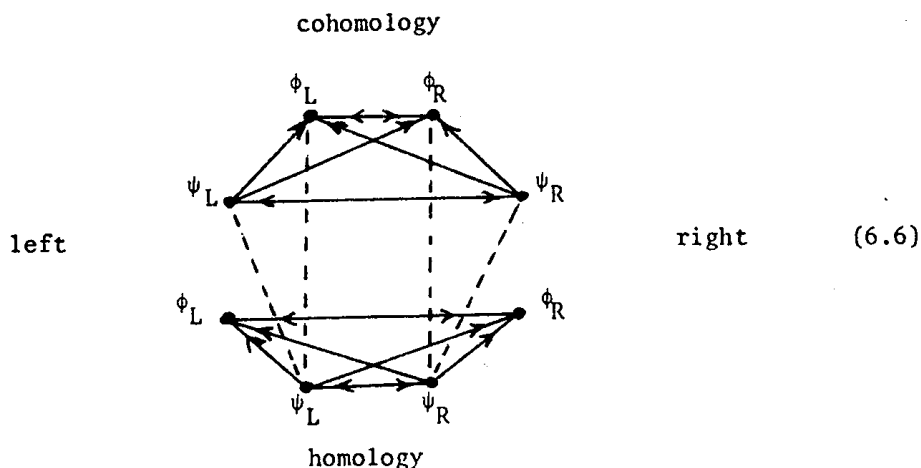
LEMMA 6.5. For Kronecker products we have

$$\langle \psi_L u, \psi_L x \rangle = \eta_L \langle u, x \rangle$$

for all  $u \in E^*X$  and  $x \in E_*X$ . (The Kronecker product on the left between  $(A \otimes E)^*X$  and  $A \otimes E_*X$  is formed in the obvious way, and takes values in the coefficient ring  $A$ .)

PROOF. We have a map  $\psi_L: E \rightarrow A \otimes E$  of ring spectra, with induced coefficient ring homomorphism  $\eta_L: \pi \rightarrow A$ .

We summarize by revising (1.1). Dashed lines indicate relationships that in general fail to deduce either structure from the other.



REMARK. In summary, conjugation  $C$  works all the time and partial duality  $D'$  half the time. Given the two left coactions on homology and cohomology, we can readily recover all the other structures. This is the precise sense in which the coactions are preferable to the actions. The multiplicativity of the coactions is also very transparent and useful.

REMARK. Our notation definitely favors the left coactions, following Adams [1]. Since conjugation works so well, one may well wonder why right actions or coactions are ever used. Historically, we find that all the major work was done, presumably unknowingly, in terms of the right coaction on cohomology. This is why our title is "eightfold way" rather than "fourfold way", quite apart from incidental connotations.

§7. THE CLASSICAL CASE, CONCLUDED. We return to the classical case  $E = H(F_p)$ , the mod  $p$  Eilenberg-MacLane spectrum, to consider the rest of Milnor's paper [7]. In §4 he explicitly introduced the right coaction on cohomology in studying the Hopf algebra  $A$ , and so must we. As before, we write the cohomology and homology groups as  $H^*X$  and  $H_*X$ . However,  $X$  need no longer be finite, if we use part (c) instead of part (b) of Theorem 2.9. Since  $\pi = \pi_*E = F_p$ , all  $\pi$ -module structures are forced and we may safely omit the formal conjugation  $c$

on  $H^*X$  and  $H_*X$  when dealing with right coactions.

First we take  $p = 2$ . The natural test space is infinite real projective space  $P = P_\infty(R)$ . Its absolute cohomology is the polynomial algebra  $H^*(P, \emptyset) = F_2[t]$  on one generator  $t \in H^1(P, \emptyset)$ , and we may identify the reduced group  $H^*P$  with the ideal  $(t)$ . Being multiplicative, the right coaction  $\psi_R: H^*P \rightarrow H^*P \hat{\otimes} A$  is determined by  $\psi_R t$ , which must have the form  $\sum_i t^i \otimes a_i$  for certain well-defined elements  $a_i \in A$  of degree  $i - 1$ . Since  $P$  is an Eilenberg-MacLane space  $K(F_2, 1)$ , this formula must hold on any class  $t \in H^1X$  for any space  $X$ . In particular, we take  $X = P \times P$ . Then  $\psi_R(t+u) = \psi_R t + \psi_R u$  in  $H^*(X, \emptyset) = F_2[t, u]$  yields

$$\sum_i (t+u)^i \otimes a_i = \sum_i t^i \otimes a_i + \sum_i u^i \otimes a_i$$

in  $F_2[t, u] \hat{\otimes} A$ , which implies that  $a_i = 0$  unless  $i$  is a power of 2. We therefore renumber, and proceed as in Milnor's Lemma 6.

DEFINITION 7.1. We define elements  $\xi_i \in A$  of degree  $2^i - 1$  for all  $i \geq 0$  by the identity

$$\psi_R t = \sum_i t^{2^i} \otimes \xi_i \in H^*X \hat{\otimes} A$$

for any  $t \in H^1X$ , for any space  $X$ .

Since  $(1 \otimes \epsilon)\psi_R t = t \otimes 1$  by (5.3b), the counit  $\epsilon$  of  $A$  clearly satisfies  $\epsilon \xi_0 = 1$  and  $\epsilon \xi_i = 0$  for all  $i > 0$ .

THEOREM 7.2. (Milnor)  $A = F_2[\xi_1, \xi_2, \xi_3, \dots]$  and  $\xi_0 = 1$ .

PROOF. This is Milnor's Theorem 2 for the case  $p = 2$ .

We break up  $\psi_R x$  according to the obvious monomial basis of  $A$ .

DEFINITION 7.3. We define the Milnor operations  $Sq^\alpha: H^*X \rightarrow H^*X$  for each multiindex  $\alpha$  on any space or spectrum  $X$  by the identity

$$\psi_R x = \sum_\alpha Sq^\alpha x \otimes \xi^\alpha \in H^*X \hat{\otimes} A.$$

In particular, the Steenrod squares are recovered as  $Sq^i = Sq^{i,0,0,\dots}$ . We have referred to  $\psi_R$  (or was it  $\psi_L$ ?) as the giant Steenrod square. One can distinguish any two operations by evaluating on products  $t_1 t_2 \dots t_k$  of classes of codegree 1 on  $P \times P \times \dots \times P$ , since

$$\psi_R t_1 t_2 \dots t_k = \prod_i (t_i \otimes 1 + t_i^2 \otimes \xi_1 + t_i^4 \otimes \xi_2 + t_i^8 \otimes \xi_3 + \dots)$$

involves all monomials  $\xi^\alpha$  nontrivially as  $k$  varies. The (generalized) Cartan formula is simply the statement that  $\psi_R$  is multiplicative.

To find the comultiplication  $\psi$  in  $A$  we simply evaluate the diagram (5.3a) (conjugated) on the fundamental class  $t \in H^1P$ , since 7.1 defines  $\xi_i$  in terms of  $\psi_R t$ . We find

$$(\psi_R \otimes 1)\psi_R t = (\psi_R \otimes 1) \sum_i t^{2^i} \otimes \xi_i = \sum_i (\psi_R t)^{2^i} \otimes \xi_i = \sum_{i,j} t^{2^{i+j}} \otimes \xi_j^{2^i} \otimes \xi_i$$

and

$$(1 \otimes \psi)\psi_R t = \sum_k t^{2^k} \otimes \psi \xi_k.$$

Equating coefficients yields the standard coproduct formula,

$$\psi \xi_k = \sum_{i+j=k} \xi_j^{2^i} \otimes \xi_i, \quad (7.4)$$

just as in Milnor's Theorem 3. To compose Milnor operations we expand the general identity (5.3a),  $(\psi_R \otimes 1)\psi_R x = (1 \otimes \psi)\psi_R x$ , by 7.3 to obtain

$$\sum_{\alpha, \beta} \text{Sq}^\beta \text{Sq}^\alpha x \otimes \xi^\beta \otimes \xi^\alpha = \sum_\gamma \text{Sq}^\gamma x \otimes \psi \xi^\gamma \quad (7.5)$$

Then  $\text{Sq}^\beta \text{Sq}^\alpha x$  is given by picking out all terms involving  $\xi^\beta \otimes \xi^\alpha$  on the right with the help of (7.4), just as in Milnor's Theorem 4B.

Similarly we can deduce the conjugations in  $A$  and  $A^*$ . We extend  $\psi_R: H^*X \rightarrow H^*X \hat{\otimes} A$   $A$ -linearly in the obvious way to a (continuous) homomorphism  $\Psi: H^*X \hat{\otimes} A \rightarrow H^*X \hat{\otimes} A$  that is readily seen to be an isomorphism.

**THEOREM 7.6.** (a) The inverse  $\Psi^{-1}: H^*X \hat{\otimes} A \rightarrow H^*X \hat{\otimes} A$  is the continuous  $A$ -linear homomorphism given by  $\Psi^{-1}(x \otimes 1) = \sum_\alpha (c\text{Sq}^\alpha)x \otimes \xi^\alpha$ ;

(b) For  $t \in H^1P$  we have  $\Psi^{-1}(t \otimes 1) = \sum_i t^{2^i} \otimes c\xi_i$ .

**PROOF.** For (a) we have

$$\begin{aligned} \Psi(\sum_\alpha (c\text{Sq}^\alpha)x \otimes \xi^\alpha) &= \sum_{\alpha, \beta} \text{Sq}^\beta (c\text{Sq}^\alpha)x \otimes \xi^{\alpha+\beta} \\ &= \sum_\gamma (\sum_{\alpha+\beta=\gamma} \text{Sq}^\beta (c\text{Sq}^\alpha)x) \otimes \xi^\gamma. \end{aligned}$$

By the definition [8] of  $c$  this reduces to  $x \otimes 1$ . For (b) we use

$$\Psi(\sum_i t^{2^i} \otimes c\xi_i) = \sum_k t^{2^k} \otimes \sum_{i+j=k} \xi_j^{2^i} \cdot c\xi_i = t \otimes 1.$$

For odd  $p$  there are some extra complications. The appropriate test space is now the infinite lens space  $L = K(F_p, 1)$ , whose cohomology is  $H^*(L, \mathbb{Z}) = E(t) \otimes_{F_p} [\beta t]$ , where  $E(t)$  denotes an exterior algebra and  $\beta$  is the Bockstein operation. This time,  $\psi_R$  is determined by its values on  $t$  and  $\beta t$ .

**DEFINITION 7.7.** We define elements  $\xi_i \in A$  of degree  $2p^i - 2$  and  $\tau_i \in A$  of degree  $2p^i - 1$  for all  $i \geq 0$ , also an element  $\omega \in A$  of degree 0, by the identities

$$\psi_R t = t \otimes \omega + \sum_i (\beta t)^{p^i} \otimes \tau_i; \quad \psi_R \beta t = \sum_i (\beta t)^{p^i} \otimes \xi_i;$$

where  $t \in H^1X$  and  $X$  is any space.

As before, the identity  $\psi_R(t+u) = \psi_R t + \psi_R u$  in  $H^*(L \times L, \mathbb{Z}) \hat{\otimes} A$  shows that  $\psi_R t$  must take this special form. Similarly for  $\psi_R \beta t$ , except that taking  $t \in H^1S^1$  shows there can be no term in  $t$ . Again, we read off the co-unit homomorphism as  $\epsilon \xi_0 = 1$ ,  $\epsilon \xi_i = 0$  for  $i > 0$ ,  $\epsilon \tau_i = 0$  for all  $i$ , and  $\epsilon \omega = 1$ .

**THEOREM 7.8.** (Milnor) We have  $\xi_0 = 1 = \omega$  and

$$A = F_p[\xi_1, \xi_2, \xi_3, \dots] \otimes E(\tau_0, \tau_1, \tau_2, \dots).$$

PROOF. This is Milnor's Theorem 2 of [7].

Again, we may define the Milnor operations as the coefficients in  $\psi_R x$  with respect to the monomial basis of  $A$ . Also, we compute the comultiplication in  $A$ . Evaluation of (5.3a) on  $t$ ,

$$(\psi_R \otimes 1)\psi_R t = t \otimes 1 \otimes 1 + \sum_i (\beta t)^{p^i} \otimes \tau_i \otimes 1 + \sum_{i,j} (\beta t)^{p^{i+j}} \otimes \xi_j^{p^i} \otimes \tau_i$$

and

$$(1 \otimes \psi)\psi_R t = t \otimes 1 \otimes 1 + \sum_n (\beta t)^{p^n} \otimes \psi \tau_n,$$

yields the second formula below,

$$\psi \xi_n = \sum_{i+j=n} \xi_j^{p^i} \otimes \xi_i; \quad \psi \tau_n = \tau_n \otimes 1 + \sum_{i+j=n} \xi_j^{p^i} \otimes \tau_i \quad (7.9)$$

and the first follows from  $(\psi_R \otimes 1)\psi_R \beta t = (1 \otimes \psi)\psi_R \beta t$  exactly as for  $p = 2$ . Hence composition of Milnor operations analogously to (7.5), and conjugations as in Theorem 7.6.

§8. THE THOM SPECTRUM MU. In this section we study the universal operation on the cohomology theory  $MU^*(-)$  defined by the unitary Thom spectrum MU. The coefficient ring  $\pi = \pi_* MU$  is a well-known polynomial ring over  $\mathbb{Z}$  and Theorem 5.4 applies with  $A = MU_* MU$ . Since our purpose is to exhibit definitions and structure, we refer to Adams [1] for detailed proofs.

We follow the same plan as §7. The appropriate test space is infinite complex projective space  $P = P_\infty(\mathbb{C})$ , whose absolute cohomology  $MU^*(P, \emptyset)$  is the ring  $\pi[[x]]$  of formal power series in the Conner-Floyd Chern class  $x = c_1(\gamma)$  of the complex Hopf line bundle  $\gamma$  over  $P$ . We may identify  $MU^*P$  with the maximal ideal  $(x)$  in  $\pi[[x]]$ . Again we use the right coaction, so that we need formal conjugation  $c$  to keep the various  $\pi$ -module structures straight.

DEFINITION 8.1. We define elements  $b_i \in A$  of degree  $2i$  for  $i \geq 0$  by the identity

$$\psi_R cx = \sum_{i=0}^{\infty} cx^{i+1} \otimes b_i \quad \text{in } cMU^*P \otimes_{\pi} A,$$

where  $x = c_1(\gamma) \in MU^2P$ .

We read off the augmentations  $\epsilon b_0 = 1$  and  $\epsilon b_i = 0$  for  $i > 0$  from (5.3b).

THEOREM 8.2. (Adams) We have  $A = \pi[b_1, b_2, b_3, \dots]$  as ring and left  $\pi$ -module, and  $b_0 = 1$ .

We break up  $\psi_R$  according to the monomial basis of  $A$ .

DEFINITION 8.3. We define the Landweber-Novikov cohomology operation  $s_\alpha: MU^*Y \rightarrow MU^*Y$  for each multiindex  $\alpha$  and any  $Y$  by the identity

$$\psi_R cy = \sum_{\alpha} cs_{\alpha} y \otimes b^{\alpha} \text{ in } cMU^*Y \otimes_{\pi} A. (y \in MU^*Y)$$

The behavior of these operations on products is immediate from the multiplicativity of  $\psi_R$ , which leads to the usual Cartan formula. To determine the comultiplication in  $A$  we apply (5.3a) to the Chern class  $x$ , conjugating of course. We have

$$(\psi_R \otimes 1)\psi_R cx = \sum_i (\psi_R cx)^{i+1} \otimes b_i = \sum_i (\sum_j cx^{j+1} \otimes b_j)^{i+1} \otimes b_i.$$

Hence (5.3a) reduces to the identity

$$\sum_k cx^{k+1} \otimes \psi b_k = \sum_i (\sum_j cx^{j+1} \otimes b_j)^{i+1} \otimes b_i \quad (8.4)$$

from which we read off  $\psi b_k$  by picking out the coefficient of  $cx^{k+1}$  on the right. Unlike ordinary cohomology, the resulting formula is not simple. To compose Landweber-Novikov operations we proceed as for (7.5) by applying (5.3a) to a general class  $y$ , using Definition 8.3, to obtain

$$\sum_{\alpha, \beta} c(s_{\beta} s_{\alpha} y) \otimes b^{\beta} \otimes b^{\alpha} = \sum_{\gamma} cs_{\gamma} y \otimes \psi b^{\gamma} \quad (8.5)$$

and picking out those terms for which  $b^{\beta} \otimes b^{\alpha}$  appears in  $\psi b^{\gamma}$ .

Because the Hopf line bundle is universal, the formula of Definition 8.1 remains valid for the Chern class of any complex line bundle.

In §7, universality led to a simplification of the formula for  $\psi_R t$ , but here the effect is quite different. If  $\xi$  and  $\omega$  are two complex line bundles over  $X$  with Chern classes  $x = c_1(\xi)$  and  $y = c_1(\omega)$ , the Chern class of the tensor product line bundle  $\xi \otimes \omega$  is not  $x + y$  but a certain formal power series  $F(x, y) = \sum_{i, j} a_{ij} x^i y^j$  with coefficients  $a_{ij} \in \pi_{2i+2j-2} MU$ , called the formal group law or formal product for  $MU$ . This follows from the universal case, in which  $X = P \times P$  with  $\xi$  and  $\omega$  the two bundles induced from  $\gamma$  by the two projections  $P \times P \rightarrow P$  and  $MU^*(P \times P, \emptyset) = \pi[[x, y]]$ . Applying  $\psi_R$  to the equation  $c_1(\xi \otimes \omega) = F(x, y)$  gives, after conjugating,

$$\psi_R cc_1(\xi \otimes \omega) = \sum_{i, j} (\psi_R cx^i \cdot \psi_R cy^j) \psi_R a_{ij} \text{ in } cMU^*X \otimes_{\pi} A.$$

By Definition 8.1 the left side is

$$\sum_k cF(x, y)^{k+1} \otimes b_k = \sum_k c(\sum_{i, j} a_{ij} x^i y^j)^{k+1} \otimes b_k,$$

which on expansion involves the  $\eta_L a_{ij} \in A$  after we transfer  $a_{ij}$  across the  $\otimes$  sign. On the right side we use the fact that  $\psi_R$  is a right  $\pi$ -homomorphism. The end result is more cleanly expressed by introducing the formal power series  $b(z) = \sum_i b_i z^{i+1}$  for any  $z$  and working purely algebraically, as

$$b(\sum_{i, j} \eta_L a_{ij} x^i y^j) = \sum_{i, j} \eta_R a_{ij} b(x)^i b(y)^j \text{ in } A[[x, y]]. \quad (8.6)$$

We could write this more succinctly as  $b(F_L(x, y)) = F_R(b(x), b(y))$ , or yet

$b(x +_L y) = b(x) +_R b(y)$ . Equating the coefficients of  $x^m y^n$  expresses  $\psi_R^{a_{mn}}$  inductively in terms of the  $\eta_L a_{ij}$  and the  $b_i$ . Now Quillen proved [9] (or see Adams [1]) that the elements  $a_{ij}$  generate the ring  $\pi$ , so that we have expressed the homomorphism  $\eta_R: \pi \rightarrow \Lambda$  in terms of  $\eta_L$  and  $b_i$ .

The generators  $a_{ij}$  of  $\pi$  are clearly not very practical. It is more convenient to work rationally by using the rational Hurewicz map  $h: MU = MU_S \rightarrow MU_H = MU_* H \otimes_Q H$  (we apply Lemma 2.8) of ring spectra, where we write  $H = H(Q)$  for the rational Eilenberg-MacLane spectrum. On coefficient groups it induces by definition the rational Hurewicz homomorphism  $h_*: \pi \rightarrow MU_* H$ . This is a rational isomorphism, which we use to identify  $MU_* H$  with the rationalization  $\pi_\emptyset$  of  $\pi$ . Then  $h$  induces the rational Hurewicz natural transformations  $h: MU_* X \rightarrow \pi_\emptyset \otimes_Q H_* X$  and  $h: MU^* X \rightarrow \pi_\emptyset \hat{\otimes}_Q H^* X$ , where of course we are writing  $H^* X$  and  $H_* X$  for cohomology and homology with rational coefficients. In homology  $h$  always induces an isomorphism  $(MU_* X)_\emptyset \cong \pi_\emptyset \otimes_Q H_* X$ , and in cohomology we have a monomorphism when there is no torsion.

Now  $hx$  remains a Chern class for  $\gamma$ , and  $\pi_\emptyset \hat{\otimes}_Q H^*(P, \emptyset) = \pi_\emptyset[[hx]]$ . But  $MU_H$  inherits another Chern class  $x' = c_1^H(\gamma)$  from ordinary cohomology, which we therefore express in terms of  $hx$ . That is, we define elements  $m_i \in \pi_\emptyset$  of degree  $2i$  for all  $i \geq 0$  by the identity

$$c_1^H(\gamma) = \log(hc_1(\gamma)) = \sum_{i=0}^{\infty} m_i \otimes hc_1(\gamma)^{i+1} \text{ in } \pi_\emptyset \hat{\otimes}_Q H^* P, \quad (8.7)$$

which defines the formal logarithmic series  $\log z = \sum_{i=0}^{\infty} m_i z^{i+1}$  over  $\pi_\emptyset$ . Closer examination of  $h$  and the Conner-Floyd Chern class reveals that  $m_0 = 1$  and that  $\pi_\emptyset$  is the polynomial ring  $Q[m_1, m_2, m_3, \dots]$ . By the universality of  $\gamma$ , (8.7) extends to any line bundle over any space. Then the identity  $c_1^H(\xi \otimes \omega) = c_1^H(\xi) + c_1^H(\omega)$  in ordinary cohomology, combined with (8.7) and the formal group law, yields the identity

$$\log F(x, y) = \log x + \log y \quad \text{in } \pi_\emptyset[[x, y]].$$

This identity gives the logarithmic series its name and allows one to solve for the  $a_{ij}$  in terms of the  $m_i$ , for example

$$a_{11} = -2m_1, \quad a_{21} = -3m_2 + 4m_1^2, \quad a_{31} = -4m_3 + 12m_1 m_2 - 8m_1^3, \quad a_{22} = -6m_3 + 24m_1 m_2 - 20m_1^3, \dots$$

(Here and later, the dots mean that one can compute as far as one wishes, not that one can write down the answers in advance.)

The commutative square of maps of ring spectra

$$\begin{array}{ccc} MU = MU_S & \xrightarrow{1 \wedge i} & MU \wedge MU \\ \downarrow h & & \downarrow 1 \wedge h \\ MU_H = MU_S \wedge H & \xrightarrow{1 \wedge i \wedge 1} & MU \wedge MU_H \end{array}$$

induces, by Lemma 2.8, Theorem 5.4 and the remarks above, the commutative square of ring homomorphisms

$$\begin{array}{ccc} \text{MU}^*P & \xrightarrow{\psi_L^P} & \Lambda_{\pi}^{\otimes} \text{MU}^*P \\ \downarrow h & & \downarrow 1 \otimes h \\ \pi_{\emptyset}^{\otimes} \text{H}^*P & \xrightarrow{\psi_L H \otimes 1} & \Lambda_{\pi}^{\otimes} \pi_{\emptyset}^{\otimes} \text{H}^*P \cong \Lambda_{\emptyset}^{\otimes} \text{H}^*P. \end{array} \quad (8.8)$$

We may identify the bottom line with  $\eta_L \otimes 1$ , where  $\eta_L: \pi_{\emptyset} \rightarrow \Lambda_{\emptyset}$  denotes the rationalization of  $\eta_L: \pi \rightarrow \Lambda$ . We evaluate (8.8) on the MU-Chern class  $x$ . By Definition 8.1,  $\psi_L x = \text{cb}(x) = \sum_i \text{cb}_i \otimes x^{i+1}$ , so that by commutativity,  $\psi_L H \otimes 1$  takes  $hx$  to  $\text{cb}(hx)$ . It therefore takes  $\log hx$  to  $\log_L \text{cb}(hx)$ , where we write formally  $\log_L z = \sum_i \eta_L m_i z^{i+1} = \eta_L \log z$ , and it plainly takes  $1 \otimes x'$  to  $1 \otimes 1 \otimes x'$ , where  $x' \in H^2 P$  denotes the Chern class in ordinary cohomology. On each side we now use  $1 \otimes x' = \log hx$ . Then in  $\Lambda_{\emptyset}^{\otimes} \text{H}^*P$  we find  $\log_R hx = \log_L \text{cb}(hx)$ , where similarly  $\log_R z = \eta_R \log z$ . Finally, we replace  $hx$  by the formal indeterminate  $z$  and conjugate to obtain

$$\log_L z = \log_R b(z) \quad \text{in } \Lambda_{\emptyset}[[z]] \quad (8.9)$$

Equating coefficients of  $z^{n+1}$  then expresses  $\eta_R m_n$  inductively in terms of the  $\eta_L m_i$  and  $b_i$ , as required.

If instead of conjugating  $\log_R z = \log_L \text{cb}(z)$  we replace  $z$  by the series  $b(z)$  and compare with (8.9), we see that  $\log_L \text{cb}(b(z)) = \log_L z$  and hence  $\text{cb}(b(z)) = z$ . In other words,  $\text{cb}(z)$  is the inverse series to  $b(z)$  with coefficients

$$\text{cb}_1 = -b_1, \text{cb}_2 = -b_2 + b_1^2, \text{cb}_3 = -b_3 + 5b_1 b_2 - 5b_1^3, \text{cb}_4 = -b_4 + 6b_1 b_3 + 3b_2^2 - 21b_1^2 b_2 + 14b_1^4, \dots \quad (8.10)$$

and we have the conjugation in  $\Lambda$ .

§9. THE BROWN-PETERSON SPECTRUM BP. In this section we study the universal operation for BP-cohomology, where BP denotes the Brown-Peterson spectrum for the prime  $p$ . The plan differs somewhat from §7. For the rest of this paper all tensor products are taken over  $\pi = \pi_* \text{BP}$  unless otherwise indicated.

We recall some elementary facts about BP. In [9], Quillen constructed it as a summand of the localization at  $p$  of MU, and the canonical map  $\text{MU} \rightarrow \text{BP}$  takes  $m_i \in H_* \text{MU}$  to an element we call  $m_j \in H_* \text{BP}$  if  $i = p^j - 1$ , or to 0 if  $i$  is not of this form. Then  $H_* \text{BP} = Z_{(p)}[m_1, m_2, \dots]$ . The canonical map equips BP-theory with a Chern class, whose formal logarithmic series is therefore

$$\log z = z + m_1 z^p + m_2 z^{p^2} + m_3 z^{p^3} + \dots$$

The Hurewicz homomorphism embeds  $\pi_* \text{BP}$  in  $H_* \text{BP}$ . Hazewinkel constructed [6] convenient polynomial generators  $v_i \in \pi_* \text{BP}$  over  $Z_{(p)}$  of degree  $2(p^i - 1)$

for all  $i > 0$  by the formula

$$p \log z = pz + \sum_{i=1}^{\infty} \log v_i z^{p^i},$$

or, expanding and equating coefficients,

$$v_n = pm_n - \sum_{j=1}^{n-1} m_j v_{n-j}^{p^j}. \quad (9.1)$$

For example, for  $p = 2$  we find

$$v_1 = 2m_1, \quad v_2 = 2m_2 - 4m_1^3, \quad \dots \quad \text{and} \quad m_1 = v_1/2, \quad m_2 = v_2/2 + v_1^3/4, \quad \dots$$

Quillen constructed a map of ring spectra  $r: BP \rightarrow BP[t_1, t_2, \dots]$ , where  $t_i$  also has degree  $2(p^i - 1)$ , by requiring  $r_*: H_*BP \rightarrow H_*BP[t_1, t_2, \dots]$  to be given by the formula

$$r_* \log z = \sum_{i=0}^{\infty} \log(z^{p^i} t_i), \quad \text{where } t_0 = 1. \quad (9.2)$$

He did not state whether this "coaction" of  $Z[t_1, t_2, \dots]$  was to be considered a left coaction or right coaction.

On the other hand, Adams defined [1, Theorem 16.1, p.112] elements  $t_i \in A = BP_*BP$  by the formula

$$\log_R z = \sum_{i=0}^{\infty} \log_L t_i z^{p^i} \quad \text{in } A_0[[z]] \quad (9.3)$$

where, just as in §8, we work rationally and extend the two unit homomorphisms  $\eta_L$  and  $\eta_R$  to  $H_*BP \rightarrow A_0$  to define the series  $\log_L z$  and  $\log_R z$ . Of course, he had to prove that  $t_i$  actually lies in  $A$  rather than  $A_0$ .

**THEOREM 9.4.** (Adams)  $A = \pi[t_1, t_2, t_3, \dots]$  and  $t_0 = 1$ .

We have the obvious problem of reconciling the two different sets of  $t_i$ , which lie in different groups. Consider the right coaction  $\psi_R: H_*BP \rightarrow H_*BP \otimes A$ . By (9.3) we have

$$\psi_R \log z = \log_R z = \sum_i \log_L (1 \otimes t_i) z^{p^i},$$

which we compare with (9.2).

**COROLLARY 9.5.** We can identify the right coaction  $\psi_R: BP \rightarrow BP \otimes_{\pi} A$  with Quillen's map  $r: BP \rightarrow BP \otimes_Z Z[t_1, t_2, \dots] = BP \otimes_{\pi} \pi[t_1, t_2, \dots]$ . (We have of course slightly extended the notation of §2 in allowing tensor products the other way round.)

From the map  $r$ , Quillen obtained cohomology operations by taking coefficients of the monomials  $t^\alpha$ , and showed that they give rise to all operations. We recast these as in Definitions 7.3 and 8.3.

**DEFINITION 9.6.** For each multiindex  $\alpha$  we define the Quillen cohomology operation  $r_\alpha: BP^*X \rightarrow BP^*X$  by the identity

$$\psi_R cy = \sum_{\alpha} c(r_\alpha y) \otimes t^\alpha \quad \text{in } cBP^*X \hat{\otimes} A.$$

Just as in (7.5) and (8.5), we can compose Quillen operations as soon as

we know the comultiplication  $\psi$  in  $A$ . This may be found by applying the bi-module homomorphism  $\psi: A \rightarrow A \otimes A$  to (9.3), once we know  $\eta_R$  in terms of  $\eta_L$  and the  $t_i$ . Let us continue to write  $v_i = \eta_L v_i \in A$  and introduce the notation  $w_i = \eta_R v_i \in A$ . Then (9.1) and (9.3) express the  $w_i$  in terms of the  $v_i$  and  $t_i$ , and for  $p = 2$  we find

$$w_1 = v_1 + 2t_1, \quad w_2 = v_2 - 3v_1^2 t_1 - 5v_1 t_1^2 + 2t_2 - 4t_1^3, \dots \quad (9.7)$$

Alternatively, one sometimes needs  $v_n$  in terms of the  $w_i$  and  $t_i$ , for example (if  $p = 2$ )

$$v_1 = w_1 - 2t_1, \quad v_2 = w_2 + 3t_1 w_1^2 - 7t_1^2 w_1 - 2t_2 + 6t_1^3, \dots \quad (9.8)$$

For  $p = 2$  the results for the comultiplication are

$$\begin{aligned} \psi t_1 &= t_1 \otimes 1 + 1 \otimes t_1 \\ \psi t_2 &= t_2 \otimes 1 + t_1 \otimes t_1^2 - v_1 t_1 \otimes t_1 + 1 \otimes t_2 = t_2 \otimes 1 + 2t_1^2 \otimes t_1 + 3t_1 \otimes t_1^2 - t_1 \otimes t_1 w_1 + 1 \otimes t_2. \end{aligned}$$

Many more formulae are given by Giambalvo [5].

The conjugation  $c$  in  $A$  may be computed by conjugating (9.3) to give  $\log_L z = \sum_i \log_R (ct_i) z^{p^i}$  and expanding. For  $p = 2$  the results are

$$ct_1 = -t_1, \quad ct_2 = -t_2 - t_1^3 - v_1 t_1^2 = -t_2 + t_1^3 - t_1^2 w_1, \dots \quad (9.10)$$

We shall need to apply the universal operation to Chern classes. Just as in Definition 8.1 for  $MU$ , we can define elements  $b_i \in A$  by the identity

$$\psi_R cx = \sum_{i=0}^{\infty} cx^{i+1} \otimes b_i \quad \text{in } BP^*X \hat{\otimes} A, \quad (9.11)$$

valid for the BP-Chern class  $x = c_1(\xi)$  of any line bundle  $\xi$ . To express  $b_i$  in terms of the previous generators of  $A$ , it is convenient to introduce the conjugates  $h_i = ct_i \in A$ , following Bendersky [3].

LEMMA 9.12. We have  $\log_R b(z) = \log_L z = \sum_i \log_R h_i z^{p^i}$ , where  $b(z) = \sum_i b_i z^{i+1}$ .

PROOF. The first equality is (8.9), which remains valid here. The second is the conjugate of (9.3).

Finally we read off the formulae for  $\psi$  and  $c$  on  $h_i$  in case  $p = 2$  from (9.9) and (9.10) as follows:

$$\begin{aligned} \psi h_1 &= h_1 \otimes 1 + 1 \otimes h_1 \\ \psi h_2 &= h_2 \otimes 1 - v_1 h_1 \otimes h_1 + 3h_1^2 \otimes h_1 + 2h_1 \otimes h_1^2 + 1 \otimes h_2 = h_2 \otimes 1 - h_1 \otimes h_1 w_1 + h_1^2 \otimes h_1 + 1 \otimes h_2. \end{aligned}$$

and

$$ch_1 = -h_1, \quad ch_2 = -h_2 + h_1^3 - v_1 h_1^2 = -h_2 - h_1^3 - h_1^2 w_1. \quad (9.14)$$

§10. UNIVERSAL UNSTABLE OPERATIONS. In this section we extend our theory of universal operations to unstable cohomology operations, with emphasis on

BP-theory. This work is of course based heavily on Ravenel-Wilson [10]. For future reference we begin more generally, with a commutative ring spectrum  $E$  having coefficient ring  $\pi = \pi_* E$ , and a second spectrum  $G$ . Tensor products are taken over  $\pi$ . Throughout this section  $X$  will denote a CW-space rather than a spectrum.

On the homotopy category of based spaces, the cohomology group functor  $G^n X$  of  $X$  is represented by a space  $\underline{G}_n$ ,  $G^n X \cong [X, \underline{G}_n]$ , where the spaces  $\underline{G}_n$  form the  $\Omega$ -spectrum corresponding to  $G$ . By the Yoneda lemma, cohomology operations  $G^n X \rightarrow E^* X$  correspond to elements of  $E^* \underline{G}_n$ . Since we no longer have additive categories, unstable operations need not be additive (for example, if  $\alpha$  and  $\beta$  are additive operations, the operation  $\gamma$  defined by  $\gamma x = \alpha x \cdot \beta x$  is rarely additive). We need to know which elements of  $E^* \underline{G}_n$  correspond to additive operations.

We assume we have duality,  $E^* \underline{G}_n \cong (E_* \underline{G}_n)^*$ . Slightly more generally, we take a free right  $\pi$ -module  $M$  and consider operations  $G^n X \rightarrow (M \otimes E)^* X$ , which are classified by  $(M \otimes E)^* \underline{G}_n$ . We assume the universal coefficient theorem 4.2 applies,  $(M \otimes E)^* \underline{G}_n \cong \text{Hom}_\pi(E_* \underline{G}_n, cM)$ , and ask which homomorphisms  $E_* \underline{G}_n \rightarrow cM$  correspond to additive cohomology operations.

Now addition in  $G^n X$  is induced by a multiplication map  $\mu$  on  $\underline{G}_n$  that makes  $\underline{G}_n$  an H-space (indeed, an infinite loop space), and  $E_*(\underline{G}_n, \emptyset)$  thus becomes a  $\pi$ -algebra with units  $E_*(o, \emptyset) \cong \pi$  inherited from the basepoint  $o$  of  $\underline{G}_n$ . The map  $\underline{G}_n \rightarrow o$  induces an augmentation  $E_*(\underline{G}_n, \emptyset) \rightarrow \pi$  of which  $E_* \underline{G}_n = E_*(\underline{G}_n, o)$  is the augmentation ideal, which allows us to consider the "indecomposables"  $QE_* \underline{G}_n = QE_*(\underline{G}_n, \emptyset)$  of the algebra  $E_*(\underline{G}_n, \emptyset)$  as a quotient of  $E_* \underline{G}_n$ .

LEMMA 10.1. Under suitable hypotheses on  $\underline{G}_n$  and  $E$ , in particular if  $G = E = \text{BP}$ ,

(a) the operation  $G^n X \rightarrow E^* X$  corresponding to a homomorphism  $f: E_* \underline{G}_n \rightarrow \pi$  is additive if and only if  $f$  factors through  $QE_* \underline{G}_n$ , so that the additive operations correspond to the dual  $(QE_* \underline{G}_n)^*$ ;

(b) the operation  $G^n X \rightarrow (M \otimes_\pi E)^* X$  corresponding to a homomorphism  $f: E_* \underline{G}_n \rightarrow cM$  is additive if and only if  $f$  factors through  $QE_* \underline{G}_n$ , so that the additive operations correspond to  $\text{Hom}_\pi(QE_* \underline{G}_n, cM)$ .

PROOF. We prove (b), of which (a) is a special case. Let  $\alpha: G^n X \rightarrow (M \otimes E)^* X$  be the operation, and assume given elements  $x, y \in G^n X$ , that is, maps of spaces  $x, y: X \rightarrow \underline{G}_n$ . The element  $\alpha(x+y)$  is the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{xy} \underline{G}_n \times \underline{G}_n \xrightarrow{\mu} \underline{G}_n \xrightarrow{\alpha} M \otimes E,$$

where  $\Delta$  is the diagonal map, while  $\alpha x$  and  $\alpha y$  are simply the composites  $\alpha \circ x$  and  $\alpha \circ y$ . The universal example is given by  $X = \underline{G}_n \times \underline{G}_n$  with

$x = p_1: X \rightarrow G_n$  and  $y = p_2: X \rightarrow G_n$  the projections to the factors, and the necessary and sufficient condition for additivity is therefore

$$\alpha \mu = \alpha p_1 + \alpha p_2: G_n \times G_n \rightarrow M \otimes E.$$

If the universal coefficient theorem 4.2 holds also for  $(M \otimes E)^*(G_n \times G_n)$ , the condition reduces to

$$f \circ \mu_* = f \circ p_{1*} + f \circ p_{2*}: E_*(G_n \times G_n) \rightarrow cM.$$

We further assume the Künneth formula, that the pairing  $E_*(G_n, \emptyset) \otimes E_*(G_n, \emptyset) \rightarrow E_*(G_n \times G_n, \emptyset)$  is an isomorphism (or at least epic). By means of the splitting  $E_*(G_n, \emptyset) \cong E_*G_n \oplus \pi$ , the condition reduces to  $f(ab) = 0$  for all  $a, b \in E_*G_n$ , as required. Results of Ravenel-Wilson [10] show that all the assumptions we made hold if  $G = E = BP$ .

We extend our previous definition 5.1 of universal operations in the obvious way to unstable operations.

DEFINITION 10.2. Given an integer  $n$  and a free right  $\pi$ -module  $R$ , we call a natural operation  $\psi_L: E^n X \rightarrow (R \otimes_\pi E)^* X$  a universal unstable operation if given any operation  $\theta: E^n X \rightarrow (M \otimes_\pi E)^* X$ , where  $M$  is a free right  $\pi$ -module, there exists a unique homomorphism  $g: R \rightarrow M$  of right  $\pi$ -modules that makes the diagram

$$\begin{array}{ccc} E^n X & \xrightarrow{\psi_L} & (R \otimes E)^* X \\ & \searrow \theta & \downarrow (g \otimes E)_* \\ & & (M \otimes E)^* X \end{array}$$

commute. If  $\psi_L$  is additive and the condition is required only for additive operations  $\theta$ , we call  $\psi_L$  a universal additive unstable operation.

If we have for each  $n$  a module  $R^n$  and an operation  $\psi_L: E^n X \rightarrow (R^n \otimes E)^* X$ , we assemble these to form the operation  $\psi_L: E^* X \rightarrow (R^* \otimes E)^* X$ , where  $R^*$  is now bigraded (not the dual of anything) and  $\psi_L$  preserves the new grading. If, further,  $R^*$  is a ring (a right  $\pi$ -algebra in the terminology of Lemma 2.7),  $R^* \otimes E$  becomes a ring spectrum and we can ask whether  $\psi_L$  is multiplicative. The definitions of the universal multiplicative unstable operation and the universal additive multiplicative unstable operation should now be clear.

As usual, each of the four kinds of universal operation is unique up to isomorphism if it exists. As in the stable case, existence is also easy if we assume enough. From now on we concentrate on the case  $E = BP$ .

THEOREM 10.3. For  $BP$  we have the following universal unstable operations:

- (a)  $\psi_L: BP^n X \rightarrow cBP_* \widehat{BP}_n \otimes_\pi BP^* X$  is universal, for each  $n$ ;
- (b)  $\psi_L: BP^* X \rightarrow cBP_* \widehat{BP}_* \otimes_\pi BP^* X$  is universal multiplicative, where  $BP_* \widehat{BP}_*$

is endowed with the circle multiplication [10];

- (c)  $\psi_L: BP^n X \rightarrow cQ_*^n \widehat{BP}_n \otimes_\pi BP^* X$  is universal additive, for each  $n$ , where we write  $Q_*^n = QBP_* \widehat{BP}_n$ ;

- (d)  $\psi_L: BP^* X \rightarrow cQ_*^* \widehat{BP}_* \otimes_\pi BP^* X$  is universal additive multiplicative.

PROOF. Results of Ravenel-Wilson [10] show that  $BP_*BP_n$  and  $Q_*^n$  are free left  $\pi$ -modules of finite type, so that the target cohomology theories are defined and satisfy Theorem 2.9(c). In (a), given an operation  $\theta$ , or map  $\theta: BP_n \rightarrow M \otimes BP$ , the homomorphism of left  $\pi$ -modules

$$g(\theta): BP_*BP_n \xrightarrow{\theta_*} BP_*(M \otimes BP) \xrightarrow{\sim} M \otimes BP_*BP \xrightarrow{1 \otimes \epsilon} M \otimes \pi = M \xrightarrow{\sim} cM$$

(using two conjugation isomorphisms) sets up the 1-1 correspondence between operations  $\theta$  and homomorphisms  $g(\theta)$  for the universal coefficient theorem 4.2. As in Theorem 5.4, we define  $\psi_L$  by requiring  $g(\psi_L)$  to be the identity homomorphism of  $BP_*BP_n$ , whence it follows by naturality that  $\psi_L$  is universal. As  $n$  varies, these are also multiplicative, because cup products in  $BP^*X$  are induced by maps of spaces  $BP_m \times BP_n \rightarrow BP_{m+n}$  which Ravenel and Wilson use to invest  $BP_*BP_*$  with the circle product structure. Because  $g(\theta)$  is evidently multiplicative whenever  $\theta$  is, we have (b).

Part (c) follows easily from (a), since Lemma 10.1 shows that everything factors through  $Q_*^n = QBP_*BP_n$ . Then (d) is similar to (b), since the circle product on  $BP_*BP_*$  passes through to make  $Q_*^*$  a bigraded algebra.

The nonadditive operations (a) and (b) appear difficult to use, and are properly handled by the Hopf ring structure on  $BP_*BP_*$  that Ravenel and Wilson set up. We shall say no more about them.

From now on we concentrate on the left coactions (c) and (d) and the relevant bigraded commutative algebra  $Q_*^*$ , whose multiplication comes from the circle product (the star product, induced by the H-space structure of  $BP_n$ , having disappeared from sight). We regard elements of  $Q_i^n$  as having degree  $i - n$  or, equivalently, codegree  $n - i$ . As usual, our machinery tends to produce left  $\pi$ -modules, when our conventions require a right module to form a tensor product. We have chosen to conjugate  $Q_*^*$  formally, but we could equally well consider the right coaction  $\psi_R: cBP^n X \rightarrow cBP^*X \hat{\otimes} Q_*^n$ , which differs only formally from  $\psi_L$ .

As in Theorem 5.2 for the stable case, the universality of  $Q_*^*$  already implies much structure.

The left unit ring homomorphism  $\eta_L: \pi \rightarrow Q_*^*$  is induced by  $i_*: BP_*S \rightarrow BP_*BP_0$  and gives each  $Q_*^n = QBP_*BP_n$  its usual free left  $\pi$ -module structure. We clearly have  $\eta_L: \pi_n BP \rightarrow Q_n^0$ . Since  $Q_*^*$  has no torsion, we define the series  $\log_L z = \eta_L \log z$  over  $Q_*^*$  rationalized, as in §8.

The right unit  $\eta_R: \pi \rightarrow Q_*^*$  is defined as

$$\eta_R = \psi_L S: \pi_n BP = BP^{-n}S \rightarrow cQ_*^{-n} \otimes BP^*S \xrightarrow{\sim} cQ_*^{-n} \xrightarrow{\sim} Q_*^{-n}.$$

In fact,  $\eta_R: \pi_n BP \rightarrow Q_0^{-n}$ , and we have a ring homomorphism as  $n$  varies, by the multiplicativity of  $\psi_L$ . It is used to make  $Q_*^*$  a right  $\pi$ -module, and to define the series  $\log_R z = \eta_R \log z$ .

REMARK. It is amusing to note that  $Q_*^*$  is also a free right  $\pi$ -module, but so far this is a theorem in search of an application.

The comultiplication  $\psi: Q_i^n \rightarrow Q_i^* \otimes Q_*^n$  is defined by the universal property of (c) to make the diagram

$$\begin{array}{ccc} BP^n X & \xrightarrow{\psi_L} & cQ_*^n \hat{\otimes} BP^* X \\ \downarrow \psi_L & & \downarrow c\psi \otimes 1 \\ cQ_*^n \hat{\otimes} BP^* X & \xrightarrow{1 \otimes \psi_L} & cQ_*^n \hat{\otimes} cQ_*^* \hat{\otimes} BP^* X \end{array}$$

commute. It involves only elements of the form  $x \otimes y$  with  $x \in Q_i^j$  and  $y \in Q_j^n$  for the same  $j$ . By the universality of (d), we find a ring homomorphism as  $n$  and  $i$  vary.

The augmentation  $\epsilon: Q_*^n \rightarrow \pi$  is defined to make the diagram

$$\begin{array}{ccc} BP^n X & \xrightarrow{\psi_L} & cQ_*^n \hat{\otimes} BP^* X \\ \downarrow & & \downarrow c\epsilon \otimes 1 \\ BP^* X & \xrightarrow{=} & \pi \otimes BP^* X \end{array}$$

commute. Again a ring homomorphism as  $n$  varies. It is a left and right counit for  $\psi$ .

In the unstable case there is extra structure. We define the suspension  $\Sigma: Q_i^n \rightarrow Q_{i+1}^{n+1}$  to make the diagram

$$\begin{array}{ccc} BP^n X & \xrightarrow{\psi_L} & cQ_*^n \hat{\otimes} BP^* X \\ \downarrow \cong & & \downarrow c \Sigma \otimes 1 \\ BP^{n+1} \Sigma X & \xrightarrow{\psi_L} & cQ_*^{n+1} \hat{\otimes} BP^* \Sigma X \cong cQ_*^{n+1} \hat{\otimes} BP^* X \end{array}$$

commute. It is of course not multiplicative, because  $\Sigma: BP^n X \cong BP^{n+1} \Sigma X$  is not. However, this may be regarded as an isomorphism of  $BP^* X$ -modules, from which it follows that  $\Sigma(a \cdot b) = (\Sigma a) \cdot b$  in  $Q_*^*$ . Therefore  $\Sigma$  is nothing but multiplication by the suspension element  $e = \Sigma 1 \in Q_1^1$ . Alternatively,  $e$  is the image of 1 under the homomorphism

$$BP_* S \xrightarrow{i_*} BP_* \underline{BP}_0 \xrightarrow{\sim} BP_* \Sigma \underline{BP}_0 \rightarrow BP_* \underline{BP}_1 \rightarrow QBP_* \underline{BP}_1,$$

where we use the structure map  $\Sigma \underline{BP}_0 \rightarrow \underline{BP}_1$  of the  $\Omega$ -spectrum  $\underline{BP}_*$ .

Finally, we may regard any stable operation as an unstable operation on  $BP^n X$  for any  $n$ , which leads to the stabilization homomorphism  $\sigma: Q_*^n \rightarrow A = BP_* BP$ , defined to make the diagram

$$\begin{array}{ccc} BP^n X & \xrightarrow{\psi_L} & cQ_*^n \hat{\otimes} BP^* X \\ \downarrow = & & \downarrow c\sigma \otimes 1 \\ BP^n X & \xrightarrow{\psi_L} & A \hat{\otimes} BP^* X \end{array}$$

commute, where of course we identify  $cA$  with  $A$  as usual. This too gives a ring homomorphism  $\sigma: Q_*^* \rightarrow A$ , which, by comparison of the definitions here

with the stable definitions in §5 and repeated use of universal properties, carries the structure of  $Q_*^*$  into the corresponding structure of  $A$ , except that  $\sigma e = 1$ .

There is no internal conjugation in  $Q_*^*$ , as will be obvious once we give the structure of  $Q_*^*$ .

We need generators for the algebra  $Q_*^*$ . The Hazewinkel generators  $v_i$  of  $\pi$  yield elements  $\eta_L v_i \in Q_*^0$ , that we continue to denote  $v_i$ , and elements  $w_i = \eta_R v_i \in Q_0^*$  (written  $[v_i]$  in Ravenel-Wilson [10]). We already defined the suspension element  $e \in Q_1^1$ . We define elements  $b_i \in Q_{2i+2}^2$  for  $i \geq 0$  as in Definition 8.1 (except that now we work unstably and use the left coaction),

$$\psi_L x = \sum_{i=0}^{\infty} c b_i \otimes x^{i+1} \quad \text{in } cQ_*^2 \hat{\otimes} BP^*X \quad (10.4)$$

where  $x = c_1(\xi)$  is the BP-Chern class of any complex line bundle  $\xi$  over  $X$ , and use them to define the formal power series  $b(z) = \sum_i b_i z^{i+1}$  over  $Q_*^*$ .

THEOREM 10.5. For  $Q_*^* = QBP_*BP_*$  we have:

(a)  $Q_*^*$  is the bigraded algebra with generators  $e, b_i, v_i$  and  $w_i$  and relations  $e^2 \log_L z = \log_R b(z)$ , in particular,  $b_0 = e^2$ ;

(b)  $\sigma: Q_*^* \rightarrow A$  is monic for all  $n$ ;

(c) we can define elements  $h_i \in Q_n^2$  (where  $n = 2p^i$ ) for all  $i \geq 0$  by  $\sigma h_i = h_i = c t_i \in A$ ;

(d)  $\sigma: Q_*^* \rightarrow A$  is the homomorphism of rings and of bimodules that carries  $e$  to 1,  $h_i$  to  $h_i$ ,  $v_i$  to  $v_i$ ,  $w_i$  to  $w_i$ , and  $b_i$  to  $b_i$ ;

(e)  $Q_*^*$  is the bigraded algebra with generators  $e, h_i, v_i$  and  $w_i$  and relations  $e^2 \log_L z = \sum_{i=0}^{\infty} \log_R h_i z^{p^i}$ , in particular,  $h_0 = e^2$ .

PROOF. This result is essentially due to Ravenel and Wilson [10]. They show (b) and that  $Q_*^*$  is torsion-free, so that we may safely work rationally. The relation in (a) is the appropriate destabilization of (8.9), either by using (b) or paralleling the proof of (8.9). Equating coefficients of  $z^{i+1}$  when  $i+1$  is not a power of  $p$  expresses  $b_i$  in terms of other generators, so that we need only those  $b_i$  with  $i$  of the form  $p^n - 1$ , which we write  $b_{(n)}$ . Equating coefficients of  $z^{p^n}$  gives

$$e^2 v_n = \sum_{i=1}^n b_{(n-i)}^{p^i} w_i + \text{less interesting terms,}$$

and Theorems 3.14 and 5.3 of [10] show in effect that these are sufficient relations.

By using the formal group law of BP, we can express  $\log_R b(z)$  in the form  $\sum_i \log_R g_i z^{i+1}$  for well defined elements  $g_i \in Q_{2i+2}^2$ , given only that  $b(z)$  is defined over  $Q_*^*$ . However, comparison with (a) shows by induction that  $g_i = 0$  unless  $i+1 = p^n$  for some  $n$ . We therefore relabel such  $g_i$  as  $h_n$ . Moreover, by construction  $b_{(i)} = h_i$  modulo decomposables, which shows that (e) follows from (a). Comparison of (e) with Lemma 9.12 shows that the

element  $h_i$  just defined does indeed stabilize to  $h_i = ct_i$  in  $A$ . The rest of the proof is now clear.

By the universal property, we may recover any additive unstable operation on  $BP^*(-)$  by composing  $\psi_L$  with a suitable homomorphism of left  $\pi$ -modules  $f: Q_*^* \rightarrow \pi$ .

EXAMPLE. Let us define a ring homomorphism  $f: Q_*^{\text{even}} \rightarrow \pi$  on generators by  $fb(z) = [p]z$ , the usual  $p$ -series (defined by  $\log [p]z = p \log z$ ),  $fv_i = v_i$ ,  $fw_i = v_i$  and  $fe^2 = p$ , which is consistent with (a), and extend additively to all of  $Q_*^*$  by  $f(ey) = fy$  for  $y \in Q_*^{\text{even}}$ . This defines Novikov's unstable  $\psi^p$  operation on  $BP^*(-)$ , such that  $\psi^p x = [p]x$  on the Chern class  $x \in BP^{2p}X$  of any line bundle over  $X$ . It is multiplicative, except for an extra factor  $p$  when two odd classes are multiplied.

REMARK. In Theorem 10.5 it is not a matter of choice that we use the elements  $h_i$  rather than  $t_i$ . One can show that  $t_i$  does not desuspend to  $Q_*^2$  for  $i > 0$ .

REMARK. Part (b) of Theorem 10.5 is extremely useful for computations. To compute  $\psi$  etc. in  $Q_*^*$  we merely have to do (or quote) the calculations in  $A$  and destabilize. For example, when  $p = 2$  we find

$$\begin{aligned} \psi h_0 &= h_0 \otimes h_0, \quad \psi h_1 = h_1 \otimes h_0 + h_0^2 \otimes h_1, \\ \psi h_2 &= h_2 \otimes h_0 + h_1^2 \otimes h_1 - h_0^2 h_1 \otimes h_0 h_1 w_1 + h_0^4 \otimes h_2, \dots \end{aligned} \quad (10.6)$$

Also, the counit  $\epsilon$  is read off directly as  $\epsilon h_0 = 1$ ,  $\epsilon v_i = v_i = \epsilon w_i$ ,  $\epsilon h_i = 0$  for  $i > 0$ ,  $\epsilon b_i = 0$  for  $i > 0$ , and  $\epsilon e = 1$ .

We pointed out in §6 that stably, the left coactions on homology and cohomology were the only ones we needed to study because all the other structures were readily deducible from these two. We close this section by pointing out that the left coaction on homology remains useful unstably.

We consider only the simplest case where  $BP_*X$  is a free  $\pi$ -module and a free coalgebra, so that it is sufficient to study the primitives,  $P(BP_*X)$ . For any left  $A$ -comodule  $M$ , define  $U(M)$  as the subgroup of  $A \otimes M$  spanned by all elements of the form  $h^\alpha \otimes m$ , where  $\deg(m) > 2 \sum \alpha_r$ ;  $M$  is called an unstable comodule if its coaction  $\psi_L: M \rightarrow A \otimes M$  factors through  $U(M)$ .

THEOREM 10.7. For  $X$  as above,  $P(BP_*X)$  is an unstable  $A$ -comodule.

PROOF. See §8 of Bendersky-Curtis-Miller [3].

§11. AN APPLICATION TO DESUSPENSION. This section demonstrates that our machinery can be used directly to produce concrete topological results. Our application is due to Wilson [11] and the methods are in principle the same, although our calculations are independent (and not guaranteed). We apply  $BP$ -theory for the prime  $p = 2$ , and all tensor products are taken over

$$\pi = \pi_* BP.$$

THEOREM 11.1. The real stunted projective space  $X = P_{16}^{26}(R)$  cannot be desuspended 11 times: that is, there does not exist a space  $Y$  with  $\Sigma^{11}Y$  homotopy-equivalent to  $X$ .

Note that although  $X$  is in the stable range,  $Y$  is highly unstable.

To calculate  $BP^*X$  we use various Atiyah-Hirzebruch spectral sequences that all obviously collapse. We know  $BP^*(P^\infty(C), \emptyset) = \pi[[x]]$ , where  $x = c_1(\gamma)$  is the Chern class of the complex Hopf line bundle  $\gamma$ . Naturality of the spectral sequence identifies  $BP^*P_8^\infty(C)$  with the ideal  $(x^8)$  in  $\pi[[x]]$ , and  $BP^*P_8^{13}(C)$  with the quotient of this group by  $(x^{14})$ . The map of spectral sequences induced by the complexification map  $X \rightarrow P_8^{13}(C)$  shows that the images of the elements  $x^i$  for  $8 \leq i \leq 13$  generate  $BP^*X$  as  $\pi$ -module. We continue to denote these images by  $x^i$ , even though products in  $BP^*X$  are trivial. Moreover,  $x^8$  generates a free summand in  $BP^*X$ , while there are relations  $2x^i = 0$  modulo higher filtration for  $i > 8$ .

To determine the exact relations in  $BP^*X$  we first consider the complexification  $P^\infty(R) \rightarrow P^\infty(C)$ . On  $P^\infty(R)$ ,  $\gamma$  is the complexification of the real Hopf line bundle and therefore  $c_1(\gamma \otimes \gamma) = 0$ . This leads to the relation  $[2]x = 0$  (see §10), where

$$[2]x = 2x - v_1x^2 + 2v_1^2x^3 - (7v_2 + 8v_1^3)x^4 + (30v_1v_2 + 26v_1^4)x^5 + \dots$$

More spectral sequences show that as  $\pi$ -algebra,  $BP^*(P^\infty(R), \emptyset)$  is generated by  $x$  with this the only relation, and  $P^\infty(R) \rightarrow P_{17}^\infty(R)$  identifies  $BP^*P_{17}^\infty(R)$  with the ideal  $(x^9)$  in  $BP^*(P^\infty(R), \emptyset)$ . Finally, the map  $X \rightarrow P_{17}^\infty(R)$  shows that the relations we seek in  $BP^*X$  are precisely  $([2]x) \cdot x^i = 0$  for  $i > 8$ . The results simplify as follows.

LEMMA 11.2. As  $\pi$ -module,  $BP^*X$  is generated by the elements  $x^i$  for  $8 \leq i \leq 13$  subject to the following relations:

$$\begin{aligned} 2x^{13} &= 0, \text{ and } BP^{26}X = Z/2, \text{ generated by } x^{13}; \\ 2x^{12} &= v_1x^{13}, \text{ and so } 4x^{12} = 0 \text{ and } BP^{24}X = Z/4, \text{ generated by } x^{12}; \\ 2x^{11} &= v_1x^{12}, \text{ so that } 8x^{11} = 0 \text{ and } BP^{22}X = Z/8, \text{ generated by } x^{11}; \\ 2x^{10} &= v_1x^{11} - v_1^3x^{13} + v_2x^{13}, \text{ so that } 16x^{10} = 0; \\ 2x^9 &= v_1x^{10} - v_1^3x^{12} + 3v_2x^{12}, \text{ so that } 32x^9 = 0. \end{aligned}$$

From now on we abbreviate by writing  $M = BP^*X$ . Because everything in  $M$  is defined (indirectly) in terms of  $\pi$  and Chern classes without ambiguity, (9.11), with the help of Lemma 9.12, gives complete information on the stable operations in  $M$ . In other words, we can compute the stable coaction

$$\psi_L: M \rightarrow A \otimes M.$$

We concentrate on the "bottom class"  $x^8$  in  $M$ . If  $X = \Sigma^{11}Y$ , there is

$$\psi_L: BP^{16}X \cong BP^5Y \rightarrow cQ_*^5 \otimes BP^*Y \cong cQ_*^5 \otimes M$$

which must stabilize under  $\sigma: Q_*^5 \rightarrow A$  to the known stable coaction on  $M$ . Thus our first question is whether  $\psi_L x^8 \in A \otimes M$  lifts to  $cQ_*^5 \otimes M$ ; such a lifting corresponds to choosing values for the unstable operations on  $x^8$  that stabilize correctly. The condition turns out to be rather weak and liftings do exist.

There is much more structure, however. If  $X$  desuspends, there are coactions  $\psi_L: BP^n X \rightarrow cQ_*^{n-11} \otimes M$  for all  $n$  that respect all the structure in §10, including the comultiplication in  $Q_*$ ; this corresponds to choosing unstable operations on all generators of  $M$  that compose correctly. Unfortunately, this is a nonlinear problem. We linearize it by composing the proposed unstable operations on  $x^8$  with only the known stable operations on  $M$ .

LEMMA 11.3. (a) There is a coaction  $\psi_L: Q_*^n \rightarrow A \otimes Q_*^n$  that makes  $\sigma: Q_*^n \rightarrow A$  a homomorphism of  $A$ -comodules;

(b) for any space  $Y$ , the diagram

$$\begin{array}{ccc} BP^n Y & \xrightarrow{\psi_L} & cQ_*^n \otimes BP^* Y \\ \downarrow \psi_L & & \downarrow c\psi_L \otimes 1 \\ cQ_*^n \otimes BP^* Y & \xrightarrow{1 \otimes \psi_L} & cQ_*^n \otimes A \otimes BP^* Y \end{array}$$

commutes;

(c) the coaction  $\psi_L: BP^n Y \rightarrow cQ_*^n \otimes BP^* Y$  factors through the cotensor product  $cQ_*^n \square_A BP^* Y \subset cQ_*^n \otimes BP^* Y$ .

PROOF. The universal property of  $\psi_L: BP^n Y \rightarrow cQ_*^n \otimes BP^* Y$  defines  $\psi_L: Q_*^n \rightarrow A \otimes Q_*^n$  to make (b) true, and shows that it is a coaction and stabilizes to  $\psi: A \rightarrow A \otimes A$ . The cotensor product is defined to make (c) equivalent to (b).

We plan to prove Theorem 11.1 by showing that  $\psi_L x^8$  is not in the image of  $c\sigma \square 1: cQ_*^5 \square_A M \rightarrow A \square_A M \cong M$ . In this case the groups are small enough that direct computation of the cotensor product is feasible, if unpleasant. To have any prospect of generalization, we clearly need a better method. We write  $cQ_*^5 \square_A M = \text{Cotor}_A^0(cQ_*^5, M)$  and compute it by resolving  $Q_*^5$  (the reverse of the usual approach in homological algebra).

We need only the first two terms of the resolution and we compute up to stable degree 10 for this example. (Our results are of course applicable more generally, to give obstructions to the existence of a 4-connected 15-dimensional space of any known stable type.) The Ravenel-Wilson basis [10] of  $Q_*^5$  in these degrees (modified as in Theorem 10.5 by using  $h_i$  rather than  $b_{(i)}$ ) consists of the 9 elements:

$$eh_0^2, eh_0 h_1, eh_1^2, eh_0 h_2, eh_0^5 w_2, eh_1 h_2, eh_0^4 h_1 w_2, eh_0 h_1 h_2 w_1, eh_0^3 h_1^2 w_2.$$

These stabilize to  $A$  in the obvious way (see Theorem 10.5) except for

$$\begin{aligned}\sigma(eh_0^5w_2) &= v_2 + 3v_1^2h_1 - 7v_1h_1^2 - 2h_2 + 6h_1^3, \\ \sigma(eh_0^4h_1w_2) &= v_2h_1 + 3v_1^2h_1^2 - 7v_1h_1^3 - 2h_1h_2 + 6h_1^4, \\ \sigma(eh_0^3h_1h_2w_1) &= v_1h_1h_2 - 2h_1^2h_2, \\ \sigma(eh_0^3h_1^2w_2) &= v_2h_1^2 + 3v_1^2h_1^3 - 7v_1h_1^4 - 2h_1^2h_2 + 6h_1^5.\end{aligned}$$

We use row reduction on the matrix of  $\sigma$  over the local ring  $Z_{(2)}$  to replace these four basis elements by more convenient ones  $q_6, q_8, q_{10}$  and  $q_{10}'$  (subscripts denote degree), defined by their stabilizations (see Theorem 10.5).

$$\sigma q_6 = 2h_1^3, \quad \sigma q_8 = 2h_1^4 + v_1h_1^3, \quad \sigma q_{10} = 2h_1^5 + v_1h_1^4, \quad \sigma q_{10}' = 2h_1^2h_2.$$

The kind of resolution we need is a minimal  $\pi$ -split resolution of  $Q_*^5$  by cofree  $A$ -comodules,

$$Q_*^5 \xrightarrow{\eta} C_0 \xrightarrow{d} C_1$$

with splitting homomorphisms  $s_0: C_0 \rightarrow Q_*^5$  and  $s_1: C_1 \rightarrow C_0$  of  $\pi$ -modules such that  $s_0 \circ \eta = 1$  and  $s_1 \circ d + \eta \circ s_0 = 1$ . We take  $C_i = A \otimes F_i$  with the obvious  $A$ -comodule structure, where  $F_i$  is a free  $\pi$ -module. Cofreeness means that given any homomorphism  $\bar{u}: N \rightarrow F_i$  of  $\pi$ -modules, where  $N$  is an  $A$ -comodule, there is a unique homomorphism  $u: N \rightarrow C_i$  of  $A$ -comodules such that  $\bar{u} = (\varepsilon \otimes 1) \circ u$ , namely

$$N \xrightarrow{\psi_L} A \otimes N \xrightarrow{1 \otimes u} A \otimes F_i = C_i \quad (11.4)$$

Thus we need specify only  $\bar{\eta}: Q_*^5 \rightarrow F_0$  and  $\bar{d}: C_0 \rightarrow F_1$ .

We take  $F_0$  to be  $\pi$ -free on generators  $f_0, f_6$  and  $f_{10}$ , with  $\bar{\eta}: Q_*^5 \rightarrow F_0$  defined by

$$\bar{\eta}(eh_0^2) = f_0, \quad \bar{\eta}q_6 = f_6, \quad \bar{\eta}q_{10} = f_{10},$$

and zero on the other basis elements. We use the comodule structure in  $Q_*^5$  (computed stably in  $A$  by (9.13)) and (11.4) to deduce  $\eta: Q_*^5 \rightarrow A \otimes F_0$  on basis elements:

$$\begin{aligned}\eta(eh_0^2) &= 1 \otimes f_0, \\ \eta(eh_0h_1) &= h_1 \otimes f_0, \\ \eta(eh_1^2) &= h_1^2 \otimes f_0, \\ \eta(eh_0h_2) &= h_2 \otimes f_0, \\ \eta q_6 &= 2h_1^3 \otimes f_0 + 1 \otimes f_6, \\ \eta(eh_1h_2) &= h_1h_2 \otimes f_0 + h_1 \otimes f_6, \\ \eta q_8 &= (2h_1^4 + v_1h_1^3) \otimes f_0 + 5h_1 \otimes f_6,\end{aligned}$$

$$\eta q_{10} = (2h_1^5 + v_1 h_1^4) \otimes f_0 - v_1 h_1 \otimes f_6 + 16h_1^2 \otimes f_6 + 1 \otimes f_{10},$$

$$\eta q'_{10} = 2h_1^2 h_2 \otimes f_0 - 2v_1 h_1 \otimes f_6 + 9h_1^2 \otimes f_6.$$

We introduced the extra elements  $f_6$  and  $f_{10}$  precisely to allow a splitting  $s_0: C_0 \rightarrow Q^5$ , given on the  $\pi$ -basis elements of  $C_0$  by:

$$\begin{aligned} s_0(1 \otimes f_0) &= eh_0^2, \quad s_0(h_1 \otimes f_0) = eh_0 h_1, \quad s_0(h_1^2 \otimes f_0) = eh_1^2, \quad s_0(h_2 \otimes f_0) = eh_0 h_2, \\ s_0(1 \otimes f_6) &= q_6, \quad s_0(h_1 h_2 \otimes f_0) = eh_1 h_2, \quad s_0(h_1 \otimes f_6) = q_8/5, \quad s_0(h_1^2 \otimes f_6) = q'_{10}/9, \\ s_0(1 \otimes f_{10}) &= q_{10}, \text{ and zero on } h_1^3 \otimes f_0, h_1^4 \otimes f_0, h_1^5 \otimes f_0 \text{ and } h_1^2 h_2 \otimes f_0. \end{aligned}$$

Next we need  $d: C_0 \rightarrow C_1 = A \otimes F_1$ , where  $F_1$  must be large enough to take care of the four basis elements of  $C_0$  that  $s_0$  kills. We take  $F_1$  free on two generators  $g_6$  and  $g_8$ , and specify  $\bar{d}(h_1^3 \otimes f_0) = g_6$  and  $\bar{d}(h_1^4 \otimes f_0) = g_8$ . The requirement  $\bar{d} \circ \eta = 0$  forces the complete specification of  $\bar{d}: C_0 \rightarrow F_1$  to be:

$$\begin{aligned} \bar{d}(h_1^3 \otimes f_0) &= g_6, \\ \bar{d}(1 \otimes f_6) &= -2g_6, \\ \bar{d}(h_1^4 \otimes f_0) &= g_8, \\ \bar{d}(h_1 \otimes f_6) &= -(2/5)g_8 - (1/5)v_1 g_6, \\ \bar{d}(h_1 h_2 \otimes f_0) &= (2/5)g_8 + (1/5)v_1 g_6, \\ \bar{d}(h_1^2 \otimes f_6) &= -(4/45)v_1 g_8 - (2/45)v_1^2 g_6, \\ \bar{d}(1 \otimes f_{10}) &= -(31/45)v_1 g_8 + (7/45)v_1^2 g_6, \end{aligned}$$

and zero on the other  $\pi$ -basis elements of  $C_0$ . Then by (11.4),

$$\begin{aligned} d(h_1^3 \otimes f_0) &= 1 \otimes g_6, \\ d(h_1^4 \otimes f_0) &= 1 \otimes g_8 + 4h_1 \otimes g_6, \\ d(h_1^5 \otimes f_0) &= 10h_1^2 \otimes g_6 + 5h_1 \otimes g_8, \\ d(h_1^2 h_2 \otimes f_0) &= -v_1 h_1 \otimes g_6 + 7h_1^2 \otimes g_6 + 2h_1 \otimes g_8. \end{aligned}$$

These four elements may be extended to a  $\pi$ -basis of  $C_1$ , on account of the presence of the terms  $1 \otimes g_6$ ,  $1 \otimes g_8$ ,  $h_1 \otimes g_8$  and  $h_1^2 \otimes g_6$  respectively. It follows that  $s_1$  exists as required. Fortunately, we need no more of  $d$ .

The cofreeness of  $C_i$  implies an isomorphism  $cC_i \square_A M \xrightarrow{\sim} cF_i \otimes M$ , induced by  $1 \otimes \psi_L: cF_i \otimes M \rightarrow cF_i \otimes A \otimes M = cC_i \otimes M$ . Hence

$$cQ^5 \square_A M = \text{Ker}(cd \square 1: cC_0 \square_A M \rightarrow cC_1 \square_A M) \xrightarrow{\sim} \text{Ker}(d_*: cF_0 \otimes M \rightarrow cF_1 \otimes M),$$

where the homomorphism  $d_*$  is defined as

$$cF_0 \otimes M \xrightarrow{1 \otimes \psi_L} cF_0 \otimes A \otimes M \xrightarrow{cd \otimes 1} cF_1 \otimes M.$$

We really want the stabilization  $\sigma: Q_*^5 \rightarrow A$  factors very easily as

$$Q_*^5 \xrightarrow{\eta} C_0 = A \otimes F_0 \xrightarrow{1 \otimes q} A \otimes \pi = A,$$

where  $q: F_0 \rightarrow \pi$  is the  $\pi$ -module homomorphism given by  $qf_0 = 1$ ,  $qf_6 = 0$  and  $qf_{10} = 0$ . Therefore in view of Lemma 11.2, candidates for the unstable  $\psi_L x^8$  have the form  $y = cf_0 \otimes x^8 + cf_6 \otimes ax^{11} + cf_{10} \otimes bx^{13}$  in  $cF_0 \otimes M$ , where  $a$  and  $b$  are integer coefficients defined mod 8 and mod 2 respectively. We use the stable coaction on  $M$ , computed by (9.11) and Lemma 9.12, to find:

$$\begin{aligned} d_*(cf_0 \otimes x^8) &= cg_8 \otimes 2x^{12}, \\ d_*(cf_6 \otimes x^{11}) &= cg_8 \otimes 2x^{12} + cg_6 \otimes 4x^{11}, \\ d_*(cf_{10} \otimes x^{13}) &= cg_8 \otimes 2x^{12} + cg_6 \otimes 4x^{11}, \end{aligned}$$

so that  $d_*y = cg_6 \otimes (4a+4b)x^{11} + cg_8 \otimes (2+2a+2b)x^{12}$ . Hence  $d_*y = 0$  if and only if  $4a + 4b = 0 \pmod{8}$  and  $2 + 2a + 2b = 0 \pmod{4}$ . These evidently admit no solutions, which establishes Theorem 11.1.

REMARK. The space  $X$  is small enough to be accessible by various bare-hand methods. For example, Theorem 11.1 can be proved by secondary operations in ordinary cohomology. However, one should note in contrast how little intelligence is used in our calculations, thanks to the richer structure of  $BP^*X$ . We use primary operations, and those only in an obvious way. We had (and still have) no idea which operation is the one that works, so in effect we computed all of them.

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DEPARTMENT OF MATHEMATICS  
JOHNS HOPKINS UNIVERSITY  
BALTIMORE, MARYLAND 21218