

EQUIVARIANT ORIENTATION OF VECTOR BUNDLES OVER DISCONNECTED BASE SPACES

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ABSTRACT. In this paper, we view the equivariant orientation theory of equivariant vector bundles from the lenses of equivariant Picard spectra. This viewpoint allows us to identify, for a finite group G , a precise condition under which an R -orientation of a G -equivariant vector bundle is encoded by a Thom class. Consequently, we are able to construct a generalization of the first Steifel-Whitney class of a “homogeneous” G -equivariant bundle with respect to an $\mathbb{E}_\infty^G$ -ring spectrum R . As an application, we show that the 2-fold direct sum of any homogeneous bundle is $H\mathcal{A}_G$ -orientable, where $\mathcal{A}_G$ is the Burnside Mackey functor. We notice that $H\mathcal{A}_G$ -orientability is equivalent to $H\mathbb{Z}$ -orientability when the order of G is odd. When the order of G is even, we show that a G -equivariant analog of the tautological line bundle over $\mathbb{RP}^\infty$ is $H\mathbb{Z}$ -orientable but not $H\mathcal{A}_G$ -orientable.

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1. Introduction

In differential topology, an orientation of a vector bundle is an orientation of each fiber which is compatible with the topology of the base space. In homotopy theory, an orientation is expressed as a Thom class in the ordinary cohomology with $\mathbb{Z}$ -coefficients of its Thom space. The concept of Thom class extends to extraordinary cohomology theories: For any ring spectrum R , an R -Thom class is a cohomology class in the R -cohomology of the Thom space ξ

$$u_\xi \in [\mathrm{Th}(\xi), \Sigma^{\dim \xi} R] = R^{\dim \xi}(\mathrm{Th}(\xi))$$

such that its restriction to the Thom space over any point is a unit in the homotopy group of R . This idea lead to many interesting ideas and results, which are now foundational in homotopy theory and its application to geometry.

Equivariant orientation theory was systematically developed in the eighties and nineties, geometrically in [CW92, CMW01], and cohomologically in [LMSM86, CW92, May96, May99]. Unlike nonequivariant theory, the geometric theory do not

coincide with the cohomological theory (see [CMW01, pg 4–5]), although a comparison appears in [May99]. One of the challenges of these developments were that many important results, such as equivariant Poincare duality (see [May96, XVI §9] and [CW16]), were best understood when restricted to G -connected spaces. This is because the equivariant dimension of a vector bundle is not well defined unless the base space is G -connected. Two different ideas came in as a remedy: Costenoble and Waner [CW92] used representations of the fundamental groupoid of the base space, instead of representation ring of G , to indexed cohomology theories, whereas May [May99] dropped the idea of a single Thom class and advocated “a family of Thom classes” instead. The difficulty in glueing these “family of Thom classes” lies in the fact that, unlike nonequivariant spaces, a G -equivariant space is not equivalent to the disjoint union of its G -connected components (see [Example 1.2](#)).

Remark 1.1 (G -connected components). The seminal work of Elmendorf [Elm83] proved that the category of G -equivariant spaces with G -equivariant continuous maps as morphisms, denote it by $\mathcal{Top}^G$, is Quillen equivalent to the category of $\mathcal{O}_G$ -spaces, i.e. the category of contravariant functors from the orbit category $\mathcal{O}_G$ to $\mathcal{Top}$. Given a G -equivariant space X we get an $\mathcal{O}_G$ -space

$$\Phi(X) : \mathcal{O}_G^{\text{op}} \longrightarrow \mathcal{Top},$$

that maps G/H to the H -fixed points of X for any subgroup H . By compatibly choosing a path component of X^H for each $H \subset G$, one can often form an $\mathcal{O}_G$ -space. The corresponding G -equivariant space (obtained using [Elm83]) is then a G -connected component of X .

Example 1.2. Let S^σ denote the one-point compactification of σ , the sign representation of C_2 . Note that the C_2 -fixed point of S^σ is S^0 which has two components. Therefore, S^σ has two different C_2 -connected components both equivalent to $\Sigma(EC_2)_+$, where EC_2 is the total space of the universal C_2 -principal bundle. The space S^σ is not equivalent to the disjoint union of its G -connected components as the underlying space of S^σ is connected.

In this paper, we propose a different approach to equivariant orientation theory, where existence of an orientation of an equivariant bundle is determined through a two-step obstruction theory. As a consequence, we are able to make explicit calculations to check whether a given equivariant bundle admits an orientation or not.

Taking advantage of the classification theory of equivariant vector bundles [tD69] (also see [May90, Zou21]), equivariant J -homomorphism (see [Remark 2.24](#)), and the recent construction of equivariant picard spectra [HHK⁺], we introduce the notion of *homogeneity* of G -equivariant vector bundles relative to any $\mathbb{E}_\infty^G$ -ring spectrum R , (see [Definition 2.20](#)). In our work, R -homogeneity is a requirement for an equivariant vector bundle to be R -orientable, hence should be thought of as the first obstruction.

Theorem 1.3. *Let G be a finite group and let R be an $\mathbb{E}_\infty^G$ -ring spectrum. Suppose that ξ is an R -homogeneous G -equivariant real vector bundle over a G -space B , then the following statements are equivalent:*

(1) The bundle ξ is R -orientable.

(2) There exists an R -Thom class

$$u_\xi \in [\mathrm{Th}(\xi), \mathcal{I}]^G$$

for some invertible R -module $\mathcal{I}$.

(3) There exists a G -equivariant weak equivalence

$$(1.4) \quad \mathrm{Th}(\xi) \wedge R \simeq B_+ \wedge \mathcal{I}$$

for some invertible R -module $\mathcal{I}$.

Remark 1.5 (Finiteness assumption on G). The work in [HHK⁺] uses the theory of parametrized ∞ -category theory which is restricted to finite groups only (see [NS]). Therefore, in Theorem 1.3, and in the rest of the paper, G will always denote a finite group.

Notation 1.6. The real regular representation of any finite group G will be denoted by ρ_G . When the underlying group G is clear from the context, we will drop the subscript.

The notion of R -homogeneity of an equivariant bundle is an extension of the concept of homogeneity (see Definition 2.8) which already existed in the literature. A G -equivariant homogeneous bundle is referred to as a “ V -bundle” in the work of Costenoble and Waner [CW16], where V is the G -representation which encodes the equivariant dimension of the bundle. There are many important and interesting examples of homogeneous G -equivariant vector bundles.

Let Π be a compact Lie group, and let $E_G \Pi$ denote the total space of universal principal G - Π bundle (see (2.14) for details). For a finite dimensional real $\Pi \times G$ representation V of the group, the G -equivariant bundle λ_V whose projection map is

$$(1.7) \quad \pi_{\lambda_V} : E_G \Pi \times_\Pi V \longrightarrow E_G \Pi \times_\Pi \mathbf{0} := B_G \Pi,$$

has a base space which is not G -connected (see [LM86] and Theorem 2.12). We show that:

Theorem 1.8. *Suppose V is a $\Pi \times G$ -representation isomorphic to*

$$V \cong n\rho_G \otimes W,$$

where W is an arbitrary finite real representation of Π , and $n \in \mathbb{N}$. Then λ_V as defined in (1.7), is a homogeneous G -equivariant vector bundle.

Remark 1.9. When $\Pi = O(n)$, the space $B_G O(n)$ is a classifying space for G -equivariant vector bundles [tD69].

Remark 1.10. When $G = C_2$, $\Pi = \Sigma_2$ and $V = \rho \otimes \tau$, the tensor product of the regular representation of C_2 and the sign representation of Σ_2 , then λ_V is a C_2 -equivariant bundle which is used in the geometric construction of the C_2 -equivariant Steenrod operations (see [BGL22, §3.2]).

Remark 1.11. When $G = C_2$, $\Pi = \Sigma_2$ and $V = U \otimes \tau$, where the C_2 -representation U is not in the form of $n\rho$, it follows from [Lemma 2.17](#) that λ_V is not homogeneous.

The second and final obstruction to R-orientability is an equivariant analog of the first Steifel-Whitney class. Classically, the first Stiefel-Whitney class is the obstruction to $H\mathbb{Z}$ -orientability of vector bundles. In [Definition 3.2](#), we define for an R-homogeneous bundle ξ , a cohomology class $w_1^R(\xi)$, which can be thought of as the first G-equivariant Steifel-Whitney class of ξ with respect to R.

Theorem 1.12. *An R-homogeneous G-equivariant bundle ξ is R-orientable if and only if $w_1^R(\xi) = 0$.*

Remark 1.13 (R-orientation order). Just like the classical first Steifel-Whitney class, our generalization also satisfies the additivity condition

$$(1.14) \quad w_1^R(\xi_1 \oplus \xi_2) = w_1^R(\xi_1) + w_1^R(\xi_2).$$

Therefore, one may define the *R-orientation order* of an R-homogeneous G-equivariant bundles as the smallest number n such that the n -fold direct sum of ξ is R-orientable.

Notation 1.15. Throughout this paper

- $\underline{A}$ denote the constant G-Tambara functor at the abelian group A,
- $\underline{A}_G$ will denote the Burnside Tambara functor of G, and
- $H\mathcal{M}$ will denote the Eilenberg MacLane spectrum corresponding to a Mackey functor $\mathcal{M}$.

Example 1.16. A Tambara functor $\mathcal{T}$ is a G-commutative monoid object in the category of Mackey functors, and therefore, the corresponding Eilenberg MacLane spectrum $H\mathcal{T}$ is an $\mathbb{E}_\infty^G$ -ring spectrum. The corresponding first Steifel-Whitney class of an $H\mathcal{T}$ -homogeneous bundle ξ over B is a class in

$$w_1^{\mathcal{T}}(\xi) := w_1^{H\mathcal{T}}(\xi) \in H^1(B; \mathcal{T}^\times),$$

where $\mathcal{T}^\times$ is the unit Mackey functor for $\mathcal{T}$.

In G-equivariant homotopy theory, $H\mathbb{Z}$ is often regarded as a generalization of the integral Eilenberg MacLane spectrum. In this paper we show that:

Theorem 1.17. *For any $H\mathbb{Z}$ -homogeneous G-equivariant vector bundle, its 2-fold direct sum is $H\mathbb{Z}$ -orientable.*

Although $H\mathbb{Z}$ is a perfectly valid generalization, many consider the Burnside Eilenberg MacLane spectrum $H\underline{A}_G$ as a more appropriate equivariant generalization of $H\mathbb{Z}$. This is because, just like $H\mathbb{Z}$ in the classical case, $H\underline{A}_G$ is the zeroth Postnikov approximation of the sphere spectrum S_G . Further, $\underline{A}_G$ is the unit in the symmetric monoidal category of Mackey functors (see [[Tam93](#), [Maz13](#)]), which is an equivariant generalization of the fact that $\mathbb{Z}$ is the unit for the tensor product of abelian groups.

Studying $H\mathcal{A}_G$ -orientation of equivariant bundles is an important problem as it is a step towards understanding the genuine stable equivalence class (see [Definition 2.1](#)) of $H\mathbb{Z}$ -orientable vector bundles. This is because the unit map for $H\mathbb{Z}$ factors through that of $H\mathcal{A}_G$

$$S_G \longrightarrow H\mathcal{A}_G \longrightarrow H\mathbb{Z}.$$

Moreover, $H\mathcal{A}_G$ -orientation of an equivariant vector bundle has a precise geometric interpretation as explained in [\[May99\]](#), because of which many important results in the subject, such as equivariant Poincare duality [\[CW16, §1.12.2\]](#), are discussed in terms of $H\mathcal{A}_G$ -cohomology. We show:

Theorem 1.18. *For any homogeneous G -equivariant vector bundle, its 2-fold direct sum is $H\mathcal{A}_G$ -orientable.*

We notice that an $H\mathbb{Z}$ -orientation is sufficient to obtain an $H\mathcal{A}_G$ -orientation when $|G|$ is odd.

Theorem 1.19. *Let G be a finite group of odd order. Then*

$$w_1^{\mathcal{A}_G}(\xi) = w_1^{\mathbb{Z}}(\xi)$$

for any $H\mathcal{A}_G$ -homogeneous G -equivariant bundle ξ .

The above theorem does not hold when G has an even order because the kernel of the map

$$\mathcal{A}_G^\times \longrightarrow \mathbb{Z}^\times$$

of G -Mackey functors is nonzero. We use this fact to define a set $\mathfrak{G}_1(-)$ (see [Definition 3.7](#)), and refer to its elements as “ghosts¹ of first Stiefel-Whitney class”. These elements can be regarded as obstructions to $H\mathcal{A}_G$ -orientability for $H\mathbb{Z}$ -orientable vector bundles:

Theorem 1.20. *An $H\mathcal{A}_G$ -homogeneous $H\mathbb{Z}$ -orientable vector bundle ξ is $H\mathcal{A}_G$ -orientable if and only if $0 \in \mathfrak{G}_1(\xi)$.*

We show that the trivial element always appears as a ghost when G acts freely on the base space of a bundle. Consequently, we have the following result.

Theorem 1.21. *Let ξ is a $H\mathcal{A}_G$ -homogeneous bundle such that G acts freely on its base space. Then ξ is $H\mathcal{A}_G$ -orientable if and only if it is $H\mathbb{Z}$ -orientable.*

In 1966, M.F. Atiyah [\[Ati66\]](#) introduced a subclass of C_2 -equivariant vector bundles whose underlying nonequivariant bundle is a complex vector bundle and the action of C_2 on the fibers is compatible with the complex conjugation action on $\mathbb{C}$. These bundles are referred to as Atiyah Real bundles. He showed that the classifying space of Atiyah Real bundles is a “genuine” C_2 -equivariant infinite loops space, and hence deloops to a genuine C_2 -spectrum kr called the connective Atiyah Real K -theory.

¹The term ghost is used to indicate that the set $\mathfrak{G}_1(\xi)$ is not defined unless $w_1^{\mathbb{Z}}(\xi)$ is dead (i.e. $w_1^{\mathbb{Z}}(\xi) = 0$).

When the tautological complex line bundle over the infinite complex projective space is equipped the Galois action of C_2 , then the resultant C_2 -equivariant bundle $\hat{\gamma}$ is called the tautological Atiyah Real line bundle. One of the results in [Ati66], which is crucial to many applications, is that the bundle $\hat{\gamma}$ is kr -orientable. This automatically implies that $\hat{\gamma}$ is orientable with respect to $H\mathbb{Z}$, the zeroth Postnikov approximation of kr . We show that:

Theorem 1.22. *The tautological Atiyah Real bundle $\hat{\gamma}$ is not $H\mathcal{A}_{C_2}$ -orientable.*

Let γ_ρ denote the ρ -dimensional bundle over $B_G \Sigma_2$ obtained by setting $\Pi = \Sigma_2$ and $V = \rho \otimes \tau$ in (1.7). In some sense, γ_ρ is the G -equivariant analog of the real tautological bundle.

When $G = C_2$ then there exist a map of C_2 -equivariant vector bundle (see [BGL22, Remark 3.20])

$$(1.23) \quad \gamma_\rho \longrightarrow \hat{\gamma},$$

thus an $H\mathbb{Z}$ -orientation of $\hat{\gamma}$ leads to an $H\mathbb{Z}$ -orientation of γ_ρ . However, Theorem 1.22 implies that $\hat{\gamma}$ is not $H\mathcal{A}_{C_2}$ -orientable. We end the paper by generalizing this result to all finite groups of even order:

Theorem 1.24. *If G is a finite group of even order, then γ_ρ is an $H\mathbb{Z}$ -orientable bundle which does not admit an $H\mathcal{A}_G$ -orientation.*

Notation 1.25. Throughout the paper, $\mathcal{Top}^G$ denotes the ∞ -category associated with simplicial model category of G -spaces, $\mathcal{Top}_*^G$ denotes the corresponding pointed category, and $\mathcal{Sp}^G$ denotes the stable ∞ -category of genuine G -spectra. Let

$$\Sigma_G^\infty : \mathbf{Ho}(\mathcal{Top}_*^G) \xrightleftharpoons{\quad} \mathbf{Ho}(\mathcal{Sp}^G) : \Omega_G^\infty$$

denote the usual loop-suspension adjoint pair of functors between their homotopy categories. For a pair of objects X, Y in either $\mathcal{Top}_*^G$ or $\mathcal{Sp}^G$, $[X, Y]^G$ will denote the homotopy class of G -equivariant map (as usual).

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Organization of the paper In [Section 2](#), we first expose the notion of homogeneity, discuss examples, and prove [Theorem 1.8](#). Then we introduce the concept of relative homogeneity, coordinate of homogeneity, and define R-orientation of equivariant bundles, with respect to a $\mathbb{E}_\infty$ -ring spectrum R . We end this section with a proof of [Theorem 1.3](#).

In [Section 3](#), we define the equivariant generalization of Stiefel-Whitney class, and prove [Theorem 1.17](#), [Theorem 1.18](#), [Theorem 1.19](#), [Theorem 1.20](#), [Theorem 1.21](#), [Theorem 1.22](#) and [Theorem 1.24](#).

The purpose of [Appendix A](#) is to record the Mackey functor structure of the units of the Burnside Tambara functor $\underline{A}_G^\times$ for a few familiar Abelian 2-groups.

2. Equivariant Orientation theory

A G -equivariant vector bundle ξ over a base space B consists of a map $\pi_\xi : E \rightarrow B$ in $\mathcal{T}op_G$ such that the fiber E_b over $b \in B$ is a real vector space, and the fiberwise action of $g \in G$

$$g : E_b \longrightarrow E_{g \cdot b}$$

is a linear isomorphism.

Definition 2.1. Two G -equivariant vector bundles, ξ_1 and ξ_2 , are *genuinely stable equivalent* if

$$\xi_1 \oplus \epsilon_V \cong \xi_2 \oplus \epsilon_V,$$

where ϵ_V is a trivial bundle at the G -representation V over B .

Genuinely stable equivalent classes of vector bundles are classified by the infinite loop space of the genuine G -equivariant real K-theory

$$\{\text{G-equivariant vector bundle over } B\} / (\sim_G) \cong [B_+, \Omega_G^\infty \mathbf{ko}_G]^G,$$

where $\sim_G$ stands for genuine stable equivalence [[Seg68](#), pg 134].

Notation 2.2. Let $f_\xi : B \rightarrow \Omega_G^\infty \mathbf{ko}_G$ denote the classifying map (in the homotopy category of $\mathcal{T}op_*$) for a G -equivariant bundle ξ over B .

When G is the trivial group, the map (in $\mathcal{S}et$)

$$(2.3) \quad \pi_0(f_\xi) : \{\text{path components of } B\} \longrightarrow \mathbb{Z}$$

assigns each path component the dimension of ξ over it. In order to identify an orientation (with respect to some ring spectrum R) using a single Thom class, we require $\pi_0(f_\xi)$ to be a constant map.

We will now introduce the notion of homogeneity for G -equivariant vector bundles, which is essentially a equivariant generalization of “ f_ξ inducing a constant map on the zeroth homotopy”.

2.1. Homogeneity of vector bundles

When G is a nontrivial group, there are a variety of morphisms in $\mathcal{O}_G\text{-Set}$ which can be regarded as constant maps. In order to describe them, we first recall that a set $\mathcal{F}$ consisting of subgroups of G is called a *family* if it is closed under inclusions up to conjugations. For a family $\mathcal{F}$, define the $\mathcal{O}_G\text{-Set}$

$$c_{\mathcal{F}}(G/H) = \begin{cases} * & \text{if } H \in \mathcal{F} \\ \emptyset & \text{otherwise.} \end{cases}$$

Definition 2.4. An $\mathcal{F}$ -point of an $\mathcal{O}_G\text{-Set}$ X is an $\mathcal{O}_G$ -map

$$p : c_{\mathcal{F}} \longrightarrow X.$$

Thus, an $\mathcal{F}$ -point p consists of a collection of points $p_H \in X(G/H)$ for each $H \in \mathcal{F}$, such that $X(\alpha)(p_{H_1}) = p_{H_2}$ whenever $H_1, H_2 \in \mathcal{F}$ and $\alpha : G/H_2 \rightarrow G/H_1$ is a morphism in $\mathcal{O}_G$.

Notation 2.5. Let $\mathcal{All}_G$ denote the family consisting of all subgroups of G .

Notation 2.6. For a G -equivariant space B , without a designated basepoint, we let

$$\pi_0(B) : \mathcal{O}_G^{\text{op}} \longrightarrow \text{Set}$$

be the $\mathcal{O}_G$ -set which assigns G/H the set of path components of B^H for any subgroup H of G .

Remark 2.7. For a spectrum $R \in \mathcal{S}p^G$, the $\mathcal{O}_G$ -set $\pi_0(\Omega_G^\infty R)$ (defined after forgetting the basepoint $\Omega_G^\infty R$) has a Mackey functor structure as it is isomorphic to the zeroth homotopy group of R given by

$$\pi_0(R)(G/H) := [\mathbb{S}_G \wedge (G/H)_+, R]^G,$$

for any subgroup H of G .

Definition 2.8. We say a G -vector bundle ξ is *homogeneous* if $\pi_0(f_\xi)$ factors through an $\mathcal{All}_G$ -point p of $\pi_0 \mathbf{ko}_G$

$$(2.9) \quad \begin{array}{ccc} & & c_{\mathcal{All}_G} \\ & \nearrow & \downarrow p \\ \pi_0(B) & \xrightarrow{\pi_0(f_\xi)} & \pi_0(\Omega_G^\infty \mathbf{ko}_G). \end{array}$$

We call p a *coordinate of homogeneity* of ξ .

Note, $\pi_0(\Omega_G^\infty \mathbf{ko}_G)$ is the real representation Mackey functor $\underline{RO}_G$ given by

$$\underline{RO}_G(G/H) \cong RO(H)$$

for any subgroup $H \subset G$. Thus, an $\mathcal{All}_G$ -point p of $\pi_0(\mathbf{ko}_G)$ is determined by $p_G \in RO(G)$. Indeed, p_H is simply the restriction of p_G to H for any subgroup H of G . Therefore, we define the *dimension* of (ξ, p) as

$$\dim(\xi, p) := p_G.$$

Remark 2.10 (Geometric interpretation). For a homogeneous G -equivariant vector bundle ξ , whose coordinate of homogeneity is p , the fiber over any point in B^H is an H -representation representing the class $p_H \in \text{RO}(H)$ (also see [CW16]).

The coordinate of homogeneity of a G -equivariant bundle may not be unique. Such situations may arise, when the H -fixed points of the base space is empty for some subgroup $H \subset G$ as demonstrated by the following example.

Example 2.11. Let ξ denote the C_2 -equivariant bundle

$$\pi_\xi : S(2\sigma) \times \sigma \longrightarrow S(2\sigma),$$

where σ is the sign representation of C_2 . Then $\pi_0(f_\xi)$ induces a map of $\mathcal{O}_{C_2}$ -Set given by the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & \text{RO}(C_2) \\ \downarrow & & \downarrow \text{res} \\ \{*\} & \longrightarrow & \mathbb{Z} \end{array}$$

that maps $*$ to 1. Thus, there are infinitely many coordinates of homogeneity.

2.2. Examples of homogeneous bundles

Products and pullbacks of homogeneous bundles are homogeneous, and therefore, the interest lies in knowing which among the universal examples of G -equivariant bundles defined (1.7) are homogeneous. The goal of this subsection is to prove Theorem 1.8 which gives a necessary condition.

We begin by recalling a result of Lashof and May [LM86] which identifies all G -connected components of the base space of these bundles. We also sketch a proof following [LM86] as the details are needed in the proof of Theorem 1.8.

Theorem 2.12 (Lashof-May). *For a subgroup H of G , the H -fixed points of $B_G \Pi$ is equivalent to*

$$(2.13) \quad (B_G \Pi)^H \simeq \bigsqcup_{[\theta] \in \Theta} B\mathcal{Z}(\theta)$$

where

- Θ is the set of Π -conjugacy classes of group homomorphism $\theta : H \rightarrow \Pi$,
- $\mathcal{Z}(\theta)$ is the subgroup of Π consisting of elements that commute with the image of θ which, up to conjugation, is independent of the choice of representative θ .

Sketch proof of Theorem 2.12. Let $\Lambda_\theta := (\theta(h), h)$ denote the graph subgroup of the group homomorphism $\theta : H \rightarrow \Pi$. The universal property of $E_G \Pi$ implies

$$(2.14) \quad (E_G \Pi)^\Lambda \simeq \begin{cases} * & \Lambda = \Lambda_\theta \text{ for some } H, \theta \\ \emptyset & \text{o.w.} \end{cases}$$

Then

$$\pi : E_G \Pi \longrightarrow B_G \Pi$$

is the universal G -equivariant principal Π bundle. The fiber $\pi^{-1}(b)$ is a $\Pi \times H$ -space for each point $b \in (B_G \Pi)^H$.

Notice that for each point $z \in \pi^{-1}(b)$, the isotropy subgroup is precisely Λ_θ for some

$$(2.15) \quad \theta : H \longrightarrow \Pi.$$

It is easy to check that the Π -conjugacy class $[\theta]$ is independent of the choice of z and therefore, is a constant on each connected component of $(B_G \Pi)^H$. As a result, we get a map

$$(2.16) \quad \alpha : \pi_0^H(B_G \Pi) \longrightarrow \Theta,$$

which is a bijection because of (2.14).

Let B be a connected component of $(B_G \Pi)^H$ and let $\theta \in \alpha([B])$. Using (2.14), one concludes that $(\pi^{-1}(B))^{\Lambda_\theta}$ is a contractible space with a free action of $\mathcal{Z}(\theta)$. Therefore $(\pi^{-1}(B))^{\Lambda_\theta} \longrightarrow B$ is a universal $\mathcal{Z}(\theta)$ -bundle and $B \simeq B\mathcal{Z}(\theta)$. $\square$

Given a $(G \times \Pi)$ -space F , one can construct a G -equivariant fiber bundle λ_F whose projection map

$$\pi_{\lambda_F} : E_G \Pi \times_\Pi F \longrightarrow B_G \Pi$$

is induced by the constant map from F to a point.

Lemma 2.17. *If $f_F : \Pi \times G \longrightarrow \text{Aut}(F)$ is the action map on F , H is a subgroup of G and $b \in (B_G \Pi)^H$, then the action of H on $F' := \pi_{\lambda_F}^{-1}(b) \cong F$ is given by the composite*

$$(2.18) \quad f_{F'} : H \xrightarrow{(\theta, \iota)} \Pi \times G \xrightarrow{f_F} \text{Aut}(F) \cong \text{Aut}(F'),$$

where $\iota : H \hookrightarrow G$ is the inclusion of H into G and $\theta \in \alpha(\text{component of } b)$, for α defined in (2.16).

Proof. Fix $z \in \pi^{-1}(b)$, and let θ be the corresponding group homomorphism as in (2.15). It follows from the construction of θ that

$$h \cdot z = \theta(z)^{-1} \cdot z.$$

Let r denote the composite

$$r : F' := \pi_{\lambda_F}^{-1}(b) \xrightarrow{\cong_{r_1}} \pi^{-1}(b) \times_\Pi F \xrightarrow{\cong_{r_2}} F$$

and set the last isomorphism in (2.18) as conjugation by r . Then the result follows from the observation that

$$\begin{aligned} h \cdot x &= r^{-1}(h \cdot r(x)) \\ &= r^{-1}(r_2(h \cdot r_1(x))) \\ &= r^{-1}(r_2(h \cdot [z, r(x)])) \\ &= r^{-1}(r_2([\theta(h)^{-1} \cdot z, h \cdot r(x)])) \\ &= r^{-1}(r_2([z, \theta(h)h \cdot r(x)])) \\ &= r^{-1}(\theta(h)h \cdot r(x)). \end{aligned}$$

$\square$

Proof of Theorem 1.8. Let $F = W \otimes V$, where W and V are real representations of Π and G respectively. Let $b \in (B_G \Pi)^H$ and let

$$\theta : H \longrightarrow \Pi$$

be a group homomorphism such that $[\theta] = \alpha(\text{component of } b)$. Then it follows from Lemma 2.17 that

$$\pi_{\lambda_F}^{-1}(b) \cong \theta^*(W) \otimes \text{Res}_H^G V$$

as an H -representation.

The regular representation ρ of any compact Lie group has the property that

$$\rho \otimes W = \bigoplus_{\dim W} \rho$$

for a finite dimensional representation W . Therefore, when V is a direct sum of regular representations of G then $\text{Res}_H^G V$ is a direct sum of regular representations of H and $\pi_{\lambda_F}^{-1}(b)$ is independent of the choice of b , for all $H \subset G$. Hence, the result follows (see Remark 2.10). $\square$

2.3. Relative homogeneity of vector bundles

The notion of homogeneity can be defined relative to any $\mathbb{E}_\infty^G$ -ring spectrum R using the genuinely stable J -homomorphism (see Remark 2.24). For any G -equivariant vector bundle ξ over B , let $f_{\xi,R}$ denote the composite

$$(2.19) \quad f_{\xi,R} : B \xrightarrow{f_\xi} \Omega_G^\infty \text{ko}_G \xrightarrow{\Omega_G^\infty \mathfrak{J}_G} \Omega_G^\infty \text{pic}(\mathbb{S}_G) \xrightarrow{\Omega_G^\infty \text{pic}(\iota_R)} \Omega_G^\infty \text{pic}(R),$$

where $\text{pic}(\mathbb{S}_G)$ and $\text{pic}(R)$ are the picard spectrum of $\mathbb{S}_G$ and R respectively (see Remark 2.23), $\mathfrak{J}_G$ is the genuine G -equivariant real J -homomorphism, and ι_R is the unit map of R .

Definition 2.20. A G -equivariant vector bundle ξ is *homogeneous relative to R* (abbrev. *R -homogeneous*) if $\pi_0(f_{\xi,R})$ factors through an $\mathcal{A}ll_G$ -point p of $\pi_0(\text{pic}(R))$

$$(2.21) \quad \begin{array}{ccc} & & \mathcal{A}ll_G \\ & \nearrow & \downarrow p \\ \pi_0(B) & \xrightarrow{\pi_0(f_{\xi,R})} & \pi_0(\text{pic}(R)). \end{array}$$

We call p a *coordinate of R -homogeneity* and set the relative dimension of the pair (ξ, p) as $\dim_R(\xi, p) := p_G$.

Remark 2.22. Any homogeneous bundle is automatically R -homogeneous for any $\mathbb{E}_\infty^G$ -ring spectrum R . Conversely, an R -homogeneous bundle ξ is homogeneous if and only if the relative coordinate admits a lift along $\pi_0(\text{pic}(\iota_R) \circ \mathfrak{J}_G)$

$$\begin{array}{ccc} & \pi_0(\text{ko}_G) & \\ & \downarrow \pi_0(\text{pic}(\iota_R) \circ \mathfrak{J}_G) & \\ \mathcal{A}ll_G & \xrightarrow[p]{} & \pi_0 \text{pic}(R). \end{array}$$

However, the map $\pi_0 \mathfrak{J}_G$ of Mackey functors is not well understood for a general group G (but see [Ang, FLM01] for partial answers).

Remark 2.23 (Equivariant Picard spectrum). The recent work of [HHK⁺] refines the definition of Picard group (see [FLM01]) to construct the Picard spectrum $\text{pic}(\mathbb{S}_G)$. Their construction involves foundations of parametrized ∞ -category introduced in [BDG⁺16, Sha]. Roughly speaking, a G - ∞ -category, or a coCartesian fibration

$$p : \mathcal{C} \longrightarrow \mathcal{O}_G^{\text{op}},$$

models genuine G -objects. A G -symmetric monoidal G - ∞ -category is a coCartesian fibration over $\underline{\text{Fin}}_*$

$$p : \underline{\mathcal{M}}^{\otimes} \longrightarrow \underline{\text{Fin}}_*^G$$

satisfying the equivariant Segal condition [HHK⁺, §7], where the category $\underline{\text{Fin}}_*$ itself is a coCartesian fibration over $\mathcal{O}_G^{\text{op}}$. The fiber $\underline{\mathcal{M}}_{[G/H]}$ over G/H spanned by invertible objects assemble to form a G - ∞ -groupoid $\underline{\text{Pic}}(\underline{\mathcal{M}})$, called the Picard groupoid of $\underline{\mathcal{M}}$. It inherits a G -symmetric monoidal structure from $\underline{\mathcal{M}}^{\otimes}$, and therefore $\underline{\text{Pic}}(\underline{\mathcal{M}})$ models an $\mathbb{E}_\infty^G$ -space which deloops to a genuine G -spectrum using the functor B in [Nar16, A.4.1] (which is a version of a equivariant delooping machine originally introduced in [GM17]). For an $\mathbb{E}_\infty^G$ -ring spectrum R , we let $\text{pic}(R)$ denote the spectrum obtained by setting $\underline{\mathcal{M}}$ as the category of R -modules.

Remark 2.24 (Genuinely stable J -homomorphism). Let $\mathcal{R}\text{ep}$ denote the symmetric monoidal topological category whose objects are $\mathbb{R}^n$ and morphisms are $\bigsqcup_{n \in \mathbb{N}} \text{Gl}_n(\mathbb{R})$. One can consider “topological G -objects” in $\mathcal{R}\text{ep}$ [HHK⁺, Construction 7.27] and obtain a G -symmetric monoidal G - ∞ -category

$$\underline{\mathcal{R}\text{ep}}^{\otimes} \longrightarrow \underline{\text{Fin}}_*^G$$

which deloops to ko_G . Explicitly, the fiber ∞ -category $\underline{\mathcal{R}\text{ep}}_{[G/H]}$ is equivalent to the nerve of the category of finite dimensional H -representations and H -equivariant linear isomorphisms. By sending an H -representation V to $\Sigma_H^\infty S^V$ via one-point compactification, we obtain the map $\mathfrak{J}_G$ of (2.19).

2.4. Orientation and Thom isomorphism

The classifying map of a G -equivariant vector bundle ξ can be regarded as a map of G - ∞ -groupoids

$$F_\xi : B \longrightarrow \underline{\mathcal{R}\text{ep}}_G,$$

where B is the fundamental ∞ -groupoid of the base space of ξ . In fact, f_ξ (as in Notation 2.2) is the homotopy class of F_ξ . Likewise, the map $f_{\xi, R}$ defined in (2.19) can also be constructed as a map of G - ∞ -groupoids

$$F_{\xi, R} : B \longrightarrow \underline{\mathcal{R}\text{ep}}_G \longrightarrow \underline{\text{Pic}}(\mathcal{S}p^G) \longrightarrow \underline{\text{Pic}}(R\text{-Mod}).$$

Definition 2.25. An R -orientation of a G -equivariant vector bundle ξ is a homotopy between the composite $F_{\xi, R}$ and a constant map.

Remark 2.26. An R -orientation of a G -equivariant bundle ξ automatically implies R -homogeneity. If c denote the constant map equivalent to $F_{\xi, R}$, then

$$\pi_0(c) : \pi_0(B) \longrightarrow \pi_0(\Omega_G^\infty \text{pic}(R))$$

serves as a coordinate of R-homogeneity of ξ .

The R-Thom spectrum of ξ is defined as the colimit

$$(2.27) \quad \mathrm{Th}(\xi, R) := \operatorname{colim}_B F_{\xi, R}.$$

The commuting diagram

$$\begin{array}{ccc} \underline{\mathrm{Pic}}(\mathcal{S}p^G) & \longrightarrow & \underline{\mathrm{Pic}}(R\text{-Mod}) \\ \downarrow & & \downarrow \\ \mathcal{S}p^G & \xrightarrow{-\wedge R} & R\text{-Mod} \end{array}$$

of G- ∞ -categories means $\mathrm{Th}(\xi, R)$ is weakly equivalent to $\mathrm{Th}(\xi, \mathbb{S}_G) \wedge R$.

Notation 2.28. We abbreviate $\mathrm{Th}(\xi) := \mathrm{Th}(\xi, \mathbb{S}_G)$.

Proof of Theorem 1.3. Let $\mathcal{I}$ denote the dimension of ξ relative to R . The proof of (1) $\Leftrightarrow$ (2) is essentially a tautology.

The proof of (2) $\Leftrightarrow$ (3) using the classical argument: An R-Thom class u_ξ can be used to construct an R-Thom isomorphism

$$\mathrm{Th}(\xi) \wedge R \xrightarrow{\Delta \wedge 1_R} \mathrm{Th}(\xi) \wedge B_+ \wedge R \xrightarrow{(2)} \mathcal{I} \wedge B_+ \wedge R \xrightarrow{(3)} B_+ \wedge \mathcal{I},$$

where Δ is the Thom diagonal map, the map (2) is induced by the R-Thom class u_ξ by , and (3) is induced by the R-module structure of $\mathcal{I}$. Conversely, an R-Thom isomorphism leads to an R-Thom-class

$$u_\xi : \mathrm{Th}(\xi) \xrightarrow{(4)} \mathrm{Th}(\xi) \wedge R \xrightarrow{(5)} B_+ \wedge \mathcal{I} \xrightarrow{(6)} \mathcal{I},$$

where the map (4) is induced by the unit of R , the map (5) is the Thom isomorphism of (1.4), and the map (6) collapses B to a point. $\square$

3. The first equivariant Stiefel Whitney class

For an R-homogeneous bundle ξ of relative dimension $\mathcal{I}$, the corresponding relative dimension zero virtual bundle is classified by the map

$$f_{\xi, R} - c_{\mathcal{I}} : B_+ \longrightarrow \mathrm{pic}(R),$$

where $c_{\mathcal{I}}$ is the constant map at $\mathcal{I}$. By construction, this map admits a lift

$$(3.1) \quad \begin{array}{ccc} & & \mathrm{bgl}_1(R) \\ & \nearrow f_{\xi, R}^{(0)} & \downarrow \\ B_+ & \xrightarrow{f_{\xi, R} - c_{\mathcal{I}}} & \mathrm{pic}(R), \end{array}$$

where $\mathrm{bgl}_1(R)$ is the 0-connected cover of $\mathrm{pic}(R)$.

Definition 3.2. For an R-homogeneous G-equivariant bundle ξ define its the *first Steifel Whitney class with respect to R* as the homotopy class

$$w_1^R(\xi) := [f_{\xi, R}^{(0)}] \in [B_+, \mathrm{bgl}_1(R)]^G.$$

Remark 3.3. Let $\underline{\text{BGL}}(\text{R-Mod})$ denote the sub G - ∞ -groupoid of $\underline{\text{Pic}}(\text{R-Mod})$ generated by R . In fact, $\underline{\text{BGL}}(\text{R-Mod})$ is symmetric monoidal and deloops to a spectrum, which we call $\text{bgl}_1(R)$. It is easy to see that $\text{bgl}_1(R)$ is the fiber of the zeroth Postnikov approximation map

$$\text{pic}(R) \longrightarrow H\mathbb{Z}_0\text{pic}(R),$$

and hence, it is the 0-connected cover of $\text{pic}(R)$.

Remark 3.4 (Uniqueness of $w_1^R(-)$). The homotopy class of the map $f_{\xi,R}^{(0)}$ is well-defined as the indeterminacies lie in $H^{-1}(B; \mathbb{Z}_0\text{pic}(R))$ (see [Remark 3.3](#)) which is a trivial group for any space G -equivariant space B .

By definition, $w_1^R(\xi)$ is the obstruction to R -orientability, and hence [Theorem 1.12](#).

3.1. $H\mathbb{Z}$ -orientability of homogenous bundles

When $R = H\mathbb{Z}$ then $\text{bgl}_1(R) \simeq \Sigma H\mathbb{F}_2$, and therefore for an $H\mathbb{Z}$ -homogeneous bundle

$$w_1^{\mathbb{Z}}(\xi) \in H^1(B; \mathbb{F}_2),$$

where B is the base space of ξ . Thus, when $H^1(B; \mathbb{F}_2) = 0$, ξ is automatically $H\mathbb{Z}$ -oriented.

Proof of Theorem 1.17. Using the additivity condition (1.14), we see that

$$w_1^{\mathbb{Z}}(\xi \oplus \xi) = w_1^{\mathbb{Z}}(\xi) + w_1^{\mathbb{Z}}(\xi) = 2w_1^{\mathbb{Z}}(\xi) = 0$$

as $H^1(B; \mathbb{F}_2)$ is an $\mathbb{F}_2$ -vector space. $\square$

3.2. $H\mathcal{A}_G$ -orientability of homogenous bundles

For any finite group G , the units of the Burnside Mackey functor $\mathcal{A}_G^\times$ consists of 2-torsions [[Mat82](#)] (also see [Theorem A.1](#)). Therefore, an argument almost identical to that in the proof of [Theorem 1.17](#) proves that the 2-fold direct sum of $H\mathcal{A}_G$ -homogeneous bundle is $H\mathcal{A}_G$ -orientable, which is essentially [Theorem 1.18](#).

Proof of Theorem 1.19. The natural map $\iota : \mathcal{A}_G \longrightarrow \mathbb{Z}$ of Tambara functors induces a map of Mackey functors on the units

$$(3.5) \quad \iota^\times : \mathcal{A}_G^\times \longrightarrow \mathbb{Z}^\times \cong \mathbb{F}_2$$

which is an isomorphism when $|G|$ is odd. Consequently, an $\mathcal{A}_G$ -homogeneous bundle is $\mathcal{A}_G$ -orientable if and only if it is $H\mathbb{Z}$ -orientable. $\square$

When $|G|$ is even then (3.5) is not an isomorphism, and therefore,

$$\mathcal{A}_G^\circ := \ker(\iota^\times)$$

is a nontrivial Mackey functor (see [Appendix A](#)). Note that $\Sigma H\mathcal{A}_G^\circ$ is the fiber of the map

$$\text{bgl}_1(\iota) : \Sigma H\mathcal{A}_G^\times \simeq \text{bgl}_1(H\mathcal{A}_G) \longrightarrow \text{bgl}_1(H\mathbb{Z}) \simeq \Sigma H\mathbb{F}_2.$$

Thus, when an $H\mathcal{A}_G$ -homogeneous G -equivariant vector bundle ξ is $H\mathbb{Z}$ -orientable, then we have a lift

$$(3.6) \quad \begin{array}{ccccc} & & B_+ & & \\ & \swarrow f_{\xi, H\mathcal{A}_G}^g & \downarrow f_{\xi, H\mathcal{A}_G}^{(0)} & \searrow f_{\xi, H\mathbb{Z}}^{(0)} & \\ \Sigma H\mathcal{A}_G^\circ & \longrightarrow & bgl_1(H\mathcal{A}_G) & \longrightarrow & bgl_1(H\mathbb{Z}) \end{array}$$

because $f_{\xi, H\mathbb{Z}}^{(0)} \simeq 0$. The map $f_{\xi, H\mathcal{A}_G}^g$ is not unique, in fact the homotopy class of $f_{\xi, H\mathcal{A}_G}^g$ can be varied by elements in the image of

$$\delta_* : H^0(B; \mathbb{F}_2) \longrightarrow H^1(B; \mathcal{A}_G^\circ),$$

where $\delta : H\mathbb{F}_2 \longrightarrow \Sigma H\mathcal{A}_G^\circ$ is the connecting map of the fiber sequence in (3.6) (the bottom horizontal row). In other words, the homotopy classes of all possible $f_{\xi, H\mathcal{A}_G}^g$ form a coset of the subgroup $D = \text{im}(\delta_*)$ within $H^1(B; \mathcal{A}_G^\circ)$.

Definition 3.7. Let ξ be a $H\mathcal{A}_G$ -homogeneous G -equivariant vector bundle. Then a *ghost* of the first Stiefel-Whitney class $w_1^{\mathbb{Z}}(\xi)$ is the homotopy class of any map $f_{\xi, H\mathcal{A}_G}^g$ that fits in the diagram (3.6). Let

$$\mathfrak{G}_1(\xi) := \{[f_{\xi, H\mathcal{A}_G}^g] \in H^1(B; \mathcal{A}_G^\circ) : \text{diagram (3.6) is homotopy commutative}\}$$

denote the D -coset consisting of all possible ghosts of the first Stiefel-Whitney class.

Proof of Theorem 1.20. It follows from (3.6) and Remark 3.4 that when $0 \in \mathfrak{G}_1(\xi)$ then $w_1^{\mathcal{A}_G}(\xi) := [f_{\xi, H\mathcal{A}_G}^{(0)}] = 0$. $\square$

Next we will show that $0 \in \mathfrak{G}_1(\xi)$ if G acts freely on the base space B of ξ .

Proof of Theorem 1.21. Let ξ denote an $H\mathcal{A}_G$ -homogeneous bundle and let B denote the base space of ξ . Since there is an $\mathbb{E}_\infty^G$ -ring map

$$H\mathcal{A}_G \longrightarrow H\mathbb{Z},$$

an $H\mathcal{A}_G$ -orientation leads to an $H\mathbb{Z}$ -orientation. We will now show that the converse is true when G acts freely on B .

As B has free action, we get a principal G -bundle $p : B \longrightarrow B/G$ whose classifying map is

$$f : B \longrightarrow EG$$

on the total space. Notice that the composite

$$B \xrightarrow{\Delta} B \times B \xrightarrow{1_B \times f} B \times EG$$

is a G -equivalence. Therefore,

$$\begin{aligned} H_G^n(B; \mathcal{A}_G^\circ) &\cong [B_+, \Sigma^n H\mathcal{A}_G^\circ]^G \\ &\cong [(B \times EG)_+, \Sigma^n H\mathcal{A}_G^\circ]^G \\ &\cong [B_+, \Sigma^n F(EG_+, H\mathcal{A}_G^\circ)]^G \end{aligned}$$

for any natural number $n \in \mathbb{N}$. By [LMSM86, II.2.2],

$$F(EG_+, H\mathcal{A}_G) \longrightarrow F(EG_+, H\mathbb{Z})$$

is a G -equivalence, which means $F(EG_+, H\mathcal{A}_G^\circ) \simeq_G *$. This means $H^n(B; \mathcal{A}_G^\circ) \cong 0$ for all $n \in \mathbb{N}$, and in particular, $\mathfrak{S}_1(\xi) = \{0\}$. Thus, the result follows from [Theorem 1.20](#). $\square$

Remark 3.8. Any Atiyah Real bundle of a given dimension is a homogeneous C_2 -equivariant bundle, and therefore it is also a $H\mathcal{A}_{C_2}$ -homogeneous bundle (see [Remark 2.22](#)).

Proof of Theorem 1.22. Recall that the C_2 -fixed points of the tautological Atiyah Real bundle $\hat{\gamma}$ is the tautological line bundle γ_1 over $\mathbb{RP}^\infty$

$$\hat{\gamma}^{C_2} \cong \gamma_1.$$

Consequently, we have a commutative diagram

$$\begin{array}{ccccc} S^1 & \xrightarrow{U_1} & \mathbb{RP}^\infty & \xrightarrow{U_2} & \Omega^\infty \mathbf{bo} \\ V_1 \downarrow & & V_2 \downarrow & & \downarrow V_3 \\ \hat{\mathbb{C}P}^1 & \xrightarrow{L_1} & \hat{\mathbb{C}P}^\infty & \xrightarrow{L_2} & \Omega_{C_2}^\infty \mathbf{br}, \end{array}$$

where V_i are the inclusion of the fixed points, U_1 and L_1 are the skeletal inclusions, and, U_2 and L_2 are the classifying maps for γ_1 and $\hat{\gamma}$ respectively. Note that

- (1) $U_2 \circ U_1$ is the classifying map for the möbius bundle and therefore represents the generator in $\pi_1 \mathbf{bo} \cong \mathbb{Z}/2$, and,
- (2) $V_{3*} : \pi_1 \mathbf{bo} \longrightarrow \pi_1^{C_2} \mathbf{br}$ is an isomorphism.

Consequently, the composition $L_2 \circ L_1 \circ V_1$ represents the nonzero element

$$\eta \in \pi_1^{C_2} \mathbf{br} \cong \mathbb{Z}/2.$$

Greenlees [[Gre05](#), pg 67] showed that

$$\pi_1^{C_2}(\mathbf{bo}_{C_2}) \cong \mathrm{RO}(C_2)/\mathrm{RU}(C_2) \cong \mathbb{Z}/2\{[1], [\sigma]\},$$

and it is a standard fact that the natural map $\Omega_{C_2}^\infty \mathbf{br} \longrightarrow \Omega_{C_2}^\infty \mathbf{bo}_{C_2}$ sends η to $[1 + \sigma]$ at the level of fundamental groups. Since the J -homomorphism induces an isomorphism $\pi_1(\mathbf{bo}_{C_2}) \cong \pi_1(\mathrm{bgl}_1(\mathbb{S}_{C_2}))$ of Mackey functors, it follows that the composition

$$\begin{array}{ccccccc} S^1 & \xrightarrow{L_1 \circ V_1} & \hat{\mathbb{C}P}^\infty & \xrightarrow{L_2} & \Omega_{C_2}^\infty \mathbf{br} & \longrightarrow & \Omega_{C_2}^\infty \mathbf{bo}_{C_2} \xrightarrow{\mathfrak{J}_{C_2}^\circ} \mathrm{bgl}_1(\mathbb{S}_{C_2}) \xrightarrow{\simeq} \mathrm{bgl}_1(H\mathcal{A}_{C_2}), \\ & & & & & & \uparrow \text{dashed arrow} \\ & & & & & & f_{\hat{\gamma}, H\mathcal{A}_{C_2}}^{(0)} \end{array}$$

is not null. Thus, $w_1^{\mathcal{A}_{C_2}}(\hat{\gamma}) := [f_{\hat{\gamma}, H\mathcal{A}_{C_2}}^{(0)}]$ cannot equal zero. $\square$

3.3. Nonexistence of $H\underline{A}_G$ -orientation of γ_ρ

The underlying nonequivariant bundle of γ_ρ is the $|G|$ -fold direct sum of the tautological line bundle γ_1 over $\mathbb{RP}^\infty$. If the order of G is odd then $\gamma_1^{\oplus |G|}$ is not $H\mathbb{Z}$ -oriented, and therefore, γ_ρ is not $H\mathbb{Z}$ -oriented. This is because by restricting the action to the trivial group results in a group homomorphism

$$H_G^1(B_G \Sigma_2; \mathbb{F}_2) \longrightarrow H^1(\mathbb{RP}^\infty; \mathbb{F}_2)$$

that maps $w_1^{\mathbb{Z}}(\gamma_\rho)$ to the nonequivariant first Stiefel Whitney class $|G| \cdot w_1^{\mathbb{Z}}(\gamma_1)$, which is nonzero.

Notation 3.9. For a group G and its subgroup H

$$\iota_H : \mathcal{T}op^G \longrightarrow \mathcal{T}op^H$$

denote the restriction functor which restricts the action of G to H . The right adjoint of ι_H is the functor $F_H(G, -)$ and let

$$\eta : \mathbb{1}_{\mathcal{T}op^G} \longrightarrow F_H(G, \iota_H(-))$$

denote the unit of the adjunction. We also use ι_H to denote the corresponding restriction functor Sp^G to Sp^H .

Notation 3.10. Let B be a G -equivariant space such that its restriction $\iota_H(B)$ is the base space of the H -equivariant bundle ξ . Then $F_H(G, \xi)$ is a G -equivariant bundle over $F_H(G, \iota_H(B))$. We let

$$\text{Ind}_H^G(\xi) := \eta_B^* F_H(G, \xi)$$

denote the G -equivariant bundle over B obtaining by pulling back $F_H(G, \xi)$ along the natural map $\eta_B : B \longrightarrow F_H(G, \iota_H(B))$.

Lemma 3.11. *Let R be an $\mathbb{E}_\infty^G$ -ring spectrum. Suppose ξ is an H -equivariant $\iota_H(R)$ -homogeneous bundle over $\iota_H(B)$ where B is a G -equivariant space. Then*

$$w_1^R(\text{Ind}_H^G(\xi)) = \text{tr}(w_1^{\iota_H(R)}(\xi)),$$

where $\text{tr} : [\iota_H(B)_+, \text{bgl}_1(\iota_H(R))]^H \longrightarrow [B_+, \text{bgl}_1(R)]^G$ is the transfer map.

Proof. The classifying map of $F_H(G, \xi)$ corresponds to that of ξ under the isomorphism

$$\begin{aligned} [F_H(G, \iota_H(B))_+, \Omega_G^\infty \text{ko}_G]^G &\cong [\Sigma_G^\infty F_H(G, \iota_H(B))_+, \text{ko}_G]^G \\ &\cong [F_H(G_+, \Sigma_H^\infty \iota_H(B_+)), \text{ko}_G]^G \\ &\cong [\Sigma_H^\infty \iota_H(B)_+, \text{ko}_H]^H \\ &\cong [\iota_H(B)_+, \Omega_H^\infty \text{ko}_H]^H, \end{aligned}$$

and its precomposition with $\eta_B : B_+ \rightarrow F_H(G, \iota_H(B))_+$ is the classifying map of $\text{Ind}_H^G(\xi)$. However, the isomorphism above composed with η_B^* is simply the transfer map

$$\text{tr} : [\iota_H(B)_+, \text{ko}_H]^H \longrightarrow [B_+, \text{ko}_G]^G,$$

of the Mackey functor $\underline{\mathbf{ko}}_G^0(B_+)$, and the commutative diagram

$$\begin{array}{ccc} [\iota_H(B)_+, \mathbf{ko}_H]^H & \xrightarrow{\text{tr}} & [B_+, \mathbf{ko}_G]^G, \\ \text{pic}(\iota_{\iota_H(R)}) \circ \mathfrak{J}_{G*} \downarrow & & \downarrow \text{pic}(\iota_R) \circ \mathfrak{J}_{G*} \\ [\iota_H(B)_+, \text{pic}(\iota_H(R))]^H & \xrightarrow{\text{tr}} & [B_+, \text{pic}(R)]^G, \end{array}$$

implies $[f_{\text{Ind}_H^G(\xi), R}] = \text{tr}([f_{\xi, \iota_H(R)}])$. Consequently, $\text{Ind}_H^G(\xi)$ is R -homogeneous when ξ is $\iota_H(R)$ -homogeneous and $w_1^R(\text{Ind}_H^G(\xi)) = \text{tr}(w_1^{\iota_H(R)}(\xi))$. $\square$

Proof of Theorem 1.24. First note that the result is true for a group of order 2 because of Theorem 1.22 and (1.23).

Now let G be an arbitrary group of even order. Note G is even it admits a subgroup of order 2. Indeed, a simple counting argument shows that there exists an element (other than 1) which is its own inverse. Fix a subgroup T of G whose order is 2. Observe that the bundle γ_{ρ_G} is isomorphic to $\text{Ind}_T^G(\gamma_{\rho_T})$ (see Notation 3.10). Since γ_T is $H\mathbb{Z}$ -orientable, it follows from Lemma 3.11 that

$$w_1^{\mathbb{Z}}(\gamma_{\rho_G}) = \text{tr}(w_1^{\mathbb{Z}}(\gamma_{\rho_T})) = \text{tr}(0) = 0,$$

thus γ_{ρ_G} is $H\mathbb{Z}$ -orientable.

Now we will show that ρ_{γ_G} is not $H\mathcal{A}_G$ -orientable. By construction γ_{ρ_G} is a homogeneous bundle, therefore it is $H\mathcal{A}_G$ -homogeneous (see Remark 2.22). By Theorem 2.12, each of the components of G -fixed points of $B_G \Sigma_2$ is isomorphic to $B\Sigma_2$. The restriction of $\gamma_{\rho_G} \cong \text{Ind}_e^G(\gamma_1)$ to the 1-skeleton of any of its component is isomorphic to $\text{Ind}_e^G(\mathfrak{m})$,

$$\begin{array}{ccc} \text{Ind}_e^G(\mathfrak{m}) & \xrightarrow{\quad\quad\quad} & \text{Ind}_e^G(\gamma_1) \\ \downarrow & & \downarrow \\ S^1 & \hookrightarrow B\Sigma_2 \hookrightarrow (B_G \Sigma_2)^G \hookrightarrow & B_G \Sigma_2 \end{array}$$

where $\mathfrak{m}$ is the mobius bundle over S^1 . To show that γ_{ρ_G} is not $H\mathcal{A}_G$ -orientable, it is enough to show $\text{Ind}_e^G(\mathfrak{m})$ is not $H\mathcal{A}_G$ -orientable.

It is well-known that $w_1^{\mathbb{Z}}(\mathfrak{m})$ is the element

$$-1 \in \mathbb{Z}^\times \cong H^1(S^1; \mathbb{F}_2) \cong \mathcal{A}_G^\times(G/e).$$

A standard formula for the norm of the sum of two elements (see [Maz13, Example 1.6.3]) implies that

$$\text{norm}(-1) \equiv -1 \pmod{\langle [G/H] : H \subsetneq G \rangle}$$

in $\mathcal{A}_G(G/G)$. In particular, $\text{norm}(-1) \neq 1$, thus $\text{tr}(-1)$ is a non-trivial element in $H_G^1(S^1; \mathcal{A}_G^\times) \cong \mathcal{A}_G^\times(G/G)$. By Lemma 3.11, the first Steifel Whitney class for $\text{Ind}_e^G(\mathfrak{m})$

$$w_1^{\mathcal{A}_G}(\text{Ind}_e^G(\mathfrak{m})) = \text{tr}(-1) \in H^1(B; \mathcal{A}_G^\times)$$

is a nontrivial element, hence $\text{Ind}_e^G(\mathfrak{m})$, and consequently γ_{ρ_G} , is not $H\mathcal{A}_G$ -orientable. $\square$

A. Units of Burnside Tambara functors

In anticipation of future applications of [Theorem 1.20](#), we record the Mackey functor structure of $\underline{A}_G^\times$ for a few familiar abelian 2-groups. The starting point of these calculations is a fundamental result due to Matsuda [\[Mat82\]](#) which identifies $\underline{A}_G^\times(G/H) := A_H^\times$, where A_H is the Burnside ring of the group H .

Theorem A.1 (Matsuda). *For a finite group G , the units $A_G^\times \cong (\mathbb{Z}/2)^{m+1}$, where m is the cardinality of the set*

$$\mathcal{I} = \{H : H \subset G \text{ has index } 2\} / (\text{conjugation}).$$

The elements $[G/H] - 1$ for $H \in \mathcal{I}$, along with -1 , form a basis for $A_G^\times$.

While the restriction maps of $\underline{A}_G^\times$ is straightforward to determine, the transfer maps are deduced from the formula of norm map $N_H^G : \underline{A}_G(G/H) \rightarrow \underline{A}_G(G/G)$ given by the formula (see [\[Maz13, Example 1.4.6\]](#) as well as [\[Nak14, Appendix A\]](#))

$$(A.2) \quad N_H^G(X) = \text{Map}_H(G, X),$$

where X is a finite H -set. The Mackey functor structure of $\underline{A}_G^\times$ can be complicated depending on the structure of G and has been the subject of research for several decades [\[Mat82, MM83, Yos90, Yn05, Bou07, Bar16, Bou21, VE23\]](#).

When $G = C_{2^n}$, then the subgroups of G are cyclic of order 2^k , in fact, there is exactly one for each $0 \leq k \leq n$. We will denote this subgroup by C_{2^k} . From [Theorem A.1](#), we get

$$\underline{A}_{C_{2^n}}^\times(C_{2^n}/C_{2^k}) \cong \mathbb{Z}/2\{\mathbf{a}_k, \mathbf{b}_k\},$$

where $\mathbf{a}_k = -1$ and $\mathbf{b}_k = [C_{2^k}/C_{2^{k-1}}] - 1$.

Proposition A.3. *The restriction and the transfer maps in the diagram*

$$\begin{array}{ccc} \underline{A}_{C_{2^n}}^\times(C_{2^n}/C_{2^k}) & & \\ \text{res}_k \left(\begin{array}{c} \nearrow \\ \searrow \end{array} \right) \text{tr}_k & & \\ \underline{A}_{C_{2^n}}^\times(C_{2^n}/C_{2^{k-1}}) & & \end{array}$$

satisfies $\text{tr}_k(\mathbf{a}_{k-1}) = \text{tr}_k(\mathbf{b}_{k-1}) = \mathbf{b}_k$, $\text{res}_k(\mathbf{a}_k) = \mathbf{a}_{k-1}$, $\text{res}_k(\mathbf{b}_k) = 0$.

Proof. The map res_k is immediate from restricting the actions. The transfer is obtained by calculating the norm in the Burnside Tambara functor $\underline{A}_G$ using the formula [\(A.2\)](#). Setting $H = \mathbf{e}$ and $G = C_{2^k}$ in [\(A.2\)](#), we get

$$\begin{aligned} N_{C_{2^{k-i}}}^{C_{2^k}}(l) &= l + \frac{l^2 - l}{2}[C_{2^k}/C_{2^{k-1}}] + \frac{l^4 - l^2}{4}[C_{2^k}/C_{2^{k-2}}] + \dots \\ &\quad \dots + \frac{l^{2^{k-i}} - l^{2^{k-i}-1}}{2^i}[C_{2^k}/C_{2^{k-i}}]. \end{aligned}$$

By setting $i = 1$ and $l = -1$ in the above equation, we get $\text{tr}_k(\mathbf{a}_{k-1}) = \mathbf{b}_k$, and by setting $i = k$ and $l = -1$, we get $\text{tr}_e^{C_{2^k}}(\mathbf{a}_0) = \mathbf{b}_k$ for all $0 \leq k \leq n$. Thus,

$$\begin{aligned} \text{tr}_k(\mathbf{b}_{k-1}) &= \text{tr}_k(\text{tr}_e^{C_{2^{k-1}}}(\mathbf{a}_0)) \\ &= \text{tr}_e^{C_{2^k}}(\mathbf{a}_0) \\ &= \mathbf{b}_k \end{aligned}$$

as desired. $\square$

When $G = C_2 \times C_2$, there are three different order 2 subgroups $G_{01} := C_2 \times \{0\}$, $G_{20} = \{0\} \times C_2$ and $G_{11} = \{(x, x) : x \in C_2\}$. Using [Theorem A.1](#), we note:

$$\begin{aligned} \underline{\mathcal{A}}_G^\times(G/G) &= \mathbb{Z}/2\{\mathbf{a}_2, \mathbf{b}_{2,10}, \mathbf{b}_{2,01}, \mathbf{b}_{2,11}\} \\ \underline{\mathcal{A}}_G^\times(G/G_{10}) &= \mathbb{Z}/2\{\mathbf{a}_{10}, \mathbf{b}_{10}\} \\ \underline{\mathcal{A}}_G^\times(G/G_{01}) &= \mathbb{Z}/2\{\mathbf{a}_{01}, \mathbf{b}_{01}\} \\ \underline{\mathcal{A}}_G^\times(G/G_{11}) &= \mathbb{Z}/2\{\mathbf{a}_{11}, \mathbf{b}_{11}\} \\ \underline{\mathcal{A}}_G^\times(G/e) &= \mathbb{Z}/2\{\mathbf{a}_0\}, \end{aligned}$$

where $\mathbf{a}_0, \mathbf{a}_{10}, \mathbf{a}_{01}, \mathbf{a}_{11}$ and $\mathbf{a}_2$ denote -1 in their respective rings, $\mathbf{b}_{ij}$ denotes $[G_{ij}] - 1$, and, $\mathbf{b}_{2,ij}$ denotes $[G/G_{ij}] - 1$.

Proposition A.4. *The restriction and transfer maps as indicated in the diagrams*

$$\begin{array}{ccc} \underline{\mathcal{A}}_G^\times(G/G) & & \underline{\mathcal{A}}_G^\times(G/G_{ij}) \\ \text{res}_{2,ij} \left(\begin{array}{c} \nearrow \\ \searrow \end{array} \right) & & \text{res}_{ij} \left(\begin{array}{c} \nearrow \\ \searrow \end{array} \right) \\ \underline{\mathcal{A}}_G^\times(G/G_{ij}) & & \underline{\mathcal{A}}_G^\times(G/e) \end{array}$$

is determined by the formulas $\text{res}_{2,ij}(\mathbf{a}_2) = \mathbf{a}_{ij}$, $\text{res}_{ij}(\mathbf{a}_{ij}) = \mathbf{a}_0$,

$$\text{res}_{2,ij}(\mathbf{b}_{2,kl}) = \begin{cases} 0 & \text{if } ij = kl, \\ \mathbf{b}_{ij} & \text{otherwise,} \end{cases}$$

$\text{tr}_{ij}(\mathbf{a}_0) = \mathbf{b}_{ij}$, $\text{tr}_{ij}(\mathbf{a}_{ij}) = \mathbf{b}_{ij}$ and $\text{tr}_{2,ij}(\mathbf{b}_{ij}) = \mathbf{b}_{2,10} + \mathbf{b}_{2,01} + \mathbf{b}_{2,11}$.

Proof. All claims, except for the formula for $\text{tr}_{2,ij}(\mathbf{b}_{ij})$, can be verified directly using arguments very similar to that in the proof of [Proposition A.3](#).

To see the formula for $\text{tr}_{2,ij}(\mathbf{b}_{ij})$, we first observe from [\(A.2\)](#) that

$$N_e^G(-1) = -1 + [G/G_{01}] + [G/G_{10}] + [G/G_{11}] - [G/e].$$

If $\text{tr}_{2,ij}(\mathbf{b}_{ij}) = \epsilon_1 \mathbf{a}_2 + \epsilon_2 \mathbf{b}_{2,10} + \epsilon_3 \mathbf{b}_{2,01} + \epsilon_4 \mathbf{b}_{2,11}$, it means

$$N_e^G(-1) = (-1)^{\epsilon_1} ([G/G_{01}] - 1)^{\epsilon_2} ([G/G_{10}] - 1)^{\epsilon_3} ([G/G_{11}] - 1)^{\epsilon_4}.$$

Comparing the two equations for $N_e^G(-1)$, we conclude $\epsilon_2 = \epsilon_3 = \epsilon_4 = 1$ and $\epsilon_1 = 0$. $\square$

[Proposition A.4](#) can be readily extended to $\mathbb{T}_n := C_2^{\times n}$. Any subgroup of $\mathbb{T}_n$ is isomorphic to $\mathbb{T}_k$ for some $k \in \{0, 1, \dots, n\}$, and there are exactly

$$\frac{(2^n - 1) \dots (2^n - 2^{k-1})}{(2^k - 1) \dots (2^k - 2^{k-1})}$$

of them. Fix a subgroup H isomorphic to $\mathbb{T}_k$. The above formula implies there are exactly $2^k - 1$ index 2 subgroups of H . Therefore, by [Theorem A.1](#),

$$\underline{\mathcal{A}}_{\mathbb{T}_n}^\times(\mathbb{T}_n/H) \cong (\mathbb{Z}/2)^{\times 2^k}$$

generated by $\mathbf{a}_H = -1$ and $\mathbf{b}_{H/K'} = [H/K'] - 1$ where K' is an index 2 subgroup of H . Likewise, for a given index 2 subgroup K of H

$$\underline{\mathcal{A}}_{\mathbb{T}_n}^\times(\mathbb{T}_n/K) \cong (\mathbb{Z}/2)^{\times 2^{k-1}}$$

generated by $\mathbf{a}_K = -1$ and $\mathbf{b}_{K/L'} - 1$ where L' is an index 2 subgroup of K .

It is easy to see that the restriction map

$$\mathrm{res}_K^H : \underline{\mathcal{A}}_{\mathbb{T}_n}^\times(\mathbb{T}_n/H) \longrightarrow \underline{\mathcal{A}}_{\mathbb{T}_n}^\times(\mathbb{T}_n/K)$$

sends $\mathbf{a}_H$ to $\mathbf{a}_K$, $\mathbf{b}_{H/K'}$ to $\mathbf{b}_{K/(K \cap K')}$ if $K' \neq K$, and $\mathbf{b}_{H/K}$ to 0. The transfer map

$$\mathrm{tr}_K^H : \underline{\mathcal{A}}_{\mathbb{T}_n}^\times(\mathbb{T}_n/K) \longrightarrow \underline{\mathcal{A}}_{\mathbb{T}_n}^\times(\mathbb{T}_n/H)$$

sends $\mathbf{a}_K$ to $\mathbf{b}_{H/K}$, and $\mathbf{b}_{K/L'}$ to $\mathbf{b}_{H/K} + \mathbf{b}_{H/K_1} + \mathbf{b}_{H/K_2}$, where K_1 and K_2 are the two index 2 subgroups of H whose intersection with K equals L' .

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