

Lecture Notes in Mathematics

Michael Artin
Barry Mazur

Etale Homotopy

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M. Artin · B. Mazur

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Authors

Michael Artin
Massachusetts Institute of Technology
Rm 2-239, Cambridge, MA 02139, USA

Barry Mazur
Harvard University
One Oxford St., Cambridge, MA 02138, USA

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These notes are an expansion of the results announced in [2]. The material was presented in a seminar at Harvard University during the academic year '65-'66.*

Our aim is to study the analogues of homotopy invariants which can be obtained from varieties by using the étale topology of Grothendieck. Using the constructions of Lubkin [21] or Verdier [3], we associate to any locally noetherian prescheme X a pro-object in the homotopy category of simplicial sets (cf. 2), which we call the étale homotopy type of the prescheme X . For a normal variety over the field of complex numbers, we show that $X_{\text{ét}}$ is a certain profinite completion of the classical homotopy type. This comparison result, together with a number of others, is in section 12.

Much of our work consists in setting up a reasonable theory of homotopy for pro-simplicial sets. This homotopy category turns out to be quite amenable to the techniques of classical algebraic topology. One may establish the analogues of Hurewicz and Whitehead theorems (4.3), (4.4), (4.5), and also one has available the techniques of Postnikov decomposition. There is

*We thank G. Borkowski for her flawless typing of the manuscript.

a direction in which this pro-category is more flexible than the classical homotopy category: It is sufficiently rich so as to admit pro-finite and p-adic completion functors (cf. (3.4)). Many of our results about schemes are phrased in terms of these functors.

For instance, let X be a connected, pointed scheme over an algebraically closed field K of characteristic zero, and let X^1, X^2 be the schemes over the complex numbers obtained from X via two embeddings of K in \mathbb{C} . Then

$$\hat{X}_{cl}^1 \approx \hat{X}_{cl}^2, \quad ,$$

where the subscript cl denotes the classical homotopy type, and where $\hat{}$ denotes pro-finite completion. We remark that there is an example due to Serre [28] of a projective nonsingular variety X over a number field K and imbeddings of K into \mathbb{C} such that X_{cl}^1 and X_{cl}^2 have non-isomorphic fundamental groups. Thus X_{cl}^1 do not necessarily have the same homotopy type. However, as Serre indicated to us, the homotopy type of a nonsingular simply connected projective surface is independent of complex imbedding. This gives an affirmative response in that case to a question raised in [2]. We do not know whether the same holds for higher dimensional simply connected nonsingular varieties.

Atiyah has suggested analogous questions for bundles: Let L/K be an extension of algebraic number fields, with K a subfield of the complex numbers. Let V denote a variety

defined over K , and E an algebraic vector bundle defined over $V_{\mathbb{K}}^{\otimes L}$. An imbedding c of L/K into \mathbb{C} over K will induce a complex vector bundle E^c over the topological space $V_{c1} = (V_{\mathbb{K}}^{\otimes L})_{c1}$. The problem is to study how the class of E^c in $K(V_{c1})$ depends on c .

As another example, let X, Y be smooth proper connected schemes over a field K of characteristic zero. Suppose given two discrete valuation rings R_p, R_q of K with residue fields of characteristics $p \neq q$ respectively, and that X, Y have isomorphic non-degenerate reductions modulo each of the valuations. Let X^1, Y^1 be the schemes over the complex numbers obtained from some imbeddings of K in \mathbb{C} , and assume finally that X^1, Y^1 are simply connected. Then

$$\hat{X}_{c1}^1 \approx \hat{Y}_{c1}^1 .$$

Note that in this case we may conclude that the classical homotopy groups of X^1 and of Y^1 are isomorphic (abstractly) since they are finitely generated abelian groups with isomorphic profinite completions

This result is in the spirit of the question originally posed by Washnitzer: Given two varieties in characteristic zero with isomorphic reductions modulo p , what additional conditions will imply that the varieties are themselves homotopic, or, to go even further, diffeomorphic? We are led to the abstract question: How much information is lost when one passes from an honest simplicial set K to its profinite completion

K ? Suppose that K is of the homotopy type of a finite CW-complex. Then we conjecture that there is at most a finite number of homotopically distinct simplicial sets whose profinite completion is isomorphic to K . In section 7, we prove a stable version of the conjecture, and moreover, we give a complete algebraic description of the set of distinct stable homotopy types with isomorphic profinite completions. We show by examples that there exist distinct stable homotopy types whose completions are isomorphic.

If V is a variety over a field K , and $G = \text{Gal}(\bar{K}/K)$ where \bar{K} is the separable algebraic closure, then G operates on the l -adic homotopy type of $\bar{V} = V \otimes_K \bar{K}$. If V is an abelian variety, this representation yields no further information than the operation on l -adic cohomology. However, for a more general V , it may be expected that more precise information is contained in this representation. Perhaps the special case $V = \mathbb{G}^{m,n}$, the Grassmanian of m -dimensional planes in n -space, $K = \mathbb{Q}$, is of interest.

Indeed, if $\hat{}$ denotes l -adic completion, one obtains using the comparison theorem (12.9) an operation of $G = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the pro-simplicial set $\widehat{\mathbb{G}_C^{m,n}}$. In other words, the topological space $\mathbb{G}_C^{m,n}$ is endowed with a natural action of G on its l -adic completion. The recent ideas of Quillen [13](b) with regard to the conjecture of Adams suggest that an analysis of the above action may provide fruitful results in topological K-theory.

We have not attempted to develop an analogue of topological

K-theory associated to the étale topology. Quillen's ideas cry out for the development of such a functor, however. Looking even further ahead, one may wonder whether analogues of the Novikov-Browder invariants (or the more recent theory of Sullivan) describing the diffeomorphy types representing a given tangential homotopy type are amenable to definition for an arbitrary scheme, proper and smooth over a field. This would bring one closer towards an understanding of the above problem of Washnitzer.

We should like to signal two errors in our announcement [2]. Namely, (5.4) of [2] is obvious nonsense. A correct statement is (8.18) below. Theorem (6.3) of [2] lacks a cohomological dimension hypothesis, as in (12.5).

§1. A glossary of the categories in which we shall work, and fibre resolutions.

We will make free use of the classical theory of simplicial objects with values in an arbitrary category. The principal references are [8], [12], [17], [23], [25]. If ST is the category of simplex types (objects being ordered sets $\Delta^n = [0, \dots, n]$ and morphisms are monotone maps), and ST/k is the full subcategory generated by Δ^i for $i \leq k$, then we have, for any category C , the category of simplicial objects with values in C , denoted $ST(C)$. It is the category of contravariant functors from ST to C . Similarly we have the category of simplicial objects truncated at level k , with values in C , denoted $(ST/k)(C)$. It is the category of contravariant functors from ST/k to C . The functor which "truncates at level k ,"

$$*/k: ST(C) \rightarrow (ST/k)(C)$$

will have a left (resp. right) adjoint, denoted sk_k (resp. $cosk_k$) provided C is closed under finite inductive (resp. finite projective) limits. We also consider the compositions,

$$cosk_k: ST(C) \rightarrow (ST/k)(C) \rightarrow ST(C)$$

$$sk_k: ST(C) \rightarrow (ST/k)(C) \rightarrow ST(C)$$

and note that the adjointness properties give us functorial morphisms,

$$sk_k X \rightarrow X, \text{ and } X \rightarrow cosk_k X.$$

Both of these functors (the skeleton, and coskeleton) admit understandable visualizations in the case where C is the category of sets (and for general C the situation is hardly any different). Briefly, the simplicial set $sk_k X$ is that subsimplicial set of X whose truncation at level k agrees with that of X , and which has no nondegenerate simplices in dimensions greater than k . The simplicial set $cosk_k X$ is obtainable from X by adding certain simplices in dimensions greater than k . In fact, in all dimensions n , the n -simplices of $cosk_k X$ are given by the set $\text{Hom}(sk_k \Delta^n, X)$ where in this formula Δ^n is the simplicial set, which as a functor from ST to Sets is represented by the object of the same name in ST (or in plainer language, Δ^n is the n -simplex, regarded as a simplicial set). The reader should interpret the above formula for the n -simplices of $cosk_k X$ in this way: Every time we have a simplicial map $sk_k \Delta^n \rightarrow X$, we are given a canonical way of extending it to a simplicial map $\Delta^n \rightarrow X$.

The reader should become at home with the coskeleton functor. It is the exceptionally good functorial properties of $cosk_k$ that allow one to achieve a functorial Postnikov-type decomposition for simplicial sets (and functoriality is crucial if one is to obtain, ultimately, any sort of Postnikov-type decomposition for pro-simplicial sets).

Since $cosk_k$ is defined as a (finite) projective limit, it commutes with projective limits. Similarly, sk_k commutes with inductive limits.

Let C be the category of sets. We have the notion of Kan morphism, or synonymously, fibre map (cf. [23], or for a more

general treatment, [25]). If $f: E \rightarrow B$ is a fibre map, where (B, b) is given as a pointed simplicial set, then $f^{-1}(b)$ will be referred to as the fibre of f .

The functor cosk_k preserves Kan morphisms. Also $\text{cosk}_k(\Delta^m) = \Delta^m$ if $k \geq m$. Thus $\text{cosk}_k(e) = e$, $k \geq 0$, where e is the one-point simplicial set, and $\text{cosk}_k(I) = I$, $k > 0$, where I is the one-simplex.

If

$$F \rightarrow E \xrightarrow[p]{} B$$

is a sequence of pointed simplicial sets, where p is a Kan morphism, and F is its fibre, then

$$\text{cosk}_k F \rightarrow \text{cosk}_k E \xrightarrow[p']{} \text{cosk}_k B$$

has the property that $\text{cosk}_k F$ is the fibre of p' , since cosk_k commutes with fibre products.

It is of course also true that for $k > 0$, cosk_k preserves homotopies, which suggests that cosk_k ($k > 0$) should have a simple interpretation as a functor on the homotopy category, and indeed we shall see this below.

Let S denote the category of simplicial sets. The most agreeable simplicial sets with which to work are those which satisfy the Kan condition. (A simplicial set is said to satisfy the

Kan, [17], [12] 3.1 or Extension condition if $K \rightarrow e$ is a Kan morphism. These simplicial sets will be referred to as Kan complexes.) Let S_0 denote the full subcategory of Kan complexes. The notion of homotopy has good properties only when restricted to the subcategory S_0 (see Kan, On C.S.S. Complexes [18] for more discussion of this), and the technique one has for extending homotopy notions to all of S is the construction of a certain adjoint functor: Let

$$S_0/(\sim) \xrightarrow{i} S/(\sim)$$

denote the inclusion functor of the homotopy categories. In [18], (On C.S.S. Complexes) Kan constructs a left adjoint to i , which he calls Ex^∞ . See also [12] 2.3.1 for a more complete treatment from the point of view of "Calculus of Fractions." Explicitly,

$$\text{Hom}_0(Ex^\infty K, L) \approx \text{Hom}(K, iL)$$

where the homs are taken in the homotopy categories, K a simplicial set, and L a Kan complex. Remarks: (i) Kan had not formulated the notion of adjointness yet when [18] was written, and no such explicit statement as above is made in [18]. However one easily checks this adjointness, using the morphisms of functors (loc. cit. 5.2):

$$\begin{aligned} Ex^\infty \cdot i &\xrightarrow{\approx} \text{ident.} \\ \text{ident.} &\xrightarrow{e^\infty} i \cdot Ex^\infty \end{aligned}$$

together with homotopy commutativity for any map f of the diagram

$$\begin{array}{ccc}
 K & \xrightarrow{\text{Ex}^\infty f} & \text{Ex}^\infty L \\
 e^\infty \downarrow & & \uparrow e^\infty \\
 \text{Ex}^\infty K & \xrightarrow{f} & L
 \end{array}$$

The main category in which we will do homotopy theory is \mathcal{K} , the extended homotopy category of simplicial sets. The objects of \mathcal{K} are simplicial sets, and for simplicial sets X, Y the set of morphisms from X to Y in \mathcal{K} , denoted $[X, Y]$ is defined to be the set of homotopy classes of morphisms from X to $\text{Ex}^\infty Y$. Note that

$$[X, Y] \approx [X, \text{Ex}^\infty Y] \approx [\text{Ex}^\infty X, \text{Ex}^\infty Y]$$

and the natural inclusion of the homotopy category of Kan complexes into \mathcal{K} is an equivalence of categories. For short we refer to \mathcal{K} as the simplicial category.

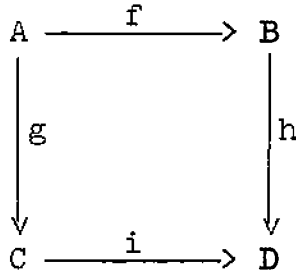
When we wish to speak of a simplicial map from one simplicial set to another (rather than a morphism in the category \mathcal{K}) and there is the possibility of confusion, we signal that we mean the former by referring to it as an actual simplicial map.

As usual, we shall have occasion to use certain closely related homotopy categories of diagrams. We define the most elaborate one that we need, first, and obtain the others as full

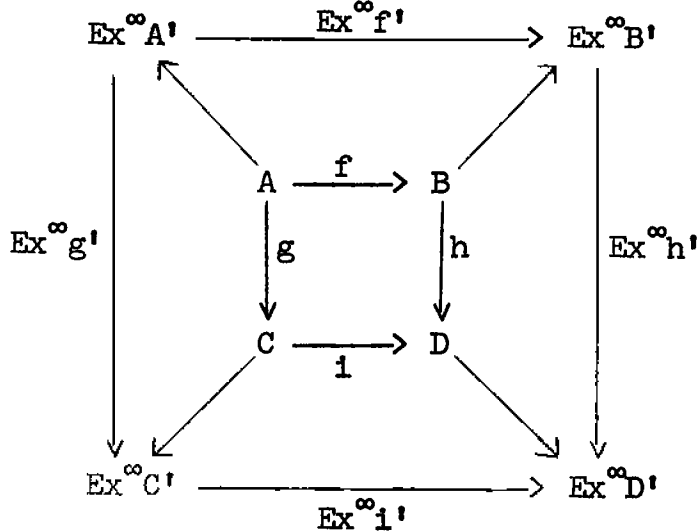
subcategories.

χ_{sq} : The extended homotopy category of simplicial squares:

Objects are actual commutative squares of simplicial sets, and actual simplicial maps:



A morphism is an equivalence class of actual commutative diagrams,



The equivalence relation is homotopy for such morphisms of diagrams.

$\chi_{o,pairs}$: The extended homotopy category of pointed connected simplicial pairs: Objects are pointed maps of pointed connected simplicial sets $X \rightarrow Y$ regarded as a square as follows:

$$\begin{array}{ccc}
 e & \xrightarrow{\quad} & e \\
 \downarrow & \quad \quad \quad & \downarrow \\
 X & \xrightarrow{\quad} & Y
 \end{array}$$

$\mathcal{K}_{\text{pairs}}$: The extended homotopy category of simplicial pairs:
 Objects are maps of simplicial sets $X \rightarrow Y$ regarded as a square as follows:

$$\begin{array}{ccc}
 \emptyset & \xrightarrow{\quad} & \emptyset \\
 \downarrow & \quad \quad \quad & \downarrow \\
 X & \xrightarrow{\quad} & Y
 \end{array}$$

\mathcal{K}_o : The pointed connected simplicial category (for short):
 Objects are maps $e \rightarrow X$ where e is the one point constant simplicial set, and X is connected; regard \mathcal{K}_o as a full subcategory of $\mathcal{K}_{\text{pairs}}$. (Thus the subscript o indicates more generally "pointed connected objects.")

$\mathcal{K}_{\text{an}_{\text{sq}}}$: The full subcategory of $\mathcal{K}_{o,\text{sq}}$ generated by those squares whose horizontal maps are fibre maps.

$\mathcal{K}_{\text{an}_{\text{pairs}}}$: The full subcategory of $\mathcal{K}_{o,\text{pairs}}$ generated by those pairs whose horizontal maps are fibre maps.

The category \mathcal{K} is equivalent to the homotopy category of CW-complexes, denoted \mathcal{K} . One passes from \mathcal{K} to \mathcal{K} by $||$, the geometric realization functor, and from \mathcal{K} to \mathcal{K} by S , the singular functor (these functors being adjoint: cf. Prop. 9.1 of [19] and also [22], [12]). As with \mathcal{K} there are closely related related categories to \mathcal{K} , which we will use.

\mathcal{K}_0 : The pointed homotopy category of connected pointed CW-complexes.

$\mathcal{K}_{\text{pairs}}$: The homotopy category of CW-pairs. Objects are continuous maps of CW-complexes, $f: X \rightarrow Y$ where morphisms are commutative squares, again,

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \downarrow & & \downarrow \\
 X' & \longrightarrow & Y'
 \end{array}$$

considered up to homotopy equivalence.

The singular functor S may be interpreted as bringing \mathcal{K}_0 to \mathcal{K}_0 and $\mathcal{K}_{\text{pairs}}$ to $\mathcal{K}_{\text{pairs}}$.

We may form the categories $\text{pro-}\mathcal{K}$, $\text{pro-}\mathcal{K}_0$, $\text{pro-}\mathcal{K}_{\text{pairs}}$, etc. These will be referred to as the pro-simplicial category, the pointed pro-simplicial category, the pro-simplicial pairs category, etc. By a pro-simplicial set we will mean an object of $\text{pro-}\mathcal{K}$. We will also pass, on occasion, to the equivalent categories $\text{pro-}\mathcal{K}$, $\text{pro-}\mathcal{K}_0$.

(The reader may wonder why we do not stick either to the category of CW-complexes or to the category of Kan complexes. The reason is this. On the one hand, we have occasion to make very strong use of fibre resolution and coskeleton functors, giving rise to Postnikov decompositions etc., and these techniques have been systematically and quite elegantly developed only for Kan complexes. On the other hand, we also use the "adjoint" techniques of attaching cells, to kill specific obstructions, etc.

which have been systematically exposed only for CW-complexes.)

Fibre Resolutions: The problem is to "replace a map, up to homotopy, by a fibre map."

Proposition 1.1: The inclusion functors

$$\mathcal{K}an \rightarrow \mathcal{K}_{0,pairs}$$

$$\mathcal{K}an_{sq} \rightarrow \mathcal{K}_{0,sq}$$

are equivalences of categories.

Proof: One may give various procedures for functorially associating to any actual simplicial map of pointed connected pairs $p: E \rightarrow B$ actually-commutative triangles,

(1.2)

$$\begin{array}{ccc}
 E & \xrightarrow[\quad a \quad]{\quad c \quad} & E' \\
 & \searrow p & \swarrow p' \\
 & & B
 \end{array}$$

such that p' is Kan, and $a: p \rightarrow p'$ is an isomorphism in $\mathcal{K}_{0,pairs}$. For example, we may take the construction (5.5.1) of [12], where a is, in their terminology, an anodyne extension, and therefore an isomorphism in $\mathcal{K}_{0,pairs}$. The construction is functorial and hence yields isomorphism-inverses to the inclusion functors appearing in the statement of proposition 1.1. Note that we do not need that E and B be connected in order for $p': E' \rightarrow B$ to be a fibre map. All that is necessary is that

$E \rightarrow B$ be surjective on connected components. The functor from pointed connected pairs to diagrams (1.2) will be referred to as fibre resolution.

The 3 by 3 Lemma: Given a diagram of pointed connected simplicial sets and actual simplicial maps,

$$(\delta) \quad \begin{array}{ccc} X_{11} & \xrightarrow{f} & X_{12} \\ \downarrow i & & \downarrow g \\ X_{21} & \xrightarrow{h} & X_{22} \end{array}$$

such that g, h induce surjections on fundamental groups, we will form, in a certain sense, a double fibre resolution of (δ) . What is meant is that we will end up with a three by three square array of simplicial sets and maps, any vertical or horizontal line of which will form, up to homotopy, a fibre triple.

To begin, let us fibre-resolve (δ) vertically (i.e. make the vertical maps fibre maps) and consider the fibre triple obtained.

$$(\delta^V) \quad \begin{array}{ccc} X_{01} & \longrightarrow & X_{02} \\ \downarrow & & \downarrow \\ X_{11}^V & \longrightarrow & X_{12}^V \\ \downarrow & & \downarrow \\ X_{21} & \longrightarrow & X_{22} \end{array}$$

The base point of X_{12} actually lies in X_{02} . Since g induces a surjection on fundamental groups, we have that X_{02} is connected. Therefore we may form the horizontal fibre resolution of the above diagram, yielding:

$$\begin{array}{ccccc}
 X_{00} & \longrightarrow & X_{01}^h & \longrightarrow & X_{02} \\
 \downarrow & & \downarrow & & \downarrow \\
 (\delta^{vh}) & & & & \\
 X_{10} & \longrightarrow & X_{11}^{vh} & \longrightarrow & X_{12}^v \\
 \downarrow & & \downarrow & & \downarrow \\
 X_{20} & \longrightarrow & X_{21}^h & \longrightarrow & X_{22}
 \end{array}$$

Lemma 1.2: Suppose that g, h induce surjections on fundamental groups. Then the left-hand row of (δ^{vh}) is, up to homotopy, a fibre triple.

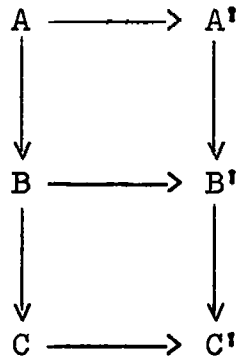
Proof: Denote (δ^{vh}) by α_1 , and form the three-by-three diagrams

$$\alpha_1 \subset \alpha_2 \subset \alpha_3 \subset \alpha_4$$

as follows: α_2 is made from α_1 , and α_4 is made from α_3 by fibre-resolving the bottom three vertical arrows. α_3 is made from α_2 by fibre-resolving the right-hand three horizontal arrows.

Denote by r_i, c_i , the upper row and left-hand column, resp. of α_i . Then r_1, r_3, c_2, c_4 are fibre-triples.

Remark: A morphism \emptyset of fibre-triples



with connected fibres A, A' which induces a homotopy equivalence on two out of three of the simplicial sets involved, actually induces a homotopy equivalence on the third. This is by the long exact homotopy sequence of a fibre triple, the five lemma and the classical Whitehead theorem.

If, when dealing with triples $A \rightarrow B \xrightarrow{g} C$, we refer to A as the "fibre" and g as the "projection," then we may say that v_i and c_i have a common "fibre," which we denote X_i . The natural maps

$$c_1 \subset c_2 \subset c_3 \subset c_4, \quad v_1 \subset v_2 \subset v_3 \subset v_4$$

induce homotopy-isomorphisms of the "projections." We may apply our remark to the inclusion maps:

$$X_1 \subset X_2 \subset X_3 \subset X_4$$

where the composite of the first two maps must be a homotopy-isomorphism since v_1 and v_3 are fibre-triples. The composite of the last two maps must be an isomorphism since c_2 and c_4

are fibre-triples. Thus all maps are homotopy-isomorphisms, concluding the proof of the lemma.

Examples: Let X be in \mathcal{X}_0 and suppose it satisfies the Kan condition. Let $n \geq 0$. Form the object $X \rightarrow \text{cosk}_n X$ in $\mathcal{X}_{0,\text{pairs}}$, and fibre resolve it, to obtain the fibre triple,

$$X^{(n)} \rightarrow X' \rightarrow \text{cosk}_n X$$

giving us the canonical $(n-1)$ -connected fibre decomposition of X . If X is a general object in \mathcal{X}_0 , we first make it satisfy the Kan condition, by applying Ex^∞ to it, and then follow the above procedure to obtain its canonical $(n-1)$ -connected fibre decomposition. To indicate that we have applied Ex^∞ , we denote the sequence obtained:

$$X^{(n)} \rightarrow X' \xrightarrow{p'} \text{cosk}_n X' .$$

We also obtain sequences,

$$X(n) \longrightarrow \text{cosk}_{n+1} X' \xrightarrow{p_{n+1}} \text{cosk}_n X'$$

where $p_{n+1} = \text{cosk}_{n+1} p'$, using that $\text{cosk}_{n+1} \text{cosk}_n = \text{cosk}_n$.

This latter sequence, is the n -th fibre triple in the canonical Postnikovoff decomposition of X . Clearly $X(n)$ is an Eilenberg-Maclane space of type $(\pi_n(X), n)$.

This being functorial, we have the identical situation in $\text{pro-}\mathcal{X}_0$. Thus, for X in $\text{pro-}\mathcal{X}_0$ we have objects of $\text{pro-}\mathcal{X}_{\text{an}}$,

$$X^{(n)} \rightarrow X' \rightarrow \text{cosk}_n X'$$

$$X(n) \rightarrow \text{cosk}_{n+1} X' \rightarrow \text{cosk}_n X'$$

for all $n \geq 0$, where, again $X(n)$ has homotopy pro-groups trivial except in dimension n , where $\pi_n(X(n)) \xrightarrow{\cong} \pi_n(X)$.

§2. Pro-objects in the homotopy category.

We work in this section with \mathcal{K}_0 , the category whose objects are connected pointed CW-complexes and whose morphisms are pointed homotopy classes of maps between them. Denote the set of morphisms from X to Y by $[X, Y]$. We are going to recall some functorial constructions in \mathcal{K}_0 which yield (cf. App.) analogous constructions in $\text{pro-}\mathcal{K}_0$:

To begin with, the homotopy and homology groups are functors on \mathcal{K}_0 , hence if $X = \{X_i\}$ is a pro-object, its homotopy and its homology with values in some abelian group $A \in (\text{Ab})$ are naturally defined as pro-groups and/or pro-abelian groups:

$$(2.1) \quad \begin{aligned} \pi_n(X) &= \{\pi_n(X_i)\} \\ H_n(X, A) &= \{H_n(X_i, A)\} \quad , \quad A \in (\text{Ab}) . \end{aligned}$$

For our purposes, it is preferable to take the actual limit group when defining cohomology. Since cohomology is contravariant, and since \varinjlim is an exact functor, this works well. Thus the cohomology is

$$(2.2) \quad H^n(X, A) = \varinjlim_i H^n(X_i, A) \quad A \in (\text{Ab})$$

which is an abelian group.

Homology and cohomology with values in twisted abelian groups can be defined in the obvious way: Suppose given an operation of $\pi_1(X) = \{\pi_1(X_i)\}$ on A , i.e., a map

$$\pi_1(X) \xrightarrow{\tau} \text{Aut}(A) \quad \text{in pro-(grps)} .$$

Then

$$(2.3) \quad H^n(X, A_\tau) = \lim_{\substack{\rightarrow \\ (i, \phi)}} H^n(X_i, A_\phi)$$

$$H_n(X, A_\tau) = \{H_n(X_i, A_\phi)\}_{(i, \phi)}$$

where the index category consists of pairs (i, ϕ) such that ϕ is a homomorphism $\pi_1(X_i) \rightarrow \text{Aut}(A)$ representing τ . It is immediately seen that this category is cofinal with the original index category $\{i\}$, hence that twisted coefficients may for practical purposes be thought of as obtained from a compatible family of maps $\pi_1(X_i) \rightarrow \text{Aut}(A)$.

Although fibre sequences and related constructions are not functorial in \mathcal{K}_0 in general, certain important cases are functorial. For instance, as remarked in §1, there is the coskeleton. For $X \in \mathcal{K}_0$, its coskeleton is characterized by the following universal property:

(2.4) The homotopy groups of $\text{cosk}_n X$ vanish in dimensions $> (n-1)$, and the canonical map

$$X \rightarrow \text{cosk}_n X$$

is universal with respect to maps into objects whose homotopy is zero in dimension $> (n-1)$.

Thus we can define $\text{cosk}_n X$ for $X = \{X_i\} \in \text{pro-}\mathcal{K}_0$ as the

pro-object $\{\text{cosk}_n X_i\}$.

Let $X \rightarrow Y$ be a map in \mathcal{K}_0 which (i) induces an isomorphism on homotopy in dimensions $< n$, and (ii) such that Y has trivial homotopy in dimensions $> n$. Then if we represent the map up to homotopy by a fibration, its fibre F is $(n-1)$ -connected. The homotopy classes of maps of any $(n-1)$ -connected W to the loop space ΩY are trivial since ΩY has zero homotopy in dimension $> (n-1)$. Therefore the exact sequence of homotopy classes of maps ([29] §7.1)

$$[W, \Omega Y] \rightarrow [W, F] \rightarrow [W, X]$$

shows that F is characterized in \mathcal{K}_0 as follows:

(2.5) F is $(n-1)$ -connected, the composed map $F \rightarrow X \rightarrow Y$ is trivial, and $F \rightarrow X$ is universal for these properties.

Thus if $X_i \rightarrow Y_i$ is a filtering inverse system of maps in \mathcal{K}_0 each of which satisfies the above assumptions (i), (ii) for some fixed n , then there is a canonically determined inverse system of maps

$$F_i \rightarrow X_i \rightarrow Y_i$$

where F_i is the fibre of the map $X_i \rightarrow Y_i$, etc...

This situation arises for instance when $Y_i = \text{cosk}_n X_i$, so that Postnikov decompositions and related constructions extend naturally to the pro-category, with the obvious definition that

(2.6) $K(G, n) = \{K(G_i, n)\}$ $G = \{G_i\} \in \text{pro-(grps)}$.

A similar notion is that of covering space. Suppose $H \subset \pi_1(X)$ is a subgroup, where for the moment X is a CW-complex. Then the covering space X_H corresponding to the subgroup H is characterized in the homotopy category \mathcal{K}_0 by the following property, as is easily verified:

(2.7) For any W the map $X_H \rightarrow X$ identifies $[W, X_H]$ with the subset of $[W, X]$ of maps which carry $\pi_1(W)$ into the subgroup H of $\pi_1(X)$.

Of course if H is a normal subgroup, then X_H is just the fibre (2.5) of the map

$$X \rightarrow K(\pi_1(X)/H, 1) .$$

Now let $X = \{X_j\}$ be in $\text{pro-}\mathcal{K}$, and let $H \hookrightarrow \pi_1(X)$ be a sub-pro-group. Replacing X, H by isomorphic objects, we may assume (App. 3.3) that H and X have the same filtering category and that we are given compatible injections

$$H_j \rightarrow \pi_1(X_j)$$

representing the map. Via (2.7), we obtain an inverse system of maps

$$(X_j)_{H_j} \rightarrow X_j ,$$

hence a map of pro-objects

$$X_H \rightarrow X .$$

For any $W = \{W_i\} \in \text{pro-}\mathcal{K}_0$, a map $W \rightarrow X_j$ factors through $(X_j)_{H_j}$ iff some representing map $W_i \rightarrow X_j$ does, and this will occur (2.7) iff the map $\pi_1(W) \rightarrow \pi_1(X_j)$ factors through H_j . Since $(W, X_H) = \varprojlim_j (W, (X_j)_{H_j})$, we obtain

Corollary (2.8): Let $X \in \text{pro-}\mathcal{K}_0$, and let $H \rightarrow \pi_1(X)$ be a monomorphism of pro-groups. There is an $X_H \in \text{pro-}\mathcal{K}_0$ together with a map $X_H \rightarrow X$ which is characterized by the property that for each $W \in \text{pro-}\mathcal{K}_0$, $[W, X_H]$ is carried to the subset of $[W, X]$ of maps such that the induced map $\pi_1(W) \rightarrow \pi_1(X)$ factors through H .

Note that in particular, this characterizes X_H independently of the choices we made in the description. We will, naturally, refer to X_H as the covering space determined by H .

§3. Completions.

Definition (3.1): A class \mathcal{C} of groups is a full subcategory of (groups) satisfying

(o) $0 \in \mathcal{C}$.

(i) A subgroup of a \mathcal{C} -group is in \mathcal{C} . Moreover, if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence of groups, then $B \in \mathcal{C}$ iff $A, C \in \mathcal{C}$.

A class \mathcal{C} is called a complete class if in addition the following axiom holds

(ii) If $A, B \in \mathcal{C}$, then the product A^B of A with itself indexed by B is in \mathcal{C} .

Of course, the index B is just to give a bound on the cardinality which is required.

We will be primarily interested in the case that \mathcal{C} is the class of finite groups, or of finite groups whose orders are products of primes coming from a given set of primes. These are clearly complete classes. However, as we shall see, the case that \mathcal{C} consists of all groups has some interest for us.

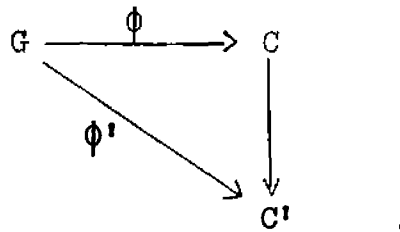
Corollary (3.2): A complete class \mathcal{C} has the following properties:

(i) If $G \supset N \supset K$ are groups such that N is normal in G and K is normal in N , and if $G/N, N/K$ are \mathcal{C} -groups, then K contains a subgroup H normal in G such that $G/H \in \mathcal{C}$.

(ii) If $G \in \mathcal{C}$, and if A is an abelian G -module such that $A \in \mathcal{C}$, then the cohomology groups $H^q(G,A)$ are in \mathcal{C} .

To see (i), note that the normalizer of K in G contains N . Therefore the set $\{K^\sigma\}$ of conjugates of K in G is indexed by a \mathcal{C} -group C . Thus $N/\cap K^\sigma$ is a subgroup of $(N/K)^C$, hence is a \mathcal{C} -group by (2.1)(ii). By (3.1)(i), $G/\cap K^\sigma$ is a \mathcal{C} -group too. For (ii), it suffices to remark that the terms in the standard cochain complex are of the form $A^{G \times \dots \times G}$, and to apply (3.1)(ii).

Let $G = \{G_i\}$ be a pro-group, and consider the category of maps $G \xrightarrow{\phi} C$, where C varies in a class \mathcal{C} , and where a map $\phi \rightarrow \phi'$ is a commutative triangle



It follows easily from axiom (i) above that the opposite category of this category is filtering; and it obviously has a small cofinal sub-category. Therefore the range \mathcal{C} determines a pro-group which we denote by \hat{G} and call the \mathcal{C} -completion of G . Clearly \hat{G} is a pro-object in the category of \mathcal{C} -groups, and the canonical map $G \rightarrow \hat{G}$ is universal with respect to maps into pro- \mathcal{C} -groups. We restate this fact as

Corollary (3.3): Let \mathcal{C} be a class of groups. The inclusion

of $\text{pro-}\mathcal{C}$ into $\text{pro}(\text{grps})$ has an adjoint

$$\wedge: \text{pro}(\text{grps}) \rightarrow \text{pro-}\mathcal{C} .$$

Equivalently, for any $G \in \text{pro}(\text{grps})$, the functor $\text{Hom}(G, \cdot)$ restricted to the category \mathcal{C} is pro-representable in \mathcal{C} .

If \mathcal{C} is a class of groups, we will denote by \mathcal{CN}_0 the full subcategory of \mathcal{N}_0 consisting of pointed CW-complexes whose homotopy groups are all in \mathcal{C} . Our object in this section is to prove the analogue of (3.2) for \mathcal{CN}_0 :

Theorem (3.4): Let $X \in \text{pro-}\mathcal{N}_0$. There is an object $\hat{X} \in \text{pro-}\mathcal{CN}_0$ and a map $X \rightarrow \hat{X}$, called the \mathcal{C} -completion of X , which is universal with respect to maps from X to elements of $\text{pro-}\mathcal{CN}_0$.

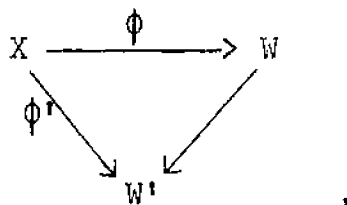
Equivalently,

(3.4'): The inclusion of $\text{pro-}\mathcal{CN}_0$ in $\text{pro-}\mathcal{N}_0$ has an adjoint $\wedge: \text{pro-}\mathcal{N}_0 \rightarrow \text{pro-}\mathcal{CN}_0$.

(3.4''): Let $X \in \text{pro-}\mathcal{N}_0$, and consider the covariant functor (X, \cdot) restricted to the category \mathcal{CN}_0 . This functor is pro-representable in \mathcal{CN}_0 .

The equivalence of the three assertions follows immediately from the definitions of pro-objects. Of course, the object in $\text{pro-}\mathcal{CN}_0$ which represents the functor (X, \cdot) of (3.4'') is just the \mathcal{C} -completion \hat{X} of (3.4). Thus we may think intuitively of \hat{X} as obtained from X by ignoring all information except that pertaining to maps to objects of \mathcal{CN}_0 .

We will prove the theorem in the form of (3.4"). What has to be shown (App. 1.2) is that given $X \in \text{pro-}\mathcal{K}_0$, the opposite category \mathcal{J} to the category of maps $X \xrightarrow{\phi} W$ is a filtering category, where a morphism $\phi \rightarrow \phi'$ is a commutative triangle

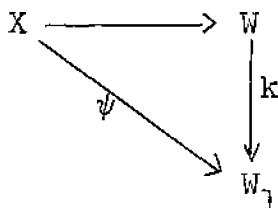


and that \mathcal{J} has a small cofinal subcategory. Since $\text{Hom}(X,W) = \varinjlim \text{Hom}(X_i,W)$ when $X = \{X_i\} \in \text{pro-}\mathcal{K}_0$ and $W \in \mathcal{K}_0$, one sees immediately that it suffices to prove the theorem for each of the X_i , i.e., in the case that $X \in \mathcal{K}_0$.

Now it follows from (3.1, (i)) that \mathcal{C} is closed under products of groups. Hence if W_1, W_2 are in $\mathcal{C}\mathcal{K}_0$, so is their product $W_1 \times W_2$. Thus \mathcal{J} satisfies (1.2a). It remains to show (1.2b). This axiom reads as follows in our situation: If

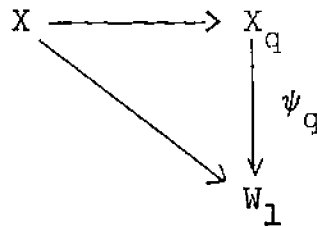
$$(3.5) \quad X \xrightarrow{\psi} W_1 \begin{array}{c} \xrightarrow{i} \\ \xrightarrow{j} \end{array} W_2$$

is a diagram in \mathcal{K}_0 with $i\psi = j\psi$ and with $W_1, W_2 \in \mathcal{C}\mathcal{K}_0$, then there is a factorization



with $ik = jk$ such that $W \in \mathcal{C}\mathcal{X}_0$.

Let us represent (3.5) by a diagram of actual maps of CW-complexes such that $i\psi$ is homotopic to $j\psi$. (We use the same letters for the representing maps). Choose such a homotopy $h: i\psi \sim j\psi$. We are going to construct inductively a sequence of factorizations



for each q such that $\pi_n(X_q) \in \mathcal{C}$ for $n \leq q$, and that $i\psi_q \sim j\psi_q$. More precisely, we take $X = X_0$. For $q > 0$, X_q is to be obtained from X_{q-1} by the addition of certain $(q+1)$ -cells, and the homotopy $h_q: i\psi_q \sim j\psi_q$ is to extend the homotopy $h_{q-1}: i\psi_{q-1} \sim j\psi_{q-1}$. Then the direct limit W of the system of X_q 's, has the required properties.

To prove the existence of X_q , we may by induction assume that X_{q-1} is already constructed. Replacing X by X_{q-1} , we may assume that $\pi_n(X) \in \mathcal{C}$ for $n < q$, and we are to construct X_q by the addition of some $(q+1)$ -cells to X .

Denote by N the kernel of the homomorphism $\phi: \pi_q(X) \rightarrow \pi_q(W_1)$, and let $\eta: F \rightarrow N$ be a surjective map of a free group F on a set S of generators of N onto N . For each $s \in S$, let $\alpha_s: \partial D^{q+1} \rightarrow X$ be a pointed continuous map representing $\eta(s)$. Since $\phi\eta(s) = 0$, we may find a continuous pointed map $\beta_s: D^{q+1} \rightarrow W_1$ extending $\psi\alpha_s$. Fix such a β_s for each $s \in S$.

Construction 1: If $d \in D^{q+1}$ denotes the base point, form the pointed CW-complex Z which is the $(q+1)$ -skeleton of $D^{q+1} \times I$. We may describe Z as follows:

$$Z = \partial(D^{q+1} \times I) = (D^{q+1} \times 0) \cup (\partial D^{q+1} \times I) \cup (D^{q+1} \times 1) .$$

Thus Z is homeomorphic with S^{q+1} . Fix such a (pointed homeomorphism. For $s \in S$, define a pointed map $f_s: Z \rightarrow W_2$ as follows:

(i) On $D^{q+1} \times 0$, let $f_s = i\beta_s$.

(ii) On $\partial D^{q+1} \times I$, let $f_s = h\alpha_s$, where h is the homotopy between $i\psi$ and $j\psi$.

(iii) On $D^{q+1} \times 1$, let $f_s = j\beta_s$.

After the identification of Z with S^{q+1} , f_s determines a class in $\pi_{q+1}(W_2)$ which represents the obstruction to extending the homotopy h to $XU_{\alpha_s} D^{q+1}$. Since S freely generates F , we obtain in this way a homomorphism $f: F \rightarrow \pi_{q+1}(W_2)$.

Construction 2: Let γ be an integer greater than 0.

Let $(A, B)_{\gamma}$ denote the pointed CW pair

$$\begin{aligned} A_{\gamma} &= D_1^{q+1} \vee D_2^{q+1} \vee \dots \vee D_{\gamma}^{q+1} \\ B_{\gamma} &= S_1^q \vee S_2^q \vee \dots \vee S_{\gamma}^q . \end{aligned}$$

We must fix an explicit pointed map

$$c_{\gamma}: (D^{q+1}, S^q) \rightarrow (A, B)_{\gamma} .$$

This map should have the property that the class $\partial c_\gamma \in \pi_q(B)$ represents $\epsilon_1 + \epsilon_2 + \dots + \epsilon_\gamma$ in the case where q is greater than 1 , and it represents the product $\epsilon_1 \epsilon_2 \dots \epsilon_\gamma$ when $q = 1$. Here ϵ_i is the class in $\pi_q(B)$ represented by $S_i^q \subset B$.

Let d denote the base point of A_γ . Define, in analogy with construction 1, $Z(\gamma)$ to be the $(q+1)$ -skeleton of $A_\gamma \times I$. Then we may also describe $Z(\gamma)$ as follows:

$$Z(\gamma) = Z_1 \vee Z_2 \vee \dots \vee Z_\gamma$$

where Z_i is the $(q+1)$ -skeleton of $D_i^{q+1} \times I$. The map c_γ may then be interpreted as a map $c_\gamma: Z \rightarrow Z(\gamma)$. This c_γ represents the class $\delta_1 + \delta_2 + \dots + \delta_\gamma$ in $\pi_{q+1}(Z(\gamma))$, where δ_i is represented by $Z_i \subset Z(\gamma)$.

Construction 3: Given an element $z \in F$, we may write z as a minimal word in S , in a unique manner:

$$z = s_1^{\epsilon_1} \dots s_\gamma^{\epsilon_\gamma}$$

where the ϵ_i are ± 1 . Define maps

$$\alpha_z: B_\gamma \rightarrow X, \beta_z: A_\gamma \rightarrow W_1, f_z: Z(\gamma) \rightarrow W_2$$

as follows:

(i) If $\epsilon_i = 1$, let $\alpha_z = \alpha_{s_i}$ on $S_i^q \subset B_\gamma$ let $\beta_z = \beta_{s_i}$ on $D_i^{q+1} \subset A_\gamma$ and let $f_z = f_{s_i}$ on $Z_i \subset Z(\gamma)$.

(ii) If $\epsilon_i = -1$, let $\alpha_z = \alpha_{s_i}^J$ on S_i^q , let $\beta_z = \beta_{s_i}^J$

on D_i^{q+1} , and let $f_z = f_{s_i} J$ on Z_i , where J denotes the evident pointed linear involution of D_i^{q+1} , S_i^q or Z_i about a hyperplane.

By construction, the map $f_z c_\gamma: Z \rightarrow W_2$ represents the element $f(z) \in \pi_{q+1}(W_2)$.

Choose a set of generators for the image group ηG of G in $\pi_q(X)$, where $G = \ker(f: F \rightarrow \pi_{q+1}(W_2))$, and let $Z \subset G$ be a set of representatives of this generating set. For each z in Z define a map $c_z: S^q \rightarrow X$ by $c_z = \alpha_z c_\gamma$. Let X_q denote the pointed CW-complex built from X by attaching $(q+1)$ -cells D_z^{q+1} for all $z \in Z$, via the maps c_z . Define $\psi_q: X_q \rightarrow W_1$ as the pointed map extending ψ on X given by $\beta_z c_\gamma$ on D_z^{q+1} .

Since $z \in G$, the map $f_z c_\gamma: Z \rightarrow W_2$ is homotopic to zero. Let $h_z: (D^{q+1} \times I) \rightarrow W_2$ be an extension of $f_z c_\gamma$. Define $h_q: X_q \times I \rightarrow W_2$ to be the pointed homotopy between $i\psi_q$ and $j\psi_q$ which is an extension of h , and which is given on $D_z^{q+1} \times I \subset X_q \times I$ by h_z .

We claim that X_q, ψ_q, h_q satisfy the necessary properties, and it remains only to check that $\psi_q(X_q) \in \mathcal{C}$. Now since $\pi_{q+1}(W_2) \in \mathcal{C}$, and since F/G is isomorphic to a subgroup of this group, F/G is in \mathcal{C} . The group $\pi_q(X_q)$ is obtained from $\pi_q(X)$ by annihilating the elements of the image ηZ of Z under η . Let $K \subset \pi_q(X_q)$ be the normal subgroup generated by ηZ . Then $\pi_q(X) \subset N \subset K$. The quotient N/K is a quotient of F/G , hence is in \mathcal{C} , and $\pi_q(X)/N$ is a subgroup of $\pi_q(W_q)$, hence in \mathcal{C} . Thus $\pi_q(X_q) = \pi_q(X)/K \in \mathcal{C}$. This completes the proof

that the category J is filtering, and it is clear from the construction of W above by spanning in some cells, that J has a small cofinal subcategory, which proves (3.4).

We want to state explicitly the construction of the cofinal index category:

Corollary (3.6): Let \mathcal{C} be a class of groups and $X \in \text{pro-}\mathcal{K}_0$. The following construction yields maps $X \rightarrow W$ with $W \in \mathcal{CK}_0$ which are cofinal among maps to elements of \mathcal{CK}_0 :

Suppose first that $X \in \mathcal{K}_0$ is an actual CW-complex. Define X_q recursively as follows: $X_0 = X$.

Choose a normal subgroup $N \subset \pi_q(X_{q-1})$ such that the quotient group is in \mathcal{C} , and a set $S \subset N$ which generates N . Let X_q be obtained from X_{q-1} by spanning a $(q+1)$ -cell D_s^{q+1} into some sphere representing s in $\pi_q(X_{q-1})$ for each $s \in S$. Now let W be the limit of the X_q 's, so that X maps canonically to W .

If $X \in \text{pro-}\mathcal{K}_0$, say $X = \{X_i\}$, first choose an i and then apply the above construction, replacing X by X_i .

From this explicit description, we get immediately

Corollary (3.7): In the notation of (3.3, 3.4),

$$\pi_1(\hat{X}) \approx \widehat{\pi_1(X)} .$$

Notice that moreover for $X \in \mathcal{K}_0$ the maps $X \rightarrow W$ described in corollary (3.6) induce surjections on fundamental groups and

yield a pro-object $\{X \rightarrow W\}$ in the category of homotopy classes of maps, $(\mathcal{K}_0, \text{pairs})$ such that the range $\{W\}$ is isomorphic to \hat{X} . Let us denote this pro-object by $X \rightarrow \hat{X}$. It is easily seen to extend naturally to a functor from \mathcal{K}_0 to $\text{pro}-(\mathcal{K}_0, \text{pairs})$. Thus we obtain

Corollary (3.8): There is a functor

$$\text{pro-}\mathcal{K}_0 \rightarrow \text{pro}-(\mathcal{K}_0, \text{pairs})$$

carrying an $X \in \text{pro-}\mathcal{K}_0$ to an object $X \rightarrow \hat{X}$ whose range is the \mathcal{C} -completion of X .

Remark (3.9): For $X \in \text{pro-}\mathcal{K}_0$, $X = \{X_i\}_{i \in I}$, we have

$$\hat{X} = \{\hat{X}_i\}_{i \in I}$$

where $\hat{}$ denotes \mathcal{C} -completion, and the right-hand side is given a sense as in App. (4.4).

§4. Cohomological criteria for \mathcal{H} -isomorphism.

It is not true that a map $X \rightarrow Y$ of objects in $\text{pro-}\mathcal{K}_0$ which induces isomorphisms on homotopy $\pi_q(X) \approx \pi_q(Y)$ is itself an isomorphism. To see this, let $X = \{X_i\}$ be in $\text{pro-}\mathcal{K}_0$. The various coskeletons $\text{cosk}_n X_i$ form a pro-object indexed by pairs (i,n) , with the obvious canonically induced maps between them. Let us denote this functorially determined pro-object by $X^{\mathcal{H}}$:

$$(4.1) \quad X^{\mathcal{H}} = \{\text{cosk}_n X_i\} \quad \text{when } X = \{X_i\} \in \text{pro-}\mathcal{K}_0 .$$

It is clear that the canonical map $X \rightarrow X^{\mathcal{H}}$ induces isomorphisms on all homotopy and homology. But if X is an actual CW-complex, then

$$[X^{\mathcal{H}}, X] = \varinjlim_n [\text{cosk}_n X, X] ,$$

and it follows immediately that the canonical map $X \rightarrow X^{\mathcal{H}}$ cannot be invertible unless all the homotopy groups of X vanish above a certain point, i.e., unless $X = \text{cosk}_n X$ for large n .

Definition (4.2): A map $f: X \rightarrow Y$ in $\text{pro-}\mathcal{K}_0$ is called a \mathcal{H} -isomorphism if the induced map $f^{\mathcal{H}}: X^{\mathcal{H}} \rightarrow Y^{\mathcal{H}}$ is an isomorphism.

It is clear that this is equivalent with the assertion that for every n , the induced map $\text{cosk}_n X \rightarrow \text{cosk}_n Y$ is an isomorphism.

The theorem below gives cohomological criteria for a map $f: X \rightarrow Y$ which determine for a complete class \mathcal{C} whether or

not the induced map \hat{f} of \mathcal{C} -completions is a \mathcal{H} -isomorphism. We will use the following terminology: If $X \in \text{pro-}\mathcal{K}_0$, a \mathcal{C} -twisted coefficient group $M \in \mathcal{C}$ on X is an abelian group $M \in \mathcal{C}$ and a map $\pi_1(X) \rightarrow \text{Aut } M$ in $\text{pro}-(\text{grps})$ which factors through some subgroup of $\text{Aut } M$ contained in \mathcal{C} . Such coefficient groups M therefore correspond to operations of $\widehat{\pi_1(X)} = \pi_1(\hat{X})$ (3.7) on M , i.e., to twisted coefficient groups $M \in \mathcal{C}$ on \hat{X} .

Let $\hat{H} \subset \pi_1(\hat{X})$ be a sub-pro-group. Its inverse image $H = \hat{H} \times_{\pi_1} \pi_1$ in $\pi_1(X)$ determines a covering space (2.7) X_H of X . Such a covering space will be called a \mathcal{C} -covering space. Thus \mathcal{C} -covering spaces are classified by $\widehat{\pi_1(X)} = \pi_1(\hat{X})$.

Theorem (4.3): Let \mathcal{C} be a complete class of groups, and $f: X \rightarrow Y$ a morphism in $\text{pro-}\mathcal{K}_0$. The following assertions are equivalent:

- (i) $\hat{f}: \hat{X} \rightarrow \hat{Y}$ is a \mathcal{H} -isomorphism.
- (ii) $\widehat{\pi_1(X)} \cong \widehat{\pi_1(Y)}$, and for every \mathcal{C} -twisted abelian coefficient group $M \in \mathcal{C}$,

$$H^q(Y, M) \xrightarrow{\sim} H^q(X, M) \quad \text{for all } q .$$

- (iii) $\widehat{\pi_1(X)} \cong \widehat{\pi_1(Y)}$ and for every induced map of corresponding \mathcal{C} -covering spaces $X' \rightarrow Y'$ and every (untwisted) abelian $A \in \mathcal{C}$,

$$H^q(Y', A) \xrightarrow{\sim} H^q(X', A) \quad \text{for all } q .$$

If we let \mathcal{C} be the class of all groups, we obtain the following corollaries:

Corollary (4.4): Let $f: X \rightarrow Y$ be a map in $\text{pro-}\mathcal{K}_0$. The following are equivalent:

- (i) f is a \mathcal{K} -isomorphism.
- (ii) $\text{cosk}_n f: \text{cosk}_n X \rightarrow \text{cosk}_n Y$ is an isomorphism for each n .
- (iii) $\pi_n(f): \pi_n(X) \rightarrow \pi_n(Y)$ is an isomorphism for each n .

The equivalence of (4.4)(i) and (ii) is trivial, and so is the implication (i) \implies (iii). It remains to show (iii) \implies (ii), and we will verify condition (ii) of (4.3). This is an exercise in passing to the limit: We may suppose (App. 3.3) given compatible maps $f_i: X_i \rightarrow Y_i$ with a single index category I representing f . Consider the morphisms of fibre sequences

$$\begin{array}{ccccc}
 K(\pi_n(X_i), n) & \longrightarrow & \text{cosk}_{n+1} X_i & \longrightarrow & \text{cosk}_n X_i \\
 \downarrow & & \downarrow & & \downarrow \\
 K(\pi_n(Y_i), n) & \longrightarrow & \text{cosk}_{n+1} Y_i & \longrightarrow & \text{cosk}_n Y_i \quad .
 \end{array}$$

Let us denote it by

$$\begin{array}{ccccc}
 K_i' & \longrightarrow & E_i' & \longrightarrow & B_i' \\
 \downarrow & & \downarrow & & \downarrow \\
 K_i & \longrightarrow & E_i & \longrightarrow & B_i \quad .
 \end{array}$$

We may assume by induction that (4.3)(ii) is proved for cosk_n . To prove it for cosk_{n+1} , let M be a twisted abelian coefficient group. We may assume given compatible operations of $\pi_1(X_i)$, $\pi_1(Y_i)$ on M (2.3). Then we obtain a filtering system of morphisms of spectral sequences indexed by I

$$\begin{array}{ccc} H^p(B_i, H^q(K_i, M)) & \implies & H^{p+q}(E_i, M) \\ \downarrow & & \downarrow \\ H^p(B_i^!, H^q(K_i^!, M)) & \implies & H^{p+q}(E_i^!, M) \end{array} ,$$

which converges to a morphism of limit spectral sequences whose abutment is the map

$$H^{p+q}(\text{cosk}_{n+1} Y, M) \longrightarrow H^{p+q}(\text{cosk}_{n+1} X, M) .$$

Hence it suffices to show that the limit is bijective on E_2^{pq} . Consider the index category of maps $i \rightarrow j$ in I . We have

$$\begin{aligned} \lim_j H^p(B_j, H^q(K_j, M)) &\approx \lim_{(i \rightarrow j)} H^p(B_i, H^q(K_j, M)) \\ &\approx \lim_j \left(\lim_{i | i \rightarrow j} H^p(B_i, H^q(K_j, M)) \right) \quad (\text{App. 1.11(b)}) \\ &= \lim_j H^p(B, H^q(K_j, M)) \quad \text{where } B = \text{cosk}_n Y . \\ &\approx \lim_j H^p(B^!, H^q(K_j, M)) \quad \text{where } B^! = \text{cosk}_n X \end{aligned}$$

the last isomorphism being by the induction hypothesis. Hence

we need only show

$$\varinjlim_j H^p(B', H^q(K_j, M)) \approx \varinjlim_j H^p(B', H^q(K'_j, M)) .$$

But

$$H^p(B', H^q(K(\epsilon, n), M))$$

is a functor of ϵ , and the two pro-objects $\pi_n(X_i)$ and $\pi_n(Y_i)$ are isomorphic, which completes the proof.

Corollary (4.5): (Hurewicz Theorem in the pro-category).

Let $X \in \text{pro-}\mathcal{K}_0$ and suppose that $\pi_q(X) = 0$ for $q < n$, where n is an integer > 1 . Then the canonical map

$$\pi_n(X) \rightarrow H_n(X)$$

is bijective.

For, say $X = \{X_i\}$, and consider the fibre sequence

$$F_i \rightarrow X_i \rightarrow \text{cosk}_n X_i .$$

Put $F = \{F_i\}$. Since the Hurewicz theorem holds for each F_i , it does for F as well. Thus it suffices to show that the map $F \rightarrow X$ is a \mathcal{K} -isomorphism. Now the pro-object $\text{cosk}_n X$ has trivial homotopy, hence is trivial by (4.4), and thus has trivial cohomology. Therefore for $A \in (\text{Ab})$, the cohomology spectral sequence of the above fibre sequence for the coefficient group A collapses to yield

$$H^q(X,A) \cong H^q(F,A) \quad \text{for all } q .$$

Since X is simply connected and $A \in (\text{Ab})$ is arbitrary, we can apply (4.3)(ii).

Proof of (4.3): (ii) \Rightarrow (i).

Suppose (ii) holds. Notice first of all that to show $\hat{f}: \hat{X} \rightarrow \hat{Y}$ is a h -isomorphism is equivalent with showing that for every $W \in \mathcal{CN}_0$ which is of finite homotopy dimension, i.e., whose homotopy vanishes in high dimensions, the composition with the map f induces a bijection

$$[X,W] \xrightarrow{\sim} [Y,W] .$$

For, if we let W have homotopy dimension $< n$, then

$$[X,W] = [\hat{X},W] = [\text{cosk}_n \hat{X},W]$$

and similarly for Y . Therefore, since $\text{cosk}_n \hat{X}$ is characterized by its maps to objects of \mathcal{CN}_0 of homotopy dimension $< n$, this will show

$$\text{cosk}_n \hat{X} \xrightarrow{\sim} \text{cosk}_n \hat{Y}$$

for each n .

The problem is the following: Given a map

$$X \xrightarrow{\phi} W ,$$

W as above, to factor it through Y, i.e., to find a commutative diagram

(4.6)

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 & \searrow \phi & \swarrow \\
 & & W
 \end{array}$$

and to show that it is unique. By (App. 3.3), we may suppose given a filtering category I of diagrams in \mathcal{K}

(4.7)

$$\begin{array}{ccc}
 X_i & \xrightarrow{f_i} & Y_i \\
 & \searrow \phi_i & \\
 & & W
 \end{array}$$

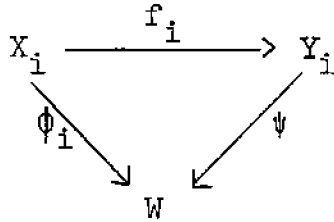
Since by (4.3)(ii), $\widehat{\pi_1(X)} \simeq \widehat{\pi_1(Y)}$, the map from $\pi_1(X)$ to $\pi_1(W)$ factors uniquely through $\pi_1(Y)$. Therefore we may (App. 3.3) further suppose given compatible commutative triangles for each i

$$\begin{array}{ccc}
 \pi_1(X_i) & \longrightarrow & \pi_1(Y_i) \\
 & \searrow & \swarrow \\
 & & \pi_1(W)
 \end{array}$$

where of course the map $\pi_1(X_i) \rightarrow \pi_1(W)$ is the one induced by ϕ_i .

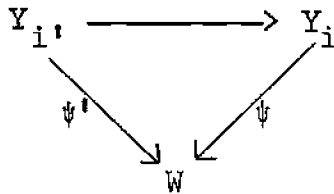
Let J be the category of pairs (i, ψ) such that ψ is a map making the triangle

(4.8)



commute (up to homotopy), where a morphism $(i, \psi) \rightarrow (i', \psi')$ is a map $i \rightarrow i'$ in I such that

(4.8')



commutes. The following lemma will prove both the existence and the uniqueness of (4.6), hence will complete the proof that (ii) \Rightarrow (i):

Lemma (4.9): J is filtering and is cofinal with I .

We leave it to the reader to verify that this is sufficient.

For our particular index categories, the assertions of (4.9) are implied by the following two facts, as is easily seen:

(4.10): (a) For any $i \in I$, there is an $i \rightarrow i'$ in I such that ϕ_i factors through $f_{i'}: X_{i'} \rightarrow Y_{i'}$.

(b) Given two maps $\psi_1, \psi_2: Y_i \rightarrow W$ making a commutative triangle (4.8)', there is an $i \rightarrow i'$ in I such that the composed maps $\psi_1^i, \psi_2^i: Y_i \rightarrow W$ are equal (in the homotopy category).

We shall show (a) by an inductive argument on the homotopy-dimension of W , and an application of obstruction theory:

Suppose $\pi_q(W)$ vanishes for $q > n$. Suppose that (4.10) has already been demonstrated for $\text{cosk}_{n-1}W$. Since $\pi_1(X_i) \rightarrow \pi_1(W)$ factors through $\pi_1(Y_i)$, we may also suppose $n \geq 2$. Thus we have a homotopy-commutative diagram,

$$\begin{array}{ccc} X_i & \longrightarrow & Y_i \\ \downarrow & & \downarrow \\ W & \longrightarrow & \text{cosk}_{n-1}W \end{array} .$$

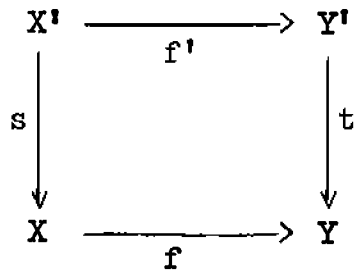
Our problem is to find a morphism $i \rightarrow i''$ such that one has a homotopy-lifting, as indicated:

$$\begin{array}{ccc} X_{i''} & \longrightarrow & Y_{i''} \\ \downarrow & \nearrow & \downarrow \\ W & \longrightarrow & \text{cosk}_{n-1}W \end{array} .$$

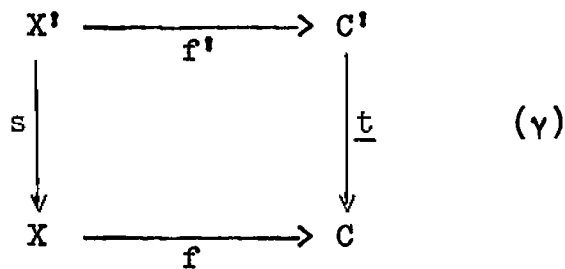
1. An elementary construction:

A continuous map $f: X \rightarrow Y$ of CW-complexes will be called cellular ([29] page 403) if $f(\text{Sk}_n X) \subset \text{Sk}_n Y$ for all n .

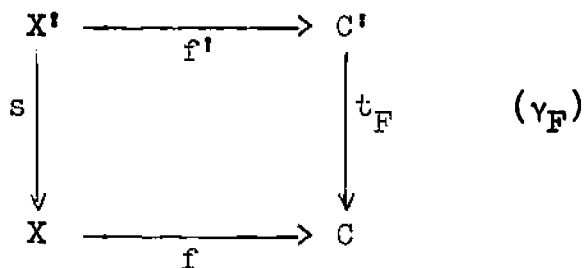
One has a notion of cellularity for homotopies, as well (loc. cit.) and the well known fact (loc. cit., chap. 7, sec. 6, thm. 17) that if f is cellular when restricted to a subcomplex $A \subset X$, then f is homotopic rel A to a cellular map. Let



be homotopy-commutative, with a given homotopy-commutativity, $F: tf' \sim fs$. Suppose that f, f' are cellular. Then we may form the CW-complex mapping cylinders of f, f' to get the diagram



where the horizontal arrows are now the natural inclusions as CW-complexes. Using F , we shall describe how to modify \underline{t} , to obtain a new map t_F and an actually-commutative diagram,



Recall that the mapping cylinder C is the fibre sum indicated in the diagram below, in the category of CW-complexes and continuous maps, (where $X \times I$ in the diagram below is given its

natural CW-structure (loc. cit. p. 401, ex. 10)):

$$\begin{array}{ccc}
 X & \xrightarrow{x\{0\}} & X \times I \\
 \downarrow f & & \downarrow i \\
 Y & \xrightarrow{j} & C
 \end{array}
 .$$

We now consider the map $\tau: X' \times I \rightarrow C$ given as follows:

- (i) $\tau(x', r) = (s(x'), 2r-1)$ for $1/2 \leq r \leq 1$.
- (ii) $\tau(x', r) = jF(x', 2r)$ for $0 \leq r \leq 1/2$.

Since

$$\begin{array}{ccc}
 X' & \xrightarrow{x\{0\}} & X' \times I \\
 \downarrow f' & & \downarrow \tau \\
 Y' & \xrightarrow{j't} & C
 \end{array}$$

is commutative, we get a unique map $t_F: C' \rightarrow C$ which, by construction makes (γ_F) above actually-commutative. If, in addition, s is cellular, we might modify F to a cellular homotopy, retaining the actual-commutativity of (γ_F) , making t_F cellular. If F is a homotopy relative to the q -skeleton of X' (for short: a homotopy rel q), the above may itself be taken to be a homotopy relative to $Sk_q X' \times I$.

Remark: The homotopy-commutative diagram (γ) induces no homomorphism on relative cohomology,

$$H^*(f,G) \rightarrow H^*(f',G) ,$$

but (γ_F) certainly does:

$$\begin{array}{ccc} H^*(f,g) & \longrightarrow & H^*(f',G) \\ || & & || \\ H^*(C,X;G) & \longrightarrow & H^*(C',X';G) \end{array}$$

Thus, a homotopy-commutative diagram (γ) together with a choice of homotopy-commutativity F determine a morphism on relative cohomology.

2. Killing obstruction classes:

Having replaced

$$\begin{array}{ccc} X_i & \longrightarrow & Y_i \\ \downarrow & & \downarrow \\ W & \longrightarrow & \text{cosk}_{n-1}W \end{array}$$

by some actually-commutative diagram, we may apply classical obstruction theory ([29], Ch. VIII) to the fibre bundle $W \rightarrow \text{cosk}_{n-1}W$, whose fibre is isomorphic to $K(\pi_n(W),n)$. The obstruction to the existence of the homotopy-lifting,

$$\begin{array}{ccc}
 X_i & \longrightarrow & Y_i \\
 \downarrow & & \downarrow \\
 W & \longrightarrow & \text{cosk}_{n-1}W
 \end{array}$$

is a class $\varrho(i) \in H^{n+1}(Y_i, X_i; \overline{\Gamma})$ where $\overline{\Gamma}$ is the induced local system of coefficients, $\{\pi_n(W)\}$. By our assumptions, $\overline{\Gamma}$ is a \mathcal{C} -twisted system of \mathcal{C} -groups.

Consider the long exact sequence,

$$H^n(Y_i, \overline{\Gamma}) \rightarrow H^n(X_i, \overline{\Gamma}) \xrightarrow{\delta} H^{n+1}(Y_i, X_i; \overline{\Gamma}) \xrightarrow{\eta} H^{n+1}(Y_i, \overline{\Gamma}) \rightarrow H^{n+1}(X_i, \overline{\Gamma})$$

By hypothesis, $H^{n+1}(Y, \overline{\Gamma}) \cong H^{n+1}(X, \overline{\Gamma})$. So there is a morphism $\alpha: i \rightarrow i'$ such that the image of $\eta\varrho(i)$ vanishes in $H^{n+1}(Y_{i'}, \overline{\Gamma})$. But the "elementary construction" allows us to represent α , after the choice of a homotopy-commutativity F , by an actually commutative diagram

$$\begin{array}{ccc}
 X_{i'} & \longrightarrow & Y_{i'} \\
 \downarrow & & \downarrow \\
 X_i & \longrightarrow & Y_i
 \end{array} \quad (\gamma_F)$$

inducing a morphism on relative cohomology. We have that $\gamma_F\varrho(i) = \varrho(i')$, and therefore, $\eta\varrho(i') = 0$. Let x' be a class in $H^n(X_{i'}, \overline{\Gamma})$ such that $\delta x' = \varrho(i')$. Find a morphism $i' \rightarrow i''$ such that the image of x' , $x'' \in H^n(X_{i''}, \overline{\Gamma})$ comes from some $y'' \in H^n(Y_{i''}, \overline{\Gamma})$. This is possible, since $H^n(Y, \overline{\Gamma}) \approx H^n(X, \overline{\Gamma})$.

Use the "elementary construction" to represent $i' \rightarrow i''$ by another actually-commutative diagram, $(\gamma_{F'})$.

By construction, the obstruction class $e(i)$ goes to zero under the map,

$$\gamma_F, \gamma_F: H^{n+1}(Y_i, X_i, \overline{\quad}) \rightarrow H^{n+1}(Y_i'', X_i'', \overline{\quad}) .$$

Thus we obtain the required extension,

$$\begin{array}{ccc} X_i'' & \longrightarrow & Y_i'' \\ \downarrow & \swarrow & \downarrow \\ W & \longrightarrow & \text{cosk}_{n-1} W . \end{array}$$

The proof of (b) is similar, and we leave it to the reader.

(iii) \Rightarrow (ii):

Assume that (iii) holds, and let $M \in \mathcal{C}$ be a \mathcal{C} -twisted abelian coefficient group on X and Y , given by a map

$$\pi_1(X) \approx \pi_1(Y) \rightarrow \text{Aut } M .$$

We can suppose (App. 3.3) given a filtering system of maps

$$X_i \xrightarrow{f_i} Y_i$$

in \mathcal{K}_0 representing f , and a compatible sequence of maps

$$\pi_1(X_i) \rightarrow \pi_1(Y_i) \rightarrow \text{Aut } M$$

representing the operation of $\pi_1(Y)$ on $\text{Aut } M$, where the first arrow is the map induced by f_i . Let H_i, G_i (resp. h_i, g_i) denote the kernel and image of $\pi_1(X_i)$ (resp. $\pi_1(Y_i)$) in $\text{Aut } M$, and let $X_i^! \rightarrow X_i, Y_i^! \rightarrow Y_i$ be the covering spaces induced by H_i and h_i respectively. The $X_i^!$ (resp. $Y_i^!$) form a pro-object

X' (resp. Y') which is the covering space of X (resp. Y) determined by the kernel of the map $\pi_1(X) \rightarrow \text{Aut } M$ (resp. $\pi_1(Y) \rightarrow \text{Aut } M$).

Using the covering X'_i of X_i , one constructs a canonical resolution of M in the well known way: If $X'_i \rightarrow X_i$ is represented by an actual covering space of CW-complexes, the resolution is the one obtained from the simplicial object

$$X_i \longleftarrow X'_i \begin{array}{c} \longleftarrow \\ \longleftarrow \end{array} (X'_i)_{\times_{X_i}} (X'_i) \begin{array}{c} \longleftarrow \\ \longleftarrow \end{array} \dots$$

by taking the direct images on X_i of the coefficient groups M on the various fibre powers of X'_i . Let us denote this resolution by

$$0 \rightarrow M \rightarrow L_i^0 \rightarrow L_i^1 \rightarrow \dots$$

Each L_i^v is the direct image of an untwisted coefficient group on X'_i , namely

$$L_i^v \approx M^{G_i^v}$$

where M is untwisted on X'_i by construction, and $M^{G_i^v}$ is the product of M with itself, indexed by elements of $G_i^v = G_i \times \dots \times G_i$.

Denote by

$$0 \rightarrow M \rightarrow K_i^0 \rightarrow K_i^1 \rightarrow \dots$$

the resolution of M on Y_i obtained in the corresponding way from the covering Y'_i , so that K_i is the direct image of the coefficient group

$$K_i^!{}^\nu = M_i^{\mathcal{G}_i^\nu} .$$

By functoriality, there is a filtering inductive system of morphisms of spectral sequences relating the spectral sequences of these two resolutions:

$$\begin{array}{ccc} H^p(H^q(Y_i, K_i^!)) & \Longrightarrow & H^{p+q}(Y_i, M) \\ \downarrow & & \downarrow \\ H^p(H^q(X_i, L_i^!)) & \Longrightarrow & H^{p+q}(X_i, M) \end{array} ,$$

hence a morphism of the limit spectral sequences, whose abutment is just the map

$$H^{p+q}(Y, M) \rightarrow H^{p+q}(X, M) .$$

We are to show that this map is bijective. Therefore it suffices to show that for each q and ν , the map

$$(*) \quad \lim_{\substack{\rightarrow \\ i}} H^q(Y_i, K_i^!{}^\nu) \rightarrow \lim_{\substack{\rightarrow \\ i}} H^q(X_i, L_i^!{}^\nu)$$

is bijective. Now we have

$$H^q(Y_i, K_i^!{}^\nu) \approx H^q(Y_i^!, K_i^!{}^\nu)$$

and

$$H^q(X_i, L_i^!{}^\nu) \approx H^q(X_i^!, L_i^!{}^\nu) .$$

Consider the index category of maps $i \rightarrow j$ in I . Clearly we have isomorphisms

$$\begin{aligned} & \varinjlim_j H^q(Y'_j, K'_j{}^v) \\ & \approx \varinjlim_{i \rightarrow j} H^q(Y'_i, K'_i{}^v) \quad (\text{where } K'_j = M^{\mathcal{G}_j^v}) \\ & \approx \varinjlim_j (\varinjlim_{i \rightarrow j} H^q(Y'_i, M^{\mathcal{G}_j^v})) \quad (\text{App. 1.11}) \\ & \approx \varinjlim_j (H^q(Y', M^{\mathcal{G}_j^v})) . \end{aligned}$$

Now (4.3)(iii) asserts that $H^q(Y', M^{\mathcal{G}_j^v}) \approx H^q(X', M^{\mathcal{G}_j^v})$. Hence we are reduced for (*) to showing that

$$(**) \quad \varinjlim_j H^q(X', M^{\mathcal{G}_j^v}) \simeq \varinjlim_j H^q(X', M^{G_j^v}) .$$

But $\{\mathcal{G}_j\}$, $\{G_j\}$ are isomorphic as pro-groups, since they represent the images of the corresponding maps $\pi_1(\hat{Y}) \rightarrow \text{Aut } M$ and $\pi_1(\hat{X}) \rightarrow \text{Aut } M$ and since $\pi_1(\hat{Y}) \approx \pi_1(\hat{X})$. Since $H^q(X', M^{F^v})$ is a functor of F , (*) follows.

(i) \Rightarrow (iii):

This implication follows from the following result:

Theorem (4.11): Let \mathcal{C} be a complete class and let $X \in \text{pro-}\mathcal{K}_0$. Let $\hat{H} \subset \pi_1(\hat{X})$ be a sub-pro-group and let $H \subset \pi_1(X)$ be its inverse image. Then the \mathcal{C} -completion \hat{X}_H of the covering space X_H of X determined by H is canonically \mathcal{C} -isomorphic to

the covering \hat{X}_H of \hat{X} .

To obtain the implication (i) \Rightarrow (iii) from (4.11), we may apply the theorem to the corresponding \mathcal{C} -covering spaces X', Y' of X, Y respectively. Since $\hat{X} \approx \hat{Y}$, it implies that also $\hat{X}' \approx \hat{Y}'$. Hence we are reduced to the case $X = X'$, i.e., to showing that if f is a \mathcal{H} -isomorphism, then

$$H^q(Y, A) \xrightarrow{\sim} H^q(X, A) \quad \text{for } A \in \mathcal{C} .$$

But

$$\begin{aligned} H^q(Y, A) &\approx [Y, K(A, q)] \\ &= [\hat{Y}, K(A, q)] \quad \text{since } K(A, q) \in \mathcal{C}\mathcal{K}_0 \\ &= [X, K(A, q)] \\ &= H^q(X, A) . \end{aligned}$$

This completes the proof of (4.3).

Proof of (4.11): Of course, the map $X_H \rightarrow \hat{X}_H$ is induced by the canonical map $X_H \rightarrow \hat{X}_H$ since $\hat{X}_H \in \text{pro-}\mathcal{C}\mathcal{K}_0$. Thus we want to show that \hat{X}_H is the \mathcal{C} -completion of X_H . We may (App. 3.3) represent the given data by a compatible filtering family of sub-pro-groups

$$\hat{H}_i \subset \pi_1(\hat{X}_i)$$

where $X = \{X_i\}$, $X_i \in \mathcal{K}_0$. Let H_i be the sub-pro-group which is the inverse image of \hat{H}_i in $\pi_1(X_i)$, so that $H = \{H_i\}$. Since (2.8) $X_H = \{(X_i)_{H_i}\}$ and (3.9) $\widehat{\{(X_i)_{H_i}\}} = X_H$, it follows

that it is enough to show that for each i

$$\widehat{(X_i)_{H_i}} \xrightarrow{\sim} (\widehat{X_i})_{\widehat{H_i}} .$$

Hence we may replace X by X_i , i.e., suppose X is in \mathcal{K}_0 .

Write $\pi_1(\widehat{X}) = \{g_j\}$ ($g_j \in \mathcal{C}$) as a pro- \mathcal{C} -group, where the g_j are quotient groups of $\pi_1(X)$, in such a way that (App. 5.2) \widehat{H} is represented by a filtering family of subgroups

$$h_j \subset g_j \quad h_j \in \mathcal{C} .$$

Then if \widehat{h}_j denotes the inverse image of h_j in $\pi_1(\widehat{X})$, it is clear that

$$\widehat{H} = \{\widehat{h}_j\} .$$

Thus, applying (2.8), (3.9) again, we may replace \widehat{H} by \widehat{h}_j , i.e., assume that

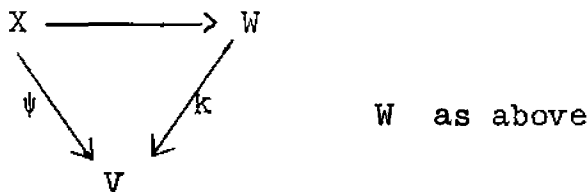
(4.12). H is a subgroup of $\pi_1(X)$, and is the inverse image of a subgroup h of some quotient group g of $\pi_1(X)$ in \mathcal{C} .

Recall (3.4) that \widehat{X} may be regarded as an inverse system I of objects W each of which is obtained from X by spanning in cells in successive dimensions so that the resulting homotopy groups are all in \mathcal{C} . Thus (2.8) $\widehat{X}_{\widehat{H}}$ will be obtained as an inverse system of the corresponding covering spaces W' (again indexed by I) of these W , if W is "large enough" so that the map $\pi_1(X) \rightarrow g$ factors through $\pi_1(W)$. This covering space W'

is obtained from $X' = X_H$ by spanning in some cells, and it is in $\mathcal{C}\mathcal{K}_0$. Therefore the theorem in our case is equivalent with the assertion that the category of such W' (indexed by I) is cofinal with the category of all maps $X' \rightarrow Y$ such that $Y \in \mathcal{C}\mathcal{K}_0$. Thus we need to verify App.(1.5) in this context. The conditions read:

(4.13). (a) Given a map $X' \xrightarrow{\phi} Y$ with $Y \in \mathcal{C}\mathcal{K}_0$, there is a map $X \rightarrow W$, W as above, such that ϕ factors through $X' \rightarrow W'$.

(b) Given a map $X \xrightarrow{\psi} V$, $V \in \mathcal{C}\mathcal{K}_0$ as above, and a pair of maps $V' \xrightarrow[i]{j} Y$ with $Y \in \mathcal{C}\mathcal{K}_0$ such that $i\psi' \sim j\psi'$ (where $\psi': X' \rightarrow V'$), there exists a diagram



such that $ik' \sim jk'$.

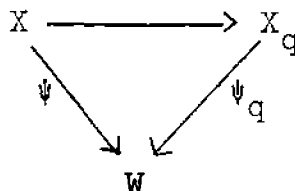
To prove (a), we construct W as the limit of a compatible sequence of CW-complexes X_q , where ϕ factors through X_q' and $\pi_n(X_q) \in \mathcal{C}$ for $n \leq q$. Given X_{q-1} , the problem is to construct X_q by spanning in some $(q+1)$ -cells in such a way that the map $X_{q-1}' \rightarrow Y$ extends to X_q' . Since X_q' is also obtained from X_{q-1}' by spanning in $(q+1)$ -cells, the condition is clearly that the map $\pi_q(X_{q-1}') \rightarrow Y$ factor through $\pi_q(X_q')$. Let $N \subset \pi_q(X_{q-1}')$

be the kernel of this map.

We treat first the case $q = 1$, setting $X_0 = X$. Let $K \subset H \subset \pi_1(X')$ be the kernel of the map $\pi_1(X) \rightarrow \mathcal{G}$ above. Then $K \subset N$ is a normal subgroup of H , hence of K . We have an injection $K/K \cap N \subset H/N \hookrightarrow \pi_1(Y)$, hence $K/K \cap N \in \mathcal{C}$. Therefore by (3.2)(i), $K \cap N$ contains a subgroup F normal in $\pi_1(X)$ and with $\pi_1(X)/F \in \mathcal{C}$. We can let X_1 be obtained by spanning in 2-cells to kill the subgroup F .

Now suppose $q > 1$. Then $\pi_1(X_{q-1})$ is a \mathcal{C} -group, and it operates on $\pi_q(X_{q-1})$. Let F be the intersection of the conjugates of N in $\pi_q(X_{q-1})$ under this action. Then $\pi_q(X_{q-1})/F$ is isomorphic to a subgroup of $(\pi_q/N)^{\pi_1}$, hence is a \mathcal{C} -group. Suppose we span a $(q+1)$ -cell into X_{q-1} whose boundary S represents the element $s \in \pi_q(X_{q-1})$. The homotopy classes of $\pi_q(X_{q-1}) = \pi_q(X'_{q-1})$ represented by the various spheres of X'_{q-1} lying over S in X_{q-1} (by choosing a path to the base point) are all translates of s under the action of π_1 . Therefore if $s \in F$, then so are all these other classes. Thus we may span in cells to kill F in X_{q-1} .

Consider (4.13)(b). The proof of this assertion follows closely that of (a). We again construct W as a limit of a sequence of factorizations



such that $\pi_n(X_q) \in \mathcal{C}$ for $n \leq q$, where $X_0 = X$ and X_q is

obtained from X_{q-1} by spanning in $(q+1)$ -cells, and where the homotopy

$$h_{q-1}: i\psi'_{q-1} \sim j\psi'_{q-1}$$

extends to a homotopy

$$h_q: i\psi'_q \sim j\psi'_q \quad .$$

Following the notation of (3.4), replace X by X_{q-1} , let N be the kernel of the map $\phi: \pi_q(X) \rightarrow \pi_q(V)$, and let $\eta: F \rightarrow N$ be a map of a free group on a set of generators $S \subset N$ onto N . Let $\alpha_s: D^{q+1} \rightarrow X$ be a pointed continuous map representing $\eta(s)$, and extend $\psi\alpha_s$ to a map $\beta_s: D^{q+1} \rightarrow V$. We obtain a factorization of ψ through $X \cup_{\alpha_s} D^{q+1} = X_s$. The covering X'_s of X_s corresponding to X' is obtained by spanning cells into each of the spheres α'_{sv} of X' lying over α_s , where the index v represents the choice of a point p_v above the base point of X . Choose for each v a path from $i(p_v)$ to the base point of Y .

For each v , we get a map

$$f_{sv}: Z \rightarrow Y$$

as in construction 1 of (3.4), replacing α_s, β_s by $\alpha'_{sv}, \beta'_{sv}$ respectively, and, using the chosen path above, the map f_{sv} determines a class \bar{f}_{sv} in $\pi_{q+1}(Y)$. The obstruction to extending the homotopy h to X'_s is represented by the vector

$$(\dots, \bar{f}_{sv}, \dots) \in (\pi_{q+1}(Y))^P$$

where $P = \{v\}$ is the index set. Thus we obtain a homomorphism $f_s: F \rightarrow (\pi_{q+1}(Y))^P$.

The important thing is that since $X' \rightarrow X$ is a \mathcal{C} -covering, and $\pi_{q+1}(Y) \in \mathcal{C}$, the group $(\pi_{q+1}(Y))^P$ is again a \mathcal{C} -group. Therefore if we replace the group $\pi_{q+1}(W_2)$ of the proof of (3.4) by $(\pi_{q+1}(Y))^P$, the same argument applies. Hence (4.13)(b) holds, which completes the proof of (4.11).

Example: (a) (Uniqueness of Eilenberg-MacLane spaces)

Corollary 4.14: Let X be in $\text{pro-}\mathcal{K}_0$ and suppose the pro-groups $\pi_q(X)$ vanish for $q \neq n$, and $\pi_n(X) = \pi$. Then X is \simeq -isomorphic to $K(\pi, n)$; $\text{cosk}_m X$ is isomorphic with $K(\pi, n)$, if $m \geq n$.

Proof: An application of (4.4) gives that $X \simeq \text{cosk}_n X$, and that $\text{cosk}_{n-1} X$ is contractible. (4.14) then follows from the fibre triple:

$$\text{cosk}_{n-1} X \rightarrow \text{cosk}_n X \rightarrow K(\pi, n).$$

(b) (Uniqueness of the n -sphere)

Corollary 4.15: Suppose \mathcal{C} is the class of finite groups whose orders have prime divisors in a fixed set P of primes. Suppose X is in $\text{pro-}\mathcal{C}\mathcal{K}_0$, is simply connected, and $H_q(X, Z) = 0$ if $q \neq n$, and $H_q(X, Z) = \hat{Z}$ if $q = n$ for some fixed $n \geq 2$. Then there is a \simeq -isomorphism,

$$\begin{array}{c} \hat{S}^n \\ \simeq \\ \simeq \\ \xrightarrow{\quad} X \end{array}$$

Proof: Let γ denote a topological generator of the projective limit of the pro-abelian group \hat{Z} . An application of (4.4) enables us to represent X by a system $\{X_i\}$ such that each X_i is $n-1$ connected, and for which there is a compatible system

$$\gamma_i \in H_n(X_i, Z)$$

which represents the element

$$\gamma \in \varprojlim H_n(X_i, Z) = \varprojlim (\hat{Z})$$

We have that $H_n(X_i, Z) \approx \pi_n(X_i)$ by the ordinary Hurewicz theorem, and hence the $\gamma_i \in \pi_n(X_i)$ may be interpreted as producing a morphism

$$S^n \xrightarrow{\psi} X .$$

in $\text{pro-}\mathcal{K}_0$. This morphism is clearly an isomorphism on all homology groups H_q for $q \neq n$.

We shall show that $\widehat{H}_n(S^n) \xrightarrow{\psi_n} H_n(X)$ is an isomorphism as well, for if we evaluate, using our assumption that $H_n(X) \approx \hat{Z}$, we find that ψ_n is a morphism

$$\hat{Z} \xrightarrow{\psi_n} \hat{Z}$$

such that

$$\varprojlim \psi_n: \varprojlim (\hat{Z}) \rightarrow \varprojlim (\hat{Z})$$

is an isomorphism. This latter fact implies that ψ_n itself is an isomorphism, and (4.15) follows from (4.3, iii).

§5. Completions and fibrations.

Throughout this section, we let \mathcal{C} denote a class (3.1) consisting of finite groups.

Let $X \in \text{pro-}\mathcal{K}$, and consider the pro-object pair $X \rightarrow \hat{X}$ described in (3.8). We can replace this object by a canonically isomorphic fibre resolution, as in (1.1). Let us write it the same way.

Proposition (5.1): The fibre χ of $X \rightarrow \hat{X}$ has the property that $\hat{\chi}$ is \mathcal{K} -contractible (4.2).

Proof: The inverse system $X \rightarrow \hat{X}$ is represented by maps $X \rightarrow W$ which induce isomorphisms of fundamental groups (3.8) and thus each object in the inverse system of χ is connected. Denote by $\tilde{\hat{X}}$ the universal covering of \hat{X} , and by \tilde{X} the induced covering of X . By (4.11), $\tilde{\hat{X}}$ is \mathcal{K} -isomorphic with the completion $\hat{\tilde{X}}$ of \tilde{X} , and the fibre of $\tilde{X} \rightarrow \tilde{\hat{X}}$ is \mathcal{K} -isomorphic with that of $\tilde{X} \rightarrow \hat{\tilde{X}}$. Thus we may suppose \hat{X} simply connected, and in fact that the inverse system comprising \hat{X} consists of simply connected objects.

We need to show that $\pi_1(\hat{\chi}) = 0$ and that for $A \in \mathcal{C}$ the cohomology groups $H^q(\chi, A)$ vanish for all $q > 0$. Let us write the inverse system as $X \rightarrow \hat{X} = \{X_i \rightarrow \hat{X}_i\}_i$, and $\chi = \{\chi_i\}$. We obtain a directed system of spectral sequences

$$(5.2) \quad (E_2^{pq})_i = H^p(\hat{X}_i, H^q(\chi_i, A)) \implies H^{p+q}(X_i, A)$$

where the coefficient groups $H^q(\chi_i, A)$ are untwisted since \hat{X}_i

is simply connected. Now \mathcal{C} is a class of finite groups, and $\pi_q(\hat{X}_i) \in \mathcal{C}$ for all q . Hence \hat{X}_i is homotopic to a CW-complex which is finite in each dimension. Therefore the cohomology of \hat{X}_i commutes with direct limits of coefficient groups. Passing to the limit of the E_2^{pq} term of (5.2) we obtain

$$\begin{aligned} \lim_i H^p(\hat{X}_i, H^q(\chi_i, A)) &= \lim_{i \rightarrow j} H^p(\hat{X}_i, H^q(\chi_j, A)) = \\ \lim_j H^p(\hat{X}, H^q(\chi_j, A)) &= H^p(\hat{X}, \lim_j H^q(\chi_j, A)) = \\ &H^p(\hat{X}, H^q(\chi, A)) . \end{aligned}$$

Therefore the limit of (5.2) is a spectral sequence

$$E_2^{pq} = H^p(\hat{X}, H^q(\chi, A)) \implies H^{p+q}(X, A) .$$

Since the coefficient groups are untwisted, we have $E_2^{0q} = H^q(\chi, A)$. But by (4.3), the map

$$E_2^{p0} = H^p(\hat{X}, A) \rightarrow H^p(X, A)$$

is an isomorphism for all p , which implies $E_2^{0q} = 0$, $q > 0$, and hence $E_2^{pq} = 0$ for $q > 0$ by induction on q .

It remains to show that $\pi_1(\hat{\chi}) = 0$. Consider the exact sequence

$$\pi_2(\hat{X}) \rightarrow \pi_1(\chi) \rightarrow \pi_1(X) \rightarrow 0 .$$

Let G be a \mathcal{C} -group, and suppose given a (say) surjective map

$$(*) \quad \pi_1(\chi) \rightarrow G .$$

Since $\widehat{\pi_1(X)} = \pi_1(\widehat{X}) = 0$, and since $\pi_1(X)$ maps to the group $G/\text{im}(\pi_2(\widehat{X}))$, it follows that $\pi_2(\widehat{X})$ maps onto G . Therefore G is abelian and the map $(*)$ represents an element of $\text{Hom}(\chi, G) = H^1(\chi, G)$ which was shown above to be zero.

Lemma (5.3): A pro-object $X \rightarrow Y$ in the homotopy category of maps of CW-complexes gives rise in a functorial way to a pro-object

$$(5.4) \quad \begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \widehat{X} & \longrightarrow & \widehat{Y} \end{array}$$

in the category of squares, where the vertical arrows are isomorphic to the ones introduced in (3.4).

Proof: It clearly suffices to treat the case of a pair $X \subset Y$ of CW-complexes. We let the pro-object (5.4) be given by the category of diagrams

$$(5.5) \quad \begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

where $U, V \in \mathcal{C}\mathcal{K}_0$, U is obtained from X by spanning cells in successive dimensions as in (3.4), and V is obtained from Y by spanning in cells including at least those spanned into X to obtain U . A morphism is a map of squares leaving $X \rightarrow Y$ fixed, taken up to homotopy. It is clear that the left vertical arrow yields a pro-object cofinal with $X \rightarrow \hat{X}$ as in (3.4), and that moreover any map $Y \rightarrow W$, $W \in \mathcal{C}$ factors through a V coming from a square (5.5). To show the right arrow cofinal with $Y \rightarrow \hat{Y}$ and that the resulting pro-object (5.4) is functorial, one reduces immediately to proving that given a diagram of CW-complexes

$$\begin{array}{ccccc}
 X & \longrightarrow & U & \xrightarrow{\quad} & W \\
 \downarrow & & \downarrow & & \downarrow \\
 Y & \longrightarrow & V & \xrightarrow{\quad} & W'
 \end{array}$$

and a homotopy h between the two squares

$$\begin{array}{ccc}
 X & \longrightarrow & Y \\
 \downarrow & & \downarrow \\
 W & \longrightarrow & W'
 \end{array} ,$$

there is a square

(5.6)

$$\begin{array}{ccc}
 X & \longrightarrow & Y \\
 \downarrow & & \downarrow \\
 U' & \longrightarrow & V'
 \end{array}$$

dominating (5.5) such that h extends to a homotopy on $U' \rightarrow V'$. To do this, we first choose V_1 so that the homotopy extends from Y to V_1 , as in (3.4). Then we choose any square

$$(5.7) \quad \begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U_1 & \longrightarrow & V_1 \end{array}$$

mapping to (5.5). Replacing (5.5) to (5.7), we are reduced to the case that h extends to V . Choose $X \rightarrow U'$ mapping to $X \rightarrow U$ so that h extends to U' , as in (3.4). Then we can add the cells spanned into X to obtain U' to V , obtaining a diagram (5.6) mapping to (5.5); and it is clear by construction that h extends.

Let $X \rightarrow Y \in \text{pro}(\mathcal{K}_{\text{pairs}})$ be as in the above discussion. If we fibre resolve the square (5.4), we obtain a pro-object of diagrams

$$(5.8) \quad \begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ \bar{F} & \longrightarrow & \hat{X} & \longrightarrow & \hat{Y} \end{array}$$

where F and \bar{F} are the fibres of the horizontal maps.

Theorem (5.9): With the above notation, suppose $Y = \{Y_i\}$ is a pro-object such that each Y_i is simply connected. Then $\hat{F} \rightarrow \bar{F}$ is a \sphericalangle -isomorphism.

Proof: Applying (1.2), we may imbed the square (5.8) in a projective system in the homotopy category of diagrams

$$(5.10) \quad \begin{array}{ccccc} \phi & \longrightarrow & \chi & \longrightarrow & \psi \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & X & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ \bar{F} & \longrightarrow & \hat{X} & \longrightarrow & \hat{Y} \end{array}$$

where the vertical and horizontal arrows are fibrations, up to homotopy. Say we index the diagrams making up (5.10) by $i \in I$. Now if we recall the construction (3.4) of a cofinal system of maps $Y \rightarrow W$ ($W \in \mathcal{CN}$) by spanning in cells, it is clear that for a simply connected Y we need only consider those W obtained by spanning in cells of dimension ≥ 3 , and then $\pi_2(Y) \rightarrow \pi_2(W)$ is surjective. Therefore in the diagram (5.10), we have

$$(5.11) \quad \pi_1(\psi) = 0 .$$

We want to show that ϕ is \mathcal{C} -acyclic, and since \mathcal{C} consists of finite groups, it suffices to treat twisted coefficient groups $A \approx \mathbb{Z}/p \in \mathcal{C}$, p a prime. Consider the inductive system of spectral sequences

$$H^p(\psi_i, H^q(\phi_i, A)) \implies H^{p+q}(\chi_i, A) .$$

We have

$$\begin{aligned} \lim E_2^{pq} &= \lim_i H^p(\psi_i, H^q(\phi_i, A)) \\ &= \lim_{i \rightarrow j} H^p(\psi_i, H^q(\phi_j, A)) . \end{aligned}$$

Since ψ is simply connected, we can take the second limit over those maps $i \rightarrow j$ which induce the zero map $\pi_1(\psi_i) \rightarrow \pi_1(\psi_j)$. Then the induced action on $H^q(\phi_j, A)$ is trivial, and so

$$\lim E_2^{pq} = \lim_j H^p(\psi, H^q(\phi_j, A)) .$$

Since $A \approx \mathbb{Z}/p$, the coefficient groups $H^q(\phi_j, A)$ are vector spaces over \mathbb{Z}/p . Moreover, ψ is \mathcal{C} -acyclic by (5.1), hence its cohomology with values in $H^q(\phi_j, A)$ is trivial. (For, its cohomology with values in a product of \mathbb{Z}/p is trivial, since cohomology commutes with products. In the exact sequence

$$0 \rightarrow \oplus \mathbb{Z}/p \rightarrow \prod \mathbb{Z}/p \rightarrow V \rightarrow 0$$

V is again a sum of (\mathbb{Z}/p) s, hence induction on q shows what we want.) Thus we conclude

$$\lim_i E_2^{pq} = 0 \quad \text{if } p > 0 .$$

This limit is the E_2^{pq} term of a spectral sequence abutting to $H^{p+q}(\chi, A)$ which is zero for $p + q > 0$ by (5.1). Therefore

$$\lim_i E_2^{0q} = \lim_j H^q(\phi_j, A) = H^q(\phi, A) = 0 , \quad q > 0 ,$$

as well, as was to be shown.

Now let $A \in \mathcal{C}$ be a coefficient group on \overline{F} , possibly twisted. Since \widehat{X}, \widehat{Y} are in $\text{pro-}\mathcal{C}$, they are projective limits of CW-complexes homotopic to ones finite in each dimension, and hence the same is true for \overline{F} . Thus cohomology on \overline{F} commutes with direct limits, and the inductive system of spectral sequences

$$E_2^{pq} = H^p(\overline{F}_i, H^q(\phi_i, A)) \implies H^{p+q}(F_i, A)$$

has as limit

$$\lim_i E_2^{pq} = \lim_j H^p(\overline{F}, H^q(\phi_j, A)) = H^p(\overline{F}, H^q(\phi, A)) = 0 \text{ if } q > 0 .$$

Therefore the spectral sequence collapses and yields isomorphisms

$$H^p(\overline{F}, A) \xrightarrow{\sim} H^p(F, A)$$

for all p .

To complete the proof that \widehat{F} and \overline{F} are \mathcal{H} -isomorphic, it suffices by (4.3) to show that $\widehat{\pi_1(F)} \xrightarrow{\sim} \pi_1(\overline{F})$. Consider the diagram

(5.12)

$$\begin{array}{ccccc}
 \widehat{\pi_2(\Psi)} & \xrightarrow{a} & \widehat{\pi_1(\Phi)} & \longrightarrow & 0 \\
 \downarrow b & & \downarrow c & & \\
 \widehat{\pi_2(\dot{Y})} & \longrightarrow & \widehat{\pi_1(\dot{F})} & & \\
 \downarrow d & & \downarrow e & & \\
 \widehat{\pi_2(\hat{Y})} & \longrightarrow & \widehat{\pi_1(\hat{F})} & & \\
 \downarrow & & \downarrow & & \\
 0 & & 0 & &
 \end{array}$$

Since Y is simply connected, $\widehat{\pi_1(\chi)} = 0$, and $\widehat{}$ is a right exact functor (meaning that it commutes with direct limits, since it has the appropriate adjointness property), the rows and columns of this diagram are exact, in an obvious sense. Since Y, \hat{Y} are simply connected, we have

$$H^2(Y, A) = \text{Hom}(\pi_2(Y), A) = \text{Hom}(\pi_2(\hat{Y}), A) = H^2(\hat{Y}, A)$$

for $A \in C$, and so the map d of (5.12) is an isomorphism. Hence b is the zero map, and since a is surjective, c is zero. Thus e is an isomorphism, as was to be shown. This completes the proof.

To conclude this section, we apply (5.9) to study the loop-space functor Ω . We revert from \mathcal{K}_0 to \mathcal{K}_0 , and suppose $e \rightarrow K$ a pointed, connected, simply connected simplicial set. If we fibre-resolve the inclusion $e \rightarrow K$, we obtain a fibre triple:

$$\Omega K \rightarrow e' \rightarrow K$$

which is actually functorial for pointed maps,

$$\begin{array}{ccccc}
 \Omega K_1 & \longrightarrow & e_1' & \longrightarrow & K_1 \\
 \downarrow \Omega f & & \downarrow & & \downarrow f \\
 \Omega K_2 & \longrightarrow & e_2' & \longrightarrow & K_2
 \end{array}$$

and if $f \sim f'$, we obtain $\Omega f \sim \Omega f'$ (1.1) and thus we may interpret Ω as a functor from the connected simply connected full subcategory of \mathcal{K}_0 to \mathcal{K}_0 .

Corollary 5.13: If $e \rightarrow K$ is a connected, simply connected pointed simplicial set, then

$$\widehat{\Omega(K)} \longrightarrow \Omega(\widehat{K})$$

is a h -isomorphism.

§6. Homotopy groups of completions.

In this section, we discuss the relationships between the homotopy groups of an $X \in \text{pro-}\mathcal{K}_0$ and of its \mathcal{C} -completion \hat{X} for a complete class \mathcal{C} . Since X maps to \hat{X} , we are given a map

$$(6.1) \quad \widehat{\pi_n(X)} \rightarrow \pi_n(\hat{X})$$

for each n , and we saw in (3.7) that it is an isomorphism when $n = 1$. More generally, we have

Corollary (6.2): Suppose that $\pi_q(X) = 0$ for $1 \leq q < n$.

Then

$$\widehat{\pi_n(X)} \xrightarrow{\sim} \pi_n(\hat{X}) .$$

This is analogous to (4.5). If say $X = \{X_i\}$, let $X' = \{X'_i\}$ be the fibre of the map $\{X_i \rightarrow \text{cosk}_{n-1} X_i\}$. It follows from (4.4 (iii)) that $X' \rightarrow X$ is a h -isomorphism and so we may replace X by X' , i.e., we may suppose that $\pi_q(X_i) = 0$ for $1 \leq q < n$ and for each i . Hence it suffices to treat the case $X = X_1$. Then in the construction (3.6) of $X \rightarrow \hat{X}$, it suffices to take maps $X \rightarrow W$ where W is obtained by spanning in cells of dimension at least $(n+1)$, from which the assertion is clear.

In addition, one can describe π_2 in complete generality as follows:

Proposition (6.3): Let $X = \{X_i\} \in \text{pro-}\mathcal{K}_0$, and let $A \in \mathcal{C}$ be abelian. Let J be the index category of pairs $j = (i, H)$

where $H \subset \pi_1(X_i)$ is a normal subgroup such that $\pi_1(X_i)/H \in \mathcal{C}$.
 Let $(X_i)_H$ be the covering space of X_i determined by H .
 Then the groups

$$H^2((X_i)_H, A)$$

form a directed system indexed by J , and $\pi_2(\hat{X})$ is characterized by the property that

$$(6.4) \quad \text{Hom}(\pi_2(\hat{X}), A) \approx \lim_j H^2((X_i)_H, A) .$$

Proof: Let \tilde{X} be the universal covering of \hat{X} , and let \tilde{X} be the induced covering space of X . Then by (4.11), $\hat{X} \underset{\neq}{\approx} \tilde{X}$. We have

$$\lim_j H^2((X_i)_H, A) = H^2(\tilde{X}, A) ,$$

and $\pi_2(\hat{X}) = \pi_2(\tilde{X})$. Hence we may replace X by \tilde{X} , i.e., we may assume $\pi_1(X)$ has no non-zero maps to \mathcal{C} -groups. Then (6.4) is just the assertion that

$$H^2(\hat{X}, A) \xrightarrow{\cong} H^2(X, A) ,$$

which follows from (4.3).

Following Serre [27], we introduce the notion of good group:

Definition (6.5): Let \mathcal{C} be a complete class, and $G \in$ pro-(groups). G is said to be \mathcal{C} -good if for every twisted abelian \hat{G} -module $A \in \mathcal{C}$,

$$H^q(\hat{G}, A) \xrightarrow{\sim} H^q(G, A) \quad \text{for all } q ,$$

where \hat{G} is the \mathcal{C} -completion of G .

Corollary (6.6): $G = \{G_i\}$ is \mathcal{C} -good if and only if the \mathcal{C} -completion of $K(G, 1) = \{K(G_i, 1)\}$ is \mathcal{H} -isomorphic to $K(\hat{G}, 1)$.

For, the completion of the map $K(G, 1) \rightarrow K(\hat{G}, 1)$ is a \mathcal{H} -isomorphism iff. G is \mathcal{C} -good, by (4.3).

Applying (4.4), we can now prove

Theorem (6.7): Let \mathcal{C} be a class of finite groups, and let $X \in \text{pro-}\mathcal{N}_0$ be simply connected. Then

$$\widehat{\pi}_q(X) \xrightarrow{\sim} \pi_q(\hat{X}) \quad \text{for } q \leq n$$

if $\pi_q(X)$ is \mathcal{C} -good for $q \leq n-1$.

Proof: Suppose $\pi_q(X)$ is \mathcal{C} -good for $q \leq n-1$. Applying (4.3) to the fibre of the map $X \rightarrow \text{cosk}_2 X$, one reduces to the case that $X = \{X_i\}$ has $\pi_1(X_i) = 0$ for all i . Suppose that actually $\pi_q(X_i) = 0$ for $q < r$ and all i . We proceed by descending induction on $r \geq 2$, the theorem being true by (6.2) when $r \geq n-1$:

Let $G = \pi_r(X)$, and consider the fibration

$$(6.8) \quad F \rightarrow X \rightarrow K(G, r) .$$

By (5.9), the \mathcal{C} -completion of (6.8) is \mathcal{H} -isomorphic with a fibre sequence, and we may suppose the theorem true for F by induction.

Thus we are done by the exact homotopy sequence arising from the completion of (6.8) if we prove the following:

Proposition (6.9): Let \mathcal{C} be a class of finite groups and $r \geq 1$ an integer. A pro-abelian group G is \mathcal{C} -good iff the map $\widehat{K(G,r)} \rightarrow K(\widehat{G},r)$ is a η -isomorphism.

Proof: For $r = 1$, this is (6.1). For $r > 1$, we may apply (6.6), (5.9) to the fibration

$$K(G,r-1) \rightarrow E \rightarrow K(G,r)$$

where E is contractible, and induction.

Corollary (6.10): Let $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ be an exact sequence of pro-abelian groups, and \mathcal{C} a class of finite groups. Suppose A'' is \mathcal{C} -good. Then the sequence $0 \rightarrow \widehat{A}' \rightarrow \widehat{A} \rightarrow \widehat{A}'' \rightarrow 0$ is exact.

This follows from the above proposition, and (5.9), applied to the fibration

$$K(A',2) \rightarrow K(A,2) \rightarrow K(A'',2) .$$

Example (6.11): Theorem (6.7) is false for non-simply connected X , even when $n = 2$. For example, let X be a wedge of a 2-sphere and a 1-sphere, and let \mathcal{C} be the class of all finite groups. Then $\pi_1(X) = \mathbb{Z}$, and $\pi_2(X)$ is a countable sum of copies of \mathbb{Z} . For any finite quotient group of $\pi_1(X)$, the corresponding covering space is a wedge of S^1 and finitely many

S^2 . We apply (6.3). It is immediately seen that while

$$\text{Hom}(\pi_2(X), A) = \prod_{-\infty}^{\infty} A ,$$

the group

$$\text{Hom}(\pi_2(\hat{X}), A)$$

is the subgroup of this group consisting of vectors

$$(\dots, a_0, a_1, \dots) \quad a_i \in A$$

which are periodic, with some period.

Example (6.12): Take \mathcal{C} to be the category of finite groups. Then the additive group of rational numbers, Q , is \mathcal{C} -good. Thus $\widehat{K(Q, n)}$ is \mathcal{H} -contractible. Apply (5.9) to

$$0 \rightarrow Z \rightarrow Q \rightarrow Q/Z \rightarrow 0$$

to obtain that $\widehat{K(Q/Z, n)}$ is \mathcal{H} -isomorphic with $\widehat{K(Z, n+1)}$, if $n \geq 2$.

An application of (5.13) gives the above for $n = 1$ as well.

§7. Stable results.

In this section we study the question of classification, up to stable homotopy equivalence, of all pointed CW-complexes of a fixed stable pro-finite homotopy type. This problem is reduced to a purely algebraic one (Cor. (7.15), below). It is shown that there are only a finite number of distinct stable CW-complexes having a given stable pro-finite completion.

If A, B are pointed CW-complexes, let $S^n A$ denote n^{th} iterated reduced suspension. For any integer k , and nonnegative integers n_j ($j = 1, 2$), two maps, $f_j: S^{n_j}(A) \rightarrow S^{n_j+k}(B)$, ($j = 1, 2$), will be referred to as "equivalent maps of degree k " if there is an integer n such that $S^{n-n_1}(f_1)$ is homotopic to $S^{n-n_2}(f_2)$. The set of equivalence classes of maps of degree k forms an abelian group which we denote $(A, B)_k$. By stable homotopy category we will mean the category $\mathcal{S}\mathcal{W}$ whose objects are pointed CW-complexes, and morphisms from A to B are given by the set $(A, B) = (A, B)_0$. As we have defined it, the stable homotopy category is an additive category. Although there is a standard technique for enlarging this category slightly to a triangulated category, (in the terminology of Verdier, cf. [16]) we shall not bother to do this. (For general treatments of stable theory, see [1], [24].)

If A is a subcomplex of X , and if we denote by X/A the CW-complex obtained by constructing the cone over A in X , we obtain a triple

$$(\Delta) A \rightarrow X \rightarrow X/A$$

where the arrows denote the natural inclusions. Anything

isomorphic to such a triple (Δ) will be referred to as a cofibration. The cofibrations play the role of the "triangles" of our category.

We have the standard facts that any continuous map of CW-complexes $f: X \rightarrow Y$ may be continued to a cofibration,

$$(\Delta_f) \quad X \xrightarrow{f} Y \rightarrow Z$$

and this cofibration is unique dependent only upon f up to (non-canonical) isomorphism. Also, functors of the form $(\ , W)$ and $(V, \)$ applied to cofibrations yield three-term exact sequences of groups, etc.

In this section let $\pi_n(A)$ denote the functor (S^n, A) , the stable homotopy group functor.

Lemma (7.1): If A, B are finite CW-complexes, then the groups $(A, B)_k$ are finitely generated abelian groups for all k . Consequently so are $\pi_n(B)$.

Proof: Well known, and easy. Here is one way of seeing it: By the Freudenthal suspension theorem, $\pi_n(B)$ may be interpreted as the ordinary homotopy groups of $S^\ell B$ for a sufficiently large choice of ℓ . But this $S^\ell B$ is a simply-connected finite CW-complex. It therefore has finitely generated integral homology groups, and the relative Hurewicz theorem applies, assuring that it has finitely generated homotopy groups. The spectral sequence relating $H^m(A, \pi_n(B))$ to $(A, B)_k$ allows one to then deduce lemma (7.1).

What is the relationship between suspension and completion? Let $\hat{}$ denote completion with respect to a class \mathcal{C} . Let \mathcal{K}_0 be the pointed homotopy category of CW-complexes. Let $\Sigma, \Delta: \mathcal{K}_0 \rightarrow \mathcal{K}_0$ be functors, Σ left adjoint to Δ and suppose, further, that both functors preserve the subcategory $\mathcal{C}\mathcal{K}_0$.

Lemma (7.2): The natural map $\hat{\Sigma X} \xrightarrow{\alpha} \Sigma(\hat{X})$ is an isomorphism.

Proof: To obtain a two-sided inverse to α , form the composite

$$X \rightarrow \Delta(\Sigma X) \rightarrow \Delta(\hat{\Sigma X})$$

where the first map comes from adjointness of Δ and Σ . Since $\Delta(\hat{\Sigma X})$ is in $\text{pro-}\mathcal{C}\mathcal{K}_0$ the above composite extends to a map, $\hat{X} \rightarrow \Delta(\hat{\Sigma X})$ which yields (by adjointness of Σ, Δ in $\text{pro-}\mathcal{K}_0$), a map $\beta: \Sigma(\hat{X}) \rightarrow \hat{\Sigma X}$. One checks that α and β are two-sided inverses of one another.

Corollary (7.3): If $\hat{}$ denotes p-completion or profinite completion, the natural map $\alpha: \widehat{S^k X} \rightarrow S^k(\hat{X})$ is an isomorphism. (S^k denotes k-fold reduced suspension.)

From now on, unless explicitly indicated, $\hat{}$ will denote profinite completion.

Corollary (7.4): Let X be a finite CW-complex. Then $\pi_n(X) \rightarrow \pi_n(\hat{X})$ is pro-finite completion.

Proof: Note that $\pi_n(X)$ is the ordinary homotopy groups of $S^k X$ for some k (with a shift in dimension), and these groups are \mathcal{C} -good where \mathcal{C} is the class of all finite groups. We may therefore use theorem (6.7).

Consider two objects of $\text{pro-}\mathcal{A}_0$, $X = (X_i)_{i \in I}$, and $Y = (Y_j)_{j \in J}$. Define the pro-abelian group

$$(X, Y)_k = \varinjlim_i (X_i, Y_j)_k \quad \text{indexed over } j \in J.$$

Notice that

$$\varprojlim_j (X, Y) = \text{Hom}(X, Y)$$

the latter group being Hom in the category $\text{pro-}\mathcal{A}_0$.

Corollary (7.5): Let A, B be finite CW-complexes. Then the natural homomorphism of abelian groups

$$(A, B) \rightarrow (\hat{A}, \hat{B})$$

is profinite completion.

Proof: Suppose $A_1 \rightarrow A_2 \rightarrow A_3$ is a cofibration. Then the Puppe sequence gives us a morphism of long exact sequences.

$$\begin{array}{cccccccc} \dots & \rightarrow & (A_3, B)_k & \rightarrow & (A_2, B)_k & \rightarrow & (A_1, B)_k & \rightarrow & (A_3, B)_{k-1} & \rightarrow & \dots \\ & & \downarrow \eta_3 & & \downarrow \eta_2 & & \downarrow \eta_1 & & \downarrow \eta_3 & & \\ \dots & \rightarrow & (A_3, \hat{B})_k & \rightarrow & (A_2, \hat{B})_k & \rightarrow & (A_1, \hat{B})_k & \rightarrow & (A_3, \hat{B})_{k-1} & \rightarrow & \dots \end{array}$$

for all $k \in \mathbb{Z}$. The top line consists in finitely generated abelian groups, and the bottom line in pro-abelian groups. Note that if we take the profinite completion of the top line we obtain

a long exact sequence in pro-Ab . Suppose $\hat{\eta}_1, \hat{\eta}_3$ were isomorphisms for all k . Then applying the five-lemma in the abelian category pro-ab, we would obtain that $\hat{\eta}_3$ is an isomorphism for all k . This observation, coupled with Cor.(7.4) and the fact that any finite CW-complex A may be built up from a finite sequence of cofibrations $j_{A_1} \rightarrow j_{A_2} \rightarrow j_{A_3}$ for $j = 1, \dots, r$ gives us Cor.(7.5). q.e.d. Note: "built up" means the following.

- (a) $j_{A_1}^1$ and $j_{A_3}^j$ are spheres of various dimensions, $j = 1, \dots, r$.
- (b) $j_{A_1}^{j+1} = j_{A_2}^j$ for $j = 1, \dots, r-1$.
- (c) $j_{A_2}^r = A$.

A model for the profinite completion of a double suspension.

Suppose $A = SB$, where B is a pointed finite CW-complex. Thus $A = BxI / (bxI \cup Bx\partial I)$. Consider, for any integer n , the map $f_n: I/\partial I \rightarrow I/\partial I$ which is multiplication by n . That is, after the identification of $I/\partial I$ with R/Z , $f_n(x) = nx$ for all $x \in R/Z$. This map gives rise to a map $f_n: A \rightarrow A$ by setting $f_n(\beta, x) = (\beta, f_n(x))$ for all $\beta \in B$ and $x \in I$. Note that this f_n represents the n -th power map of the group-valued functor $[A,]$ into itself.

Suppose now that $A = S^2B$, so that $[A,]$ actually takes values in the category of abelian groups. Continuing the f_n to cofibrations of finite complexes, we obtain

$$(\Delta_n) \quad A \xrightarrow{f_n} A \xrightarrow{g_n} A/n$$

Since $f_{mn} = f_m \circ f_n$ we obtain compatible maps,

$$\begin{array}{ccccc}
 A & \xrightarrow{f_{mn}} & A & \longrightarrow & A/mn \\
 \downarrow f_m & & \downarrow \text{id} & & \downarrow \\
 A & \xrightarrow{f_n} & A & \longrightarrow & A/n
 \end{array}$$

Thus we may interpret $\tilde{A} = (A/n)_{n \in J}$ as a pro-object of \mathcal{K}_0 . Applying the homology functor to (Δ_n) note that A/n has finite homology groups in each dimension. Since A is simply connected, so is A/n . Consequently by the relative Hurewicz theorem, A/n has finite homotopy groups in each dimension. Thus \tilde{A} is actually in $\text{pro-}\mathcal{K}_0$, where \mathcal{C} is the class of finite groups.

If W is in \mathcal{K}_0 , then $[A, W]$ the group of homotopy classes of continuous maps is a finite abelian group. Abelian since A is a double suspension; finite by induction on the number of cells of B .

Proposition (7.6): $A \rightarrow \tilde{A}$ is profinite completion.

Proof: We need only show for any W in \mathcal{K}_0 that the induced map $[\tilde{A}, W] \rightarrow [A, W]$ is an isomorphism. This follows by considering the long exact sequences,

$$[SB, W] \xrightarrow{n} [SB, W] \rightarrow [A/n, W] \rightarrow [A, W] \xrightarrow{n} [A, W]$$

and using that $[A, W]$ is finite.

If p is a prime the above argument gives us a very useful model for the p -completion of A . It is given by the pro-object:

$$\dots \rightarrow A/p^{r+1} \rightarrow A/p^r \rightarrow \dots \rightarrow A/p$$

Stable prohomotopy equivalence.

If A, B are CW-complexes we call stable prohomotopy equivalence between A and B an isomorphism $\varphi: \hat{A} \rightarrow \hat{B}$ in the category $\text{pro-S}\mathcal{K}_0$.

Let A be a finite CW-complex. Then $(A, A) = E(A)$ may be given the structure of a ring by composition. It is of finite type as a \mathbb{Z} -module, by lemma (7.1).

If B is a finite CW-complex, then (A, B) is a left $E(A)$ -module and a right- $E(B)$ -module. Of course, it is of finite type over both of those rings, since it is in fact of finite type over \mathbb{Z} .

If \hat{A} is the profinite completion of A , let $E(\hat{A})$ denote the ring, $\text{Hom}(\hat{A}, \hat{A}) = \varprojlim (\hat{A}, \hat{A})$. Since (\hat{A}, \hat{A}) is a pro-object in the category of finite abelian groups, we may regard the module $\text{Hom}(\hat{A}, \hat{A})$ as a profinite abelian group. Thus $E(\hat{A})$ is a topological ring and, after cor. (7.5), may be described as the profinite completion of $E(A)$; also $E(\hat{A}) = E(A) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$.

Corollary (7.7): Let A, B be finite CW-complexes. Then

$$\text{Hom}(\hat{A}, \hat{B}) = \text{Hom}(A, B) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} = \text{Hom}(A, B) \otimes_{E(B)} E(\hat{B}).$$

Lemma (7.8): Let $\epsilon: A \rightarrow B$ be a morphism of finite CW-complexes having the property that $\hat{\epsilon}: \hat{A} \rightarrow \hat{B}$ is a h -isomorphism. Then ϵ is an isomorphism.

Proof: Let us begin by seeing that the analogous statement holds for finitely generated abelian groups. Namely, if $\varphi: C \rightarrow C'$ is a homomorphism of finitely generated abelian groups such that $\hat{\varphi}$ is an isomorphism, then φ is, since $\hat{}$ is a faithfully flat functor on the category of finitely generated abelian groups.

Now consider the two homomorphisms,

$$\text{Hom}(B, A) \xrightarrow{\alpha} E(A)$$

$$\text{Hom}(B, A) \xrightarrow{\beta} E(B)$$

induced by composition on the right (resp. left) by ϵ . Since $\hat{\epsilon}$ is a \mathcal{H} -isomorphism, and A is a finite complex, $\hat{\alpha}$ and $\hat{\beta}$ are isomorphisms. Consequently α and β are themselves isomorphisms by our opening remarks. If $1_A \in E(A)$, $1_B \in E(B)$ are the identity maps then $\alpha^{-1}(1_A)$ and $\beta^{-1}(1_B)$ are left and right inverses to ϵ , respectively. This concludes lemma (7.8).

Corollary (7.9): Let A, B be finite CW-complexes which are of the same stable prohomotopy type. Suppose that $\text{Hom}(A, B)$ is a free module of rank one over $E(A)$. Then A is isomorphic to B in the stable homotopy category.

Proof: Let $\epsilon: A \rightarrow B$ denote a generator of the free $E(A)$ -module, $\text{Hom}(A, B)$. Then $\hat{\epsilon}$ is an isomorphism, and lemma (7.8) applies.

Let Λ denote a ring (with unit, not necessarily commutative) of finite type as a module over Z . Let $\hat{\Lambda}$ denote the profinite completion, $\hat{\Lambda} = \Lambda \hat{\otimes} Z$. Denote by $P(\Lambda)$ the set of isomorphism

classes of modules M of finite type over Λ which have the property that $\hat{M} = M \otimes_{\Lambda} \hat{\Lambda}$ is free of rank one over $\hat{\Lambda}$.

A useful fact concerning completion of such rings Λ is the following:

Lemma (7.10): Let M, N be Λ -modules, with M of finite type. Then

$$\eta_M: \text{Hom}_{\Lambda}(M, N) \otimes_{\Lambda} \hat{\Lambda} \rightarrow \text{Hom}_{\hat{\Lambda}}(\hat{M}, \hat{N})$$

is an isomorphism. Consequently a module P of finite type over Λ is projective over Λ if and only if \hat{P} is projective over $\hat{\Lambda}$.

Proof: Find a resolution of Λ -modules,

$$0 \rightarrow R \rightarrow F \rightarrow M \rightarrow 0$$

with F free of finite type over Λ . It follows that R is of finite type over Λ since F is a finitely generated free abelian group. Form the diagram of exact sequences,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_{\Lambda}(M, N) \otimes_{\Lambda} \hat{\Lambda} & \longrightarrow & \text{Hom}_{\Lambda}(F, N) \otimes_{\Lambda} \hat{\Lambda} & \longrightarrow & \text{Hom}_{\Lambda}(R, N) \otimes_{\Lambda} \hat{\Lambda} \\ & & \downarrow \eta_M & & \downarrow \eta_F & & \downarrow \eta_R \\ 0 & \longrightarrow & \text{Hom}_{\hat{\Lambda}}(\hat{M}, \hat{N}) & \longrightarrow & \text{Hom}_{\hat{\Lambda}}(\hat{F}, \hat{N}) & \longrightarrow & \text{Hom}_{\hat{\Lambda}}(\hat{R}, \hat{N}) \end{array}$$

and since F is free of finite type, η_F is an isomorphism. Therefore η_A is injective. Since M was supposed to be a quite general Λ -module of finite type, we may conclude that η_R is

also injective. Diagram-chasing then gives η_M surjective.

The final statement of lemma 7.10 then follows in the non-trivial direction, for if $N_1 \rightarrow N_2 \rightarrow 0$ is a surjective morphism of Λ -modules then $\text{Hom}_\Lambda(P, N_1) \otimes_{\hat{Z}} \rightarrow \text{Hom}_\Lambda(P, N_2) \otimes_{\hat{Z}}$ is surjective since η_P is an isomorphism, and $\text{Hom}_\Lambda(P, N_1) \rightarrow \text{Hom}_\Lambda(P, N_2)$ is surjective since \hat{Z} is faithfully flat over Z .

Let A, B be finite CW-complexes of the same stable prohomotopy type. Then $\text{Hom}(A, B) = M$ is a module of finite type over $E(A) = \Lambda$ such that \hat{M} is free of rank one over $\hat{\Lambda}$ by cor. (7.7). Thus M is a projective module over $\hat{\Lambda}$ by lemma (7.10), and represents a class in $P(\Lambda)$. Let us write $P(\Lambda) = P(A)$ and denote this class $c(A, B)$. The results obtained above assure us that $c(A, B)$ plays the role of an obstruction to A and B being of the same stable homotopy type. Note that $P(\Lambda)$ has a distinguished element $0 \in P(\Lambda)$ represented by the isomorphism class of the free module of rank one over Λ .

Corollary (7.11): Let A, B be finite CW-complexes of the same stable pro-homotopy type. Then $c(A, B) = 0$ if and only if A, B have the same stable homotopy type.

It is easily seen that a module M of finite type such that \hat{M} is free of rank one over $\hat{\Lambda}$ may also be described as a (projective) Λ -module such that $M_p = M \otimes_{Z_p}$ is free of rank one over $\Lambda_p = \Lambda \otimes_{Z_p}$, for all primes p , where Z_p denotes the integers localized at the prime p . (An application of Nakayama's lemma). Thus $P(\Lambda)$ is exactly the set $P_1(\Lambda)$ defined in [4], (10.1).

According to a result of Jordan-Zassenhaus [32] there are

only a finite number of isomorphism classes of modules M over Λ satisfying the conditions:

- (i) M is a free Z -module.
- (ii) $M \otimes_Z \mathbb{Q}$ is a free $\Lambda \otimes_Z \mathbb{Q}$ -module of finite rank.

We are grateful to Bass for calling our attention to this result, and pointing out that this implies that $P_1(\Lambda)$ is a finite set.

Twisting finite CW-complexes.

Let us consider a category \mathcal{C} which is the full subcategory of the stable homotopy category generated by simply connected finite CW-complexes. Let X be a finite, simply connected CW-complex, and let τ be a class in $P(X)$. We shall describe a finite simply connected CW-complex $Y = X^\tau$ which is of the same stable prohomotopy type as X and such that $c(X, Y) = \tau$.

Let $\varphi: \mathcal{C}^0 \rightarrow \text{Ab}$ be the functor represented by X ; thus, $\varphi(A) = (A, X)$. Of course, the abelian group (A, B) may be regarded in a natural manner as a right $E(X)$ -module. Let M be a left $E(X)$ -module representing the class τ . Define the functor $\varphi^\tau: \mathcal{C}^0 \rightarrow \text{Ab}$ as follows:

$$\varphi^\tau(A) = (A, X) \otimes_{E(X)} M$$

(we may write symbolically, $\varphi^\tau = \varphi \otimes_{E(X)} M$).

Definition: A finite cohomological functor $\alpha: \mathcal{C}^0 \rightarrow \text{Ab}$ is one such that

- (i) $\alpha(S^q)$ is finitely generated for all $q (\geq 2)$
- (ii) if $A_1 \rightarrow A_2 \rightarrow A_3$ is a cofibration in \mathcal{C} , then

$$\alpha(A_1) \rightarrow \alpha(A_2) \rightarrow \alpha(A_3)$$

is exact.

(iii) There is an integer N such that $\text{Hom}_{\text{funct}}(\alpha, W) = 0$ for all N -connected objects W (where in the above, we have identified W with the functor which it represents).

Note that if φ is a functor representable in \mathcal{C} , then φ is a finite cohomological functor. For, if Y is the object representing φ , then (i) expresses the fact that $\pi_q(Y)$ is finitely generated for all q , (ii) is implied by the Puppe sequence, and (iii) expresses the fact that Y has finite cohomological dimension. Actually we have the stronger result:

Lemma (7.12): Let $\varphi: \mathcal{C}^0 \rightarrow \text{Ab}$ be represented by X , and let τ be a class in $P(X)$. Then φ^τ is a finite cohomological functor.

Proof: Let M be a left- $E(X)$ -module representing τ . Then (i) follows because M is of finite type over $E(X)$, and (ii) follows because M is projective over $E(X)$. Since φ^τ satisfies both (i) and (ii), using an argument similar to that of lemma (7.1), one sees that $\varphi^\tau(A)$ is a finitely generated abelian group for all objects A of \mathcal{C} .

Now let N be the integer for which φ satisfies (iii). We show that φ^τ satisfies (iii) for this N . Suppose not.

Then there would be an N -connected W and a nontrivial morphism of functors $\varphi^T \xrightarrow{Y} W$. Consider the functor $\widehat{\varphi^T}$ defined by $\widehat{\varphi^T}(A) = \widehat{\varphi^T(A)}$. Applying the operation $\widehat{}$, we obtain a morphism of functors $\widehat{\gamma}: \widehat{\varphi^T} \rightarrow \widehat{W}$. Note that $\widehat{\gamma}$ is not the trivial morphism, for if A is an object such that $\gamma(A): \varphi^T(A) \rightarrow (A, W)$ is not the zero homomorphism, then $\widehat{\gamma}(A): \widehat{\varphi^T}(A) \rightarrow \widehat{(A, W)}$ is not the zero homomorphism either. This uses that both domain and range of $\gamma(A)$ are finitely generated abelian groups. Note also that $\widehat{\varphi^T}$ is isomorphic to the functor $\widehat{\varphi}$. Thus $\text{Hom}_{\text{funct}}(\widehat{\varphi^T}, \widehat{W}) = \text{Hom}_{\text{funct}}(\widehat{X}, \widehat{W}) = \text{Lim}_{\leftarrow} (X, W)$. But $(X, W) = 0$, by assumption, which gives the contradiction sought.

We now state and prove Ed Brown's representability theorem in our context (cf. [7]).

Representability theorem.

If α is a finite cohomological functor, then α is representable in \mathcal{C} .

Proof. We shall construct, below, a sequence

$$Y_0 \subset Y_1 \subset Y_2 \subset \dots \quad (\Sigma)$$

of simply connected finite CW-complexes such that Y_q is built from Y_{q-1} by adjunction of cells of dimension q , and $q+1$. We shall also find compatible morphisms of functors $\varphi_q \rightarrow \alpha$, where $\varphi_q = (\ , Y_q)$.

The morphism of functors $\varphi_q \rightarrow \alpha$ will have the property

that $\varphi_q(S^i) \rightarrow \alpha(S^i)$ is an isomorphism for $i \leq q$. If we let Y denote the direct limit of the CW-complexes Y_q , then Y would represent the limit functor $\varphi = \lim(\varphi_q)$, were Y in the category \mathcal{C} . This means that for any object X of \mathcal{C} , $\varphi(X) = \varinjlim \varphi_q(X) = (X, Y)$ where the latter group is Hom in the stable homotopy category of CW-complexes. Note also that these limits are actually stationary in the sense that $\varphi(X) = \varphi_N(X)$ for a large enough choice of N , depending of course on X . We obtain a morphism of functors, $\varphi \rightarrow \alpha$ which now has the property that $\varphi(S^i) \rightarrow \alpha(S^i)$ is an isomorphism for all i . It follows from (ii) and the fact that every X in \mathcal{C} may be built up from spheres by a finite number of cofibrations that $\varphi \rightarrow \alpha$ is an isomorphism of functors.

Our assumption (iii) then assures us that Y has finite cohomological dimension, and (i) says that the homotopy groups are finitely generated in each dimension. These two facts put together allow us to conclude that Y has finitely generated integral homology groups, which vanish beyond a certain dimension. It then follows that Y may be replaced by a finite CW-complex Y' . ([30], prop. (4.1).) It is here that we use that Y is simply connected. This finite CW-complex Y' being, of course, again simply connected, is an object of our category \mathcal{C} , and represents the functor α .

We shall achieve (Σ) inductively. Take $Y_0 = Y_1$ to be a point. Suppose Y_{q-1} given. For simplicity we let an object of \mathcal{C} stand for the contravariant functor which it represents. Let b_1, \dots, b_m generate the group $\alpha(S^q)$. Use the same letters to

denote the corresponding morphisms of functors, $b_j: S^q \rightarrow \alpha$. Let $Z = Y_{q-1} \vee (S^q)_1 \vee (S^q)_2 \vee \dots \vee (S^q)_m$. That is Z is the finite CW-complex obtained from Y_{q-1} by wedging m copies of S^q . Extend the functorial morphism $Y_{q-1} \rightarrow \alpha$ to a morphism $Z \rightarrow \alpha$ by setting it to be b_j on $(S^q)_j$ for $j = 1, \dots, m$. It then follows that $Z(S^q) \rightarrow \alpha(S^q)$ is surjective. Let c_1, \dots, c_k be elements in $Z(S^q)$ which generate the kernel of the above homomorphism. Take continuous maps $f_j: S^q \rightarrow Z$ which are representatives of c_j for $j = 1, \dots, k$ and define $Y_q = Z \cup_{f_j} (D^{q+1})_j$. Since α satisfies (ii), the morphism $Z \rightarrow \alpha$ extends to a morphism $Y_q \rightarrow \alpha$, and moreover $Y_q(S^q) \rightarrow \alpha(S^q)$ is an isomorphism. Since we have added to Y_{q-1} only wedges of q -spheres, and cells of dimension $q+1$ we have not disturbed the inductive hypothesis, giving us also: $Y_q(S^i) \rightarrow \alpha(S^i)$ is an isomorphism for $i < q$. q.e.d.

A consequence of the representability theorem is that φ^τ defined above is representable by a finite simply connected CW-complex, X^τ .

Proposition (7.13): (i) X^τ has the same stable prohomotopy type as X .

(ii) $c(X, X^\tau) = \tau$

(iii) If Y is a finite simply connected CW-complex having the same stable prohomotopy type as X , and such that $c(X, Y) = \tau$, then $Y \approx X^\tau$.

Proof: (i) follows from cor. (7.7) together with the fact that \hat{M} is free of rank one over $\hat{\Lambda}$. (ii) is clear:

$$\text{Hom}(X, X^\tau) = \varphi^\tau(X) = (X, X)_{\mathbb{E}(X)}^M \approx M .$$

(iii) comes from this fact: The natural homomorphism

$$(7.14) \quad \text{Hom}(A, X)_{\mathbb{E}(X)} \otimes \text{Hom}(X, Y) \xrightarrow{Y} \text{Hom}(A, Y)$$

coming from composition is an isomorphism for all A , provided X and Y are of the same stable prohomotopy type. To prove this we need only show that $\hat{\vee}$ is an isomorphism. But this is true using cor. (7.7).

Corollary (7.14): Let X be a finite simply connected CW-complex. The set of distinct isomorphism classes of finite CW-complexes of the same stable prohomotopy type as X may be put in a one:one correspondence with the set $P(X)$ by associating to every such Y the class $c(X, Y) \in P(X)$.

Of course, our methods made use of simple connectivity in only one place, namely to insure that X^τ be a finite CW-complex, and so we may state more generally:

Corollary (7.15): Let X be a finite CW-complex. The set of distinct stable homotopy types of CW-complexes Y of the same stable prohomotopy type as X may be put in a one:one correspondence with the set $P(X)$ by associating to every such Y the class $c(X, Y) \in P(X)$.

Since $P(X)$ is a finite set by Jordan-Zassenhaus, we obtain:

Corollary (7.16): Let X be a finite CW-complex. There are

only a finite number of stably inequivalent CW-complexes of the same stable prohomotopy type as X .

Examples: 1. Fix a continuous map $\gamma: S^m \rightarrow S^n$ representing a class $\gamma \in \pi_m(S^n)$, of order p , where p is a prime. For any integer i , consider the CW-complex,

$$X_i = S^n \cup_{(i\gamma)} D^{m+1} .$$

Since n must be greater than one, X_i is simply connected.

Lemma (7.17): If $i \not\equiv \pm j \pmod{p}$, then X_i and X_j are of distinct stable homotopy types.

Proof: The idea is to start with the finite complex X_i regarded as an object of \mathcal{C} and to try to recover, a priori, the classes $\{i\gamma\} \subset \pi_m(S^n)$ as an invariant of X_i . We do this by considering all cofibrations in the category \mathcal{C} of the form

$$(7.18) \quad \Delta: S^n \xrightarrow{f} X_i \xrightarrow{g} S^{m+1} .$$

Note that there is one by the construction of X_i . Now, if Δ is any cofibration of the above form, f must represent a generator of $\pi_n(X_i) = \mathbb{Z}$. This may be seen by applying the long exact stable homotopy sequence to Δ . Thus, up to homotopy, $f: S^n \rightarrow S^n \cup_{(i\gamma)} D^{m+1}$ is either the canonical inclusion or the canonical inclusion composed with the automorphism of S^n of degree -1 . From this we may conclude that up to isomorphism in \mathcal{C} there is a unique cofibration Δ of the form (7.18).

Continuing this cofibration one stage further, forming the cofibration induced by g , we obtain:

$$(7.19) \quad S^n \xrightarrow{f} X_i \xrightarrow{g} S^{m+1} \xrightarrow{h} S^{n+1}$$

where $h = \pm S(i\gamma)$, the sign depending on the orientations chosen for the S^n and S^{m+1} occurring. Desuspending h we obtain the classes $\pm i\gamma \in \pi_n(S^m)$, and this was obtained in an "a priori manner," concluding the proof of the lemma.

We state without giving the proof:

Lemma (7.20): If i, j are not congruent to zero mod p , then $\hat{X}_i \approx \hat{X}_j$. Note that to show the profinite completions of X_i and X_j are the same, it suffices to show that their q -completions are the same for all q . (This is true since X_i, X_j are finite and simply connected (cf. (12.13)).) But for all $q \neq p$, the q -completions of X_i split, in the sense that $(X_i)_q \approx (S^n)_q \vee (S^{m+1})_q$ for all i . Therefore one need only concentrate on the p -completion.

§8. Hypercoversings.

In this section we review, with some extensions, the theory of Verdier ([3] SGA 4, fascicule 1, exposé 5, appendice). It will be convenient to make the following preliminary definitions, generalizing Kan's notion of a free simplicial group: Let $S(n,m)$ denote the set of surjective monotone maps $s: \Delta^n \rightarrow \Delta^m$. Set

$$S(n) = \bigsqcup_{m=1}^n S(n,m) = \bigsqcup_{m=1}^{\infty} S(n,m) .$$

Definition (8.1): A splitting up to level n of a simplicial object K . with values in a category C consists of subobjects $N_k \subset K_k$ for every $k \leq n$ satisfying the following condition: For every $j = 0, \dots, n$, the canonical map

$$\bigsqcup_{s \in S(j)} N_s \rightarrow K_j$$

is an isomorphism, where for $s \in S(j,k)$, N_s denotes a copy of N_k , its mapping to K_j being induced by $s: K_k \rightarrow K_j$. The object N_k will be called the nondegenerate part of K_k and the above direct sum decomposition will be called a degeneracy decomposition of K . We shall refer to a simplicial object with values in C with a given splitting as a split simplicial object.

Remark: In the category of sets, any simplicial object has a unique splitting. More generally, using (§9) below, let C be a locally connected and distributive category. If a splitting exists, it is unique. If C is the category of etale, separated

schemes over a scheme X , then every simplicial object with values in C has a unique splitting. (We do not use this fact.)

Lemma (8.2): Let C be a category closed under finite co-products, and let K_\bullet be a simplicial object of C which admits a splitting up to level m . Then $Sk_m K_\bullet$ is representable.

Proof: Set

$$S(n)/m = \bigsqcup_{j=0}^m S(n, j) .$$

Then we write

$$(Sk_m K_\bullet)_n = \bigsqcup_{t \in S(n)/m} N_t .$$

Now for $r: \Delta^i \rightarrow \Delta^n$, we seek a definition of the operator r on $(Sk_m K_\bullet)_n$, and we do this inductively with respect to n and on each direct summand N_t separately.

(a) If r is surjective, take r to be the identity map $N_t \rightarrow N_{tr}$.

(b) If r is not surjective, then construct the following commutative square of monotone maps:

$$\begin{array}{ccc} \Delta^n & \xrightarrow{t} & \Delta^j \\ \uparrow r & & \uparrow v \\ \Delta^i & \xrightarrow{u} & \Delta^k \end{array}$$

There is a unique such square with u surjective and v injective. Since r is not surjective, we have $k < n$. Thus by inductive

hypothesis, the operation of v is defined, and by (a), that of u is. Set

$$r \mid N_t = u \mid N ,$$

and one easily sees that $Sk_m K.$ so defined is a simplicial object representing the functor after which it is named.

Lemma (8.3): Let C be a category closed under finite direct sums, and $K.$ a simplicial object with values in C , given a splitting up to level $n-1$. To extend this splitting to a splitting up to level n , it is necessary and sufficient to give a subobject $N_n \subset K_n$ such that the natural map below is an isomorphism:

$$(Sk_{n-1} K.)_n \cup N_n \rightarrow K_n .$$

Proof: Evident from the construction of $Sk_{n-1} K.$ above.

Now let C be a site ([3], I.2) (belonging to some universe U). We will assume for convenience that C is closed under finite products and fibred products, and under finite coproducts. Moreover, we always assume that the topology of C is "weaker than the canonical topology," i.e., that for $x \in C$, $\text{Hom}(\cdot, x)$ is a sheaf on C . A point of C is a morphism of sites ([3], III, 2) $(\text{Sets}) \rightarrow C$, i.e., a functor on the underlying categories

$$p: C \rightarrow (\text{Sets})$$

which carries coverings to surjective families, and which is exact

in the sense that it commutes with finite fibred products and arbitrary direct limits. A pointed site is one together with the choice of a point. A pointed simplicial object X . with values in a pointed site C is a simplicial object together with a choice of point in the set $p(X_0)$.

Definition (8.4): Let C be a (pointed) site. A hypercovering X . of C is a (pointed) simplicial object with values in C satisfying the following conditions for all n :

(surj₀) Let e be the final object of C . The map

$$X_0 \rightarrow e$$

is a covering.

(surj_n) The canonical morphism

$$X_{n+1} \rightarrow (\text{Cosk}_n X.)_{n+1}$$

is a covering.

A truncated hypercovering of level n is a truncated simplicial object satisfying (surj_q) for $q \leq n$.

If X is any object of C , we will refer to a hypercovering of the site C/X as simply a hypercovering of X .

Note that a morphism of sites carries hypercoverings to hypercoverings.

Enlightenment (8.5): The reader dismayed by the austerity of the above should work things out carefully in these particular

cases:

(a) Let C be the site (sets), where as always coverings are surjective families of maps. A hypercovering X is then a non-empty simplicial set, by (surj_0) . Taking into account the construction of coskeletons (§1), the condition (surj_n) is explained as follows: For every map of the simplicial set $\partial\Delta^n$, the boundary of the n -simplex, to X , there is at least one map of the n -simplex Δ^n to X inducing it. It follows easily that such a simplicial set satisfies the Kan condition, and is contractible. In fact hypercoverings are characterized by these two properties.

(b) Let C be any site, Y an object of C , and $U \rightarrow Y$ a covering morphism. Regarding U as a simplicial object truncated at level 0 with values in C/Y , we may form $\text{Cosk}_0 U$. This is just, in plainer language, the canonical simplicial object

$$\begin{array}{ccccccc}
 U & \begin{array}{c} \curvearrowright \\ \longleftarrow \\ \longleftarrow \end{array} & U \times_Y U & \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longleftarrow \end{array} & U \times_Y U \times_Y U & \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow \end{array} & \dots
 \end{array}$$

It is a hypercovering of Y . This gives the clue as to the novel feature, and the ultimate use, of hypercoverings. Namely, if we start out with an old-fashioned covering $U \rightarrow Y$ and proceed to the computation of, say, cech cohomology, then we must mechanically form the simplicial object above, whose n -simplexes are determined by the original covering. In the construction of hypercoverings, however, we are given much greater freedom. In the choice of n -simplices K_{n+1} , the mechanical procedure would be to take $K_{n+1} = (\text{Cosk}_n K.)_{n+1}$, when say $K./n$ is already determined. Here we are given the option of "refining" $(\text{Cosk}_n K.)_{n+1}$

further (see below), since we require only that condition $(\text{sur}j_n)$ holds. It is this option of refining further in each dimension that makes hypercoverings useful in cases where the category of mere coverings is too coarse.

We shall now prove some elementary lemmas about hypercoverings:

Lemma 8.6: Let $K.$ be a hypercovering of C , and $(L./n) \xrightarrow{(\varphi/n)} (K./n)$ a morphism of truncated (pointed) simplicial objects, such that $L./n$ is a hypercovering of C truncated at level n . Then there is a hypercovering $L.$ of C , and a morphism of simplicial objects, $L. \xrightarrow{\varphi} K.$ whose truncation at level n is the morphism φ/n .

In fact, we may take

$$L. = (\text{Cosk}_n L.) \times_{(\text{Cosk}_n K.)} K. .$$

The verification that this works is easy, and we omit it.

Lemma (8.7): Let C be a (pointed) site and $K.$ a hypercovering truncated at level n , split up to level $n-1$. There is a morphism of hypercoverings truncated at level n , $L. \rightarrow K.$, which induces an isomorphism on $(n-1)$ -skeleta, and such that $L.$ is split up to level n .

Proof: We take $L/n-1 = K./n-1$, and

$$L_n = (\text{Sk}_{n-1} K.)_n \amalg_{K_n} K_n .$$

The boundary operators $L_n \rightarrow L_{n-1}$ are the natural ones on each summand, and the degeneracy maps $L_{n-1} \rightarrow L_n$ are those coming from the simplicial object $Sk_{n-1}K.$:

$$L_{n-1} = K_{n-1} \xrightarrow{S} (Sk_{n-1}K.)_n \hookrightarrow L_n$$

(the inclusion being as first summand). This yields a simplicial object which maps to $K.$ by sending L_n to K_n via the natural map on each summand. So defined, $L.$ clearly satisfies the condition surj_n . q.e.d.

We therefore obtain:

Lemma 8.8: Let $K.$ be a hypercovering, which is split as a simplicial object up to level n . There is a morphism, $K! \rightarrow K.$ of simplicial objects such that

- (a) It induces an isomorphism on Sk_n .
- (b) $K!$ is a split simplicial object, which is a hypercovering for C . We shall call such a thing a split hypercovering.

The above lemma allows us to replace hypercoverings by split hypercoverings, if we wish.

Lemma 8.9: Let $K.$ be a hypercovering of C , split up to level n :

$$K_n = (Sk_{n-1}K.)_n \amalg N_n,$$

and let $M \rightarrow N_n$ be a covering morphism. There is a hypercovering $L.$ of C and a morphism $L. \rightarrow K.$ which induces an isomorphism

on $n-1$ skeleta, and such that $L_n = (\text{Sk}_{n-1} K.)_n \sqcup M$ and maps in the expected way to K_n .

Proof: The truncation of L at level n is taken as defined. Now apply (8.7).

The above lemma assures us that we may "refine the non-degenerate part" of a split hypercovering arbitrarily. Moreover, a refinement at level n entails no change on the $n-1$ skeleton. This is a quite convenient technical fact.

Lemma (8.10): Let $K., L.$ be hypercoverings, and let $\phi/n: K./n \rightarrow L./n$ be a map. There is a hypercovering $K!$ mapping to $K.$ with $K!/n = K./n$, and an extension of ϕ/n to a map $\phi': K! \rightarrow L.$. In particular, if $K., L.$ are hypercoverings with $K./n = L./n$, then there is a $K!$ dominating both, with $K!/n = K./n = L./n$.

Proof: We will construct $K!$ by induction on n . Thus it suffices to find $K!$ so that ϕ/n extends to dimension $n+1$. By lemma (8.6), it suffices to find a truncated hypercovering $K!/(n+1)$.

Let N' be the object making the following diagram cartesian

$$\begin{array}{ccc}
 N' & \xrightarrow{h} & L_{n+1} \\
 \downarrow f & & \downarrow g \\
 K_{n+1} & \longrightarrow & (\text{Cosk}_n L)_{n+1} \quad .
 \end{array}$$

The map f is a covering, since g is one. Since compositions

of coverings are coverings, the map $N' \rightarrow (\text{Cosk}_n K)_{n+1}$ obtained from f by composition is also a covering. Set $K'_q = K_q$ for $q \leq n$, and

$$K'_{n+1} = N' \amalg (\text{Sk}_n K)_{n+1} ,$$

and define the truncated object $K'_{/n+1}$ with the evident face and degeneracy maps. Now ϕ/n extends canonically to $\text{Sk}_n K$. Thus we can take for ϕ'_{n+1} this canonical map together with the map $h: N' \rightarrow L_{n+1}$ above.

For the moment, let C be any category with finite coproducts. If S is a set and $X \in C$, we denote the direct sum of X with itself indexed by S by

$$X \otimes S .$$

This is a covariant bifunctor in X and S .

The unit interval will mean the 1-simplex in the category of sets, denoted by

$$I. = \Delta^1 .$$

Given a simplicial object $K.$ with values in G , we may form the simplicial object

$$K. \otimes I. ,$$

whose n -simplex object is just

$$K_n \otimes I_n = \bigsqcup_{i \in I_n} K_n^i ,$$

where the superscript denotes the index, which is an increasing sequence of zeros and ones of length $n+1$. Thus faces and degeneracies operate on these indices, viz.

$$d_\nu: K_n^i \longrightarrow K_{n-1}^{d_\nu i}$$

$$s_\nu: K_n^i \longrightarrow K_{n+1}^{s_\nu i} .$$

Two maps

$$f^0, f^1: K. \rightarrow L.$$

of simplicial objects will be called strictly homotopic if there is a map

$$f: K. \otimes I. \rightarrow L.$$

such that

$$f \epsilon^0 = f^0 , \quad f \epsilon^1 = f^1 ,$$

where $\epsilon_\nu: K. \rightarrow K. \otimes I.$ ($\nu = 0,1$) are the evident "inclusions." Two maps are homotopic if they can be related by a chain of strict homotopies, in the usual sense. If f^0, f^1 are maps of pointed simplicial objects, then the strict homotopies will be assumed to preserve the base point.

This notion of homotopy is functorial in C .

Proposition (8.11): Let C be a (pointed) site, let $K., L.$ be hypercoverings with $K.$ split, and let $f^0, f^1: K. \rightarrow L.$ be maps between them. Suppose given a truncated strict homotopy between f^0, f^1 :

$$f/n: K.\otimes I./n \rightarrow L./n .$$

There is a refinement of $K.$, i.e., a map of hypercoverings $\phi: K! \rightarrow K.$, and an extension of $(\phi\otimes I.)f/n = f'/n$ to a strict homotopy

$$f': K!\otimes I. \rightarrow L. .$$

Moreover, $K!$ can be chosen so as to be equal to $K.$ in dimensions $\leq n-1$.

Proof: By induction, it suffices to extend the homotopy to dimension $n+1$, applying (8.9). Now using the splitting of $K.$, we have in the notation of (8.1)

$$(K.\otimes I.)_n = \bigsqcup_{\substack{i \in I_n \\ t \in S(n)}} N_t^i ,$$

from which one sees immediately that $K.\otimes I.$ acquires a splitting from that of $K.$. The degeneracies $(Sk_n(K.\otimes I.))_{n+1}$ identify with the sum of those N_t^i , $i \in I_{n+1}$, $t \in S(n+1)$ for which we can find a degeneracy operator $s_\nu: \Delta^{n+1} \rightarrow \Delta^n$ satisfying

$$i = s_\nu i' , \quad t = s_\nu t'$$

for suitable i', t' . One checks easily that the non-degenerate N_t^i are of the following two types:

(a) $t = \text{identity} \in S(n+1, n+1)$, i.e., $N_t^i = N_{n+1}$.

(b) $t = s_\nu: \Delta^{n+1} \rightarrow \Delta^n$ for some ν , and $i \neq s_\nu i'$ for any i' . Thus in this case $t \in S(n+1, n)$, and so $N_t^i = N_n$.

We have therefore

$$(K \cdot \otimes I.)_{n+1} = \text{Sk}_n(K \cdot \otimes I.)_{n+1} \cup M_{n+1}$$

where M_{n+1} is the sum of the N_t^i of types (a), (b). Now the truncated homotopy f/n extends uniquely to $\text{Sk}_n(K \cdot \otimes I.)_{n+1}$.

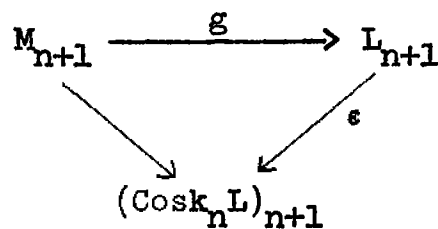
What we have to do is to refine K . so that there is a map

$g: M_{n+1} \rightarrow L_{n+1}$ commuting with the appropriate face operators.

On the other hand, we also have a unique extension of f to a

map $K \cdot \otimes I. \rightarrow \text{Cosk}_n L$. . Therefore there is a map $M_{n+1} \rightarrow$

$(\text{Cosk}_n L)_{n+1}$, and the requirement on g is just that the diagram



commute. Since ϵ is a covering by (surj_n) for L , there are coverings of the components N_n, N_{n+1} of M_{n+1} for which such a map exists. Thus two applications of lemma (8.9) complete the proof. We leave the considerations of base point to the reader.

Corollary (8.12): With the notation of (8.11), suppose that $K./n = L./n$, and that $f^0/n = f^1/n = \text{identity}$. Then a refinement

$K!$ of K . can be found with $K!/n = K./n$, such that the truncated identity homotopy at level n extends to a strict homotopy $f': K! \otimes I. \rightarrow L.$

Here a straight application of (8.11) would require a refinement differing from K . in dimension $\geq n$. This comes from the existence of non-degenerate N_t^i of type (b) in dimension $n+1$, as in the proof of (8.11). However, in our case the identity homotopy extends trivially to $Sk_n K. \otimes I.$, and the N_t^i of type (b) are summands of $(Sk_n K. \otimes I.)_{n+1}$. Thus we do not after all need to drop back a step in order to extend the homotopy of dimension $n+1$. Now an application of (8.11) yields the result.

Combining (8.11), (8.12) with lemma (8.10), we obtain

Corollary (8.13): (i) For a (pointed) site C , let $HR(C)$ denote the category whose objects are hypercoverings, and whose maps are homotopy classes of morphisms. Then the dual category $HR(C)^\circ$ is filtering. Let \bar{C} be a pointed site, and \bar{C} its underlying site. The stripping functor $HR(C)^\circ \rightarrow HR(\bar{C})^\circ$ is cofinal.

(ii) Let $K./n$ be a truncated split hypercovering, and denote by $HR(C, K./n)$ the category of hypercoverings with transition $K./n$, and with homotopy classes of morphisms. Then $HR(C, K./n)^\circ$ is filtering.

Verdier's main result ((8.16) below) is based on the following proposition:

Proposition (8.14): Let $K.$ be a hypercovering. For an object $U \in C$, we denote as in [3] by \mathbb{Z}_U the sheaf of abelian groups characterized by the property that for any abelian sheaf A we have

$$\text{Hom}(\mathbb{Z}_U, A) = A(U) .$$

Consider the simplicial sheaf $\mathbb{Z}_{K.}$ with its augmentation

$$\mathbb{Z}_{K.} \rightarrow \underline{\mathbb{Z}} = \mathbb{Z}_e$$

(e the final object of C). This augmentation is a resolution of \mathbb{Z}_e .

We will prove the proposition under the extra assumption that the site C has "sufficiently many points," so that a sequence $A \rightarrow A' \rightarrow A''$ of sheaves is exact when for every point $p: C \rightarrow$ (sets) the sequence $p^*A \rightarrow p^*A' \rightarrow p^*A''$ is exact. This will be enough for our applications, and we leave the general case to the reader.

We have $p^*(\mathbb{Z}_U) = \mathbb{Z}_{p(U)}$, where the abelian group on the right is just the direct sum of \mathbb{Z} with itself indexed by the set $p(U)$. For, if $A \in (\text{Ab})$, then

$$\text{Hom}(\mathbb{Z}_{p(U)}, A) = A^{p(U)} = p_*A(U) = \text{Hom}(\mathbb{Z}_U, p_*A) = \text{Hom}(p^*\mathbb{Z}_U, A) .$$

hence the functor p^* applied to the above augmentation yields

$$(*) \quad \mathbb{Z}_{p(X.)} \rightarrow \mathbb{Z} .$$

But $p(X.)$ is a hypercovering in (sets), whence is contractible (8.5). Therefore its homology is that of the point, which is the exactness of (*).

Corollary (8.15): Let $K.$ be a hypercovering, and F an abelian sheaf on C . There is a spectral sequence

$$E_1^{pq} = H^q(K_p, F) \implies H^{p+q}(C, F) \quad ,$$

where H denotes sheaf cohomology, and $H^*(C, F) = H^*(e, F)$.

This is a standard spectral sequence of Ext for a resolution, via the identification $H^p(*, F) = \text{Ext}^p(\mathbb{Z}_*, F)$.

Theorem (8.16): Let F be an abelian sheaf on the (pointed) site C . We have a canonical isomorphism

$$H^q(C, F) = \varinjlim_{K.} H^q(F(K.))$$

where the H on the right denotes cohomology of the cosimplicial group $F(K.)$, and where the limit is taken over the category $\text{HR}(C)^\circ$.

Proof: We pass to the limit over the terms E_2^{pq} of the spectral sequence (8.14):

$$E_2^{pq}(K.) = H^p(H^q(K., F)) \quad .$$

Let $K.$ be a split hypercovering, and α a class in $H^q(K_n, F)$.

Write $K_n = \coprod N_t$ as in (8.1). Then α is the product of classes $\alpha_t \in H^q(N_t, F)$. Since cohomology vanishes locally, there are coverings $N_t^!$ of N_t such that $\alpha_t \mapsto 0$ in $H^q(N_t^!, F)$. Applying (8.9) repeatedly, one finds that there is a refinement $K^!$ of K . such that $\alpha \mapsto 0$ in $H^q(K^!, F)$. This proves that $\varinjlim H^q(K., F) = 0$, hence that $\varinjlim E_2^{pq}(K.) = 0$, for all $q > 0$, which yields the theorem.

Definition (8.17): Let \mathcal{C} be a class of groups. An object of a site is said to have \mathcal{C} -dimension $\leq d$ if for every locally constant abelian sheaf F on C , locally isomorphic to a constant sheaf A for $A \in \mathcal{C}$, $H^q(X, F) = 0$ for all $q > d$. The site C is said to have local \mathcal{C} -dimension $\leq d$ if for every $X \in C$, there is a covering $X' \rightarrow X$ such that X' has \mathcal{C} -dimension $\leq d$.

Theorem (8.18): Let C have local \mathcal{C} -dimension $\leq d$, let $K./n$ be a truncated split hypercovering, and F an abelian locally constant \mathcal{C} -sheaf on C . Assume that the objects K_q are all of \mathcal{C} -dimension $\leq d$ for $q = 0, \dots, n$. Then the canonical map

$$\varinjlim_{K.} H^q(F(K.)) \rightarrow H^q(C, F)$$

is surjective if $q = n+d+1$, and bijective if $q > n+d+1$, where $K.$ runs through the category $HR(C, K./n)$ of (8.13).

Proof: By (8.9), we can refine the non-degenerate parts of a split hypercovering arbitrarily, in dimension $> n$, in the category $HR(C, K./n)$. Since cohomology vanishes locally, it is clear that as $K.$ runs over $HR(C, K./n)$, we have

$$\lim_{\vec{K}} H^q(K., F) = H^q(\text{Sk}_n K., F)$$

for all $q > 0$. By lemma (8.19) below, it follows that

$$\lim_{\vec{K}} H^p(H^q(K., F)) = 0$$

if $p > n$, and $q > 0$. Because the K_q ($q = 0, \dots, n$) have \mathcal{C} -dimension $\leq d$, we also have by (8.2)

$$H^q(\text{Sk}_n K., F) = 0$$

if $q > d$. Thus if we pass to the limit over the spectral sequences (8.15), we obtain

$$\lim_{\vec{K}} E_2^{pq} = \lim_{\vec{K}} H^p(H^q(K., F)) = 0$$

if $q > d$ or if $p > n$ and $q > 0$. This yields the theorem immediately.

Lemma (8.19): Let $K.$ be a split simplicial object in a category \mathcal{C} with finite coproducts, and let $T: \mathcal{C}^\circ \rightarrow (\text{Ab})$ be a functor sending coproducts to products. Then we have

$$H^p(T(\text{Sk}_n K.)) = 0 \quad \text{if } p > n.$$

Proof: Consider the inclusion $\text{Sk}_{n-1} K. \hookrightarrow \text{Sk}_n K.$. Since T carries sums to products and $K.$ is split, this inclusion yields an exact sequence of co-simplicial groups

$$(*) \quad 0 \rightarrow X. \rightarrow T(\text{Sk}_n K.) \rightarrow T(\text{Sk}_{n-1} K.) \rightarrow 0 \quad ,$$

where $X.$ can be described as follows: Let S_q^n denote the minimal simplicial pointed set representing the n -sphere, i.e., the simplicial set with two non-degenerate simplices: one 0-simplex and one n -simplex. Let $N \in C$ denote the non-degenerate part of K_n . Then X_q is the product of N with itself indexed by the pointed set S_q^n (where the base point factor is set equal to one). Thus $H^q(X.) = H^q(S_q^n, T(N))$ is the cohomology of the pointed n -sphere with values in $T(N)$, which is zero for $q \neq n$. The lemma now follows from the exact cohomology sequence of (*) and induction.

§9. The Verdier functor.

Let C be a category admitting finite fibred products. We say that C is distributive if it has an initial object \emptyset , and if the following condition holds: For every set of objects $Y_i \in C$, $i \in I$, such that the coproduct $\coprod_i Y_i$ is representable in C , for every set of maps $Y_i \rightarrow S$, and for every map $X \rightarrow S$ in C , the canonical morphism of functors

$$\coprod_i X \times_S Y_i \rightarrow X \times_S (\coprod_i Y_i)$$

is an isomorphism.

Every site C has an underlying distributive category ([3] II, 4.8b). Let C be a distributive category. An object $X \in C$ is called connected if X is not the initial object \emptyset , and if X has no non-trivial coproduct decomposition, so that $X = X_1 \sqcup X_2$ implies that $X_i = \emptyset$ for exactly one i . The category C is called locally connected if every object is a coproduct of connected objects. For brevity, we will say that C is connected if it is locally connected, and has a connected final object. Using the distributivity, it is immediately seen that the expression of an object X as a coproduct of connected objects is essentially unique. Moreover, the rule associating to an object its set of connected components is a functor. We denote this functor by

$$\pi: C \rightarrow (\text{Sets}) ,$$

and call it the connected component functor.

Now let C be a locally connected site. Then applying the functor π to the category $HR(C)$ of hypercoverings (§8), we obtain a pro-object

$$\coprod C = \{\pi(K.)\}_{K. \in HR(C)}$$

in the homotopy category \mathcal{K} of simplicial sets. The rule associating to a locally connected site C the pro-object $\coprod C$ is functorial with respect to morphisms of sites. We call it the Verdier functor.

Note that if C is a connected pointed site, then since the functor $p: C \rightarrow (\text{sets})$ is exact, it commutes with coproducts. Hence an element $x \in p(X)$ "lies in" one of the connected components of X . Thus a pointed hypercovering yields a pointed simplicial set via π_0 , and so the Verdier functor \coprod carries pointed sites to the homotopy category \mathcal{K}_0 of pointed simplicial sets. Via the geometric realization, we can identify $\coprod C$ with an object of \mathcal{K}_0 , and thus the theory developed in sections 2-6 is applicable. In particular, the homology pro-groups of a connected (pointed) site, and the homotopy pro-groups of a connected pointed site are defined, by the rule

$$(9.1) \quad \begin{aligned} H_q(C, A) &= H_q(\coprod C, A) \\ \pi_q(C) &= \pi_q(\coprod C) . \end{aligned}$$

If A is an abelian group, we may form the "constant" sheaf

$\underline{A} = A_e$ on C , where e is the final object, as in (8.14).

Lemma (9.2): Let C be a locally connected site, and let $X \in C$. Then there is a canonical isomorphism $A^\pi(X) \xrightarrow{\sim} \underline{A}(X)$.

We leave the proof as an exercise. It follows that if K_\bullet is a hypercovering, then $\underline{A}(K_\bullet) = A^\pi(K_\bullet)$, therefore that

$$H^q(\underline{A}(K_\bullet)) = H^q(\pi(K_\bullet), A)$$

where the group on the right is the cohomology of the simplicial set. Applying (8.16), we find that cohomology with constant coefficients factors through the Verdier functor:

Corollary (9.3): Let C be a locally connected site and A an abelian group. There is a canonical isomorphism

$$H^q(C, \underline{A}) \approx H^q(\coprod C, A) .$$

Example (9.4): Let X be a prescheme, and let C be the etale site over X . Recall that this site has as objects the schemes X' etale over X , and as coverings surjective families. It is clear that an object X' of C is connected if and only if it is a connected prescheme. We say that X is locally connected if the etale site is, so that every X' etale on X is a coproduct of connected schemes.

Proposition (9.5): A locally noetherian prescheme is locally connected.

cf. EGA I 6.1.9

For a locally connected prescheme X , we write

$$(9.6) \quad X_{\text{et}} \in \text{pro-}\mathcal{K}$$

to denote the pro-object $\prod C$ for the etale site C over X . We refer to X_{et} as the etale homotopy type of X . By pointed prescheme we mean a prescheme together with a geometric point. This point then gives a point for the etale site C , in the obvious way. Thus if X is a pointed prescheme, then C is a pointed site, and so X_{et} is in $\text{pro-}\mathcal{K}_0$ in this case. Notations of the type

$$(9.7) \quad \begin{aligned} & H_q(X_{\text{et}}, A) \\ & \pi_q(X_{\text{et}}) \\ & H^q(X_{\text{et}}, A) \quad A \in (\text{Ab}) \end{aligned}$$

are defined as in (2.1), (2.2).

Example (9.8): Let X be a pointed, connected, and locally connected topological space, and let C be the ordinary site whose objects are coproducts (disjoint unions) of open subsets of X . Then we obtain a definition of homotopy pro-groups for X , which we will denote by

$$\bigvee \pi_q(X) = \pi_q(C) \quad .$$

We will discuss the relation of these pro-groups with the usual homotopy groups in section 12.

Example (9.9): Let G be a group, and let C be the site

of left G -sets, where as always a covering is a surjective map. This site is canonically pointed by the stripping functor $(G\text{-sets}) \rightarrow (\text{Sets})$. Among the hypercoverings we have the one

$$(9.10) \quad G \times G \times G \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} G \times G \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} G = K. \quad .$$

We leave it as an exercise to show that since the G -set G is a projective object, this hypercovering dominates every other. Thus in the category $\text{pro-}\mathcal{K}_0$ we have

$$\pi(K.) \approx \prod C$$

whence $\prod C$ is actually in \mathcal{K}_0 . Moreover, $\pi(K.)$ is just the orbit space of $K.$ under the action of G , and this orbit space is the standard construction of the Eilenberg-MacLane space $K(G,1)$. Hence

$$\prod C = K(G,1) \quad ,$$

and

$$\begin{aligned} \pi_1(C) &= G \\ \pi_q(C) &= 0 \quad \text{if } q > 0 . \end{aligned}$$

Example (9.11): Let G be a pro-finite group, and let C be the site of finite continuous G -sets. Then via example (9.10), one sees that every truncated hypercovering $L./n$ is dominated by one of the form

$$\bar{G} \times \bar{G} \times \bar{G} \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \bar{G} \times \bar{G} \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} \bar{G} \quad ,$$

where \bar{G} is a suitable finite quotient of G . Thus we obtain

$$\prod_{\neq} C \approx K(G,1) \quad ,$$

and

$$\pi_1(C) = G$$

$$\pi_q(C) = 0 \quad \text{if } q > 0 \quad .$$

§10. The fundamental group.

Let C be a site, and \mathcal{J} a fibred category ([15], exp. 190, [13]) over C which is a category of descent (loc.cit.) for the site C , i.e., so that for $X \in C$ and $a, b \in \mathcal{J}(X)$, the functor

$$f \mapsto \text{Hom}(f^*a, f^*b)$$

for maps $f: Y \rightarrow X$ is a sheaf on the site C/X , or, so that morphisms between objects of \mathcal{J} "descend" with respect to coverings.

Let $K.$ be a hypercovering of C , and

$$x \in \mathcal{J}(K_0) .$$

By descent data for x relative to $K.$, we mean an isomorphism

$$(10.1) \quad \phi: d_0^*x \xrightarrow{\sim} d_1^*x$$

between the two pull-backs of x to $\mathcal{J}(K_1)$, satisfying the compatibility condition that in $\mathcal{J}(K_2)$ we have

$$(10.2) \quad d_1^*\phi = d_2^*\phi d_0^*\phi ,$$

with the conventions as in (loc.cit.).

Proposition (10.3): The map $K. \rightarrow \text{Cosk}_0 K.$ induces a one-one correspondence between descent data for x relative to $K.$

and descent data for x relative to $\text{Cosk}_0 K$.

Proof: First of all, since $K_2 \rightarrow (\text{Cosk}_1 K)_2$ is a covering (8.4), the compatibility condition (10.2) holds for ϕ on K iff it does on $\text{Cosk}_1 K$. Thus descent data depend only on $\text{Cosk}_1 K$, and so we may replace K by $\text{Cosk}_1 K$. Taking into account the description of Cosk_1 (§1), we may then write K_2 as the subobject of $K_1 \times K_1 \times K_1$ of triples (b_0, b_1, b_2) satisfying the identities

$$(10.4) \quad \begin{aligned} d_0 b_0 &= d_0 b_1 \\ d_1 b_0 &= d_0 b_2 \\ d_1 b_1 &= d_1 b_2 \end{aligned} .$$

Consider the following diagram:

$$(10.5) \quad \begin{array}{ccccc} & & K_1 \times (K_0 \times K_0)^{K_1} & & \\ & & \delta_0 \downarrow \quad \delta_1 \downarrow & \searrow u & \\ K_0 & \xleftarrow{d_0} & K_1 & \xleftarrow{d_0} & K_2 & = & K. \\ & \parallel & & & \downarrow \epsilon & & \downarrow \\ & & & & K_0 \times K_0 \times K_0 & = & \text{Cosk}_K K. \\ & & & & \downarrow (d_0, d_1) & & \\ K_0 & \xleftarrow{d_0} & K_0 \times K_0 & \xleftarrow{d_0} & K_0 \times K_0 \times K_0 & & \end{array}$$

The map u is defined as follows: If $(a, b) \in K_1 \times (K_0 \times K_0)^{K_1}$

(here we are using a shorthand notation in which "elements" are supposed to be replaced by maps from a "test object" into $K_1 \times (K_0 \times K_0) K_1$), so that $d_i a = d_i b$ for $i = 1, 2$, then

$$u(a, b) = (b, a, s_0 d_1 a) \in K_2 \subset K_1 \times K_1 \times K_1 .$$

The conditions (10.4) are immediately checked, so that $u(a, b)$ is indeed in K_2 .

Now given descent data for x relative to $\text{Cosk}_0 K$, its pull-back to K determines it uniquely, since (d_0, d_1) of (10.5) is a covering (8.4). Thus it remains to show that any descent data ϕ relative to K comes from one relative to $\text{Cosk}_0 K$. For this, we need only show that the two pull-backs $\delta_0^* \phi$, $\delta_1^* \phi$ are equal. For, since (d_0, d_1) is a covering, this implies that the isomorphism ϕ descends to $K_0 \times K_0$. The fact that it is descent data follows from the surjectivity of the map ϵ of (10.5).

We have

$$\begin{aligned} \delta_0 &= d_0 u \\ \delta_1 &= d_1 u \\ s_0 d_1 \delta_1 &= d_2 u . \end{aligned}$$

Therefore (10.2) implies

$$\delta_0^* \phi = (\delta_1^* d_1^* s_0^* \phi)(d_1^* \phi) .$$

Hence it suffices to show $d_1^* s_0^* \phi = \text{identity}$. But, calculating with simplicial identities, and (10.2),

$$\begin{aligned} (s_0^* d_1^* \phi) &= (s_0^* d_2^* \phi)(s_0^* d_0^* \phi) \\ \phi &= (s_0^* d_2^* \phi) \phi . \end{aligned}$$

Since ϕ is an isomorphism, $s_0^* d_2^* \phi = d_1^* s_0^* \phi = \text{id}$, as required.

Thus descent data depend only on the covering $K_0 \xrightarrow{e} e$ of the final object e of C , and on $x \in \mathcal{J}(K_0)$. They form a category in a well known way ([15], [13]), which we will denote here by $\overline{\mathcal{J}(e)}$. We think of it as a completion of $\mathcal{J}(e)$.

Let $\bar{x} \in \mathcal{J}(e)$, and put $x = e^* \bar{x}$. Then $d_0^* x$ and $d_1^* x$ are canonically isomorphic to $(ed_0)^* \bar{x} = (ed_1)^* \bar{x}$, which yields canonical descent data for $e^* \bar{x} = x$. If a descent data (x, ϕ) is isomorphic to such a canonical descent data, it is said to be effective descent data. In case every descent data is effective, the categories $\mathcal{J}(e)$ and $\overline{\mathcal{J}(e)}$ are equivalent. Otherwise, $\mathcal{J}(e)$ is a full subcategory of $\overline{\mathcal{J}(e)}$ (cf. [13] for a detailed account).

As an example, suppose C closed under arbitrary coproducts, and consider the fibred category of "trivial coverings," so that $\mathcal{J}(X)$ is the category of objects

$$\{X \otimes S \mid S \in (\text{Sets})\} ,$$

where maps are ones compatible with the projection $X \otimes S \rightarrow X$. The category $\overline{\mathcal{J}(e)}$ is then to be thought of as the category of locally trivial coverings of e . It is known that if C is the etale site of a locally noetherian scheme Y , then every object of $\overline{\mathcal{J}(Y)}$ is represented by an element of C , i.e., that descent data for a covering $X \otimes S$, $X \rightarrow Y$ an etale surjective map, is

effective in C . This is proved in ([9], X, 6).

Now suppose in addition that C is locally connected. Then it is immediately seen that the group of automorphisms of $X \otimes S$ over X is just the permutation group $P(S)$ of S if X is connected, and is $P(S)^{\pi(X)}$ in general. Thus if $K.$ is a hypercovering, then descent data for the trivial covering $K_0 \otimes S$ is given by an automorphism of $K_1 \otimes S = d_0^*(K_0 \otimes S) = d_1^*(K_0 \otimes S)$ satisfying the condition (10.2), which in turn is given by an element of $P(S)$ for each connected component of K_1 . Obviously this is nothing but a 1-cocycle of the simplicial set $\pi(K.)$ with values in the group $P(S)$. Two such data yield isomorphic elements of $\overline{\mathcal{F}(e)}$ iff the cocycles are cohomologous.

We recall that covering spaces of the geometric realization of a simplicial set $L.$ are in 1-1 correspondence up to canonical isomorphism with simplicial covering spaces, which are maps of simplicial sets $L! \rightarrow L.$ satisfying the following axiom: For any simplex $x \in L_q$, denote by L'_x the set of simplexes mapping to x . Then the face operators d_ν carry L'_x isomorphically to $L'_{d_\nu x}$. This fact is well known and elementary. Moreover, a moment's reflection shows that isomorphism classes of simplicial covering spaces $L! \rightarrow L.$ for which $\text{card}(L'_x) = \text{card}(S)$ for all x are classified by elements of $H^1(L., P(S))$. Thus we obtain the following

Corollary (10.6): Let C be a locally connected site closed under arbitrary coproducts, and let $K.$ be a hypercovering of C . The category of locally trivial coverings of the final object e of C which become trivial on K_0 is equivalent with the

category of simplicial covering spaces of $\pi(K.)$.

Now suppose that C is pointed and connected. Let G be a group, and denote by $\pi^1(C,G)$ the set of isomorphism classes of locally trivial coverings of e given with an operation of G making them into a principal fibre bundle. By the above corollary, we have

Corollary (10.7): With the above notation,

$$\pi^1(C,G) = \text{Hom}_{(\text{grps})}(\pi_1(C),G) .$$

Thus the pro-group $\pi_1(C)$ has the usual property. In case C is the etale site on a noetherian scheme, this is the "pro-groupe fondamentale enlargi" of ([9]X,6, [21]).

As another example, we can let \mathcal{F} be the fibred category of locally constant abelian sheaves, i.e., sheaves F for which there is an abelian group A and a hypercovering $K.$ such that the restriction of F to C/K_0 is the constant sheaf \underline{A} . If again C is locally connected, then a reasoning analogous to the above shows that to give such a sheaf is equivalent with giving a local coefficient group on $\pi(K.)$ (in particular, is given by a homomorphism $\pi_1(\pi(K.)) \rightarrow \text{Aut}(A)$ if C is connected and pointed). Consequently we have by (9.3)

Corollary (10.8): Let C be a locally connected site. The category of locally constant abelian sheaves on C is equivalent with the category of locally coefficient groups on $\prod C$, and

there is a canonical isomorphism

$$H^q(C, F) \approx H^q(\coprod C, F)$$

where F denotes both the corresponding sheaf and local coefficient group.

§11. A profiniteness theorem.

Let X be a locally noetherian prescheme. X is said to be geometrically unibranch if for every $x \in X$ the integral closure of the local ring $\mathcal{O}_{X,x}$ is again local (EGA IV, 6.15.1). An equivalent condition is that every X' etale over X which is connected, is in fact irreducible. Thus the notion is local for the etale topology. A normal prescheme is geometrically unibranch.

Theorem (11.1): Let X be a pointed, connected, geometrically unibranch, noetherian prescheme. Then the etale homotopy type X_{et} is pro-finite, i.e., is in $\text{pro-}\mathcal{C}\mathcal{X}_0$, where \mathcal{C} denotes the class of finite groups, and $\mathcal{C}\mathcal{X}_0$ denotes the category of $Z \in \mathcal{X}_0$ such that $\pi_q(Z) \in \mathcal{C}$ for all q .

To prove the theorem, we remark to begin with that every hypercovering $K.$ of X for the etale topology is dominated by one in which each K_q is a noetherian prescheme--we will call such a hypercovering noetherian. This is trivial. Thus the theorem will result from the following more precise assertion:

Theorem (11.2): With the notation of (11.1), let $K.$ be a noetherian etale hypercovering of X . Then $\pi(K.) \in \mathcal{C}\mathcal{X}_0$.

Proof: Since the etale sites of X and X_{red} are equivalent, ([13]VIII,1.1), we may suppose X reduced. Since X is irreducible, it has a generic point P , and every non-empty etale Y over X will cover P . If Y is connected, it is irreducible, and so Y_{X_P} will be a one point scheme. Thus the connected

components of Y and of $Y \times_X P$ are in 1-1 correspondence, whence we have

$$\pi(K.) = \pi(K. \times_X P) \quad ,$$

where $K. \times_X P$ is the induced étale hypercovering of $P = \text{Spec } k(P)$. Now the étale site over $\text{Spec } K$, K a field, is equivalent with the site of continuous G -sets, $G = \text{Gal}(\bar{K}/K)$ ([13], VIII, 2). Hence we are reduced to proving the following: Let G be a profinite group, and $K.$ a hypercovering of the site of continuous G -sets. Assume K_q a finite set for all q . Then $\pi_q(\pi(K.))$ are finite groups for all q .

Truncating $K.$, and replacing G by a suitable finite quotient \bar{G} , we are reduced to the following:

Proposition (11.3): Let G be a finite group, let C be the pointed site (§8) of finite G -sets, and let $K.$ be a hypercovering of C . Then $\pi_q(\pi(K.))$ are finite groups for all q .

Proof: The functor $\pi: C \rightarrow (\text{sets})$ is just the orbit space functor. We can also consider the point, i.e., the stripping functor $p: C \rightarrow (\text{sets})$ sending a G -set to its underlying set. This functor is a morphism of sites $(\text{sets}) \rightarrow C$. If we identify the category of sheaves on C with the category of G -modules, ([3], VIII, 2), then the inverse and direct image functors p^*, p_* correspond to the restriction of a G -module to the trivial subgroup, and the induction of a module from the trivial subgroup to G . We have for any sheaf F on C maps as with G -modules

$$(11.4) \quad F \xrightarrow{\text{res}} p_* p^* F \xrightarrow{\text{trace}} F$$

whose composition is multiplication by the order n of G .

Lemma (11.5): Let $L.$ be a simplicial object in C such that the underlying simplicial set $p(L.)$ is acyclic in dimension > 0 , i.e., that $H^q(p(K.), \mathbb{Z}) = 0$, $q > 0$. Then $H^q(\pi(L.), \mathbb{Z})$ is a finite group for every $q > 0$.

Proof: We have $H^q(\pi(L.), \mathbb{Z}) = H^q(\underline{\mathbb{Z}}(L.))$, where $\underline{\mathbb{Z}}$ is the constant sheaf on C . Denoting by \mathbb{Z} also the constant sheaf on (sets), we have

$$H^q(p_* \mathbb{Z}(L.)) = H^q(p(L.), \mathbb{Z}) = 0, \quad q > 0.$$

Since $p_* \underline{\mathbb{Z}} = \mathbb{Z}$, it follows from (11.4) that $H^q(\mathbb{Z}(K.))$ is annihilated by n . Since these groups are obviously finitely generated, the lemma follows.

Now we claim that to prove (11.2) it suffices to show that if $K.$ is a hypercovering, then $\pi_1(\pi(K.))$ is a finite group. For, suppose this shown. Let $S. \rightarrow \pi(K.)$ be the universal covering as simplicial set. We can construct a simplicial object $L.$ in C , and a map $L. \rightarrow K.$ such that $\pi(L.) = S.$, as follows: We put symbolically

$$L. = K. \otimes_{\pi(K.)} S.,$$

i.e., for every connected component x of K_q , let S_x denote the elements of S_q mapping to x . Then

$$L_q = \coprod_{x \in \pi(K_q)} x \otimes_S x .$$

Or, viewing K_q as a set with G -operation, the underlying set of L_q is

$$p(L_q) = p(K_q) \times_{\pi(K_q)} S_q ,$$

and G operates on the left factor. Via this description, it is clear that $p(L.)$ is a simplicial covering of the contractible simplicial set $p(K.)$. Therefore $p(L.)$ is acyclic in positive dimensions, and by lemma (11.5), we find that $H^q(\pi(L.), \mathbb{Z}) = H^q(S., \mathbb{Z})$ are finite groups for all $q > 0$. Since $S.$ is simply connected, it follows from the mod- \mathcal{C} Hurewicz theorem that $\pi_q(S.) = \pi_q(\pi(K.))$ are finite groups for all $q > 1$, which completes the proof of the theorem.

Thus it remains to show that $\pi_1(\pi(K.))$ is a finite group. Since this is a finite simplicial set, it will suffice to prove

Lemma (11.6): Let G be a group and let $K.$ be a hyper-covering of the site of G -sets. Then $\pi(K.)$ satisfies the Kan condition in dimension 1.

For, then Kan's construction of π_1 gives it as a quotient of the finite set K_1 .

Proof: Write $\bar{K}. = \pi(K.)$. Let $\bar{x}, \bar{y} \in \bar{K}_1$ satisfy $d_0 \bar{x} = d_0 \bar{y}$. We need to find $\bar{b} \in \bar{K}_2$ with $d_0 \bar{b} = \bar{x}$ and $d_1 \bar{b} = \bar{y}$. It is clear that we may choose representatives x, y in K_1 (strictly speaking, in $p(K_1)$) with $d_0 x = d_0 y$. By (8.4), there is a $z \in K_1$ with

$$d_0 z = d_1 x$$

$$d_1 z = d_1 y \quad .$$

Then the triple (x, y, z) identifies with an element of $(\text{Cosk}_1 K.)_2$
 $\subset K_1 \times K_1 \times K_1$ (cf. (10.4)). By (8.4), there is a $b \in K_2$ mapping
to (x, y, z) , and \bar{b} has the required property.

§12. Comparison theorems.

We begin with a topological result: Let X be a connected, pointed topological space. Assume that every open subset of X is paracompact, and that X is locally contractible, i.e., that every point $x \in X$ contains arbitrarily small contractible neighborhoods. Such a space is locally arc-wise connected. Let C be the ordinary site on X of coproducts of open subsets. We denote by $S.X$ the singular complex, i.e., the simplicial set of singular simplexes $S_q X = \{\text{maps } \Delta^q \rightarrow X\}$.

Theorem (12.1): With the above notation, let $U.$ be a hypercovering of C such that every connected component of every U_q is contractible. Then the simplicial set $\pi(U.)$ is canonically homotopic to the singular complex $S.X$ of X . Hence the pro-object $\prod C$ is canonically isomorphic to the element $S.X$ of \mathcal{K}_0 , and in particular, in the notation of (9.8), $\pi_q(X) \approx \bigvee \pi_q(X)$.

Corollary (12.2): Let X be a connected pointed algebraic scheme over \mathbb{C} . With the notation of (12.1), we define

$$X_{c1} = \prod C .$$

Then

$$X_{c1} \approx S.X \quad \text{in pro-}\mathcal{K}_0 .$$

For, the hypotheses of (12.1) on X are satisfied for an algebraic

scheme, since it is triangulable (cf. [20]).

Proof of (12.1): Let $S.U_q$ denote the singular complex of U_q . Then $S.U.$ is a bisimplicial set. We denote by $\Delta.$ its diagonal simplicial set

$$\Delta_q = S_q U_q .$$

Then we have obvious maps of simplicial sets

(12.3)

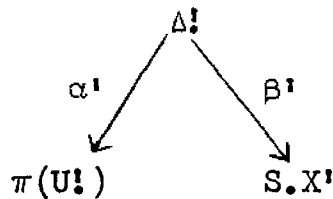
$$\begin{array}{ccc}
 & \Delta. & \\
 \alpha \swarrow & & \searrow \beta \\
 \pi(U.) & & S.X
 \end{array}
 ,$$

and we claim that these two maps are homotopy equivalences, which will prove the theorem.

First of all, to prove that the maps induce an isomorphism on π_1 , it is convenient to do this by showing that the category of simplicial covering spaces of the three objects are equivalent via α, β . Since X is locally arcwise connected, we can identify simplicial covering spaces of $S.X$ with locally trivial covering spaces of X . Now by the discussion of section 10, simplicial covering spaces of $\pi(U.)$ are given by gluing data for a covering space of X relative to the open covering U_0 . Thus it is trivial that coverings of $\pi(U.)$ and of $S.X$ correspond. It remains to show for instance that every descent data for a simplicial covering of $\Delta.$ is obtained from descent data on $\pi(U.)$ in a unique way. This is a simple exercise which we leave to the

reader.

Now let $X' \rightarrow X$ be a covering space, say connected. Then $U! = X' \times_X U$ is obviously a hypercovering of X' satisfying the assumptions of (12.1), and if we form the simplicial covering space $\pi(U!) \times_{\pi(U)} \Delta. = \Delta!$, it is in fact the diagonal simplicial set $\Delta!_q = S_q U!_q$ of $S.U!$. Moreover the covering space of $S.X$ corresponding to X' is just $S.X'$. Therefore we obtain a diagram



covering (12.3). To show that α, β are homotopy equivalences, it suffices to show that for every such X' , the maps α', β' induce isomorphisms on cohomology with arbitrary coefficients A . Thus we may as well drop the $'$.

By the Eilenberg-Zilber theorem, the two spectral sequences

$$\begin{aligned}
 {}^h E_2^{pq} &= H^p_v(H^q_h(A^{S.U.})) && (h = \text{horizontal}, \\
 {}^v E_2^{pq} &= H^p_h(H^q_v(A^{S.U.})) && v = \text{vertical})
 \end{aligned}$$

have abutments the graded groups of certain filtrations of $H^*(\Delta., A)$. Since U_q is component-wise contractible, we have $H^q(U_p, A) = {}^h E_1^{pq} = 0$ if $q > 0$. Hence the spectral sequence yields

$${}^h E_2^{p0} = H^p(H^0(U_p, A)) = H^p(\pi(U.), A) \approx H^p(\Delta., A) .$$

This shows that α is a homotopy equivalence.

To show that β induces an isomorphism on cohomology, we use the sheaf of singular cochains $s^q(\cdot, A)$ (cf. [6], p. 18). The cosimplicial sheaf $s^*(A)$ is a flabby resolution of the constant sheaf \underline{A} ([16], p. 119), and it is used to give the usual identification of sheaf cohomology with singular cohomology (loc.cit.).

Consider the surjective map ([16], p. 19) of bicomplexes

$$A \begin{matrix} S & U \\ P & U \\ q & \end{matrix} \xrightarrow{\phi} s^p(U_q, A) \longrightarrow 0 .$$

A consideration of the first spectral sequences $'E_2^{pq}$ above shows that ϕ induces an isomorphism on the cohomology of the associated total complexes. Consider the second spectral sequences: We have $H^q(s^p(U, A)) \approx H^q(X, s^p(A))$. This follows from the spectral sequence (8.15) since $s^p(A)$ is flabby, and moreover $H^q(X, s^p(A)) = 0$ if $q > 0$. Thus the map ϕ on $"E_2^{pq}$ yields

$$(12.4) \quad "E_2^{p0} = H_h^p(H_v^0(A^{S \cdot U})) \longrightarrow H^p(X, A) ,$$

and the groups $H^p(X, A)$ are the abutments of the two spectral sequences. Now we are interested in the map

$$A^{S \cdot X} \longrightarrow A^{S \cdot U} .$$

given by the projection $S \cdot U_q \longrightarrow S \cdot X$. This map induces a map

$$H^p(X, A) \longrightarrow {}''E_2^{p0} .$$

Clearly the composition of this map with (12.4) is the identity, which shows that β induces an isomorphism on cohomology, and completes the proof.

Now let f be a morphism of sites $C \rightarrow D$. Then the pull-back f^* carries hypercoverings to hypercoverings. Thus if C and D are pointed and connected, and if f is a pointed morphism then we obtain a map

$$\prod f: \prod D \longrightarrow \prod C \quad \text{in } \text{pro-}\mathcal{X}_0 .$$

We call an f-map ϕ between hypercoverings $L.$ of D and $K.$ of C a map

$$\phi: L. \rightarrow f^*K. .$$

Such a map induces a map

$$\pi(f, \phi): \pi(L.) \rightarrow \pi(K.) ,$$

and it is these maps which induce the map of pro-objects $\prod f$.

Theorem (12.5): Let \mathcal{C} be a complete class of groups, and let \wedge denote \mathcal{C} -completion. With the above notation, suppose that C, D are of local \mathcal{C} -dimension $\leq d$ for some d , and that $\widehat{\prod f}$ is a \wedge -isomorphism. Then $\prod f$ is an isomorphism.

Proof: Our problem is the following: Let $W \in \mathcal{C}\mathcal{X}_0$ be a

"test object," and suppose given a map $b: \prod D \rightarrow W$. We have to show that it factors uniquely through $\prod f$. Thus we need to show:

(i) There is an f -map $\phi: L. \rightarrow f^*K.$ of hypercoverings, and a homotopy commutative diagram

(12.6)

$$\begin{array}{ccc}
 \pi(L.) & \xrightarrow{\pi(f, \phi)} & \pi(K.) \\
 \searrow \beta & & \swarrow \alpha \\
 & W &
 \end{array}$$

in \mathcal{X}_0 , such that β represents b .

(ii) Given $\phi: L. \rightarrow f^*K.$ and two maps α_i as above, there is a "refinement" of ϕ , i.e., a homotopy commutative diagram

$$\begin{array}{ccc}
 L! & \xrightarrow{\phi'} & f^*K! \\
 \downarrow & & \downarrow f^*e \\
 L. & \xrightarrow{\phi} & f^*K.
 \end{array}$$

such that the pull-back of α_i to $\pi(K.)$ are equal in \mathcal{X}_0 .

These two problems can be treated analogously. We give a detailed treatment of (i) in steps (a)-(c) below, leaving (ii)

to the reader.

We shall make repeated use of the "elementary construction" of (4.9), which replaces homotopy-commutative diagrams by actually commutative ones, after a choice of homotopy-commutativity.

Our obstruction theory argument below is, in fact, quite similar to the argument of (4.9), except that we must now be a bit more delicate in our method of killing obstructions, so as to be able to pass to a limit. In fact we only see how to do this by an inductive process in the skeleta of $\pi(L.)$, rather than, as in (4.9), by induction on the coskeleta of W .

In the special context of schemes, our technique (and especially the somewhat convoluted manner of expressing (12.7)) might be rendered superfluous by a rigidification of the category of hypercoverings.

(a) Killing Relative Cohomology Classes by Refinement.

We adopt the following conventions: B will refer to the CW-complex which is the realization of the simplicial set $\pi(L.)$; A will refer to the CW-complex mapping cylinder of the geometric realization of $\pi(\varphi)$. Thus we have the natural inclusion $\varphi: B \rightarrow A$, allowing us to identify B with a subcomplex of A . If $L! \xrightarrow{\phi'} f*K!$ is another morphism of hypercoverings, we refer to the CW-pair associated to it by the symbols: $B' \xrightarrow{\varphi'} A'$.

Lemma (12.7): Under the hypotheses of (12.5), suppose

$x \in H^{q+1}(A, B; G)$ where G is a \mathcal{C} -twisted system of abelian \mathcal{C} -groups. There is a refinement of ϕ :

$$(\xi) \quad \begin{array}{ccc} L'' & \xrightarrow{\phi''} & f^*K'' \\ \downarrow & & \downarrow \\ L & \xrightarrow{\phi} & f^*K. \end{array}$$

which is a homotopy-commutative diagram of split hypercoverings such that:

- (1) $L''/\delta = L/\delta$; $K''/\delta = K/\delta$; $\phi''/\delta = \phi/\delta$ where $\delta = q-d-1$.
- (2) L'' , K'' have all objects of \mathcal{C} -dimension $\leq d$.
- (3) There is a homotopy-commutativity rel δ , F , of the diagram,

$$\begin{array}{ccc} B'' & \xrightarrow{\varphi''} & A'' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\varphi} & A \end{array}$$

such that the induced map,

$$H^{q+1}(A, B; G) \rightarrow H^{q+1}(A'', B''; G)$$

sends x to zero.

Proof: Consider the diagram:

$$\begin{array}{ccccccc}
 H^q(A) & \longrightarrow & H^q(B) & \longrightarrow & H^{q+1}(B,A) & \longrightarrow & H^{q+1}(A) & \longrightarrow & H^{q+1}(B) \\
 || & & || & & & & || & & || \\
 H^q(\pi K_.) & \longrightarrow & H^q(\pi L_.) & & & & H^{q+1}(\pi K_.) & \longrightarrow & H^{q+1}(\pi L_.) \\
 \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 H^q(\coprod C, G) & \xrightarrow{\approx} & H^q(\coprod D, G) & & & & H^{q+1}(\coprod C, G) & \xrightarrow{\approx} & H^{q+1}(\coprod D, G)
 \end{array}$$

The image of x in $H^{q+1}(\coprod C, G)$ is zero, hence there is a refinement $K! \rightarrow K$. such that the image in $H^{q+1}(\pi K!, G)$ is zero, and by (8.18) we may take $K!$ equal to K . in dimensions $\leq a = q-d$. Then we have a map $L./a \rightarrow f^*K!/a = f^*K./a$ and by (8.10), there is an $L! \rightarrow L$. with $L!/a = L./a$ and an extension of this map to $L! \rightarrow f^*K!$.

We obtain a diagram

$$\begin{array}{ccc}
 B' & \xrightarrow{\varphi'} & A' \\
 \downarrow & & \downarrow \\
 B & \xrightarrow{\varphi} & A
 \end{array} \quad (\xi')$$

which is commutative in dimensions $\leq a$. By (8.11) we can change $L!$ in dimensions $> a-1$ to make the truncated commutative diagram extend to a homotopy-commutative square. Let F' be a homotopy-commutativity $\text{rel}(a-1)$ for (ξ') . Then F' induces a map

$H^{q+1}(A, B; G) \rightarrow H^{q+1}(A', B'; G)$ such that the image of x is the image of a class $y \in H^q(B', G)$. We now make a second refinement,

$$(\xi'') \quad \begin{array}{ccc} L'' & \xrightarrow{\phi''} & f^*K'' \\ \downarrow & & \downarrow \\ L' & \xrightarrow{\phi'} & f^*K' \end{array}$$

such that the image of y in $H^q(B'', G)$ is in the image of $H^q(A'', G)$. This may be done, by our assumptions, and invoking (8.18), (8.11) a second time, we may obtain this with a refinement ϕ'' which when restricted to δ -skeleta is isomorphic to ϕ'/δ , and such that (ξ'') is homotopy-commutative rel δ . Clearly, for any homotopy-commutativity F'' of (ξ'') the induced map

$$H^{q+1}(A', B'; G) \rightarrow H^{q+1}(A'', B''; G)$$

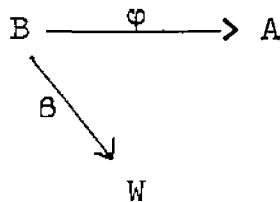
sends $\partial y = x'$ to zero. To conclude the proof of (12.7) we need only take for (ξ) the "composite" of the diagrams (ξ') and (ξ'')

$$(\xi) \quad \begin{array}{ccc} L'' & \xrightarrow{\phi''} & f^*K'' \\ \downarrow & & \downarrow \\ L' & \xrightarrow{\phi'} & f^*K' \\ \downarrow & & \downarrow \\ L & \xrightarrow{\phi} & f^*K. \end{array}$$

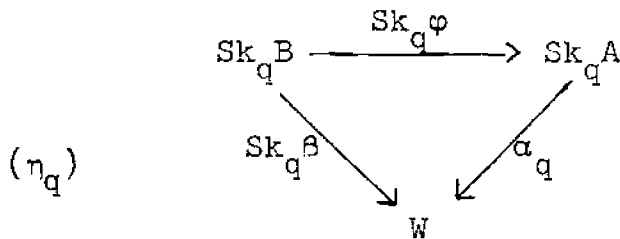
and take F to be the homotopy-commutativity rel δ , obtained by "composing" (in the evident sense) the homotopy-commutativity F' of (ξ') with any homotopy-commutativity rel δ , F'' of (ξ'') .

(b) The Inductive Step.

Let A, B be as in (a) above. Suppose we are given a diagram

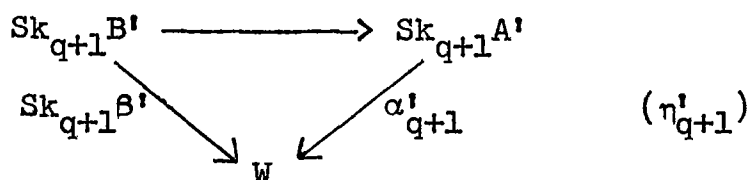


where $W \in \mathcal{C}\mathcal{A}_0$, such that for a given integer $q > 0$, there is a homotopy-commutative triangle,



Lemma 12.8: There is a refinement ϕ' of ϕ such that

- (1) $\phi'/q-d-1 = \phi/q-d-1$
- (2) $L!, K!$ have all objects of \mathcal{C} -dimension $\leq d$.
- (3) There is a homotopy-commutative diagram,



where $\alpha'_{q+1}/q-d-1 = \alpha_q/q-d-1$.

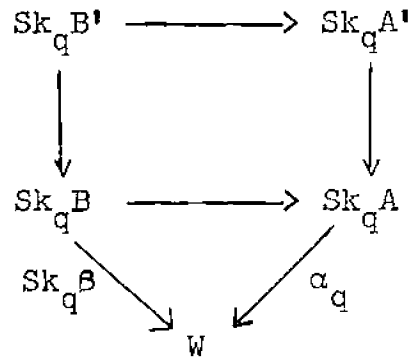
Proof: We may assume that (η_q) is actually-commutative, since $Sk_q\varphi$ is an inclusion as a subcomplex ([29], Chap. 7. sec. 6, thm. 12).

Let \circ_{q+1} denote the obstruction to finding an extension to Sk_{q+1} agreeing with (η_q) on Sk_{q-1} . The obstruction lies in a relative cohomology group, $H^{q+1}(A,B;G)$ where G is the twisted coefficient group locally isomorphic to $\pi_{q+1}(W)$, the twist being induced by the action of $\pi_1(W)$ on $\pi_{q+1}(W)$, whence it is a \mathcal{C} -twisted abelian \mathcal{C} -group.

We apply (12.7) with $x = \circ_{q+1}$, to get a refinement $\phi': L! \rightarrow f*K!$ and an actually-commutative diagram

$$\begin{array}{ccc}
 B' & \xrightarrow{\varphi'} & A' \\
 \downarrow s & & \downarrow t_F \\
 B & \xrightarrow{\varphi} & A
 \end{array}
 \quad (\xi_F)$$

satisfying the conclusions of (12.7). Since s is the geometric realization of a morphism of simplicial sets, s is cellular; φ and φ' are clearly cellular. Hence t_F may be assumed cellular. We may then consider the diagram $(Sk_q \xi_F)$ to obtain



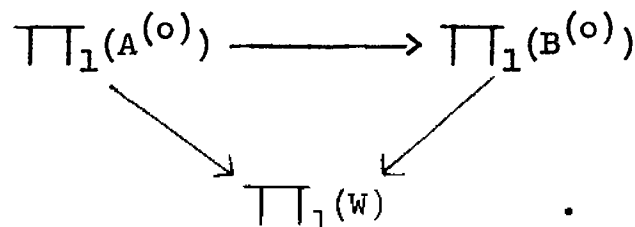
The above being actually commutative gives us an extension,

$$\beta'_q: B' \cup \text{Sk}_q A' \longrightarrow W$$

of β' , whose obstruction, o'_{q+1} , is the image of o_{q+1} via the mapping on relative cohomology induced by (ξ_F) . Thus, by our very choice of refinement (12.7) we have: $o'_{q+1} = 0$. Consequently, β'_{q-1} may be extended to β'_{q+1} . This latter map yields our diagram (η'_{q+1}) , and the three properties asserted in (12.8) come from the similarly numbered conclusions of (12.7).
q.e.d.

(c) The Proof of (i):

We find a preliminary refinement $L^{(o)} \xrightarrow{\phi^{(o)}} f^*K^{(o)}$ such that the factorization below may be found:



This is possible since $\pi_1(W) \in \mathcal{C}$ and $\widehat{\prod} f$ is a h -isomorphism. Then for $L_{\bullet}^{(0)} \xrightarrow{\phi^{(0)}} f^*K_{\bullet}^{(0)}$ we may find a diagram of the form (η_1) . Applying our inductive step (c) we obtain successive refinements,

$$\phi^{(q)}: L_{\bullet}^{(q)} \longrightarrow f^*K_{\bullet}^{(q)}$$

which admit extensions, (η_{q+1}) , for all $q \geq 1$. Since

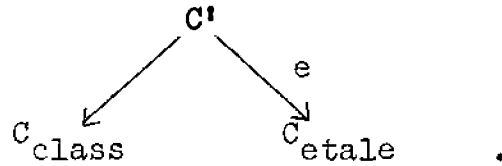
$$\phi^{(q)}/q - (d+1) = \phi^{(q+1)}/q - (d+1)$$

we may define the refinement $\phi^{(\infty)}: L_{\bullet}^{(\infty)} \longrightarrow f^*K_{\bullet}^{(\infty)}$ by $\phi^{(\infty)}/n = \phi^{(q)}/n$ for any $q \geq n+d+1$. Thus admits an extension of the form (12.6), concluding our proof of (i).

Theorem (12.9): (generalized Riemann existence theorem)

Let X be a connected, pointed prescheme of finite type over the field of complex numbers. Then there is a canonical map $\epsilon: X_{C1} \longrightarrow X_{\text{et}}$, and it induces an isomorphism on profinite completions.

Proof: We introduce in addition to the sites C_{class} (12.2) and C_{etale} (9.4) also the site C' whose objects are topological spaces X' lying over the underlying topological space of X , such that the map $X' \rightarrow X$ is a local isomorphism, i.e., that every point $x' \in X'$ has a neighborhood which is isomorphic to its image. Since an etale map of preschemes $X' \rightarrow X$ is a local isomorphism on underlying spaces, and since an open set lies in C' , we have morphisms of sites



Now it is clear from the definition of local isomorphism that every hypercovering of C' is dominated by a hypercovering of C_{class} . Thus the map $\prod C' \rightarrow X_{c1}$ is a homotopy equivalence, and so e yields the map ϵ . By the discussion of section 10 and the comparison theorems of ([3], XI, XVI, 4), this map ϵ induces an isomorphism on cohomology with twisted finite coefficients, and on $\hat{\pi}_1$ ($\hat{}$ = profinite completion). Therefore $\hat{\epsilon}$ is a \mathcal{H} -isomorphism, by (4.3). Now apply Theorem (12.5) and ([3] X, 4.3) to obtain an actual isomorphism.

Corollary (12.10): With the notation of (12.9), suppose in addition that X is geometrically unibranch. Then X_{et} is isomorphic to the profinite completion of X_{c1} .

Apply (11.1).

The same considerations as used for (12.9), together with assorted comparison results from ([3], XVI) yield the following results:

Corollary (12.11): Let k be a field admitting two imbeddings

$$\epsilon_1, \epsilon_2: k \rightarrow \mathbb{C}$$

into the complex numbers, and let X be a pointed scheme of finite type over k , X_i the schemes over \mathbb{C} obtained by the imbeddings

ϵ_i , $i = 1,2$. Then

$$\widehat{X}_{1,\text{class}} \approx \widehat{X}_{2,\text{class}}$$

where $\widehat{}$ denotes profinite completion.

Corollary (12.12): Let X be a proper pointed scheme over $\text{Spec}(k)$, where k is a separably algebraically closed field, and let $K \supset k$ be another separably closed field. Denote by X_K the scheme obtained from X by extension of scalars. Then the canonical map

$$(\widehat{X}_K)_{\text{et}} \rightarrow \widehat{X}_{\text{et}}$$

is an isomorphism, where $\widehat{}$ denotes profinite completion.

Corollary (12.13): Let R be a discrete valuation ring with separably algebraically closed residue field k , and let $f: X \rightarrow \text{Spec}(R)$ be a smooth proper scheme with connected geometric fibres X_0, X_1 , both being assumed pointed compatibly with a chosen section of $X/\text{Spec } R$. Then there is a canonical isomorphism

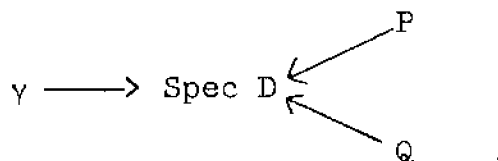
$$\widehat{X}_{1,\text{et}} \xrightarrow{\sim} \widehat{X}_{0,\text{et}}$$

where $\widehat{}$ denotes completion with respect to the class of finite groups of order prime to the characteristic p of k .

In what follows, let \mathcal{C}_p denote the class of finite p -groups, and ${}^p\mathcal{C}$ the class of finite groups whose order is not divisible

by p . If Z is in $\text{pro-}\mathcal{X}_0$, we let ${}_pZ$ and Z_p denote completion with respect to the class ${}_p\mathcal{C}$ and \mathcal{C}_p respectively. Let $\hat{}$ denote profinite completion.

Corollary (12.14): Let D be a dedekind domain, and consider a situation as follows



where γ is a geometric point with residue field \mathbb{C} lying over the generic point, and where P, Q are two geometric points of characteristics $p \neq q$ respectively. Let X be a smooth, proper scheme over $\text{Spec } D$, with a given section. Assume that $X_{\gamma, \mathbb{C}}$ is connected and simply connected. Then

$${}_p(X_{P, \text{et}}) \times (X_{Q, \text{et}})_p \approx \widehat{X_{\gamma, \mathbb{C}}}$$

Proof: We apply (12.12) and (12.9) at the local rings of the two points P, Q respectively, to conclude that

$${}_p(X_{P, \text{cl}}) \approx {}_p(X_{\gamma, \text{cl}}) \quad , \quad (X_{Q, \text{et}})_p \approx (X_{\gamma, \text{cl}})_p \quad .$$

Thus it suffices to show that if Z is any simply connected element of \mathcal{X}_0 , then

$$\hat{Z} \approx {}_pZ \times Z_p \quad .$$

In fact, if Z is in $\mathcal{C}\mathcal{X}_0$, \mathcal{C} the class of finite groups, then it is easily seen that in fact Z decomposes as a product $Z = \prod_p Z_p$ of factors in $\mathcal{C}\mathcal{X}_0$ itself. This follows for example from a consideration of the Postnikov decomposition. By the construction of Z for a simply connected Z , it is a pro-object whose members are simply connected elements of $\mathcal{C}\mathcal{X}_0$, and hence split in the canonical way, whence Z splits too.

Corollary (12.15): With the notation of (12.13), suppose X' is another scheme satisfying the hypotheses of X . If $X_P \approx X'_P$ and $X_Q \approx X'_Q$, then $\widehat{X}_{\gamma, \text{cl}} \approx \widehat{X}'_{\gamma, \text{cl}}$.

APPENDIX

§1. Limits.

We recall some facts about limits. Since the results are well known and/or routine, we will omit most of the proofs. We will work in a fixed universe U ([3], VI), so that all categories are assumed to be U -categories, but not necessarily U -small, i.e., a set of U ([3], VI, 2.1).

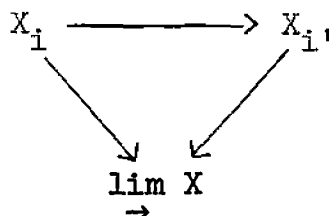
Let I be an "index" category (the word index has the empty meaning), and

(1.1) $X: I \rightarrow C$

a functor, where C is another category. For $i \in I$ denote by $X_i \in C$ the value of X on i . A direct limit

$$\lim_{\rightarrow} X = \lim_{\rightarrow} X_i$$

is an object of C together with maps $X_i \rightarrow \lim_{\rightarrow} X$ for each $i \in I$ such that for every map $i \rightarrow i'$ in I the resulting diagram

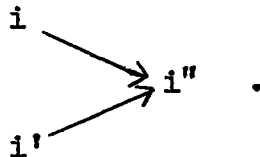


commutes, and such that $\varinjlim X$ is universal with respect to these properties. Clearly $\varinjlim X$ is unique up to unique isomorphism, if it exists. It does always exist if C is for instance the category of sets, groups, or abelian groups and if I is a U -small category.

The inverse limit $\varprojlim X$ is defined in an analogous way ([3], I).

A category I will be called filtering if

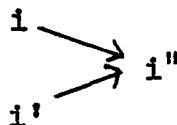
(1.2)(a) Every pair i, i' of objects of I can be embedded in a diagram



(b) (essential uniqueness of maps) If $i \twoheadrightarrow i'$ is a pair of maps in I , there is a map $i' \rightarrow i''$ such that the two composed maps $i \twoheadrightarrow i''$ are equal.

When I is a U -small filtering category, and $C = (\text{sets})$ in (1.2), then $\varinjlim X$ can be described in the familiar way as the set of equivalence classes in $\cup X_i$ for the following equivalence relation:

(1.3) Let $\alpha \in X_i$, $\alpha' \in X_{i'}$. Then $\alpha \sim \alpha'$ iff there is a diagram



such that the images of α, α' in X_{i_1} under the induced maps are equal.

For a U-small I , the functor

$$(1.4) \quad \lim_{\rightarrow} : \text{Hom}(I, (\text{sets})) \rightarrow (\text{sets})$$

always commutes with arbitrary direct limits. If I is filtering, it is also left exact in the sense that it commutes with finite inverse limits, i.e., it preserves monomorphisms and commutes with finite fibred products.

If $X: I \rightarrow (\text{grps})$ is a U-small filtering system of groups (or of abelian groups), then the set-theoretic limit has an (abelian) group structure making it into the direct limit in the category of (ab) groups. The functor \lim_{\rightarrow} is an exact functor on abelian groups.

Now let $I \xrightarrow{\phi} J$ be a functor, and suppose I filtering. We will call ϕ cofinal if it has the following properties:

(1.5)(a) For every $j \in J$, there is an $i \in I$ and a map

$$j \rightarrow \phi(i) .$$

(b) (essential uniqueness) If $j \in J$, and $i \in I$, and if $j \xrightarrow{\twoheadrightarrow} \phi(i)$ are two maps in J , there is a map $i \rightarrow i'$ in I such that the composed maps $j \xrightarrow{\twoheadrightarrow} \phi(i')$ are equal.

The condition to be cofinal is transitive. Moreover, one verifies easily that ϕ cofinal implies J filtering (where we

always assume as above that I is filtering in any case). Condition (b) can be restated in several equivalent ways, especially if J is filtering. But note that (b) does not follow from (a) even if both I and J are filtering:

Example (1.6): Let J be the category consisting of one object j , with one map $f: j \rightarrow j$ other than the identity. Suppose that $ff = f$. Then J is filtering. The one point subcategory I consisting of the object j and the identity map is also filtering, and the inclusion satisfies (a) but not (b).

A functor $X: J \rightarrow (\text{sets})$ is just a set X together with a projection map f from X to a subset $Y \subset X$. Clearly

$$\lim_{\vec{J}} X = Y$$

while

$$\lim_{\vec{I}} X|I = X.$$

(1.7) One case in which (b) follows from (a) is that J is filtering and that I is a full subcategory of J . The category I is then automatically filtering.

Proposition (1.8): Let $I \xrightarrow{\phi} J$ be cofinal, with I filtering, and let $X: J \rightarrow (\text{sets})$ be a functor. Suppose I, J U -small. Denote by X_ϕ the composed functor $i \rightsquigarrow X_\phi(i)$. Then the canonical map

$$\lim_{\vec{I}} X_\phi \rightarrow \lim_{\vec{J}} X$$

is bijective.

Proof: Apply (1.3). The surjectivity of the map follows immediately from (1.5)(a). To prove injectivity, let $\alpha \in X_{\phi(i)}$, $\alpha' \in X_{\phi(i')}$ and suppose their images are equal in $\lim_{\rightarrow} X$. This means that there is a diagram in J

$$(1.9) \quad \begin{array}{ccc} \phi(i) & & \\ & \searrow & \\ & & j \\ & \nearrow & \\ \phi(i') & & \end{array}$$

such that $\alpha = \alpha'$ in X_j . Choose $i'' \in I$ and a map $j \rightarrow \phi(i'')$ (cf. (1.5)(a)). Changing i'' if necessary, we may assume that there are maps $i \rightarrow i''$ and $i' \rightarrow i''$. Then we obtain two maps $\phi(i) \twoheadrightarrow \phi(i'')$, one factoring through j . By (1.5)(b), there is a map $i'' \rightarrow i'''$ such that the composed maps $\phi(i) \twoheadrightarrow \phi(i''')$ are equal. Replacing i'' by i''' and applying the same reasoning to i' , it follows that there exists a diagram

$$\begin{array}{ccc} i & & \\ & \searrow & \\ & & i'' \\ & \nearrow & \\ i' & & \end{array}$$

in I whose image in J fits into a commutative diagram

$$\begin{array}{ccccc} \phi(i) & & & & \\ & \searrow & & & \\ & & j & \longrightarrow & \phi(i'') \\ & \nearrow & & & \\ \phi(i') & & & & \end{array}$$

with (1.9). Hence the images of α, α' are equal in $X_\phi(i'')$, hence equal in $\varinjlim X_\phi$.

Proposition (1.10): Let I be a filtering category, and let J be the category of maps $i \xrightarrow{f} i'$ in I , where a morphism $f \rightarrow g$ in J is a commutative diagram

$$\begin{array}{ccc} i & \xrightarrow{f} & i' \\ \downarrow & & \downarrow \\ i'' & \xrightarrow{\quad} & i''' \end{array}$$

in I . Then

(a) J is filtering.

(b) The functors "domain" and "range" from J to I , taking

$$i \xrightarrow{f} i' \quad \begin{array}{l} \rightsquigarrow i \\ \rightsquigarrow i' \end{array} ,$$

are cofinal.

(c) The functor "identity map" $I \rightarrow J$ taking

$$i \rightsquigarrow i = i$$

is cofinal.

Proposition (1.11): Let I be a U-small filtering category and let J be the category of maps of I above.

(a) For fixed $i \in I$, the sub-category J_i of J consisting

of maps from i to varying $i' \in I$ is filtering.

(b) Let $X: J \rightarrow (\text{sets})$ be a functor. There is a natural functor $I \rightarrow (\text{sets})$ sending

$$i \rightsquigarrow \lim_{\substack{\rightarrow \\ J_i}} X|_{J_i} ,$$

and a canonical isomorphism

$$\lim_{\substack{\rightarrow \\ I}} (\lim_{\substack{\rightarrow \\ J_i}} X|_{J_i}) \xrightarrow{\sim} \lim_{\substack{\rightarrow \\ J}} X .$$

§2. Pro-objects and pro-representable functors.

Let C be a category. A pro-object [15] in C is a (contravariant) functor

$$X: I^{\circ} \rightarrow C$$

from some U-small filtering index category I to C . We will often use the notation

$$X = \{X_i\}_{i \in I} \quad .$$

One thinks of a pro-object X as an inverse system of objects of C , and the point is that the pro-objects of C can be made in a good way into a category

pro- C

by the rule

$$(2.1) \quad \text{Hom}(X, Y) = \varprojlim_j (\varinjlim_i \text{Hom}(X_i, Y_j))$$

when $X = \{X_i\}_{i \in I}$ and $Y = \{Y_j\}_{j \in J}$, where we leave it to the reader to elucidate the maps involved in the directed systems.

Note that the index categories are not assumed equal. Also, it is important to understand right away that the pro-object contains much more information than the inverse limit $\varprojlim X$ even if the

latter exists in C , which we do not assume.

Clearly, a functor $F: C \rightarrow C'$ induces in the obvious way a functor

$$(2.2) \quad \text{pro-F}: \text{pro-C} \rightarrow \text{pro-C}' \quad .$$

The objects of C themselves are pro-objects if the index category is taken to be the one point category, and by (2.1), C forms a full subcategory of pro-C . If $X = \{X_i\} \in \text{pro-C}$ and $Y \in C$, then

$$(2.3) \quad \text{Hom}(X, Y) = \varinjlim_i \text{Hom}(X_i, Y) \quad .$$

In this way every $X \in \text{pro-C}$ gives rise to a functor

$$(2.4) \quad \text{Hom}(X, .): C \rightarrow (\text{sets}) \quad .$$

It can be shown ([25]) that morphisms between the functors associated to pro-objects X, Y are in 1-1 correspondence with elements of $\text{Hom}(X, Y)$ defined as in (2.1). Thus pro-C is equivalent via (2.4) to a full subcategory of $\text{Hom}(C, (\text{sets}))$. The functors which are isomorphic to $\text{Hom}(X, .)$ for some pro-object X are called pro-representable functors.

Corollary (2.5): Let $X = \{X_j\}_{j \in J}$ be a pro-object and let $I \xrightarrow{\phi} J$ be a cofinal functor, with I filtering and U -small. Then the pro-object $X_\phi = \{X_{\phi(i)}\}_{i \in I}$ is isomorphic with X .

In fact, it follows from (2.1) that the functors represented by X and X_ϕ are isomorphic. Of course, the isomorphism $X_\phi \rightarrow X$ is given by the obvious element of $\varprojlim_j (\varinjlim_i \text{Hom}(X_{\phi(i)}, X_j))$. We will refer to X_ϕ as obtained from X by "re-indexing" with the index category I via ϕ .

The conditions on a functor to be pro-representable are easily understood. Recall that if $F: C \rightarrow (\text{sets})$ is a functor, then the morphisms of functors

$$\text{Hom}(Z, \cdot) \rightarrow F$$

are in 1-1 correspondence with elements of $F(Z)$. This is easily seen. Suppose we denote by J the category of pairs

$$(2.6) \quad J = \{(Z, \xi) \mid Z \in C \text{ and } \xi \in F(Z)\},$$

where a morphism

$$(Z, \xi) \rightarrow (Z', \xi')$$

is a map $Z \leftarrow Z'$ such that the induced image of ξ' in $F(Z)$ is ξ . The functor

$$J^\circ \rightarrow C$$

sending

$$(Z, \xi) \rightsquigarrow Z$$

gives rise to a functor $\mathcal{J}: C \rightarrow (\text{sets})$ by

$$\mathcal{J}(Y) = \lim_{\substack{\rightarrow \\ (Z, \xi) \in J}} \text{Hom}(Z, Y) .$$

The index category J is in general not a U -small category, so this limit is a priori in the next universe U^+ . But since ξ gives a map $\text{Hom}(Z; \cdot) \rightarrow F$ as above, we obtain a map $\mathcal{J} \rightarrow F$, and it is well known that this morphism is bijective (hence in particular that the limit can be taken in U).

Proposition (2.7): (i) Let $X = \{X_i\}_{i \in I}$ be a pro-object, and let J be the category of pairs $\{(Z, \xi) \mid \xi \in \text{Hom}(X, Z)\}$ as above. There is a functor $I \xrightarrow{\phi} J$ given by

$$i \rightsquigarrow (X_i, \xi_i)$$

where ξ_i is the image in $\text{Hom}(X, X_i)$ of $\text{id}_{X_i} \in \text{Hom}(X_i, X_i)$. This ϕ is cofinal (hence J is filtering).

(ii) Conversely, let $F: C \rightarrow (\text{sets})$ be a functor and let J be the category of pairs defined as in (2.6). Then F is pro-representable iff there is a filtering U -small category I and a cofinal functor $I \xrightarrow{\phi} J$. Equivalently (cf. [15]), F is pro-representable iff J is filtering and contains a U -small cofinal sub-category.

We omit this routine verification.

Corollary (2.8): Suppose C is itself a U -small category

which is closed under finite inverse limits, i.e., under finite products and finite fibred products. Then a functor $F: C \rightarrow (\text{sets})$ is pro-representable iff it is left exact, i.e., commutes with finite inverse limits.

§3. Morphisms of pro-objects.

Let $X = \{X_i\}_{i \in I}$, $Y = \{Y_j\}_{j \in J}$ be in pro-C . If $I = J$ and if

$$f_i: X_i \rightarrow Y_i \quad i \in I$$

is a compatible system of maps, i.e., a morphism of functors $f: X \rightarrow Y$, then f determines in an obvious way a morphism of pro-objects $X \rightarrow Y$ which we denote by the same letter f . Of course, a general morphism of pro-objects will not be of this form even if the index categories are equal. However, one can put it into such a form by re-indexing as follows:

A morphism $f: X \rightarrow Y$ is by (2.1) a compatible collection of maps $\{f_j: X \rightarrow Y_j\}_{j \in J}$, and each f_j is an element of $\varinjlim \text{Hom}(X_i, Y_j)$. We will say that a map

$$\phi: X_i \rightarrow Y_j$$

represents f if the image of ϕ in $\varinjlim \text{Hom}(X_i, Y_j)$ is f_j . This is the same as saying that the diagram in pro-C

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X_i & \xrightarrow{\phi} & Y_j \end{array}$$

commutes. A morphism $\phi \rightarrow \phi'$ between maps representing f

consists of a map $i \rightarrow i'$ in I and a map $j \rightarrow j'$ in J such that the diagram

$$\begin{array}{ccc}
 X_{i'} & \xrightarrow{\phi'} & Y_{j'} \\
 \downarrow & & \downarrow \\
 X_i & \xrightarrow{\phi} & Y_j
 \end{array}$$

commutes.

Proposition (3.1): The category M of maps ϕ representing a map $f: X \rightarrow Y$ of pro-objects is U -small and filtering, and the functors

$$M \longrightarrow I \quad \text{sending} \quad \phi: X_i \longrightarrow Y_j \rightsquigarrow i$$

and

$$M \longrightarrow J \quad \text{sending} \quad \phi: X_i \longrightarrow Y_j \rightsquigarrow j$$

are cofinal.

Corollary (3.2): A map $f: X \rightarrow Y$ of pro-objects of C can be represented, up to isomorphism, by a (U -small) filtering inverse system of maps $\{f_i: X_i \rightarrow Y_i\}_{i \in I}$, i.e., by a pro-object in the category of maps of C .

The above results can be generalized as follows.

Proposition (3.3): ("uniform approximation") Let Δ be a finite diagram with commutation relations, and suppose that

Δ has no loops, i.e., that the beginning and end of a chain of arrows are always distinct. Let D be a diagram in pro-C of the type of Δ , i.e., a morphism of Δ to pro-C . There is a filtering inverse system $\{D_i\}_{i \in I}$ of diagrams of C such that the diagram in pro-C determined by $\{D_i\}$ is isomorphic to D .

Remark (3.4): One can also make related assertions of the following type: Let $F: C \rightarrow C'$ be a functor, $X \in \text{pro-C}$ and $Y \in \text{pro-C}'$. Let $f: F(X) \rightarrow Y$ be a morphism. Then we may re-index X, Y by a single index category I so that f is represented by a compatible system of maps $f_i: F(X_i) \rightarrow Y_i$. This follows immediately from (3.3).

To prove (3.3) consider the category I of diagrams D_i of type Δ in C which represent D , i.e., each of whose maps represents the corresponding map of D . The problem is essentially to show that I is filtering and has a U -small cofinal subcategory, and that for every object X of D , $X = \{X_i\}_{i \in I}$, i.e., that the X_i are cofinal among all maps of X to objects of C .

Induction on the number of vertices of Δ . Suppose it true for $(n-1)$ vertices, and that Δ has n vertices. Choose an "initial" vertex v of Δ , i.e., one having no arrows leading to it. This is possible since Δ has no loops. Let Δ' be the diagram obtained from Δ by removing v and all arrows leading out of v . Let D' be the corresponding diagram in pro-C . By induction, D' can be approximated uniformly, say by $\{D'_j\}_{j \in J}$. Let $X = \{X_i\}_{i \in I}$ be the object of D corresponding

to v . Let K be the category of objects k consisting of a pair of indices (i,j) and a diagram D_k representing D and made up out of X_i, D_j^i and some maps, where a morphism $k \rightarrow k'$ is a map $(i,j) \rightarrow (i',j')$ such that the induced maps give a morphism of diagrams $D_{k'} \rightarrow D_k$. It remains to show that K is filtering, and that the obvious functors $K \rightarrow I, K \rightarrow J$ are cofinal. We leave this as an exercise for the reader.

Scholie (3.5): Suppose given a U -small filtering inverse system of pairs of maps

$$X_i \begin{array}{c} \xrightarrow{f_i} \\ \xrightarrow{g_i} \end{array} Y_i \quad .$$

Then by (2.1) the induced maps of pro-objects $f, g: X \rightarrow Y$ are equal iff for each i the maps $X \rightarrow Y_i$ induced by f, g are equal, i.e., iff for each i there is an i' such that the composed maps

$$X_{i'} \rightarrow X_i \begin{array}{c} \xrightarrow{f_i} \\ \xrightarrow{g_i} \end{array} Y_i$$

are equal. In particular, suppose C has a zero object 0 .

Then the maps $f: X \rightarrow Y$ is the zero map iff for every i there is an $i \rightarrow i'$ such that the composed map $X_{i'} \rightarrow X_i \rightarrow Y_i$ is the zero map. If we set $X_i = Y_i, f_i = id$, we find that an object is the zero object (which is equivalent with the assertion that the identity map is the zero map) iff for every $i \in I$ there is an $i \rightarrow i'$ such that the structure map

$$X_{i'} \rightarrow X_i$$

is the zero map.

§4. Exactness properties of the pro-category.

Proposition (4.1): Let I be a U -small filtering category and C any category. Suppose that C has finite direct (inverse) limits. Then the functor

$$\text{Hom}(I^\circ, C) \rightarrow \text{pro-}C$$

associating with an $X: I^\circ \rightarrow C$ the corresponding pro-object commutes with finite direct (inverse) limits.

Proof: Let $D = \{D_i\}_{i \in I}$ be a finite diagram of functors $I^\circ \rightarrow C$. We will denote by $X_\alpha = \{X_{\alpha i}\}_{i \in I}$ the various objects making up the diagram D (we do not bother to label the maps). Let $\underline{X} = \{\underline{X}_i\}_{i \in I}$ be the object $\lim_{\rightarrow} D = \{\lim_{\rightarrow} D_i\}_{i \in I}$. Then in $\text{pro-}C$,

$$\begin{aligned} \text{Hom}(\underline{X}, Y) &= \lim_{\leftarrow j} \lim_{\rightarrow i} \text{Hom}(\underline{X}_i, Y_j) \\ &= \lim_{\rightarrow j} \lim_{\rightarrow i} \lim_{\leftarrow \alpha} \text{Hom}(X_{\alpha i}, Y_j) \\ &= \lim_{\leftarrow j} \lim_{\leftarrow \alpha} \lim_{\rightarrow i} \text{Hom}(X_{\alpha i}, Y_j) \quad \text{because} \quad \lim_{\rightarrow i} \end{aligned}$$

commutes with finite inverse limits ([3], I, 2.8)

$$= \lim_{\leftarrow \alpha} \lim_{\leftarrow j} \lim_{\rightarrow i} \text{Hom}(X_{\alpha i}, Y_j) \quad \text{because} \quad \lim_{\leftarrow j}$$

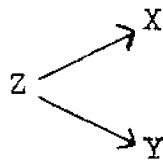
commutes with inverse limits

$$= \lim_{\leftarrow \alpha} \text{Hom}(X_\alpha, Y) \quad .$$

This shows that \underline{X} has the property required of a direct limit in the category pro-C . The proof for an inverse limit goes the same way.

Proposition (4.2): If C is closed under finite direct (inverse) limits, so is pro-C .

Proof: To show pro-C closed under arbitrary finite direct limits, it suffices to show it closed under coproducts and amalgamated coproducts ([3], I, 2) $X \amalg Y$ and $X \amalg_Z Y$. To show for instance the second, we suppose given a diagram



in pro-C . By (App. 3.3) we may represent it up to isomorphism by a filtering system



Now apply (4.1).

Proposition (4.3): If C is a U -small category which is closed under finite inverse limits, then pro-C is closed under arbitrary (U -small) direct limits.

Proof: If $\{X^\alpha\}$ is a system of objects of pro-C , the

functor

$$\lim_{\leftarrow \alpha} \text{Hom}(X^\alpha, Y) \quad Y \in C$$

is a left exact functor from C to sets, since each $\text{Hom}(X^\alpha, Y)$ is left exact. Hence the functor is pro-representable (2.8).

Proposition (4.4): For any C , $\text{pro-}C$ is closed under (U-small) filtering inverse limits.

Proof: Let $\{X^j\}_{j \in J}$ be a filtering inverse system of pro-objects of C , and say $X^j = \{X_i^j\}$ where $i \in I_j$. Let K be the category whose objects are pairs of indices (j, i) with $j \in J$ and $i \in I_j$, and where a map $(j, i) \rightarrow (j', i')$ consists of a map $j \rightarrow j'$ and a map $X_{i'}^{j'} \rightarrow X_i^j$ representing $X^{j'} \rightarrow X^j$. One verifies easily that K is filtering (and U-small), and we claim that the pro-object

$$\{X_i^j\}_{(j, i) \in K}$$

is an inverse limit of the filtering system $\{X^j\}$. In fact, it is clear that if $Z \in \text{pro-}C$, then an element $\lim_{\leftarrow j} \text{Hom}(Z, X^j)$ determines a unique element of $\text{Lim}_{(j, i)} \text{Hom}(Z, X_i^j)$, and conversely.

Proposition (4.5): Let A be an additive (abelian) category. Then $\text{pro-}A$ is again additive (abelian).

Proof: The axioms of ([11]) for $\text{pro-}A$ follow immediately from those of A using (4.1), (4.2). Note that if C is a U-small category, then $\text{pro-}A$ is equivalent by (2.7)

with the category $S_{\text{ex}}(A, (\text{ab}))$ considered by Gabriel ([11]).

Proposition (4.6): Let A be an abelian category. A monomorphism (epimorphism) in $\text{pro-}A$ can be represented by an inverse system of monomorphisms (epimorphisms).

Proof: In an abelian category, a monomorphism is a kernel of a map, and an epimorphism is a cokernel. Since these are finite limits, we can apply (4.1). We leave the assertion for $\text{pro-}(\text{grps})$ as an exercise.

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