THH **OF THE MORAVA E-THEORY SPECTRUM** E_2

SANJANA AGARWAL

ABSTRACT. The Morava E-theories, E_n, are complex-oriented 2-periodic ring spectra, with homotopy groups $W_{\mathbb{F}_{p^n}}[[u_1, u_2, ..., u_{n-1}]][u, u^{-1}]$. Here W denotes the ring of Witt vectors. E_n is a Landweber exact spectrum and hence uniquely determined by its homotopy groups as BP_*-algebra. Algebraic K-theory of E_n is a key ingredient towards analyzing the layers in the p-complete Waldhausen's algebraic K-theory chromatic tower. One hopes to use the machinery of trace methods to get results towards algebraic K-theory once the computation for $THH(E_n)$ is known.

In this paper we describe $THH(E_2)$ as part of consecutive chain of cofiber sequences where each cofiber sits in the next cofiber sequence and the first term of each cofiber sequence is describable completely in terms of suspensions and localisations of E_2. For these results, we first calculate $K(i)$-homology of $THH(E_2)$ using a Bökstedt spectral sequence and then lift the generating classes of $K(1)$-homology to fundamental classes in homotopy group of $THH(E_2)$. These lifts allow us to construct terms of the cofiber sequence and explicitly understand how they map to $THH(E_2)$.

CONTENTS

1. Introduction

The goal of this article is to study and describe the topological Hochschild homology of the second Morava E-theory spectrum, $THH(E_2)$, in terms of the suspensions and localizations of E_2-spectrum (Theorem 5.24). After p-completion, we are able to give a full description of $THH(E_2)_p^\wedge$ in these terms as follows.

Theorem 1.1 (Theorem 5.26). *We have the following diagram of $(E_2)_p^\wedge$-modules for $THH(E_2)_p^\wedge$, where $(C_{f_i})_p^\wedge$ are the cofiber maps of $(f_i)_p^\wedge$*

$$
\begin{array}{ccc}
(E_2)_p^\wedge & \xrightarrow[\quad(2.1)\quad]{(f_1)_p^\wedge:=p\text{-completed unit map}} & THH(E_2)_p^\wedge \\
& & \downarrow{\scriptstyle (C_{f_1})_p^\wedge} \\
\Sigma^{2p-1}L_1(E_2)_p^\wedge & \xrightarrow[\quad(5.12)\quad]{(f_2)_p^\wedge:=(\overline{j_1})_p^\wedge} & \overline{THH}(E_2)_p^\wedge \\
& & \downarrow{\scriptstyle (C_{f_2})_p^\wedge} \\
(\bigvee_\alpha(\Sigma^{|\alpha|}L_1E_2[u_1^{-1}]\bigvee\Sigma^{|\alpha|+2p-1}L_1E_2[u_1^{-1}]))_p^\wedge & \simeq & \overline{\overline{THH}}(E_2)_p^\wedge.
\end{array}
$$

Here $\overline{THH}(E_2)$ is the cofiber of the unit map f_1 and $\overline{\overline{THH}}(E_2)$ is the cofiber of the map of E_2-modules, f_2. Both the cofibers are L_1-local where L_1 denotes the Bousfield localization with respect to the first Johnson-Wilson theory $E(1)$. For an E_2-module M, by $M[u_1^{-1}]$ we mean the spectrum

$$\mathrm{hocolim}(M \xrightarrow{u_1} M \xrightarrow{u_1} M \xrightarrow{u_1} \cdots)$$

where the u_1-multiplication is defined by the E_2-module structure of M. See §5.3 for details on indexing, α.

The cofiber sequences in the diagram above, are possible to construct due to the homotopy classes of $THH(E_2)$ that we are able to lift from $K(i)$-homology classes of $THH(E_2)$ along the Hurewicz map. Here $K(i)$ is the ith Morava K-theory. Computing $K(i)_*THH(E_2)$ is crucial to our procedure and we show the following

Theorem 1.2 (Theorem 3.9, Theorem 3.28, Theorem 3.29, Lemma 3.30). *$THH(E_2)$ is L_2-local. And, we have following isomorphisms of $K(0)_*E(2)$, $K(1)_*E(2)$ and*

$K(2)_*E(2)$-*algebras, respectively*

$$K(0)_*THH(E_2) \cong (K(0)_*E_2) \otimes_{\mathbb{Z}[[u_1]]_{\mathbb{Q}}} (\Lambda_{\mathbb{Z}[[u_1]]_{\mathbb{Q}}} HH_1^{\mathbb{Q}}(\mathbb{Z}[[u_1]]_{\mathbb{Q}})) \otimes_{\mathbb{Q}} (\Lambda_{\mathbb{Q}} du)$$

$$K(1)_*THH(E_2) \cong (K(1)_*E_2) \otimes_{F_p[[u_1]]} (\Lambda_{\mathbb{F}_p[[u_1]]} HH_1^{\mathbb{F}_p[u_1]} \mathbb{F}_p[[u_1]]) \otimes_{\mathbb{F}_p} (\Lambda_{\mathbb{F}_p} dt_1)$$

$$K(2)_*THH(E_2) \cong K(2)_*E_2$$

where $\mathbb{Z}[[u_1]]_{\mathbb{Q}} := \mathbb{Z}_p[[u_1]] \otimes_{\mathbb{Z}} \mathbb{Q}$ *and* du, dt_1 *are as in* (2.16),(2.11).

Here, for M an R-module, $\Lambda_R M$ is the (graded) exterior R-algebra on M. And, $\Lambda_{\mathbb{Q}} du := \Lambda_{\mathbb{Q}} \mathbb{Q}\langle du \rangle$, $\Lambda_{\mathbb{F}_p} dt_1 := \Lambda_{\mathbb{F}_p} \mathbb{F}_p \langle dt_1 \rangle$.

In this section, we give some context to the question we address in this paper and comment on some unanswered questions and future directions. For terminologies and notations used in this paper, please look at Subsection 1.3.

1.1. **Motivation.** The 2002 paper of Ausoni and Rognes [AR02] initiated a program to understand algebraic K-theory of ring spectra and its relationship to chromatic phenomenon. We begin with a brief outline of this program as it has served as one of the big motivations behind many of the calculations in this area over the last 20 years.

In algebraic number theory, the arithmetic of the ring of integers in a number field is largely captured by its Picard group, unit group, etc. These groups are closely related to the algebraic K-theory groups associated to this ring $K_0(R)$, $K_1(R)$, respectively. Thus, algebraic K-theory encodes extremely rich information about the arithmetic structures of rings within itself.

There is a symmetric monoidal functor from the category of rings to the category of spectra

$$H(-) : \mathcal{R}ing \to \mathcal{S}p$$
$$R \mapsto HR$$

where HR is the Eilenberg-MacLane spectrum associated to R. Since we can extend the algebraic K-theory functor to ring spectra (see for example [EKMM97, Chapter 6]), a natural question to ask is: what structural information about ring spectra is encoded by algebraic K-theory. Waldhausen in [Wal85] showed that algebraic K-theory for ring spectra, particularly the sphere spectrum $\mathbb{S}$, is related to high dimensional differential topology. Thus, understanding $K(\mathbb{S})$ is an important and relevant question for many subjects.

Work of Dundas, Goodwillie, McCarthy [DGM13, Theorem 0.0.2] and Hesselholt, Madsen [HM97, Theorem B.1] show that the square

$$\begin{array}{ccc} K(\mathbb{S}) & \longrightarrow & K(\mathbb{S}_p) \\ \downarrow & & \downarrow \\ K(\pi_0\mathbb{S}) & \longrightarrow & K(\pi_0\mathbb{S}_p) \end{array}$$

is homotopy cartesian after p-completion ($\mathbb{S}_p$ is the p-complete sphere spectrum). So upto p-completion, it is enough to understand $K(\mathbb{S}_p)$. $\mathbb{S}_p$ is the homotopy limit of

the chromatic tower

$$\cdots \to L_n\mathbb{S}_p \to \cdots \to L_1\mathbb{S}_p \to L_0\mathbb{S}_p = H\mathbb{Q}_p$$

where L_n denotes the Bousfield localization with respect to the nth Johnson-Wilson theory $E(n)$ (see Subsection 1.3(4),(2)). The Hopkins-Ravenel chromatic convergence theorem implies that the map

$$\mathbb{S}_p \xrightarrow{\simeq} \mathrm{holim}_n \ L_n\mathbb{S}_p$$

is a weak equivalence. It also induces a tower on algebraic K-theory

$$\cdots \to K(L_n\mathbb{S}_p) \to \cdots \to K(L_1\mathbb{S}_p) \to K(L_0\mathbb{S}_p) = K(\mathbb{Q}_p)$$

with a map

$$K(\mathbb{S}_p) \to \mathrm{holim}_n \ K(L_n\mathbb{S}_p).$$

Waldhausen [Wal84] conjectured that this map should be a weak equivalence. Waldhausen's work also suggests that there should be an algebraic K-theoretic interpretation of the fibers of the maps in this sequence: the fibers should be closely related to $K(L_{K(n)}\mathbb{S}_p)$, where $L_{K(n)}$ is Bousfield localization with respect to nth Morava K-theory $K(n)$ (see Subsection 1.3(4),(1)).

From the work of Devinatz-Hopkins [DH04], $L_{K(n)}\mathbb{S}_p$ is closely related to E_n, the nth Morava E-theory (see Subsection 1.3(3)) as

$$L_{K(n)}\mathbb{S}_p \simeq E_n^{h\mathbb{G}_n}$$

where

$$\mathbb{G}_n := \mathbb{S}_n \rtimes C_n$$

is the semidirect product of nth profinite Morava stabilizer group and the cyclic group of order n. Based on this, Rognes conjectured that

$$K(L_{K(n)}\mathbb{S}_p) \to K(E_n)^{h\mathbb{G}_n}$$

is a 'nice' map in sufficiently high dimensions: in particular up to smashing with a finite p-local CW-spectrum of chromatic type $n+1$, this map is a weak equivalence. Hence, in [AR02], the authors lay out a plan to analyze $K(E_n)$. The theme is to construct "localization sequences" (cofibration sequences analogous to Quillen's algebraic K-theory localization sequence [Qui73]) in algebraic K-theory of connective commutative $\mathbb{S}$-algebras and then use the machinery of trace methods [Mad95] to compute things using topological cyclic homology, TC.

The localization sequences conjectured in [AR02] towards this project were proven to not hold in [ABG18]. Blumberg and Mandell in [BM20] prove the following localization sequences in K-theory, TC, and topological Hochschild homology THH
(1.3)
$$\cdots \to K(W\mathbb{F}_{p^n}[[u_1, \cdots, u_{n-1}]]) \to K(BP_n) \to K(E_n) \to \Sigma \cdots$$
$$\cdots \to TC(W\mathbb{F}_{p^n}[[u_1, \cdots, u_{n-1}]]) \to TC(BP_n) \to TC(BP_n|E_n) \to \Sigma \cdots$$
$$\cdots \to THH(W\mathbb{F}_{p^n}[[u_1, \cdots, u_{n-1}]]) \to THH(BP_n) \to THH(BP_n|E_n) \to \Sigma \cdots$$

where W denotes the ring of p-typical Witt vectors, and BP_n denotes the connective cover of E_n. This is exceptional in the sense that this provides us with tools for

calculating algebraic K-theory of non-connective ring spectra using localization sequences.

To implement the machineries of trace methods and localization sequences towards understanding the relationship between algebraic K-theory and the chromatic phenomenon, we therefore, first need to know THH of various spectra involved. The topological Hochschild homologies (and algebraic K-theories) of various complex-oriented Landweber theories are thus extremely useful, but not yet very well known except in few cases [MS93], [AR02], [Aus10], [AR12b], [AR12a], [Sto20]. In [AR20], Ausoni and Richter make progress towards calculating $THH(E(n))$ but these are under commutativity assumptions on $E(n)$ spectra. Recent developments in the field (for example [BSY22], [HW22], [BHLS23]) have given us big insights into this program but explicit structure for algebraic K-theory, TC, and THH of BP_n and E_n remain completely unknown and interesting for $n \geq 2$.

1.2. **Future questions and Remarks.** The computations of this paper result in some further questions:

- The methodology used for the computation in this paper does not currently extend to higher Morava E-theories E_n, $n \geq 2$. This is due to the fact that the computations of Hochschild homologies of rings

$$k[[x_1, \ldots, x_n]]$$

  (k any field) remains, to our knowledge, unknown for $n \geq 2$. The methods of [LL01], where

$$HH_*^k(k[[x]])$$

  is calculated, do not extend to these rings. Hochschild homology of these rings are of interest to us.
- In moving from $E(2)$ to E_2, one adds a lot of nice structure. But in terms of homotopy groups for us, it means working with power series rings instead of polynomial rings. Taming the power series ring somehow is necessary to make the result stronger. For this reason, continuous THH as defined in Efimov's recent work [Efi25] seems to be an appropriate invariant to work with. In upcoming work with Noah Wisdom, we explore some computations in this direction.
- The computations of this paper can be useful in the broader Rognes program due to the relationship

$$
\begin{array}{c}
THH(BP_n) \\
\downarrow \\
THH(BP_n)[u^{-1}] \overset{\simeq}{\longrightarrow} THH(E_n)
\end{array}
$$

  and the fact that $THH(BP_n)$ and $TC(BP_n)$ are crucial in the localization sequences (1.3) towards understanding $K(E_n)$. If we can lift the classes of $THH(E_2)$ to $THH(BP_2)$, we should be able to compute the cofiber $THH(BP_2|E_2)$ since we understand $THH(W\mathbb{F}_{p^2}[[u_1]])$ from [LL01]. Note that in general understanding $THH(W\mathbb{F}_{p^n}[[u_1, \cdots, u_{n-1}]])$ might require the first bullet point above.

- Given the understanding of classes $\pi_* THH(E_2)$ in this paper, one can ask if its possible to understand and say something about the cyclotomic structure of $THH(E_2)$. We have been unable to say anything in that direction yet.
- We are unable to show complete splitting of cofiber diagrams in Theorem 5.24 and Theorem 5.26. In fact, we conjecture that these sequences do not split but we do not have a proof so far.

1.3. **Some terminology and notation.**

(1) We denote the *ith Morava K-theory spectrum* as $K(i)$. These are complex oriented (but not Landweber exact) spectra with p-typical formal group laws and coefficient rings

$$K(i)_* = \mathbb{F}_p[v_i, v_i^{-1}], \text{ where } |v_i| = 2^{p^i - 1}.$$

(2) We denote the *nth Johnson-Wilson theory spectrum* as $E(n)$. These are complex oriented spectra with p-typical formal group laws and coefficient rings

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, \cdots, v_n, v_n^{-1}], \text{ where } |v_i| = 2^{p^i - 1}.$$

They are Landweber exact and hence determined by their coefficient rings as BP_*-algebras.

(3) We denote the *nth Morava E-theory spectrum* as E_n. These are complex oriented spectra with p-typical formal group laws and coefficient rings

$$E_{n*} \cong W\mathbb{F}_{p^n}[[u_1, \cdots, u_{n-1}]][u, u^{-1}], \text{ where } |u| = 2, |u_i| = 0.$$

They are Landweber exact and hence determined by their coefficient rings as BP_*-algebras. They are E_∞-ring spectra. More details on BP_*-algebra structure is discussed in Subsection 2.2.

(4) We denote by L_n, the Bousfield localization with respect to $E(n)$ and by $L_{K(n)}$, the Bousfield localization with respect to $K(n)$. Note that E_n is L_n-local spectrum.

(5) For k a commutative ring and A a k-algebra, $N_k^{cy}(A)$ denotes the cyclic bar construction of A over k. We also call this the Hochschild complex. The homotopy groups of this simplicial abelian group or the homology groups of the associated chain complex is denoted by $HH_*^k(A)$, the Hochschild homology of A relative to k. Topological Hochschild homology, denoted by THH, is the topological analogue of Hochschild homology in the category of spectra $(Sp, \mathbb{S}, \wedge)$. It is a simplicial spectrum with homotopy groups denoted as THH_*. For more details see [Mad95],[KN18].

(6) $\mathbb{Q}_p$ denotes the p-complete rationals. For a field k, $k(t)$ denotes the field of fractions of the polynomial ring $k[t]$, $k[[t]]$ represents the ring of formal power series with coefficients in k and with associated field of fractions denoted as $k((t))$.

Outline. In Section 2, we give some preliminary tools and results we need for the calculations in this paper. Section 3 is concerned with computations of the $K(i)$-homologies of $THH(E_2)$. For this, we first compute $HH_{*,*}^{K(i)_*} K(i)_* E_2$ and then use the Bökstedt spectral sequence. Next we lift these homology classes along

the Hurewicz map to $\pi_* THH(E_2)$ in Section 4. Finally, using the lifted classes in $\pi_* THH(E_2)$, we construct various cofiber sequences to prove our main result in Section 5.

2. Preliminaries

In this section, we review some preliminary mathematical results needed in the rest of the paper. These results concern the unit map from a ring to its topological Hoschschild homology (Subsection 2.1), particular homotopy and Hochschild homology classes of relevant complex oriented spectra (Subsection 2.2), the Bökstedt spectral sequence (Subsection 2.3), the Hochschild homology of power series rings (Subsection 2.4), and some results of [AR20] on the Hochschild homology of $K(i)_* E(n)$ (Subsection 2.5). We include these to make the paper more self-contained; a reader familiar with this material can skip this section and consult it as needed in the later sections.

2.1. **Unit map.** There is a unit map (of E_∞-ring spectra)

$$(2.1) \qquad\qquad E_2 \to THH(E_2)$$

which we denote by f_1 for reasons that will be apparent in Section 5. This map in induced from the map of ring spectrum

$$\mathbb{S} \to THH(E_2)$$

(which represents the unit element in $\pi_* THH(E_2)$) after taking smash product with E_2 and using the E_2-module structure on $THH(E_2)$. Since E_∞-ring structures can be rigidified to commutative ring structures [EKMM97, II.3], $THH(E_2)$ is modeled as the geometric realization of a simplicial spectrum and the unit map can also be seen as the inclusion of the 1-skeleton.

Since E_2 is a commutative ring spectrum, there is a map of commutative E_2-algebra spectra in the other direction

$$THH(E_2) \to E_2$$

given by the multiplication maps $E_2^{\wedge n+1} \to E_2$. The composite map

$$E_2 \to THH(E_2) \to E_2$$

is identity and thus we have a splitting of E_2-modules

$$(2.2) \qquad\qquad THH(E_2) \simeq E_2 \vee \overline{THH}(E_2)$$

where $\overline{THH}(E_2)$ denotes the cofiber of (2.1). The cofiber inherits the structure of a non-unital commutative E_2-algebra.

2.2. **Classes.** Throughout this paper, we work with various key homological classes that arise out of the Brown-Peterson spectrum BP, BP_*BP-theory and maps of ring spectra

$$BP \to K(n),$$
(2.3)
$$BP \to E(m) \to E_m$$

(for any n, m). We use these maps without further comment to make $K(n)_*$, $E(m)_*$, and E_{m*} into BP_*-modules and E_{m*} into an $E(m)_*$-module. More details on these spectra, these maps, and the classes defined below can be found in [Rav86].

$E(m)$ and E_m have canonical choices of formal group laws, i.e., canonical ring maps from BP. Thus, there are canonical classes

(2.4)
$$v_i \in \pi_* E$$

for $E = E(m)$ or E_m which are the images of the classes v_i in BP_*. Also, from (2.3) there are classes

(2.5)
$$v_i \in \pi_*(E \wedge E) \cong E_* E,$$

(2.6)
$$t_j \in \pi_*(E \wedge E) \cong E_* E$$

which are the image classes of

$$BP_*BP \cong \mathbb{Z}_{(p)}[v_i, t_j] \to E_* E.$$

These v_is are the left Hurewicz image, $\eta_L(v_i)$. Note that $|t_j| = |v_j| = 2^{p^i - 1}$, t_j in BP_*BP is the image of $v_j \in BP_*$ under the right unit map η_R and v_i in BP_*BP is the image of $v_i \in BP_*$ under the left unit map η_L. In the same vein, we have

(2.7)
$$v_i \in K(n)_* E,$$

(2.8)
$$t_j \in K(n)_* E$$

where (2.7) arises from the image $\eta_L(v_i)$ of $K(n)$.

There is a Hurewicz map

$$\pi_*(E \wedge E) \to K(n)_*(E \wedge E) \cong K(n)_* E \otimes_{K(n)_*} K(n)_* E$$

where the isomorphism holds since $K(n)$ satisfies the Künneth theorem. The image of the class t_j from (2.6) under this map can be found by looking at the corresponding Hurewicz map for BP,

$$\pi_*(BP \wedge BP) \to BP_*(BP \wedge BP) \cong BP_*BP \otimes_{BP_*} BP_*BP.$$

Here, the isomorphism holds since BP_*BP is flat over BP_*. Standard formulae for the Hopf algebroid diagonal and antipode [Rav86, A2.1.27] then in principle calculate the image of t_j for all j. In the case $j = 1$, we get the image class to be

$$t_1 \otimes 1 + 1 \otimes t_1.$$

Thus,

(2.9)
$$t_1 \in \pi_*(E \wedge E) \mapsto t_1 \otimes 1 + 1 \otimes t_1 \in K(n)_* E \otimes_{K(n)_*} K(n)_* E.$$

The Hochschild complex of $K(n)_*E$ over $K(n)_*$ has

$$K(n)_*E \otimes_{K(n)_*} K(n)_*E$$

as the simplicial degree 1 term. As an element here, $t_1 \otimes 1$ is a degenerate class (it is $s_0(t_1)$ where s_0 is the zeroth degeneracy map of the Hochschild complex and $t_1 \in K(n)_*E$ is a simplicial degree 0 element). Therefore, as classes in the Hochschild complex of $K(n)_*E$ over $K(n)_*$

$$(2.10) \qquad t_1 \otimes 1 + 1 \otimes t_1 = 1 \otimes t_1.$$

$1 \otimes t_1$ is in the kernel of difference of face maps $d_0 - d_1$ going down from simplicial level 1 to simplicial level 0 in the Hochschild complex. Therefore, it is an element of the first Hochschild homology:

$$(2.11) \qquad dt_1 \in HH_{(1,2p-2)}^{K(n)_*} K(n)_*E$$

denotes this homological class which is the image of $t_1 \in \pi_{2p-2}(E \wedge E)$ as in (2.9).

The u, u_i classes of E_{m*} are related to v_i classes of BP_* and $E(m)_*$ via the second map in (2.3). At the level of homotopy groups, $E(m) \to E_m$ sends

$$(2.12) \qquad \begin{aligned} v_i &\mapsto u_i \cdot u^{p^i-1}, \text{ for } i < m \\ v_m &\mapsto u^{p^m-1}. \end{aligned}$$

From the relations given by Hazewinkel formula [Rav92, B.5, Pg 167-171] we have the following equation relating u_1, v_1, t_1 and u

$$(2.13) \qquad u_1 = u^{1-p}v_1 = u^{1-p}pt_1.$$

The Hurewicz map takes the classes $u, u_i \in E_{m*}$ to classes that we denote by the same symbol

$$(2.14) \qquad u, u_i \in K(n)_*E_m.$$

Under the right Hurewicz map

$$K(n)_*E_m \to K(n)_*(E_m \wedge E_m) \cong K(n)_*E_m \otimes_{K(n)_*} K(n)_*E_m$$

we have

$$(2.15) \qquad \begin{aligned} u_i &\mapsto 1 \otimes u_i \\ u &\mapsto 1 \otimes u. \end{aligned}$$

As in the discussion above (2.11), these result in elements of first Hochschild homology, denoted

$$(2.16) \qquad du_i \in HH_{(1,0)}^{K(n)_*} K(n)_*E_m \text{ and } du \in HH_{(1,2)}^{K(n)_*} K(n)_*E_m.$$

2.3. **The Bökstedt spectral sequence.** The Bökstedt spectral sequence is an essential tool that computes the homology of $THH(R)$ from the Hochschild homology of the homology of a ring spectrum R for any homology theory that satisfies the Künneth theorem [EKMM97, Theorem IX.2.9]. Examples of such homology theories include ordinary homology theory with field coefficients and all Morava K-theories $K(i)$.

For the homology theory $K(i)$ and the ring spectrum E_2, the associated Bökstedt spectral sequence has the form

$$(2.17) \qquad \mathrm{E}^2_{r,s} = (HH^{K(i)*}_r(K(i)_* E_2))_s \implies K(i)_{r+s} THH(E_2).$$

Here $K(i)_*$ and $K(i)_* E_2$ are graded rings, and so the Hochschild homology has an internal degree (s in the display) in addition to a homological degree (r in the display). To construct the spectral sequence, we use the cyclic bar construction of THH and the fact that smash product commutes with geometric realization

$$K(i) \wedge THH(E_2) = K(i) \wedge |E_2^{\wedge(\bullet+1)}| \cong |K(i) \wedge E_2^{\wedge(\bullet+1)}|.$$

The spectral sequence arises from the simplicial filtration, using the Künneth isomorphism

$$(2.18) \qquad \pi_*(K(i) \wedge (E_2 \wedge \cdots \wedge E_2)) \cong (K(i)_* E_2) \otimes_{K(i)_*} \cdots \otimes_{K(i)_*} (K(i)_* E_2).$$

This turns the simplicial filtration of $K(i)_* THH(E_2)$ into the Hochschild complex of $K(i)_* E_2$ relative to $K(i)_*$.

(2.17) is a multiplicative spectral sequence of commutative $K(i)_* E_2$-algebras [AR05, Proposition 4.3]. The differentials in E^2-page go from

$$(2.19) \qquad\qquad\qquad\qquad \mathrm{E}^2_{r,s} \to \mathrm{E}^2_{r-2,s+1}$$

and satisfy Leibniz rule.

2.4. **Hochschild homology of H-smooth rings.** For our computations, we need to know the relative Hochschild homologies of certain rings that are not étale or smooth over the base ring but do satisfy the condition of H-smoothness of Larsen–Lindenstrauss [LL01]. Below we summarize some of the results from [LL01] and use them to conclude H-smoothness of the ring pairs we are interested in.

Definition 2.20. [LL01, Definition 1.1] Let k be a commutative ring and A a commutative k-algebra. Then the *pair (k, A) is H-smooth* (or A is H-smooth over k) if $HH_*(A)$ is flat over k and the Hochschild-Kostant-Rosenberg map [HKR62]

$$\lambda_n^{A/k} : \Lambda^n HH_1^k(A) \to HH_n^k(A)$$

is an isomorphism for all n.

For an H-smooth pair (k, A), we have by definition, the calculation of $HH_*^k(A)$ in terms of $HH_1^k(A)$. The main results of [LL01] that we need allow for identification of some useful H-smooth pairs. The following three results are the ones we need.

Proposition 2.21. [LL01, Proposition 1.3] *H-smoothness respects arbitrary base change.*

Proposition 2.22. [LL01, Corollary 1.6] *For k a commutative ring, let A be a commutative k-algebra and B a commutative A-algebra. Then if A is H-smooth over k and B is H-smooth over A, B is H-smooth over k.*

Theorem 2.23. [LL01, Theorem 1.7] *If A is an excellent ring as in [Sta25], $\mathfrak{m}$ is a maximal ideal in A, and $\hat{A}_{\mathfrak{m}}$ is the completion of A at $\mathfrak{m}$, then $\hat{A}_{\mathfrak{m}}$ is H-smooth over A.*

Remark 2.24. While the definition of an excellent ring is complicated, in practice most rings are excellent. Some examples of excellent rings include fields, Noetherian complete local rings, $\mathbb{Z}$, Dedekind domains with fraction field of characteristic zero, and finite type ring extensions of any of the above. In particular, $\mathbb{Q}$, $\mathbb{Q}_p$, $\mathbb{Z}_p[t]$, $\mathbb{F}_p$ and $\mathbb{F}_p[t]$ are excellent rings.

Observation 2.25.
(1) From Remark 2.24, Theorem 2.23 and the fact that smooth extensions are by definition H-smooth, we conclude that the following pairs are H-smooth: $(\mathbb{Q}, \mathbb{Q}_p)$, $(\mathbb{Q}_p, \mathbb{Q}_p[t])$, $(\mathbb{Z}_p[t], \mathbb{Z}_p[[t]])$, $(\mathbb{F}_p, \mathbb{F}_p[t])$ and $(\mathbb{F}_p[t], \mathbb{F}_p[[t]])$.
(2) Doing a base change on pair $(\mathbb{Z}_p[t], \mathbb{Z}_p[[t]])$ by tensoring with $\mathbb{Q}$ and using Proposition 2.21 we get $(\mathbb{Q}_p[t], \mathbb{Z}_p[[t]] \otimes \mathbb{Q})$ is H-smooth.
(3) Applying Proposition 2.22 on pairs $(\mathbb{Q}, \mathbb{Q}_p)$, $(\mathbb{Q}_p, \mathbb{Q}_p[t])$, $(\mathbb{Q}_p[t], \mathbb{Z}_p[[t]] \otimes \mathbb{Q})$ we observe $(\mathbb{Q}, \mathbb{Z}_p[[t]] \otimes \mathbb{Q})$ is H-smooth. Similarly from H-smoothness of $(\mathbb{F}_p, \mathbb{F}_p[t])$ and $(\mathbb{F}_p[t], \mathbb{F}_p[[t]])$, we conclude $(\mathbb{F}_p, \mathbb{F}_p[[t]])$ is H-smooth.

The final result below tells us the Hochschild homology of power series rings in one variable.

Proposition 2.26. [LL01, Proposition 2.5, Corollary 2.3] *Let k be a field, then there is a non-canonical isomorphism*

$$HH_*^k k[[t]] \cong HH_*^k k[t] \otimes_{k[t]} HH_*^{k[t]} k[[t]] \cong HH_*^k k[t] \otimes_{k[t]} (k[[t]] \oplus \Lambda_{k((t))}^{\geq 1} V_k)$$

where $V_k := HH_1^{k(t)} k((t))$ is an infinite dimensional vector space over $k((t))$. In the bracketed term, $k[[t]]$ is in homological degree 0 and V_k is in homological degree 1.

The latter isomorphism in Proposition 2.26 holds because from [LL01, Corollary 2.3]

$$HH_i^{k[t]} k[[t]] \cong HH_i^{k(t)} k((t))$$

for $i \geq 1$ and thus

(2.27) $$\Lambda_{k[[t]]}^{\geq 1} HH_1^{k[t]} k[[t]] \cong \Lambda_{k((t))}^{\geq 1} HH_1^{k(t)} k((t)).$$

We will be applying this result mostly in the specific case of $k = \mathbb{F}_p$.

2.5. **Hochschild homology calculations for $E(n)$.** Our later computations rely on some of the calculations of Ausoni–Richter [AR20] on the Hochschild homology of the Morava K-theory of Johnson-Wilson spectra, $HH_*^{K(i)_*}(K(i)_* E(n))$. While a lot of [AR20] requires the hypothesis that the Johnson-Wilson theories are commutative, these computations do not need this hypotheses and so hold unconditionally. We

state the results here in the slightly reformulated form using discussions in the proof of [AR20, lemma 5.1].

Theorem 2.28. [AR20, Proposition 2.3, Theorem 3.4, Page 380 line 3] *We have the following isomorphisms of $K(0)_*E(2)$, $K(1)_*E(2)$ and $K(2)_*E(2)$-algebras, respectively*

$$HH_{*,*}^{K(0)*}K(0)_*E(2) \cong K(0)_*E(2) \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}(dt_1, dt_2),$$

$$HH_{*,*}^{K(1)*}K(1)_*E(2) \cong K(1)_*E(2) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(dt_1),$$

$$HH_{*,*}^{K(2)*}K(2)_*E(2) \cong K(2)_*E(2).$$

Here the classes dt_1 in

$$HH_{(1,2p-2)}^{K(0)*}K(0)_*E(2) \text{ and } HH_{(1,2p-2)}^{K(1)*}K(1)_*E(2)$$

are induced from the class $t_1 \in \pi_{2p-1}(E(2) \wedge E(2))$ and the class

$$dt_2 \in HH_{(1,2p^2-2)}^{K(0)*}K(0)_*E(2)$$

is induced from $t_2 \in \pi_{2p^2-1}(E(2) \wedge E(2))$. Also, for $i > 2$, $K(i)_*E(2)$ is trivial.

3. $K(i)$ HOMOLOGY GROUPS OF THH(E_2)

In this section, we calculate $K(i)_*(THH(E_2))$ for all i (Theorem 3.9, Theorem 3.28, Theorem 3.29 and Lemma 3.30). First, we calculate the Hochschild homologies $HH^{K(i)*}(K(i)_*E_2)$ of $K(i)_*E_2$ relative to $K(i)_*$ using the H-smoothness results and Ausoni–Richter [AR20] calculations reviewed above. We then use the Bökstedt spectral sequence (2.17) to compute $K(i)_*THH(E_2)$.

3.1. **The case $i = 0$:** $K(0)_*THH(E_2)$. Note that $K(0)_*E_2 \cong \pi_*(E_2)_{\mathbb{Q}}$ and

$$(3.1) \qquad\qquad HH_*^{K(0)*}(K(0)_*E_2) \cong HH_*^{\mathbb{Q}}(\pi_*(E_2)_{\mathbb{Q}}).$$

To compute (3.1), we need the homotopy groups of rationalized spectrum E_2. These are given by tensoring the homotopy groups of E_2 with $\mathbb{Q}$. We get

$$\pi_*(E_2)_{\mathbb{Q}} \cong (\mathbb{Z}_p[\mu_{p^2-1}][u, u^{-1}][[u_1]]) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$\cong (\mathbb{Z}[\mu_{p^2-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[u, u^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}_p[[u_1]]) \otimes \mathbb{Q}$$

$$\cong (\mathbb{Z}_p[[u_1]] \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[\mu_{p^2-1}] \otimes_{\mathbb{Q}} \mathbb{Q}[u, u^{-1}].$$

Here μ_{p^2-1} denotes the group of $(p^2 - 1)$ roots of unity. For brevity, we denote

$$\mathbb{Z}[[u_1]]_{\mathbb{Q}} := \mathbb{Z}_p[[u_1]] \otimes_{\mathbb{Z}} \mathbb{Q}$$

in the rest of the paper. Next, taking the Hochschild homology with respect to $\mathbb{Q}$, we get

$$(3.2) \quad HH_*^{\mathbb{Q}}(\pi_*(E_2)_{\mathbb{Q}}) \cong HH_*^{\mathbb{Q}}(\mathbb{Z}[[u_1]]_{\mathbb{Q}}) \otimes_{\mathbb{Q}} HH_*^{\mathbb{Q}}\mathbb{Q}[\mu_{p^2-1}] \otimes_{\mathbb{Q}} HH_*^{\mathbb{Q}}\mathbb{Q}[u, u^{-1}].$$

We now identify each of these Hochshild homology terms.

Lemma 3.3. $HH_*^{\mathbb{Q}}(\mathbb{Q}[u, u^{-1}]) \cong \mathbb{Q}[u, u^{-1}] \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}(du).$

Proof. This is due to étaleness of $\mathbb{Q}[u, u^{-1}]$ over $\mathbb{Q}[u]$ which from [Lod98, E.1.1.8] implies

$$(3.4) \qquad HH_*^{\mathbb{Q}}(\mathbb{Q}[u, u^{-1}]) \cong \mathbb{Q}[u, u^{-1}] \otimes_{\mathbb{Q}[u]} HH_*^{\mathbb{Q}}\mathbb{Q}[u],$$

and to smoothness of $\mathbb{Q}[u]$ over $\mathbb{Q}$ which from the theorem of Hochschild-Kostant-Rosenberg [HKR62], [KN18, Example 2.5] implies

$$(3.5) \qquad HH_*^{\mathbb{Q}}\mathbb{Q}[u] \cong \Lambda_{\mathbb{Q}[u]}(du) \cong \mathbb{Q}[u] \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}(du). \qquad \square$$

Lemma 3.6. $HH_*^{\mathbb{Q}}(\mathbb{Q}[\mu_{p^2-1}]) \cong \mathbb{Q}[\mu_{p^2-1}].$

Proof. This is from étaleness of $\mathbb{Q}[\mu_{p^2-1}]$ over $\mathbb{Q}$ and from [Lod98, E.1.1.8]. We have

$$HH_*^{\mathbb{Q}}(\mathbb{Q}[\mu_{p^2-1}]) \cong \mathbb{Q}[\mu_{p^2-1}] \otimes_{\mathbb{Q}} HH_*^{\mathbb{Q}}\mathbb{Q}$$
$$\cong \mathbb{Q}[\mu_{p^2-1}]. \qquad \square$$

Lemma 3.7. $HH_*^{\mathbb{Q}}(\mathbb{Z}[[u_1]]_{\mathbb{Q}}) \cong \Lambda_{\mathbb{Z}[[u_1]]_{\mathbb{Q}}} HH_1^{\mathbb{Q}}(\mathbb{Z}[[u_1]]_{\mathbb{Q}}).$

Proof. This is from Observation 2.25(3) and Definition 2.20. $\qquad \square$

Using Lemma 3.6, Lemma 3.3 and Lemma 3.7, we rewrite (3.2) as follows.

Proposition 3.8.

$$HH_*^{K(0)*}(K(0)_* E_2) \cong K(0)_* E_2 \otimes_{\mathbb{Z}[[u_1]]_{\mathbb{Q}}} \Lambda_{\mathbb{Z}[[u_1]]_{\mathbb{Q}}} HH_1^{\mathbb{Q}}(\mathbb{Z}[[u_1]]_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}(du).$$

Here, elements in $K(0)_* E_2$ have homological degree 0, du has homological degree 1 and internal degree 2 (2.16) and elements in $HH_1^{\mathbb{Q}}(\mathbb{Z}[[u_1]]_{\mathbb{Q}})$ have homological degree 1 and internal degree 0.

Proof.

$$HH_*^{K(0)*}(K(0)_* E_2)$$
$$\cong \Lambda_{\mathbb{Z}[[u_1]]_{\mathbb{Q}}} HH_1^{\mathbb{Q}}(\mathbb{Z}[[u_1]]_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{Q}[\mu_{p^2-1}] \otimes_{\mathbb{Q}} \mathbb{Q}[u, u^{-1}] \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}(du)$$
$$\cong (\mathbb{Z}[[u_1]]_{\mathbb{Q}} \otimes \mathbb{Q}[\mu_{p^2-1}] \otimes \mathbb{Q}[u, u^{-1}]) \otimes_{\mathbb{Z}[[u_1]]_{\mathbb{Q}}} \Lambda_{\mathbb{Z}[[u_1]]_{\mathbb{Q}}} HH_1^{\mathbb{Q}}(\mathbb{Z}[[u_1]]_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}(du)$$
$$\cong K(0)_* E_2 \otimes_{\mathbb{Z}[[u_1]]_{\mathbb{Q}}} \Lambda_{\mathbb{Z}[[u_1]]_{\mathbb{Q}}} HH_1^{\mathbb{Q}}(\mathbb{Z}[[u_1]]_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}(du).$$

$\qquad \square$

This then leads to the computation of $K(0)_* THH(E_2)$.

Theorem 3.9. *We have the following isomorphism of $K(0)_* E_2$-algebras*

$$K(0)_* THH(E_2) \cong K(0)_* E_2 \otimes_{\mathbb{Z}[[u_1]]_{\mathbb{Q}}} \Lambda_{\mathbb{Z}[[u_1]]_{\mathbb{Q}}} HH_1^{\mathbb{Q}}(\mathbb{Z}[[u_1]]_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}} du.$$

Proof. Proposition 3.8 gives us the E^2 page of the Bökstedt spectral sequence (2.17). The multiplicative generators lie in homological degree (column) 1. From the discussion in (2.19), we conclude there are no non-zero differentials for degree reasons and the spectral sequence collapses on the E^2 page.

Since the E$^\infty$ page is free as a graded commutative $K(0)_*E_2$ algebra, there are no possible additive or multiplicative extensions. We get an isomorphism of $K(0)_*E_2$-algebras

$$K(0)_*THH(E_2) \cong K(0)_*E_2 \otimes_{\mathbb{Z}[[u_1]]_\mathbb{Q}} \Lambda_{\mathbb{Z}[[u_1]]_\mathbb{Q}} HH_1^\mathbb{Q}(\mathbb{Z}[[u_1]]_\mathbb{Q}) \otimes_\mathbb{Q} \Lambda_\mathbb{Q} du. \qquad \square$$

3.2. **The case** $i = 1$ **and** $i = 2$: $K(1)_*THH(E_2)$ **and** $K(2)_*THH(E_2)$. The calculations for $i = 1$ and $i = 2$ use similar techniques so we present them together.

Since

$$K(i)_*E_2 \cong E_{2*}K(i)$$

and from Landweber exactness of E_2

$$E_{2*}K(i) \cong E_{2*} \otimes_{BP_*} BP_*(K(i))$$

we have

$$K(i)_*E_2 \cong K(i)_*(BP) \otimes_{BP_*} (E_2)_*.$$

Using the fact that E_{2*} is flat over $E(2)_*$ and the fact that $p = 0$ in $K(i)_*$, the second map in (2.3) induces isomorphisms

$$(3.10) \qquad K(i)_*E_2 \cong K(i)_*E(2) \otimes_{E(2)_*} E_{2*} \cong K(i)_*E(2) \otimes_{E(2)_*/p} E_{2*}/p$$

for $i = 1, 2$. This gives

$$(3.11) \qquad HH_*^{K(i)_*}(K(i)_*E_2) \cong HH_*^{K(i)_*}(K(i)_*E(2) \otimes_{E(2)_*/p} E_{2*}/p).$$

Observation 3.12. The map $E_{2*}/p \to E(2)_*/p$ induced from the second map in (2.3) is flat and at the π_*-level, does three things on the usual presentations

(1) adding a $(p^2 - 1)$th root of unity, μ_{p^2-1},
(2) adding u, the $(p^2 - 1)$th root of v_2 (2.12), and using the generators u, u_1,
(3) completing at the ideal (u_1).

Next, we need the compuation of the right hand side of (3.11) in terms of the Hochschild homology of the $\mathbb{F}_p$-algebras $E(2)_*/p$ and E_{2*}/p. The following lemma is proved in several steps.

Lemma 3.13. *For* $i = 1, 2$

$$HH_*^{K(i)_*}(K(i)_*E(2) \otimes_{E(2)_*/p} E_{2*}/p) \cong HH_*^{K(i)_*}(K(i)_*E(2)) \otimes_{HH_*^{\mathbb{F}_p}(E(2)_*/p)} HH_*^{\mathbb{F}_p}(E_{2*}/p).$$

The main step we need is the following theorem. Theorem 3.19 below proves that the hypotheses of Theorem 3.14 apply in our case.

Theorem 3.14. *Let* $k \to R \to S \to A$ *and* $k \to S \to B$ *be maps of commutative rings. Regard* $A \otimes_S B$ *as an* R *algebra via the map* $R \cong R \otimes_S S \to A \otimes_S B$. *If* B *is flat over* S *and* $HH_*^k(B)$ *is flat over* $HH_*^k(S)$, *then*

$$HH_*^R(A \otimes_S B) \cong HH_*^R(A) \otimes_{HH_*^k(S)} HH_*^k(B).$$

Proof. We use the notation of cyclic bar complex as mentioned in Subsection 1.3,(5). Note that as simplicial abelian groups

$$(3.15) \qquad (N^{cy}_{\otimes_R})_\bullet(A \otimes_S B) \cong (N^{cy}_{\otimes_R})_\bullet(A) \otimes_{(N^{cy}_{\otimes_k})_\bullet(S)} (N^{cy}_{\otimes_k})_\bullet(B).$$

This is a quick check, one sees that the natural map at each simplicial level

$$(A \otimes_S B) \otimes_R \ldots \otimes_R (A \otimes_S B) \to (A \otimes_R \ldots \otimes_R A) \otimes_{(S \otimes_k \ldots \otimes_k S)} (B \otimes_k \ldots \otimes_k B)$$

is an isomorphism and that these maps respect simplicial structure.

Since B is flat over S, the right hand side in (3.15) represents the derived tensor product in simplicial modules

$$(N^{cy}_{\otimes_R})_\bullet(A) \otimes^{\mathbb{L}}_{(N^{cy}_{\otimes_k})_\bullet(S)} (N^{cy}_{\otimes_k})_\bullet(B)$$

and hence also the derived tensor product in dg-modules

$$(3.16) \qquad N^{cy}_{\otimes_R}(A) \otimes^{\mathbb{L}}_{N^{cy}_{\otimes_k}(S)} N^{cy}_{\otimes_k}(B).$$

Using this formula and taking homotopy groups on both sides of (3.15), we see

$$(3.17) \qquad HH^R_*(A \otimes_S B) \cong H_*(N^{cy}_{\otimes_R}(A) \otimes^{\mathbb{L}}_{N^{cy}_{\otimes_k}(S)} N^{cy}_{\otimes_k}(B)).$$

Using Lemma 3.18 below with M, N and C being $N^{cy}_{\otimes_R}(A)$, $N^{cy}_{\otimes_k}(B)$ and $N^{cy}_{\otimes_k}(S)$ respectively, we have a spectral sequence

$$\mathrm{Tor}^{HH^k_* S}_{*,*}(HH^R_* A, HH^k_* B) \Longrightarrow H_*(N^{cy}_{\otimes_R}(A) \otimes^{\mathbb{L}}_{N^{cy}_{\otimes_k}(S)} N^{cy}_{\otimes_k}(B)).$$

Since $HH^k_*(B)$ is flat over $HH^k_*(S)$, all the higher Tor-terms vanish and the spectral sequence collapses, giving us

$$HH^R_* A \otimes_{HH^k_* S} HH^k_* B \cong H_*(N^{cy}_{\otimes_R}(A) \otimes_{N^{cy}_{\otimes_k}(S)} N^{cy}_{\otimes_k}(B)). \qquad \square$$

We need an algebraic analogue of the Eilenberg-Moore spectral sequence as in [EKMM97, IV.6] which is a consequence of Tor spectral sequence. We are using the same proof techniques as in [EKMM97, IV.4.1] and [LM06, Section 6]. We expect this spectral sequence is known to experts but we could not find it in literature.

Lemma 3.18. *Let C be a simplicial or dg-ring, M a right C-module, and N a left C-module. Then there exists an algebraic Eilenberg-Moore spectral sequence*

$$Tor^{H_* C}_{p,q}(H_* M, H_* N) \Longrightarrow H_{p+q}(M \otimes^{\mathbb{L}}_C N).$$

Proof. Let Y be a cofibrant approximation of N so that $M \otimes_C Y$ represents the derived tensor product $M \otimes^{\mathbb{L}}_C N$. Choose a free $H_* C$ resolution of $H_* Y$

$$\cdots \to F_n \xrightarrow{\overline{d_n}} \cdots \to F_1 \xrightarrow{\overline{d_1}} F_0 \xrightarrow{\overline{d_0}} H_* Y.$$

We can realize $\overline{d_0}$ as H_* of map of C-modules

$$\bigoplus_{\text{gens of } F_0} \sum^{n_{\alpha_0}} C \xrightarrow{d_0} Y$$

where F_0 is graded, n_α is the internal degree, and the direct sum is over a chosen set of generators of F_0. Let

$$Y_{-1} = Y \text{ and } Y_0 = C_{d_0},$$

the homotopy cofiber of the map d_0. From surjectivity of $\overline{d_0}$ on homology, we get

$$H_*(Y_0) \cong \sum ker(\overline{d_0}),$$

the suspension of the kernel module. Next we realize $\overline{d_1}$, similarly, as a map of C-modules

$$\bigoplus_{\text{gens of } F_1} \sum^{n_{\alpha_1}} C \xrightarrow{d_1} Y_0$$

which induces surjectivity after taking homology since $\overline{d_1} \circ \overline{d_0} = 0$. Let $Y_1 = C_{d_1}$. Then

$$H_*(Y_1) \subset \sum ker(\overline{d_1}).$$

Iterating this process gives a filtration

$$Y_{-1} \to Y_0 \to Y_1 \to \cdots$$

Let $Y_\infty = \text{hocolim } Y_i$, then $Y_\infty \simeq 0$. The filtration on $X \otimes_N Y_\infty \simeq 0$ gives a spectral sequence converging to

$$H_*(X \otimes Y_{-1}) \cong M \otimes_C^{\mathbb{L}} N$$

where the associated graded is $X \otimes (\bigoplus \sum^{n_\alpha} N)$, the E_1-term is $H_*X \otimes_{H_*N} F_i$ and the E_2-term is the required Tor term. $\square$

The next statement says that Theorem 3.14 can be applied on

$$HH_*^{K(i)*}(K(i)_*E(2) \otimes_{E(2)_*/p} E_{2*}/p).$$

Theorem 3.19. $HH_*^{\mathbb{F}_p}(E_{2*}/p)$ *is flat over* $HH_*^{\mathbb{F}_p}(E(2)_*/p)$. *Moreover,* $HH_*^{\mathbb{F}_p}(E_{2*}/p)$ *is isomorphic to*

$$\left(HH_*^{\mathbb{F}_p}(E(2)_*/p) \otimes_{\mathbb{F}_p[v_1][v_2,v_2^{-1}]} (\mathbb{F}_p[\mu_{p^2-1}][u_1][u,u^{-1}])\right) \otimes_{\mathbb{F}_p[u_1]} \left(\Lambda_{\mathbb{F}_p[[u_1]]} HH_1^{\mathbb{F}_p[u_1]} \mathbb{F}_p[[u_1]]\right).$$

Proof. Note

$$(3.20) \qquad E(2)_*/p \cong \mathbb{F}_p[v_1][v_2,v_2^{-1}], \text{ and } E_{2*}/p \cong \mathbb{F}_p[\mu_{p^2-1}][[u_1]][u,u^{-1}].$$

From Observation 3.12 we have the following sequence of labelled maps

$$\mathbb{F}_p[v_1][v_2,v_2^{-1}] \xrightarrow{1} \mathbb{F}_p[\mu_{p^2-1}][v_1][v_2,v_2^{-1}] \xrightarrow{2} \mathbb{F}_p[\mu_{p^2-1}][u_1][u,u^{-1}] \xrightarrow{3} \mathbb{F}_p[\mu_{p^2-1}][[u_1]][u,u^{-1}].$$

Map 1 is unramified and flat, hence étale. From [Lod98, E.1.1.8], we have

$$HH_*^{\mathbb{F}_p}(\mathbb{F}_p[\mu_{p^2-1}][v_1][v_2,v_2^{-1}])$$

$$(3.21) \qquad \cong HH_*^{\mathbb{F}_p}(\mathbb{F}_p[v_1][v_2,v_2^{-1}]) \otimes_{\mathbb{F}_p[v_1][v_2,v_2^{-1}]} \mathbb{F}_p[\mu_{p^2-1}][v_1][v_2,v_2^{-1}]$$

$$\cong HH_*^{\mathbb{F}_p}(\mathbb{F}_p[v_1][v_2,v_2^{-1}]) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\mu_{p^2-1}].$$

We conclude

$$(3.22) \qquad HH_*^{\mathbb{F}_p}(\mathbb{F}_p[v_1][v_2, v_2^{-1}]) \to HH_*^{\mathbb{F}_p}(\mathbb{F}_p[\mu_{p^2-1}][v_1][v_2, v_2^{-1}])$$

is flat.

From (2.12), map 2 sends

$$v_1 \mapsto u_1 u^{p^2-1} \text{ and } v_2 \mapsto u^{p^2-1}.$$

In other words, its attaching a (p^2-1)-root of a unit and replacing a variable with a unit times that variable. Hence map 2 is étale and gives

$$
\begin{aligned}
& HH_*^{\mathbb{F}_p}(\mathbb{F}_p[\mu_{p^2-1}][u_1][u, u^{-1}]) \\
(3.23) \quad & \cong HH_*^{\mathbb{F}_p}(\mathbb{F}_p[\mu_{p^2-1}][v_1][v_2, v_2^{-1}]) \otimes_{\mathbb{F}_p[\mu_{p^2-1}][v_1][v_2, v_2^{-1}]} \mathbb{F}_p[\mu_{p^2-1}][u_1][u, u^{-1}] \\
& \cong HH_*^{\mathbb{F}_p}(\mathbb{F}_p[\mu_{p^2-1}][v_1][v_2, v_2^{-1}]) \otimes_{\mathbb{F}_p[\mu_{p^2-1}][v_1][v_2, v_2^{-1}]} \mathbb{F}_p[\mu_{p^2-1}][v_1][u, u^{-1}].
\end{aligned}
$$

Since u is a p^2-1 root of the image of v_2, this a free module extension of

$$(3.24) \qquad HH_*^{\mathbb{F}_p}(\mathbb{F}_p[\mu_{p^2-1}][v_1][v_2, v_2^{-1}])$$

and hence flat.

Map 3 is the completion at ideal (u_1) and hence flat. Taking Hochschild homology relative to $\mathbb{F}_p$ on both sides of this map, we have
$$(3.25)$$
$$HH_*^{\mathbb{F}_p}(\mathbb{F}_p[u_1]) \otimes HH_*^{\mathbb{F}_p}(\mathbb{F}_p[\mu_{p^2-1}][u, u^{-1}]) \to HH_*^{\mathbb{F}_p}(\mathbb{F}_p[[u_1]]) \otimes HH_*^{\mathbb{F}_p}(\mathbb{F}_p[\mu_{p^2-1}][u, u^{-1}])$$

where the tensors are over $\mathbb{F}_p$. Applying Proposition 2.26

$$(3.26) \qquad HH_*^{\mathbb{F}_p}\mathbb{F}_p[[u_1]] \cong (HH_*^{\mathbb{F}_p}\mathbb{F}_p[u_1]) \otimes_{\mathbb{F}_p[u_1]} (\mathbb{F}_p[[u_1]] \oplus \Lambda_{\mathbb{F}_p[[u_1]]}^{\geq 1} V_{\mathbb{F}_p})$$

where $V_{\mathbb{F}_p}$ is a vector space over $\mathbb{F}_p((u_1))$ and $\mathbb{F}_p((u_1))$ is a localization of $\mathbb{F}_p[[u_1]]$, we conclude

$$(3.27) \qquad HH_*^{\mathbb{F}_p}\mathbb{F}_p[u_1] \to (HH_*^{\mathbb{F}_p}\mathbb{F}_p[u_1]) \otimes_{\mathbb{F}_p[u_1]} (\mathbb{F}_p[[u_1]] \oplus \Lambda_{\mathbb{F}_p[[u_1]]}^{\geq 1} V_{\mathbb{F}_p})$$

is a map into an infinite vector space of a localization and thus flat.

From (3.22), (3.24) and (3.27), $HH_*^{\mathbb{F}_p}(E_{2*}/p)$ is flat over $HH_*^{\mathbb{F}_p}(E(2)_*/p)$. In fact,

$$
\begin{aligned}
& HH_*^{\mathbb{F}_p}\mathbb{F}_p[\mu_{p^2-1}][[u_1]][u, u^{-1}] \\
& \cong HH_*^{\mathbb{F}_p}\mathbb{F}_p[\mu_{p^2-1}][u, u^{-1}] \otimes_{\mathbb{F}_p} HH_*^{\mathbb{F}_p}\mathbb{F}_p[[u_1]] \\
& \overset{(3.26)}{\cong} HH_*^{\mathbb{F}_p}\mathbb{F}_p[\mu_{p^2-1}][u_1][u, u^{-1}] \otimes_{F_p[u_1]} \Lambda_{\mathbb{F}_p[[u_1]]} HH_1^{\mathbb{F}_p[u_1]}\mathbb{F}_p[[u_1]]
\end{aligned}
$$

Applying (3.21) and (3.23) on the last term, we have the result. $\qquad \square$

From Theorem 3.19 and Theorem 3.14, we see Lemma 3.13 holds.

Finally, we compute $K(1)$ and $K(2)$ homologies of $THH(E_2)$ in the results below.

Theorem 3.28. *We have the following isomorphism of $K(1)_* E_2$-algebras*

$$K(1)_* THH(E_2) \cong K(1)_* E_2 \otimes_{\mathbb{F}_p[[u_1]]} \Lambda_{\mathbb{F}_p[[u_1]]} HH_1^{\mathbb{F}_p[u_1]}\mathbb{F}_p[[u_1]] \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p} dt_1.$$

Proof. From (3.11), Lemma 3.13 and Theorem 3.19, $HH_*^{K(1)*}(K(1)_*E_2)$ is isomorphic to

$$(HH_*^{K(1)*}K(1)_*E(2)\otimes_{\mathbb{F}_p[v_1][v_2,v_2^{-1}]}\mathbb{F}_p[\mu_{p^2-1}][u_1][u,u^{-1}])\otimes_{F_p[u_1]}\Lambda_{\mathbb{F}_p[[u_1]]}HH_1^{\mathbb{F}_p[u_1]}\mathbb{F}_p[[u_1]],$$

which from Theorem 2.28 is isomorphic to

$$((K(1)_*E(2)\otimes\Lambda dt_1)\otimes_{\mathbb{F}_p[v_1][v_2,v_2^{-1}]}\mathbb{F}_p[\mu_{p^2-1}][u_1][u,u^{-1}])\otimes_{F_p[u_1]}\Lambda_{\mathbb{F}_p[[u_1]]}HH_1^{\mathbb{F}_p[u_1]}\mathbb{F}_p[[u_1]].$$

To use (3.10), we rewrite this as

$$((K(1)_*E(2)\otimes\Lambda dt_1)\otimes_{\mathbb{F}_p[v_1][v_2,v_2^{-1}]}\mathbb{F}_p[\mu_{p^2-1}][[u_1]][u,u^{-1}])\otimes_{F_p[[u_1]]}\Lambda_{\mathbb{F}_p[[u_1]]}HH_1^{\mathbb{F}_p[u_1]}\mathbb{F}_p[[u_1]]$$

and then use the equation to get

$$HH_*^{K(1)*}(K(1)_*E_2)\cong K(1)_*E_2\otimes_{F_p[[u_1]]}\Lambda_{\mathbb{F}_p[[u_1]]}HH_1^{\mathbb{F}_p[u_1]}\mathbb{F}_p[[u_1]]\otimes_{\mathbb{F}_p}\Lambda_{\mathbb{F}_p}dt_1.$$

Here, elements of $K(1)_*E_2$ have homological degree 0, dt_1 is as in (2.11) and has homological degree 1 and internal degree $2p^2-2$ and the elements of $HH_1^{\mathbb{F}_p[u_1]}\mathbb{F}_p[[u_1]]$ have homological degree 1 and internal degree 0.

As in the proof of Theorem 3.9, the multiplicative generators live in homological degree (column) 1 and so the Bökstedt spectral sequence collapses on E^2-page due to degree reasons. This is again a free graded commutative $K(1)_*E_2$ algebra on the first column, so there are no possible additive or multiplicative extensions and we have an isomorphism of $K(1)_*E_2$-algebras as in the statement of the result.   □

Theorem 3.29. *The unit map* (2.1) *is a $K(2)$-equivalence, i.e., it induces an isomorphism of $K(2)_*E_2$-algebras*

$$K(2)_*E_2\cong K(2)_*THH(E_2).$$

Proof. As above, we use (3.11), Lemma 3.13 and Theorem 3.19, to get $HH_*^{K(2)*}(K(2)_*E_2)$ isomorphic to

$$(K(2)_*E(2)\otimes_{\mathbb{F}_p[v_1][v_2,v_2^{-1}]}\mathbb{F}_p[\mu_{p^2-1}][u_1][u,u^{-1}])\otimes_{\mathbb{F}_p[u_1]}\Lambda_{\mathbb{F}_p[[u_1]]}HH_1^{\mathbb{F}_p[u_1]}\mathbb{F}_p[[u_1]].$$

Extending scalars in the middle tensor factor from $\mathbb{F}_p[u_1]$ to $\mathbb{F}_p[[u_1]]$, we see that this is isomorphic to

$$(K(2)_*E(2)\otimes_{\mathbb{F}_p[v_1][v_2,v_2^{-1}]}\mathbb{F}_p[\mu_{p^2-1}][[u_1]][u,u^{-1}])\otimes_{\mathbb{F}_p[[u_1]]}\Lambda_{\mathbb{F}_p[[u_1]]}HH_1^{\mathbb{F}_p[u_1]}\mathbb{F}_p[[u_1]].$$

Using (3.10) and (3.20), we can rewrite this as

$$K(2)_*E_2\otimes_{\mathbb{F}_p[[u_1]]}\Lambda_{\mathbb{F}_p[[u_1]]}HH_1^{\mathbb{F}_p[u_1]}\mathbb{F}_p[[u_1]].$$

Using the formulas in [Rav92, B.5, Pg 167-171], we see that $\eta_R(v_1)=0$ in $K(2)_*E_2$, and it follows that u_1 acts by 0 on $K(2)_*E_2$. Since $HH_1^{\mathbb{F}_p[u_1]}\mathbb{F}_p[[u_1]]$ is a $[u_1,u_1^{-1}]$-module (above (6)), we have

$$K(2)_*E_2\otimes_{\mathbb{F}_p[[u_1]]}\Lambda_{\mathbb{F}_p[[u_1]]}^n HH_1^{\mathbb{F}_p[u_1]}\mathbb{F}_p[[u_1]]=0$$

for $n\geq 1$, and so

$$K(2)_*E_2\otimes_{\mathbb{F}_p[[u_1]]}\Lambda_{\mathbb{F}_p[[u_1]]}HH_1^{\mathbb{F}_p[u_1]}\mathbb{F}_p[[u_1]]\cong K(2)_*E_2\otimes_{\mathbb{F}_p[[u_1]]}\mathbb{F}_p[[u_1]]\cong K(2)_*E_2.$$

We conclude
$$HH_*^{K(2)*}(K(2)_*E_2) \cong K(2)_*E_2.$$

In the Bökstedt spectral sequence, only the zeroth column is non-zero, and therefore the sequence collapses to give

$$K(2)_*THH(E_2) \cong HH_*^{K(2)*}(K(2)_*E_2) \cong K(2)_*E_2.$$

Since this map is induced by the unit map $E_2 \to THH(E_2)$, we conclude that the unit map is a $K(2)$-equivalence. $\qquad\square$

3.3. **The case $i > 2$:** $K(i)_*THH(E_2) = 0$. Since E_2 is L_2-local, for $i > 2$

$$K(i)_*E_2 = 0.$$

Therefore, for $i > 2$,

$$HH_{*,*}^{K(i)*}K(i)_*E_2 = 0.$$

And from the Bökstedt spectral sequence, we get the result:

Lemma 3.30. *For $i > 2$, $K(i)_*THH(E_2) = 0$. $THH(E_2)$ is L_2-local.*

4. Lifting $K(i)_*THH(E_2)$ classes to $\pi_*THH(E_2)$ classes

The purpose of this section is to lift certain $K(i)$-homology classes of $THH(E_2)$ to homotopy classes along the Hurewicz map

$$\pi_*THH(E_2) \to K(i)_*THH(E_2).$$

We lift the classes represented by elements of $K(i)_*E_2$ in Subsection 4.1, the elements of $HH_1^{\mathbb{F}_p[u_1]}\mathbb{F}_p[[u_1]] \in K(1)_*THH(E_2)$ in Proposition 4.5 and the class $dt_1 \in K(1)_*THH(E_2)$ in Proposition 4.3.

To do the lifting, we need the commutative diagram below for $i = 0, 1, 2$.

(4.1)
$$
\begin{array}{ccc}
\pi_*(E_2) \otimes \pi_*(E_2) & \xrightarrow{\ \theta\ } & \pi_*(E_2 \wedge E_2) \\
{\scriptstyle \Psi_{E_2} \otimes \Psi_{E_2}}\downarrow & & \downarrow{\scriptstyle \Psi_{E_2 \wedge E_2}} \\
K(i)_*E_2 \otimes_{K(i)_*} K(i)_*E_2 & \xrightarrow{\ \cong\ } & K(i)_*(E_2 \wedge E_2)
\end{array}
$$

The horizontal maps are the usual homology pairing maps, and the vertical maps are the Hurewicz maps. Henceforth, we write just Ψ for the Hurewicz maps when the source and target are clear. The isomorphism in the diagram above is from (2.18) and these isomorphic terms are column 1 of E^1-page of Bökstedt spectral sequence (2.17). Note also, that $\pi_*(E_2 \wedge E_2)$ is column 1 of E^1-page of filtration spectral sequence

$$\mathrm{E}^1_{s,t} := \pi_t(E_2^{\wedge s+1}) \Longrightarrow \pi_{s+t}THH(E_2)$$

arising from the simplicial filtration of $THH(E_2)$. Ψ extends to a map

(4.2)
$$\pi_t(E_2^{\wedge s+1}) \to K(i)_t(E_2^{\wedge s+1})$$

for each s, t and it commutes with the respective differentials of the spectral sequences associated to each term, thus making it a map of E^1-onwards pages of spectral sequences. Due to naturality, this map converges to the Hurewicz map

$$\pi_* THH(E_2) \to K(i)_* THH(E_2).$$

4.1. **Lifting** $K(i)_* E_2$. $K(i)_* E_2$ is the 0th homology term in $K(i)_* THH(E_2)$ for $i = 0, 1, 2$. From Subsection 2.1, this is induced from the unit map which splits. The homological classes $K(i)_* E_2$ correspond exactly to the homotopy classes of E_2 in $\pi_* THH(E_2)$.

4.2. **Lifting the class** dt_1 **from** $K(1)_* THH(E_2)$. From (2.6), (2.9), (2.10), (2.11), we have

$$t_1 \in \pi_{2p-2}(E_2 \wedge E_2)$$

which maps along Ψ to

$$1 \otimes t_1 \in K(i)_* E_2 \otimes_{K(i)_*} K(i)_* E_2$$

and is denoted by

$$dt_1 \in K(i)_{2p-1} THH(E_2).$$

We show

Proposition 4.3. *There exists a class* $\lambda_1 \in THH_{2p-1}(E_2)$ *such that under Hurewicz homomorphism*

$$\lambda_1 \mapsto dt_1 \in K(1)_{2p-1} THH(E_2).$$

Further,

$$\Psi(\lambda_1) \in K(0)_{2p-1} THH(E_2)$$

is a $K(0)_* E_2$-*linear combination of classes* du *and* du_1 *as in* (2.16).

Proof. Using (4.1) and the discussion in (4.2), we have a commutative diagram

$$
\begin{array}{ccc}
t_1 \in \pi_{2p-2}(E_2 \wedge E_2) & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\!\!\!\to & THH_{2p-1}(E_2) \\
\Big\downarrow{\scriptstyle \Psi} & & \Big\downarrow{\scriptstyle \Psi} \\
1 \otimes t_1 \in K(1)_* E_2 \otimes_{K(1)_*} K(1)_* E_2 & \cdots\cdots\to & dt_1 \in K(1)_{2p-1} THH(E_2)
\end{array}
$$

where the horizontal dotted arrows depict how surviving classes of a spectral sequence map from the first page of spectral sequence to the final one and then to a class of the object the spectral sequence converges to. Since the square commutes, the image of t_1 along the top horizontal arrow is the required lift. We denote this class as λ_1 following [MS93]. λ_1 also maps to the class

$$dt_1 \in K(0)_{2p-1} THH(E_2).$$

This class is related to the classes

$$du, du_1 \in K(0)_* THH(E_2)$$

via (2.13). We get

$$du_1 = d(u^{1-p} p t_1) = (1 - p) p t_1 u^{-p} du + p u^{1-p} dt_1$$

in $K(0)_*THH(E_2)$ giving us the claimed linear combination. $\qquad\square$

4.3. **Lifting the classes $HH_1^{\mathbb{F}_p[u_1]}\mathbb{F}_p[[u_1]]$ from $K(1)_*THH(E_2)$.** Due to (2.27), we work with

$$HH_1^{\mathbb{F}_p(u_1)}\mathbb{F}_p((u_1)).$$

From [LL01, Example 3.3], this is an infinite dimensional vector space over $\mathbb{F}_p((u_1))$ with dimension the cardinality of the continuum. Let $\{s\}_I$ be a basis set of $\mathbb{F}_p((u_1))$ over $\mathbb{F}_p(u_1)$, then we can choose a basis set for

$$\mathbb{F}_p((u_1)) \otimes_{\mathbb{F}_p(u_1)} \mathbb{F}_p((u_1))$$

over $\mathbb{F}_p((u_1))$ (where the latter acts on former the way it would act on each level of the Hochschild complex, in our case on the left tensor factor) from the set $\{1 \otimes s\}_I$. Without loss of generality, we choose $\{s\}_I$ so that each $s \in \{s\}_I$ satisfies

$$s \in \mathbb{F}_p[[u_1]]$$

and hence has (non-unique) lifts to $\mathbb{Z}[[u_1]]$ and $\mathbb{Z}_p[[u_1]]$. We denote such a lift of s by s'. Further, $HH_1^{\mathbb{F}_p(u_1)}\mathbb{F}_p((u_1))$ is a quotient of $\mathbb{F}_p((u_1)) \otimes_{\mathbb{F}_p(u_1)} \mathbb{F}_p((u_1))$, so we have a basis

(4.4) $$\mathfrak{B} \subset \{1 \otimes s\}_I$$

of $HH_1^{\mathbb{F}_p(u_1)}\mathbb{F}_p((u_1))$ over $\mathbb{F}_p((u_1))$. Choose a lift s' for each s such that $(1 \otimes s) \in \mathfrak{B}$. Then there are classes

$$1 \otimes s' \in \pi_0 E_2 \otimes \pi_0 E_2$$

since

$$\pi_0 E_2 \otimes \pi_0 E_2 \cong \mathbb{Z}_p[\mu_{p^2-1}][[u_1]] \otimes \mathbb{Z}_p[\mu_{p^2-1}][[u_1]].$$

These classes map to

$$\Psi(1 \otimes s') \in K(1)_0 E_2 \otimes_{K(1)_0} K(1)_0 E_2$$

on E_1-page of (2.17) and further down to $[1 \otimes s] \in HH_{(1,0)}^{K(1)_*} K(1)_* E_2$ on E^2-page. We lift each element of $\mathfrak{B}$.

Proposition 4.5. *For each $(1 \otimes s) \in \mathfrak{B}$, there exists a class $\tilde{s} \in THH_1 E_2$, such that under Hurewicz homomorphism each $\tilde{s}$ maps to the corresponding class*

$$(1 \otimes s) \in K(1)_1 THH(E_2).$$

Proof. From (4.1) and the discussion in (4.2), we have a commutative diagram

$$
\begin{array}{ccc}
\theta(1 \otimes s') \in \pi_0(E_2 \wedge E_2) & \dashrightarrow & THH_1(E_2) \\
\downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} \\
\Psi(1 \otimes s') \in K(1)_0 E_2 \otimes_{K(1)_0} K(1)_0 E_2 & \dashrightarrow & \in K(1)_1 THH(E_2)
\end{array}
$$

where, as before, the horizontal dotted arrows depict how surviving classes of a spectral sequence map from the first page of spectral sequence to the final one and to a class of the object the spectral sequence converges to. Since the class

$$[1 \otimes s] \in K(1)_1 THH(E_2)$$

is non-zero, $1 \otimes s'$ survives the filtration spectral sequence representing a non-zero class in $THH_1(E_2)$. We denote this class as $\tilde{s}$. □

5. $THH(E_2)$ AND $THH(E_2)_p^{\wedge}$

In this section we prove our main statements Theorem 5.24, Theorem 5.26. This essentially amounts to constructing a bunch of cofiber sequences using the lifted classes from Subsection 4.1, Proposition 4.3 and Proposition 4.5. We do this in Subsection 5.1, Subsection 5.2 and Subsection 5.3, respectively.

To construct the cofiber sequences, we need the following remarks.

Remark 5.1. For

$$A \to B \to C$$

a cofiber sequence of spectra with A, B both L_n-local (Subsection 1.3,(4)), we have

(1) The cofiber, C, is L_n-local.
(2) If $A \to B$ is a $K(n)$-equivalence, then C is L_{n-1}-local.

Remark 5.2. In places in this section, we work in terms of $E_2[u_1^{-1}]$-modules: by [EKMM97, §VIII.4], we can invert the element u_1 in $\pi_0 E_2$ to produce an E_∞ ring spectrum $E_2[u_1^{-1}]$ with

$$\pi_*(E_2[u_1^{-1}]) \cong E_{2*}[u_1^{-1}].$$

Likewise we can invert the action of u_1 on an E_2-module M to produce an $E_2[u_1^{-1}]$-module $M[u_1^{-1}]$ with

$$\pi_*(M[u_1^{-1}]) \cong M_*[u_1^{-1}].$$

More generally, for any homology theory on E_2-modules, $(-)[u_1^{-1}]$ has the effect of inverting the action of u_1, and so in particular, we have

$$K(i)_*(E_2[u_1^{-1}]) \cong (K(i)_*E_2)[u_1^{-1}] \quad \text{and} \quad K(i)_*(M[u_1^{-1}]) \cong (K(i)_*M)[u_1^{-1}].$$

Homotopically, the underlying E_2-module of $M[u_1^{-1}]$ is modeled as a homotopy colimit by multiplication by u_1:

$$(5.3) \qquad\qquad M[u_1^{-1}] \simeq \mathrm{hocolim}(M \xrightarrow{u_1} M \xrightarrow{u_1} M \xrightarrow{u_1} \cdots).$$

Note that if M is an E_2-module on which u_1 is already invertible, then as E_2-modules

$$M \simeq M[u_1^{-1}];$$

in such a case (when working in the homotopy category of E_2-modules), we will regard M as an $E_2[u_1^{-1}]$-module by abuse of notation.

5.1. **The cofiber of unit map.** We denote the unit map from (2.1) as f_1 and consider the associated cofiber sequence

$$(5.4) \qquad E_2 \xrightarrow{f_1} THH(E_2) \xrightarrow{C_{f_1}} \overline{THH}(E_2).$$

Here we are writing $\overline{THH}(E_2)$ for the cofiber of f_1, which we study in this subsection. We have two immediate observations from Remark 5.1, Remark 5.2.

Observation 5.5. E_2 is L_2-local and from Lemma 3.30, $THH(E_2)$ is L_2-local. Using Remark 5.1, we conclude $\overline{THH}(E_2)$ is L_1-local.

Observation 5.6. The following isomorphism

$$K(1)_* E_2 \otimes_{\mathbb{F}_p[[u_1]]} HH_1^{\mathbb{F}_p(u_1)} \mathbb{F}_p((u_1)) \cong K(1)_* E_2[u_1^{-1}] \otimes_{\mathbb{F}_p((u_1))} HH_1^{\mathbb{F}_p(u_1)} \mathbb{F}_p((u_1))$$

implies that $\mathfrak{B}$ (4.4) is a basis of

$$K(1)_* E_2[u_1^{-1}] \otimes_{F_p((u_1))} HH_1^{\mathbb{F}_p(u_1)} \mathbb{F}_p((u_1))$$

as a $K(1)_* E_2[u_1^{-1}]$-module.

Remark 5.7. Note that since

$$K(i)_* THH(E_2) \cong HH_{*,*}^{K(i)_*} K(i)_* E_2,$$

we can give $K(i)_* THH(E_2)$ a grading using the extra homological grading on the right hand side. We refer to this grading on $K(i)_* THH(E_2)$ as the *homological grading* and the degree of elements in this grading as their *homological degree*. In the case $i = 0$ and $i = 1$, Theorem 3.9 and Theorem 3.28 write the right hand side as an exterior algebra in certain classes of homological degree 1 (that depend on i). In terms of these classes, the homological grading is therefore a homogeneous grading in the intrinsic multiplication. In the case $i = 2$, Theorem 3.29 asserts that the homological grading is concentrated in degree zero.

We can now reorganize the computation of $K(i)_* THH(E_2)$ in terms of the homological grading of the previous remark. The following theorems are just restatements of Theorem 3.28 and Theorem 3.9. $\mathfrak{B}$ is as in (4.4). The first result uses Observation 5.6 to write the homological classes as internal direct sum, no similar analogue exists for the second result.

Theorem 5.8 (Theorem 3.28)**.** *In terms of the homological grading of Remark 5.7, the homological degree n part of $K(1)_* THH(E_2)$ is the $K(1)_* E_2$-module described as follows:*

- $K(1)_* E_2$ *in homological degree* 0
- *The internal direct sum of the $K(1)_* E_2$-modules $K(1)_* E_2(dt_1)$ and $K(1)_* E_2[u_1^{-1}](1 \otimes s)$, for all $(1 \otimes s) \in \mathfrak{B}$, in homological degree 1*
- *The internal direct sum of the $K(1)_* E_2$-modules $K(1)_* E_2[u_1^{-1}](1 \otimes s_1)(1 \otimes s_2) \cdots (1 \otimes s_n)$ for any n distinct choice of basis elements in $\mathfrak{B}$ and $K(1)_* E_2[u_1^{-1}](1 \otimes s_1)(1 \otimes s_2) \cdots (1 \otimes s_{n-1}) dt_1$ for any $n-1$ distinct choice of basis elements in $\mathfrak{B}$, in homological degree n, $n \geq 2$.*

Theorem 5.9 (Theorem 3.9). *In terms of the homological grading of Remark 5.7, the homological degree n part of $K(0)_* THH(E_2)$ is the $K(0)_* E_2$-module described as follows:*

- *$K(0)_* E_2$ in homological degree 0,*
- *The internal direct sum of $K(0)_* E_2(du)$ and $K(0)_* E_2 \otimes_{\mathbb{Z}[[u_1]]_{\mathbb{Q}}} HH_1^{\mathbb{Q}} \mathbb{Z}[[u_1]]_{\mathbb{Q}}$ in homological degree 1,*
- *$K(0)_* E_2$ module generated by products of n distinct degree 1 classes in homological degree n.*

This allows us to know the homological classes of $\overline{THH}(E_2)$.

Observation 5.10. From Observation 5.5, $\overline{THH}(E_2)$ has only $K(0)$ and $K(1)$-homology classes. f_1 maps on to homological degree 0 classes in $K(i)_* THH(E_2)$ for $i = 0, 1$ from Theorem 5.8, Theorem 5.9. The remaining classes from Theorem 5.9 and Theorem 5.8 correspond to respective $K(0)$ and $K(1)$-homology classes of $\overline{THH}(E_2)$, respectively.

5.2. **Second cofiber sequence.** Next we use the analysis of $\overline{THH}(E_2)$ above to construct a second cofiber sequence of E_2-modules

$$X_2 \xrightarrow{f_2} \overline{THH}(E_2) \xrightarrow{C_{f_2}} \overline{\overline{THH}}(E_2)$$

where C_{f_2} is the cofiber map and $\overline{\overline{THH}}(E_2)$ denotes the cofiber. Then, we analyze $\overline{\overline{THH}}(E_2)$.

From Proposition 4.3, we have

$$\lambda_1 : \mathbb{S}^{2p-1} \to THH(E_2).$$

Smashing with E_2, using the E_2-module structure of $THH(E_2)$, and composing with C_{f_1}, we get E_2-module map

$$j_1 : \Sigma^{2p-1} E_2 \simeq E_2 \wedge \mathbb{S}^{2p-1} \to E_2 \wedge THH(E_2) \to THH(E_2) \to \overline{THH}(E_2).$$

Since $\overline{THH}(E_2)$ is L_1-local, j_1 factors through the following map of E_2-modules

$$\overline{j_1} : \Sigma^{2p-1} L_1 E_2 \to \overline{THH}(E_2).$$

By construction of j_1, the induced map at $K(1)_*$ level sends the generator of $K(1)_*(\Sigma^{2p-1} E_2)$ to the class

(5.11) $$dt_1 \in K(1)_* \overline{THH}(E_2)$$

and hence the same is true for the map $\overline{j_1}$. Let

(5.12) $$X_2 := \Sigma^{2p-1} L_1 E_2, \ \ f_2 := \overline{j_1} : \Sigma^{2p-1} L_1 E_2 \to \overline{THH}(E_2).$$

Observation 5.13. Similar to Observation 5.10, from (5.12) we conclude that $\overline{\overline{THH}}(E_2)$ is L_1-local and its $K(1)$-homology classes are the ones generated by all homological degree 1 classes of $\overline{THH}(E_2)$ except (5.11). From Observation 5.6 and Theorem 5.8, $K(1)_* \overline{\overline{THH}}(E_2)$ is now a $K(1)_* E_2[u_1^{-1}]$-module.

5.3. Third cofiber sequence.

There is a natural map

$$(5.14) \qquad \overline{\overline{THH}}(E_2) \to \overline{\overline{THH}}(E_2)[u_1^{-1}].$$

From Remark 5.2, Observation 5.13

$$(5.15) \qquad K(1)_*(\overline{\overline{THH}}(E_2)[u_1^{-1}]) \cong (K(1)_*\overline{\overline{THH}}(E_2))[u_1^{-1}] \cong K(1)_*\overline{\overline{THH}}(E_2).$$

From $K(1)$-equivalence and L_1-locality of

$$\overline{\overline{THH}}(E_2)[u_1^{-1}] \text{ and } \overline{\overline{THH}}(E_2)$$

we conclude, the cofiber of (5.14) is rational using Remark 5.1. Let C_∞ denote this cofiber.

Consider λ_1 and $\tilde{s}$ from Proposition 4.3 and Proposition 4.5. Let $\tilde{\mathfrak{B}}$ denote the chosen lifts $\tilde{s}$ of elements $(1 \otimes s) \in \mathfrak{B}$ from Proposition 4.5. We denote by

$$(5.16) \qquad q_\alpha : \mathbb{S}^{|\alpha|} \to THH(E_2),$$

the elements of the form

$$\Pi_\alpha \tilde{s} \in \pi_{|\alpha|} THH(E_2),$$

where $\tilde{s}$ are all distinct elements of set α, α ranges over all non-empty finite subsets of $\tilde{\mathfrak{B}}$ and $|\alpha|$ denotes the cardinality of these subsets. Similarly, let

$$(5.17) \qquad q_{\alpha,\lambda_1} : \mathbb{S}^{|\alpha|+2p-1} \to THH(E_2)$$

denote elements

$$\Pi_\alpha \tilde{s} \cdot \lambda_1 \in \pi_{|\alpha|+2p-1} THH(E_2)$$

for all α as above.

Composing (5.16), (5.17) with C_{f_1}, C_{f_2} and (5.14) we get maps from suspensions of sphere spectrum to $\overline{\overline{THH}}(E_2)[u_1^{-1}]$ which is an $E_2[u_1^{-1}]$-module. Smashing these maps with $E_2[u_1^{-1}]$ and using the $E_2[u_1^{-1}]$-module structure of $\overline{\overline{THH}}(E_2)[u_1^{-1}]$, we get $E_2[u_1^{-1}]$-module maps

$$(5.18) \qquad \begin{aligned} &\overline{q_\alpha} : \Sigma^{|\alpha|} E_2[u_1^{-1}] \simeq E_2[u_1^{-1}] \wedge \mathbb{S}^{|\alpha|} \to \overline{\overline{THH}}(E_2)[u_1^{-1}] \\ &\overline{q_{\alpha,\lambda_1}} : \Sigma^{|\alpha|+2p-1} E_2[u_1^{-1}] \simeq E_2[u_1^{-1}] \wedge \mathbb{S}^{|\alpha|+2p-1} \to \overline{\overline{THH}}(E_2)[u_1^{-1}]. \end{aligned}$$

If we repeat the above process without composing with (5.14) and using the E_2-module structure of $\overline{\overline{THH}}(E_2)$, we have E_2-module maps

$$(5.19) \qquad \begin{aligned} &\widetilde{q_\alpha} : \Sigma^{|\alpha|} E_2 \simeq E_2 \wedge \mathbb{S}^{|\alpha|} \to E_2 \wedge \overline{\overline{THH}}(E_2) \to \overline{\overline{THH}}(E_2) \\ &\widetilde{q_{\alpha,\lambda_1}} : \Sigma^{|\alpha|+2p-1} E_2 \simeq E_2 \wedge \mathbb{S}^{|\alpha|+2p-1} \to E_2 \wedge \overline{\overline{THH}}(E_2) \to \overline{\overline{THH}}(E_2). \end{aligned}$$

Note $\overline{q_\alpha}$, $\overline{q_{\alpha,\lambda_1}}$ are u_1 localizations of $\widetilde{q_\alpha}$ and $\widetilde{q_{\alpha,\lambda_1}}$, i.e., $\overline{q_\alpha} = \widetilde{q_\alpha}[u_1^{-1}]$ and $\overline{q_{\alpha,\lambda_1}} = \widetilde{q_{\alpha,\lambda_1}}[u_1^{-1}]$. From L_1-locality of $\overline{\overline{THH}}(E_2)$ and $\overline{\overline{THH}}(E_2)[u_1^{-1}]$ these maps factor through the following E_2-module and $E_2[u_1^{-1}]$-module maps, respectively:

$$\widetilde{q_\alpha}' : \Sigma^{|\alpha|} L_1 E_2 \to \overline{\overline{THH}}(E_2), \ \widetilde{q_{\alpha,\lambda_1}}' : \Sigma^{|\alpha|+2p-1} L_1 E_2 \to \overline{\overline{THH}}(E_2)$$

$$\overline{q_\alpha}' : \Sigma^{|\alpha|} L_1 E_2[u_1^{-1}] \to \overline{\overline{THH}}(E_2)[u_1^{-1}], \ \overline{q_{\alpha,\lambda_1}}' : \Sigma^{|\alpha|+2p-1} L_1 E_2[u_1^{-1}] \to \overline{\overline{THH}}(E_2)[u_1^{-1}].$$

We are ready to define the third cofiber sequence.

Construction 5.20. Let $\widetilde{Q}$ be the map of E_2-modules

$$\bigvee_\alpha (\widetilde{q_\alpha}' \bigvee \widetilde{q_{\alpha,\lambda_1}}') : \bigvee_\alpha (\Sigma^{|\alpha|} L_1 E_2 \bigvee \Sigma^{|\alpha|+2p-1} L_1 E_2) \to \overline{\overline{THH}}(E_2)$$

and $\overline{Q}$ be the map of $E_2[u_1^{-1}]$-modules

$$\overline{Q} : \bigvee_\alpha (\overline{q_\alpha}' \bigvee \overline{q_{\alpha,\lambda_1}}') : \bigvee_\alpha (\Sigma^{|\alpha|} L_1 E_2[u_1^{-1}] \bigvee \Sigma^{|\alpha|+2p-1} L_1 E_2[u_1^{-1}]) \to \overline{\overline{THH}}(E_2)[u_1^{-1}].$$

Further, define

$$X_3 := \bigvee_\alpha (\Sigma^{|\alpha|} L_1 E_2 \bigvee \Sigma^{|\alpha|+2p-1} L_1 E_2).$$

Note that this makes

$$\bigvee_\alpha (\Sigma^{|\alpha|} L_1 E_2[u_1^{-1}] \bigvee \Sigma^{|\alpha|+2p-1} L_1 E_2[u_1^{-1}]) = X_3[u_1^{-1}].$$

Then our third cofiber sequence is given by

$$(5.21) \qquad\qquad X_3 \xrightarrow{f_3 := \widetilde{Q}} \overline{\overline{THH}}(E_2).$$

We have the following comparision of $\widetilde{Q}$ and $\overline{Q}$:

Theorem 5.22. *We have a commutative diagram as follows*

$$(5.23)$$

$$
\begin{array}{ccccc}
X_3 & \longrightarrow & X_3[u_1^{-1}] & & \\
\downarrow{\scriptstyle \widetilde{Q}} & & \downarrow{\scriptstyle \overline{Q}} & & \\
\overline{\overline{THH}}(E_2) & \longrightarrow & \overline{\overline{THH}}(E_2)[u_1^{-1}] & \longrightarrow & C_\infty \\
& & \downarrow & & \\
& & C_{\overline{Q}} & &
\end{array}
$$

where $C_{\overline{Q}}$ denotes the cofiber of $\overline{Q}$. Moreover, the map $\overline{Q}$ is a $K(1)_*$-isomorphism and both C_∞ and $C_{\overline{Q}}$ are rational.

Proof. The commutativity of the diagram and rationality of C_∞ follows from their construction in this section. All the terms in the square part of the diagram are L_1-local and hence rationality of $C_{\overline{Q}}$ follows from Remark 5.1 once we show $\overline{Q}$ induces a $K(1)_*$-isomorphism.

By construction, q_α and q_{α,λ_1} are the lifts of $K(1)$-homology classes of $THH(E_2)$ to $\pi_* THH(E_2)$. Using Observation 5.13 and (5.15), they can be extended to lifts of $K(1)$-homology classes of $\overline{\overline{THH}}(E_2)[u_1^{-1}]$ to $\pi_* \overline{\overline{THH}}(E_2)[u_1^{-1}]$. Thus after taking $K(1)_*$ on each side of $\overline{Q}$ we have a wedge of

$$\Sigma^{|\alpha|} K(1)_*(L_1 E_2[u_1^{-1}]) \cong K(1)_*(L_1 E_2[u_1^{-1}]) \wedge \mathbb{S}^{|\alpha|} \cong K(1)_* E_2[u_1^{-1}] \wedge \mathbb{S}^{|\alpha|} \text{ and}$$

$$\Sigma^{|\alpha|+2p-1} K(1)_*(L_1 E_2[u_1^{-1}]) \cong K(1)_*(L_1 E_2[u_1^{-1}]) \wedge \mathbb{S}^{|\alpha|+2p-1} \cong K(1)_* E_2[u_1^{-1}] \wedge \mathbb{S}^{|\alpha|+2p-1}$$

on the source side and under $K(1)_* \overline{Q}$ they map to the wedge of

$$K(1)_* E_2[u_1^{-1}]\Pi_\alpha(1 \otimes s) \text{ and } K(1)_* E_2[u_1^{-1}]\Pi_\alpha(1 \otimes s)dt_1$$

which is the target, giving us the desired equivalence. $\qquad\square$

5.4. **Proofs of the main results.** From (5.4), (5.12), 5.20 and Theorem 5.22 we have the following result.

Theorem 5.24. *There is a diagram of E_2-modules*

(5.25)

$$
\begin{array}{ccc}
E_2 & \xrightarrow{f_1} & THH(E_2) \\
& & \downarrow{\scriptstyle C_{f_1}} \\
X_2 & \xrightarrow{f_2} & \overline{THH}(E_2) \\
& & \downarrow{\scriptstyle C_{f_2}} \\
X_3 & \xrightarrow{f_3} & \overline{\overline{THH}}(E_2)
\end{array}
$$

such that

$$E_2 \xrightarrow{f_1} THH(E_2) \xrightarrow{C_{f_1}} \overline{THH}(E_2)$$

$$X_2 \xrightarrow{f_2} \overline{THH}(E_2) \xrightarrow{C_{f_2}} \overline{\overline{THH}}(E_2)$$

are cofiber sequences, and the map induced by f_3 on

$$X_3[u_1^{-1}] \to C_{f_2}[u_1^{-1}]$$

is a $K(1)_*$-isomorphism with a rational cofiber. Further, X_2 and X_3 are explicitly identifiable in terms of suspensions and localizations of E_2, and X_2, X_3, $\overline{THH}(E_2)$ and $\overline{\overline{THH}}(E_2)$ are all L_1-local.

Upon p-completion, the rational cofiber vanishes. We show for p-complete THH of E_2 spectrum there is a complete description in terms of cofiber sequences made of terms that are suspensions and localizations of E_2 (upto p-completion):

Theorem 5.26. *We have the following diagram of $(E_2)_p^\wedge$-modules for $THH(E_2)_p^\wedge$, where $(C_{f_i})_p^\wedge$ are the cofiber maps of $(f_i)_p^\wedge$*

$$
\begin{array}{ccc}
(E_2)_p^\wedge & \xrightarrow{(f_1)_p^\wedge := p\text{-completed unit map}} & THH(E_2)_p^\wedge \\
& & \downarrow{\scriptstyle (C_{f_1})_p^\wedge} \\
\Sigma^{2p-1} L_1 (E_2)_p^\wedge & \xrightarrow{(f_2)_p^\wedge := (\overline{\jmath_1})_p^\wedge} & \overline{THH}(E_2)_p^\wedge \\
& & \downarrow{\scriptstyle (C_{f_2})_p^\wedge} \\
(\bigvee_\alpha (\Sigma^{|\alpha|} L_1 E_2[u_1^{-1}] \bigvee \Sigma^{|\alpha|+2p-1} L_1 E_2[u_1^{-1}]))_p^\wedge & \simeq & \overline{\overline{THH}}(E_2)_p^\wedge
\end{array}
$$

Proof. This is mostly just the p-completion of the diagram in Theorem 5.24. The only different thing is the equivalence. We get this by taking the p-completion of (5.23). Since C_∞ and $C_{\overline{Q}}$ are rational, their p-completion is a point. We get a commutative square

$$
\begin{array}{ccc}
(\bigvee_\alpha(\Sigma^{|\alpha|}L_1E_2 \vee \Sigma^{|\alpha|+2p-1}L_1E_2))^\wedge_p & \longrightarrow & (\bigvee_\alpha(\Sigma^{|\alpha|}L_1E_2[u_1^{-1}] \vee \Sigma^{|\alpha|+2p-1}L_1E_2[u_1^{-1}]))^\wedge_p \\
\downarrow{\widetilde{Q}^\wedge_p} & & \simeq\downarrow{\overline{Q}^\wedge_p} \\
\overline{\overline{THH}}(E_2)^\wedge_p & \xrightarrow{\quad\simeq\quad} & \overline{\overline{THH}}(E_2)[u_1^{-1}]^\wedge_p
\end{array}
$$

giving us the required equivalence. $\qquad\square$

References

[ABG18]   Benjamin Antieau, Tobias Barthel, and David Gepner. On localization sequences in the algebraic K-theory of ring spectra. *J. Eur. Math. Soc. (JEMS)*, 20(2):459–487, 2018. 4

[AR02]   Christian Ausoni and John Rognes. Algebraic K-theory of topological K-theory. *Acta Mathematica*, 188(1):1 – 39, 2002. 3, 4, 5

[AR05]   Vigleik Angeltveit and John Rognes. Hopf algebra structure on topological hochschild homology. *Algebraic & Geometric Topology*, 5(3):1223–1290, October 2005. 10

[AR12a]   Christian Ausoni and John Rognes. Algebraic k-theory of the first morava k-theory. *Journal of the European Mathematical Society*, 14(4):1041–1079, June 2012. 5

[AR12b]   Christian Ausoni and John Rognes. Rational algebraic$k$–theory of topological$k$–theory. *Geometry & Topology*, 16(4):2037–2065, September 2012. 5

[AR20]   Christian Ausoni and Birgit Richter. Towards topological Hochschild homology of Johnson-Wilson spectra. *Algebr. Geom. Topol.*, 20(1):375–393, 2020. 5, 7, 11, 12

[Aus10]   Christian Ausoni. On the algebraic k-theory of the complex k-theory spectrum. *Inventiones mathematicae*, 180(3):611–668, March 2010. 5

[BHLS23]   Robert Burklund, Jeremy Hahn, Ishan Levy, and Tomer M. Schlank. k-theoretic counterexamples to ravenel's telescope conjecture, 2023. 5

[BM20]   Andrew J. Blumberg and Michael A. Mandell. Localization for $THH(ku)$ and the topological Hochschild and cyclic homology of Waldhausen categories. *Mem. Amer. Math. Soc.*, 265(1286):v+100, 2020. 4

[BSY22]   Robert Burklund, Tomer M. Schlank, and Allen Yuan. The chromatic nullstellensatz, 2022. 5

[DGM13]   Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy. *The local structure of algebraic K-theory*, volume 18 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2013. 3

[DH04]   Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topology*, 43(1):1–47, 2004. 4

[Efi25]   Alexander I. Efimov. Localizing invariants of inverse limits, 2025. 5

[EKMM97]   A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole. 3, 7, 10, 15, 22

[HKR62]   G. Hochschild, Bertram Kostant, and Alex Rosenberg. Differential forms on regular affine algebras. *Transactions of the American Mathematical Society*, 102, 03 1962. 10, 13

[HM97]   Lars Hesselholt and Ib Madsen. On the K-theory of finite algebras over Witt vectors of perfect fields. *Topology*, 36(1):29–101, 1997. 3

[HW22]   Jeremy Hahn and Dylan Wilson. Redshift and multiplication for truncated brown-peterson spectra, 2022. 5

[KN18]   Achim Krause and Thomas Nikolaus. Lectures on topological hochschild homology and cyclotomic spectra. *preprint*, 2018. 6, 13

[LL01]   Michael Larsen and Ayelet Lindenstrauss. Topological Hochschild homology and the condition of Hochschild-Kostant-Rosenberg. *Comm. Algebra*, 29(4):1627–1638, 2001. 5, 10, 11, 21

[LM06]   L. Gaunce Lewis and Michael A. Mandell. Equivariant universal coefficient and künneth spectral sequences. *Proceedings of the London Mathematical Society*, 92(2):505–544, March 2006. 15

[Lod98]   Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1998. Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili. 13, 16

[Mad95]   Ib Madsen. Algebraic k-theory and traces. *Current Developments in Mathematics*, 1995, 01 1995. 4, 6

[MS93]   J. E. McClure and R. E. Staffeldt. On the topological Hochschild homology of bu. I. *Amer. J. Math.*, 115(1):1–45, 1993. 5, 20

[Qui73]   Daniel Quillen. Higher algebraic K-theory. I. Algebr. K-Theory I, Proc. Conf. Battelle Inst. 1972, Lect. Notes Math. 341, 85-147 (1973)., 1973. 4

[Rav86]   Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1986. 8

[Rav92]   Douglas C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith. 9, 18

[Sta25]   The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2025. 11

[Sto20]   Bruno Stonek. Higher topological hochschild homology of periodic complex k-theory, 2020. 5

[Wal84]   Friedhelm Waldhausen. Algebraic k-theory of spaces, localization, and the chromatic filtration of stable homotopy. In Ib H. Madsen and Robert A. Oliver, editors, *Algebraic Topology Aarhus 1982*, pages 173–195, Berlin, Heidelberg, 1984. Springer Berlin Heidelberg. 4

[Wal85]   Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 318–419. Springer, Berlin, 1985. 3

Department of Mathematics, Indiana University Bloomington, 831 E Third St, Bloomington, IN 47405

Email address: sanjagar@iu.edu