

From Operads to Dendroidal Sets

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ABSTRACT. Dendroidal sets offer a formalism for the study of ∞ -operads akin to the formalism of ∞ -categories by means of simplicial sets. We present here an account of the current state of the theory while placing it in the context of the ideas that led to the conception of dendroidal sets. We briefly illustrate how the added flexibility embodied in ∞ -operads can be used in the study of A_∞ -spaces and weak n -categories in a way that cannot be realized using strict operads.

1. Introduction

This work aims to be a conceptually self-contained introduction to the theory and applications of dendroidal sets, surveying the current state of the theory and weaving together ideas and results in topology to form a guided tour that starts with Stasheff's work [37] on H -spaces, goes on to Boardman and Vogt's work [5] on homotopy invariant algebraic structures followed by the generalization [1, 2, 3] of their work by Berger and Moerdijk, arrives at the birth of dendroidal sets [34, 35] and ends with the establishment, by Cisinski and Moerdijk in [8, 9, 10], of dendroidal sets as models for homotopy operads. With this aim in mind we adopt the convention of at most pointing out core arguments of proofs rather than detailed proofs that can be found elsewhere.

We assume basic familiarity with the language of category theory and mostly follow [30]. Regarding enriched category theory we assume little more than familiarity with the definition of a category enriched in a symmetric monoidal category as can be found in [21]. The elementary results on presheaf categories that we use can be found in [31]. Some comfort of working with simplicial sets is needed for which the first chapter of [14] suffices. Some elementary understanding of Quillen model categories is desirable with standard references being [17, 18].

Operads arose in algebraic topology and have since found applications across a wide range of fields including Algebra, Theoretical Physics, and Computer Science. The reason for the success of operads is that they offer a computationally effective formalism for treating algebraic structures of enormous complexity, usually involving some notion of (abstract) homotopy. As such, the first operads to be

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introduced were topological operads and most of the other variants are similarly enriched in other categories. However, the presentation we give here of operads treats them as a rather straightforward generalization of the notion of category. It is that viewpoint that quite naturally leads to defining dendroidal sets to serve as the codomain category for a nerve construction for operads, extending the usual nerve of categories.

The path we follow is the following one. We first examine the expressive power of non-enriched symmetric operads. We find that by considering operad maps it is possible to classify a wide range of strict algebraic structures such as associative and commutative monoids and to show that operads carry a closed monoidal structure that, via the internal hom, internalizes algebraic structures. We show that the rather trivial fact that algebraic structures can be transferred along isomorphisms, which we call the isomorphism invariance property, is a result of symmetric operads supporting a Quillen model structure compatible with the monoidal structure. We then turn to the much more challenging homotopy invariance property for algebraic structures in the presence of homotopy notions. We show how the theory of operads is used to adequately handle this more subtle situation, however at a cost. The internalization of these so-called weak algebras, via an internal hom construction, is lost. The sequel can be seen as a presentation of the successful attempt to develop a formalism for weak algebraic structures in which the internalization of algebras is restored. This formalism is given by dendroidal sets and a suitable Quillen model structure which can be used to give a proof of the homotopy invariance property (which is completely analogous to the case of non-enriched operads). The consequences and applicability of the added flexibility of dendroidal sets is portrayed by considering n -fold A_∞ -spaces and weak n -categories.

Section 2 introduces in the first half non-enriched symmetric operads and presents their basic theory. The second half is concerned with enriched operads and the Berger-Moerdijk generalization of the Boardman-Vogt W -construction. Section 3 is a parallel development of the ideas in Section 2. The first half introduces dendroidal sets and presents their basic theory while the second half is concerned with the homotopy coherent nerve construction with applications to A_∞ -spaces and weak n -categories. Section 4 is devoted to the Cisinski-Moerdijk model structure on dendroidal sets and the way it is used to prove the homotopy invariance property. Section 5 closes this work with a brief presentation of a planar dendroidal Dold-Kan correspondence and discusses the yet unsolved problem of obtaining a satisfactory geometric realization for dendroidal sets.

Remark. Below we work in a convenient category of topological spaces Top . In some places it is important that this category be closed monoidal, in which case the category of compactly generated Hausdorff spaces would suffice. We will not remark about such issues further.

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2. Operads and algebraic structures

Remark. The reader already familiar with operads who reads this section just to familiarize herself with the notation is strongly advised to look at Remark 2.19 and

Fact 2.30 below. For her convenience the opening paragraph below recounts the contents of the entire section.

Our journey starts with non-enriched symmetric operads, also known as symmetric multicategories (originating in Lambek's study of deductive systems in logic [23]) or symmetric coloured operads (e.g., [3, 24]). In the literature on operads these structures are underrepresented probably due to the fact that the first operads, introduced by May in [33], were enriched in topological spaces and many of the most important uses of operads require enrichment. The point of view of operads we adopt is that operads generalize categories. Consequently, just as a study of categories starts with non-enriched categories, with enrichment usually treated at some later stage, we first present non-enriched symmetric operads. The operadically versed reader will immediately recognize that our definition of algebra differs slightly from the standard one. We define the Boardman-Vogt tensor product of symmetric operads and the notion of natural transformations for symmetric operads that endows the category of symmetric operads with the structure of a symmetric closed monoidal category. We then address the isomorphism invariance property and treat it in the context of a suitable Quillen model structure on symmetric operads. We then turn to the much more subtle and interesting case of the homotopy invariance property and give an expository treatment of the theory developed by Berger and Moerdijk relevant for the rest of the presentation.

2.1. Trees. Symmetric (also called 'non-planar') rooted trees are useful in the study of symmetric operads. There is no standard definition of 'tree' that is commonly used (Ginzburg and Kapranov in [12] use a topological definition while Leinster in [24] uses a combinatorial one) but all approaches are essentially the same. More recently, Joachim Kock in [22] established a close connection between trees and polynomial functors, offering yet another formalism of trees while shedding a different light on the symbiosis between operads and trees.

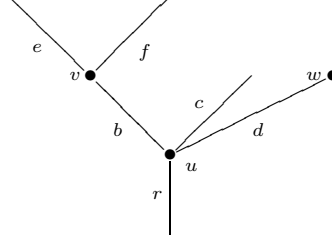
We present here the formalism of trees we use and introduce terminology for commonly occurring trees as well as grafting of trees. We end the section by presenting a generalization of posets that shows trees to be analogues of finite linear orders.

2.1.1. Symmetric rooted trees.

Definition 2.1. A *tree* (short for symmetric rooted tree) is a finite poset (T, \leq) which has a smallest element and such that for each $e \in T$ the set $\{y \in T \mid y \leq e\}$ is linearly ordered. The elements of T are called *edges* and the unique smallest edge is called the *root*. Part of the information of a tree is a subset $L = L(T)$ of maximal elements, which are called *leaves*. An edge is *outer* if it is either the root or it belongs to L , otherwise it is called *inner*.

Given edges $e, e' \in T$ we write e/e' if $e' < e$ and if for any $x \in T$ for which $e' \leq x \leq e$ holds that either $x = e'$ or $x = e$. For a non-leaf edge e the set $in(e) = \{t \in T \mid t/e\}$ is called the set of *incoming edges* into e . For such an edge e the set $v = \{e\} \cup in(e)$ is called the *vertex* above e and we define $in(v) = in(e)$ and $out(v) = e$ which are called, respectively, the set of *incoming edges* and the *outgoing edge* associated to v . The *valence* of v is equal to $|in(v)|$ and could be 0. Note that there is no vertex associated to a leaf. We will draw trees by the graph dual of their Hasse diagrams with the root at the bottom and will use a •

for vertices. For example, in the tree



there are three vertices of valence 2, 3, and 0 and three leaves $L = \{e, f, w\}$. The outer edges are e, f, c , and r , where r is the root. The inner edges are then b and d .

2.1.2. *Some common trees.* The following types of trees appear often enough in the theory of dendroidal sets to merit their own notation.

Definition 2.2. For each $n \geq 0$, a tree L_n of the form

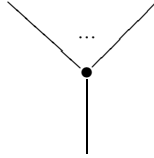


with one leaf and n vertices, all unary (i.e., each vertex has valence equal to 1), will be called a *linear tree of order n* . The special case of the tree L_0



consisting of just one edge and no vertices is called the *unit* tree. We denote this tree by η , or η_e if we wish to explicitly name its unique edge. In this tree, the only edge is both the root and a leaf.

Definition 2.3. For each $n \geq 0$, a tree C_n of the form



that has just one vertex and n leaves will be called an *n -corolla*. Note that the case $n = 0$ results in a tree different than η .

2.1.3. Grafting.

Definition 2.4. Let T and S be two trees whose only common edge is the root r of S which is also one of the leaves of T . The *grafting*, $T \circ S$, of S on T along r is the poset $T \cup S$ with the obvious poset structure and set of leaves equal to $(L(S) \cup L(T)) - \{r\}$.

Pictorially, the grafted tree $T \circ S$ is obtained by putting the tree S on top of the tree T by identifying the output edge of S with the input edge r of T . By repeatedly grafting, one can define a full grafting operation $T \circ (S_1, \dots, S_n)$ in the obvious way.

We now state a useful decomposition of trees that allows for inductive proofs on trees. The proof is trivial.

Proposition 2.5. *Let T be a tree. Suppose T has root r and $\{r, e_1, \dots, e_n\}$ is the vertex above r . Let T_{e_i} be the tree that contains the edge e_i as root and everything above it in T . Then*

$$T = T_{root} \circ (T_{e_1}, \dots, T_{e_n})$$

where T_{root} is the n -corolla consisting of r as root and $\{e_1, \dots, e_n\}$ as the set of leaves.

2.1.4. Trees and dendroidally ordered sets. The trees we defined above are going to be the objects of the category Ω whose presheaf category Set_Ω is the category of dendroidal sets. Recall that the simplicial category Δ (whose presheaf category is the category of simplicial sets) can be defined as (a skeleton of) the category of totally ordered finite sets with order preserving maps. In this section we present an extension of the notion of totally ordered finite sets closely related to trees. The content of this section is not used anywhere in the sequel and is presented for the sake of completeness. Consequently we give no proofs and refer the reader to [38] for more details.

First we extend the notion of a relation and that of a poset to what we call broad relation and broad poset. For a set A we denote by $A^+ = (A^+, +, 0)$ the free commutative monoid on A . A *broad relation* is a pair (A, R) where A is a set and R is a subset of $A \times A^+$. As is common with ordinary relations, we use the notation $aR(a_1 + \dots + a_n)$ instead of $(a, (a_1 + \dots + a_n)) \in R$.

Definition 2.6. A *broad poset* is a broad relation (A, R) satisfying:

- (1) Reflexivity: aRa holds for any $a \in A$.
- (2) Transitivity: For all $a_0, \dots, a_n \in A$ and $b_1, \dots, b_n \in A^+$ such that $a_i R b_i$ for $1 \leq i \leq n$, holds that if $a_0 R (a_1 + \dots + a_n)$ then $aR(b_1 + \dots + b_n)$.
- (3) Anti-symmetry: For all $a_1, a_2 \in A$ and $b_1, b_2 \in A^+$ if $a_1 R (a_2 + b_2)$ and $a_2 R (a_1 + b_1)$ then $a_1 = a_2$.

When (A, R) is a broad poset we denote R by \leq . The meaning of $<$ is then defined in the usual way.

A *map* of broad posets $f : A \rightarrow B$ is a set function preserving the broad poset structure, that is if $a \leq (a_1 + \dots + a_n)$ then $f(a) \leq (f(a_1) + \dots + f(a_n))$.

Definition 2.7. We denote by $BrdPoset$ the category of all broad posets and their maps.

Let \star be a singleton set $\{\star\}$ with the broad poset structure given by $\star \leq \star$. Note that \star is not a terminal object in $BrdPoset$.

Lemma 2.8. (*Slicing lemma for broad posets*) *There is an isomorphism of categories between $\text{BrdPoset}/\star$ and the category Poset of posets and order preserving maps. Moreover, along this isomorphism one obtains a functor $k_! : \text{Poset} \rightarrow \text{BrdPoset}$ which has a right adjoint $k^* : \text{BrdPoset} \rightarrow \text{Poset}$ which itself has a right adjoint $k_* : \text{Poset} \rightarrow \text{BrdPoset}$.*

As motivation for the following definition recall that a finite ordinary poset A is linearly ordered if, and only if, it has a smallest element and for every $a \in A$ the set $a_\uparrow = \{x \in A \mid a < x\}$ is either empty or has a smallest element.

Definition 2.9. A finite broad poset A is called *dendroidally ordered* if

- (1) There is an element $r \in A$ such that for every $a \in A$ there is $b \in A^+$ such that $r \leq a + b$.
- (2) For every $a \in A$ the set $a_\uparrow = \{b \in A^+ \mid a < b\}$ is either empty or it contains an element $s(a) = a_1 + \cdots + a_n$ such that every $b \in a_\uparrow$ can be written as $b = b_1 + \cdots + b_n$ with $a_i \leq b_i$ for all $1 \leq i \leq n$.
- (3) For every $a_0, \dots, a_n \in A$, if $a_0 \leq a_1 + \cdots + a_n$ then for $i \neq j$ there holds $a_i \neq a_j$.

Trees are related to finite dendroidally ordered sets as follows. Given a tree T define a broad relation on the set $E(T)$ of edges by declaring $e \leq e_1 + \cdots + e_n$ precisely when there is a vertex v such that $\text{in}(v) = \{e_1, \dots, e_n\}$ (without repetitions) and $\text{out}(v) = e$. The transitive closure of this broad relation is then a dendroidally ordered set. This construction can be used to give an equivalence of categories between the full subcategory DenOrd of BrdPoset spanned by the dendroidally ordered sets and the dendroidal category Ω defined below. It is easily seen that DenOrd , upon slicing over \star , is isomorphic to the category of all finite linearly ordered sets and order preserving maps.

2.2. Operads and algebras. We now present symmetric operads viewed as a generalization of categories where arrows are allowed to have domains of arity n for any $n \in \mathbb{N}$. We then define the notion of \mathcal{P} -algebras for a symmetric operad \mathcal{P} which are often referred to as the *raison d'être* of operads. We deviate here from the common definition of algebras noting that our definition encompasses the standard one. We define an algebra to simply be a morphism between symmetric operads, the difference being purely syntactic. The assertion that symmetric operads exist in order to define algebras thus agrees with the idea that in any category the objects' *raison d'être* is to serve as domains and codomains of arrows.

Definition 2.10. A *planar operad* \mathcal{P} consists of a class \mathcal{P}_0 whose elements are called the *objects* of \mathcal{P} and to each sequence $P_0, \dots, P_n \in \mathcal{P}_0$ a set $\mathcal{P}(P_1, \dots, P_n; P_0)$ whose elements are called *arrows* depicted by $\psi : (P_1, \dots, P_n) \rightarrow P_0$. With this notation (P_1, \dots, P_n) is the *domain* of ψ , P_0 its *codomain*, and n its *arity* (which is allowed to be 0). The domain and codomain are assumed to be uniquely determined by ψ . There is for each object $P \in \mathcal{P}_0$ a chosen arrow $\text{id}_P : P \rightarrow P$ called the *identity* at P . There is a specified composition rule: Given $\psi_i : (P_1^i, \dots, P_{m_i}^i) \rightarrow P_i$, $1 \leq i \leq n$, and an arrow $\psi : (P_1, \dots, P_n) \rightarrow P_0$ their composition is denoted by $\psi \circ (\psi_1, \dots, \psi_n)$ and has domain $(P_1^1, \dots, P_{m_1}^1, P_1^2, \dots, P_{m_2}^2, \dots, P_1^n, \dots, P_{m_n}^n)$ and codomain P_0 . The composition is to obey the following unit and associativity laws:

- Left unit axiom: $\text{id}_P \circ \psi = \psi$
- Right unit axiom: $\psi \circ (\text{id}_{P_1}, \dots, \text{id}_{P_n}) = \psi$

- Associativity axiom: the composition

$$\psi \circ (\psi_1 \circ (\psi_1^1, \dots, \psi_{m_1}^1), \dots, \psi_n \circ (\psi_1^n, \dots, \psi_{m_n}^n))$$

is equal to

$$(\psi \circ (\psi_1, \dots, \psi_n)) \circ (\psi_1^1, \dots, \psi_{m_1}^1, \dots, \psi_1^n, \dots, \psi_{m_n}^n).$$

The morphisms of planar operads are the obvious structure preserving maps. A map of operads will also be referred to as a *functor*.

Definition 2.11. A *symmetric operad* is a planar operad \mathcal{P} together with actions of the symmetric groups in the following sense: for each $n \in \mathbb{N}$, objects $P_0, \dots, P_n \in \mathcal{P}_0$, and a permutation $\sigma \in \Sigma_n$ a function $\sigma^* : \mathcal{P}(P_1, \dots, P_n; P_0) \rightarrow \mathcal{P}(P_{\sigma(1)}, \dots, P_{\sigma(n)}; P_0)$. We write $\sigma^*(\psi)$ for the value of the action of σ on $\psi : (P_1, \dots, P_n) \rightarrow P_0$ and demand that for any two permutations $\sigma, \tau \in \Sigma_n$ there holds $(\sigma\tau)^*(\psi) = \tau^*\sigma^*(\psi)$. Moreover, these actions of the permutation groups are to be compatible with compositions in the obvious sense (see [24, 33] for more details). *Functors* of symmetric operads $\mathcal{P} \rightarrow \mathcal{Q}$ are functors of the underlying planar operads that respect the actions of the symmetric groups.

When dealing with operads we make a distinction between small and large ones according to whether the class of objects is, respectively, a set or a proper class. If more care is needed and size issues become important we implicitly assume working in the formalism of Grothendieck universes ([6]) similarly to the way such issues are avoided in category theory. We now obtain the category Ope_π of small planar operads and their functors as well as the category Ope of small symmetric operads and their functors. There is clearly a forgetful functor $Ope \rightarrow Ope_\pi$ which has an easily constructed left adjoint $S : Ope_\pi \rightarrow Ope$ called the *symmetrization* functor.

Remark 2.12. We note that our symmetric operads are also called symmetric multicategories (see e.g., [24]) as well as symmetric coloured operads. The composition as given above is sometimes called full \circ composition. Using the identities in an operad we can then define what is known as the \circ_i composition as follows. Given an arrow ψ of arity n and $1 \leq i \leq n$ one can compose an arrow φ onto the i -th place of the domain of ψ , provided the object at the i -th place is equal to the codomain of φ , by means of $\psi \circ_i \varphi = \psi \circ (id, \dots, id, \varphi, id, \dots, id)$ with φ appearing in the i -th place. Some authors consider operads defined in terms of the \circ_i operations rather than the full \circ composition. In the presence of identities there is no essential difference but if identities are not assumed than one obtains a slightly weaker structure called a *pseudo-operad* (see [32]). The operads we consider always have identities so that the full \circ and partial \circ_i compositions differ only cosmetically and will be used interchangeably as convenient.

Operads are closely related to categories. Indeed, one trivially sees that a category is an operad where each arrow has arity equal to 1.

A slightly less trivial and more useful fact is the following. Call a symmetric operad *reduced* if it has no 0-ary operations. We denote by \star an operad with one object and only the identity arrow on it.

Lemma 2.13. (*Slicing lemma for symmetric operads*) *There is an isomorphism between the category Cat of small categories and the slice category Ope/\star . Moreover, there are functors $j_! : Cat \rightarrow Ope$ and $j^* : Ope \rightarrow Cat$ such that j^* is right adjoint to $j_!$. The functor j^* does not preserve pushouts and thus does not have a*

right adjoint. However, the restriction of j_* to the subcategory of reduced operads does have a right adjoint. Under the isomorphism $Cat \cong Ope/\star$ the functor $j_!$ is the forgetful functor $Cat = Ope/\star \rightarrow Ope$.

PROOF. We explicitly describe the functors, omitting any details. Given a category \mathcal{C} the operad $j_!(\mathcal{C})$ has $j_!(\mathcal{C})_0 = \mathcal{C}_0$ (here \mathcal{C}_0 stands for the class of objects of the category \mathcal{C}) and the arrows in $j_!(\mathcal{C})$ are given for $P_0, \dots, P_n \in j_!(\mathcal{C})_0$ as follows:

$$j_!(\mathcal{C})(P_1, \dots, P_n; P_0) = \begin{cases} \mathcal{C}(P_1, P_0) & \text{if } n = 1 \\ \emptyset & \text{if } n \neq 1 \end{cases}$$

The composition is the same as in \mathcal{C} . The right adjoint j^* is given for an operad \mathcal{P} as follows. $j^*(\mathcal{P})_0 = \mathcal{P}_0$ and the arrows in $j^*(\mathcal{P})$ are given for $C, D \in j^*(\mathcal{P})_0$ by:

$$j^*(\mathcal{P})(C, D) = \mathcal{P}(C; D).$$

The composition is the same as in \mathcal{P} . Finally, the functor j_* , right adjoint to the restriction of j^* to reduced operads, is defined for a category \mathcal{C} as follows. $j_*(\mathcal{C})_0 = \mathcal{C}_0$ and the arrows in $j_*(\mathcal{C})$ are given for $P_0, \dots, P_n \in j_*(\mathcal{C})_0$ as follows:

$$j_*(\mathcal{C})(P_1, \dots, P_n; P_0) = \begin{cases} \mathcal{C}(P_1, P_0) & \text{if } n = 1 \\ \{(P_1, \dots, P_n; P_0)\} & \text{if } n \neq 1 \end{cases}$$

Composition of unary arrows is given as in \mathcal{C} . Composition of two arrows at least one of which is not unary is uniquely determined since the hom set of where that arrow is to be found consists of just one object. It is therefore automatic that the composition so defined is associative. \square

Remark 2.14. The construction of the three functors above follows from general abstract nonsense and is related to locally cartesian closed categories. Indeed, if \mathcal{C} is a category with a terminal object $*$, then for any object $A \in \mathcal{C}_0$ the unique arrow $A \rightarrow *$ induces a functor between the slice categories $F_! : \mathcal{C}/A \rightarrow \mathcal{C}/*$. It is then a general result that $F_!$ has a right adjoint F^* if, and only if, \mathcal{C} admits products with A . Moreover, F^* has a right adjoint F_* if, and only if, A is exponentiable in \mathcal{C} . The case we had at hand is when \mathcal{C} is the category of symmetric operads, or its subcategory of reduced symmetric operads, and $A = \star$.

Due to this intimate connection between symmetric operads and categories we will employ category theoretic terminology in the context of symmetric operads. For example, we will refer to morphisms of operads as functors, and feel free to use category theoretic terminology within the 'category part' $j^*(\mathcal{P})$ of an operad \mathcal{P} . So the notion of a unary arrow f in \mathcal{P} being, for instance, an isomorphism, a monomorphism, or a split idempotent simply means that f has the same property in the category $j^*(\mathcal{P})$. In this spirit we give the following definition of equivalence of operads.

Definition 2.15. Let \mathcal{P} and \mathcal{Q} be symmetric operads and $F : \mathcal{P} \rightarrow \mathcal{Q}$ a functor. We say that F is an *equivalence of operads* if F is fully faithful (which means that it is bijective on each hom-set) and essentially surjective (which means that $j^*(F)$ is an essentially surjective functor of categories).

Remark 2.16. We make a few remarks to emphasize differences and similarities between the categories Ope and Cat :

- Ope is small complete and small cocomplete.
- There is a unique initial operad which is, of course, equal to $j_!(\emptyset)$.
- For the operad \star above and a terminal category $*$ there holds that $\star \cong j_!(*)$ and $*$ $\cong j^*(\star)$.
- \star is not terminal but is exponentiable in the category of reduced symmetric operads.
- The terminal object in Ope is the operad $Comm = j_*(*)$ consisting of one object and one n -ary operation for every $n \in \mathbb{N}$.
- The subobjects of the terminal operad $Comm$ are all of the following form. An operad with one object and for every $n \geq 0$ at most one arrow of arity n such that if an arrow of arity m and an arrow of arity k exist then there is also an arrow of arity $m + k - 1$.

A typical example of category is obtained by fixing some mathematical object and considering the totality of those objects and their naturally occurring morphisms. In many cases these objects also have a notion of 'morphism of several variables' in which case the totality of objects and their multivariable arrows will actually form an operad. One case in which this is guaranteed is the following.

Lemma 2.17. *Let $(\mathcal{E}, \otimes, I)$ be a symmetric monoidal category and consider for every $x_0, \dots, x_n \in \mathcal{E}_0$ the set $\hat{\mathcal{E}}(x_1, \dots, x_n; x_0) = \mathcal{E}(x_1 \otimes \dots \otimes x_n, x_0)$. With the obvious definitions of composition and identities this construction defines a symmetric operad $\hat{\mathcal{E}}$ with $(\hat{\mathcal{E}})_0 = \mathcal{E}_0$.*

PROOF. The associativity of the composition in $\hat{\mathcal{E}}$ is a result of the coherence in \mathcal{E} . \square

Remark 2.18. Certainly not every symmetric operad is obtained in that way from a symmetric monoidal category (e.g., any of the proper subobjects of $Comm$ or any operad of the form $j_!(\mathcal{C})$ for a category \mathcal{C}). It is possible to internally characterize those symmetric operads that do arise in that way from symmetric monoidal categories, as is explained in detail in [16] and indicated in [24].

Another type of category that arises naturally is one that encodes some properties of arrows abstractly. For example, the free-living isomorphism $0 \rightleftharpoons 1$ is a category with two distinct objects and, except for the two identities, two other arrows between the objects, each of which is the inverse of the other. A functor from the free-living isomorphism to any category \mathcal{C} corresponds exactly to a choice of an isomorphism in \mathcal{C} and can be seen as the abstract free-living isomorphism becoming concrete in the category \mathcal{C} . A similar phenomenon is true in operads, where one readily sees the much greater expressive power of operads compared to categories. Consider for example the terminal operad $Comm$, for which it is straightforward to prove that any functor of operads $Comm \rightarrow \hat{\mathcal{E}}$ is the same as a commutative monoid in \mathcal{E} , for any symmetric monoidal category \mathcal{E} . There is no category \mathcal{C} with the property that functors $\mathcal{C} \rightarrow \mathcal{E}$ correspond to commutative monoids in \mathcal{E} .

Remark 2.19. To distinguish between symmetric operads such as $Comm$ thought of as encoding properties of arrows and symmetric operads such as $\hat{\mathcal{E}}$ thought of as environments where operads \mathcal{P} are interpreted concretely we will use letters near \mathcal{P} for abstract symmetric operads and letters near \mathcal{E} for symmetric operads as environments (whether they come from a symmetric monoidal category or not). We

will also call symmetric operads \mathcal{E} *environment operads*. The distinction is purely syntactic.

The utility of operads is in their ability to codify quite a wide range of algebraic structures in the way described above. The usual terminology one uses is that of an *algebra* of an operad. The following definition of algebra is more general than the usual one (e.g., [32, 33]).

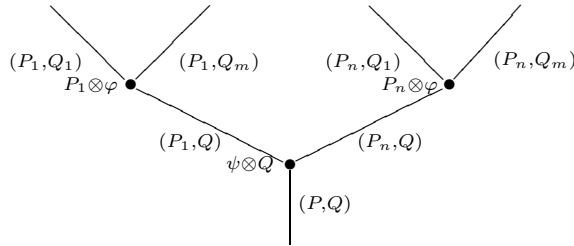
Definition 2.20. Let \mathcal{P} and \mathcal{E} be symmetric operads and consider a functor $F : \mathcal{P} \rightarrow \mathcal{E}$. If $F_0 : \mathcal{P}_0 \rightarrow \mathcal{E}_0$ is the object part of the functor F we say that F is a \mathcal{P} -*algebra* structure on the collection of objects $\{F_0(P)\}_{P \in \mathcal{P}_0}$ in the environment operad \mathcal{E} .

Many basic properties of \mathcal{P} -algebras are captured efficiently by the introduction of a closed monoidal structure on *Ope*. The appropriate tensor product of symmetric operads is the Boardman-Vogt tensor product which was first introduced in [5] for (certain structures that are essentially equivalent to) symmetric operads enriched in topological spaces. The construction is general enough that it can be performed for operads enriched in other monoidal categories and certainly also in the non-enriched case, which is the version we give now.

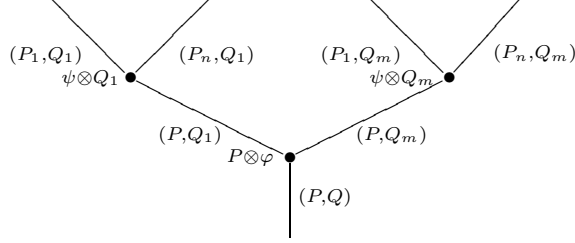
Definition 2.21. Let \mathcal{P} and \mathcal{Q} be two symmetric operads. Their *Boardman-Vogt tensor product* is the symmetric operad $\mathcal{P} \otimes \mathcal{Q}$ with $(\mathcal{P} \otimes \mathcal{Q})_0 = \mathcal{P}_0 \times \mathcal{Q}_0$ given in terms of generators and relations as follows. For each $Q \in \mathcal{Q}_0$ and each operation $\psi \in \mathcal{P}(P_1, \dots, P_n; P)$ there is a generator $\psi \otimes Q$ with domain $(P_1, Q), \dots, (P_n, Q)$ and codomain (P, Q) . For each $P \in \mathcal{P}_0$ and an operation $\varphi \in \mathcal{Q}(Q_1, \dots, Q_m; Q)$ there is a generator $P \otimes \varphi$ with domain $(P, Q_1), \dots, (P, Q_m)$ and codomain (P, Q) . There are five types of relations among the arrows (σ and τ below are permutations whose roles are explained below):

- 1) $(\psi \otimes Q) \circ ((\psi_1 \otimes Q), \dots, (\psi_n \otimes Q)) = (\psi \circ (\psi_1, \dots, \psi_n)) \otimes Q$
- 2) $\sigma^*(\psi \otimes Q) = (\sigma^*\psi) \otimes Q$
- 3) $(P \otimes \varphi) \circ ((P \otimes \varphi_1), \dots, (P \otimes \varphi_m)) = P \otimes (\varphi \circ (\varphi_1, \dots, \varphi_m))$
- 4) $\sigma^*(P \otimes \varphi) = P \otimes (\sigma^*\varphi)$
- 5) $(\psi \otimes Q) \circ ((P_1 \otimes \varphi), \dots, (P_n \otimes \varphi)) = \tau^*((P \otimes \varphi) \circ ((\psi, Q_1), \dots, (\psi, Q_m)))$

By the relations above we mean every possible choice of arrows $\psi, \varphi, \psi_i, \varphi_j$ for which the compositions are defined. The relations of type 1 and 2 ensure that for any $Q \in \mathcal{Q}_0$, the map $P \mapsto (P, Q)$ naturally extends to a functor $\mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{Q}$. Similarly, the relations of type 3 and 4 guarantee that for each $P \in \mathcal{P}_0$, the map $Q \mapsto (P, Q)$ naturally extends to a functor $\mathcal{Q} \rightarrow \mathcal{P} \otimes \mathcal{Q}$. The relation of type 5 can be visualized as follows. The left hand side can be drawn as



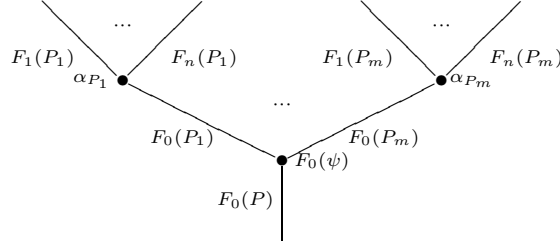
while the right hand side can be drawn as



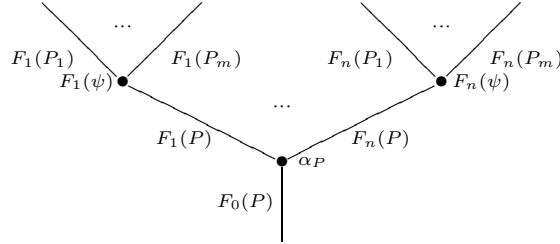
As given, the operations cannot be equated since their domains do not agree. There is however an evident permutation τ that equates the domains and it is that permutation τ that is used in the equation of type 5 above.

Theorem 2.22. *The category $(\text{Ope}, \otimes, \star)$ is a symmetric closed monoidal category.*

PROOF. The internal hom operad $[\mathcal{P}, \mathcal{Q}]$ has as objects all morphisms of operads $F : \mathcal{P} \rightarrow \mathcal{Q}$ and the arrows with domain F_1, \dots, F_n and codomain F_0 are analogues of natural transformations as follows. A *natural transformation* α from (F_1, \dots, F_n) to F_0 is a family $\{\alpha_P\}_{P \in \mathcal{P}_0}$, with $\alpha_P \in \mathcal{Q}(F_1(P), \dots, F_n(P); F_0(P))$, satisfying the following property. Given any operation $\psi \in \mathcal{P}(P_1, \dots, P_m; P)$ consider the following diagrams in \mathcal{Q} :



and



and let φ_1 and φ_2 be their respective compositions. Then $\varphi_2 = \sigma^*(\varphi_1)$, where σ is the evident permutation equating the domain of φ_1 with that of φ_2 . The interested reader is referred to [38] for more details on horizontal and vertical compositions of natural transformations leading to the construction of the strict 2-category of small operads in which the strict 2-category of small categories embeds. \square

We now return to our general notion of \mathcal{P} -algebras in \mathcal{E} and notice the very simple result:

Lemma 2.23. *Let \mathcal{P} and \mathcal{E} be symmetric operads. The internal hom $[\mathcal{P}, \mathcal{E}]$ is rightfully to be called the operad of \mathcal{P} -algebras in \mathcal{E} in the sense that the objects of $[\mathcal{P}, \mathcal{E}]$ are the \mathcal{P} -algebras in \mathcal{E} , the unary arrows are the morphisms of such algebras, and the n -ary arrows are 'multivariable' morphisms of algebras (with 0-ary morphisms thought of as constants).*

It is trivial to verify for example that $[Comm, Set]$ is isomorphic to the operad obtained from the symmetric monoidal category $CommMon(Set)$ of commutative monoids in Set by means of the construction given in Lemma 2.17. Here Set can be replaced by any symmetric monoidal category. This motivates the following definition.

Definition 2.24. Let \mathcal{E} be a symmetric operad and S some notion of an algebraic structure on objects of \mathcal{E} together with a notion of (perhaps multivariable) morphisms between such structures. We call a symmetric operad \mathcal{P} a *classifying* operad for S (in \mathcal{E}) if the operad $[\mathcal{P}, \mathcal{E}]$ satisfies that $[\mathcal{P}, \mathcal{E}]_0$ is precisely the set of S -structures in \mathcal{E} and the arrows in $[\mathcal{P}, \mathcal{E}]$ correspond precisely to the notion of morphisms between such structures.

Example 2.25. The symmetric operad $Comm$ is a classifying operad for commutative monoids in a symmetric operad \mathcal{E} . There is a symmetric operad As that classifies monoids in \mathcal{E} (i.e., an object with an associative binary operation with a unit) which the reader is invited to find. A *magma* is a set together with a binary operation, not necessarily associative, and there is a symmetric operad that classifies magmas. There is also a symmetric operad that classifies non-unital monoids as well as one that classifies non-unital commutative monoids. It is a rather unfortunate fact that there is no symmetric operad that classifies all small categories. However, given a fixed set A consider the category Cat_A of *categories over A* , in which the objects are categories having A as set of objects and where the arrows are functors between such categories whose object part is the identity. Then there is a symmetric operad C_A that classifies categories over A . Similarly, with the obvious definition, there is a symmetric operad O_A that classifies symmetric operads over A .

Remark 2.26. In general, there can be two non-equivalent operads \mathcal{P} and \mathcal{Q} that classify the same algebraic structure. We will not get into the question of detecting when two symmetric operads have equivalent operads of algebras.

A well-known phenomenon in category theory is the interchangeability of repeated structures. Thus, for example, a category object in Grp is the same as a group object in Cat . With the formalism of symmetric operads we have thus far we can easily prove a whole class of such cases (but in fact not the case just mentioned, since group objects cannot be classified by symmetric operads).

Lemma 2.27. *Let \mathcal{P}_1 and \mathcal{P}_2 be two symmetric operads and let \mathcal{E} be an environment operad. Then \mathcal{P}_1 -algebras in \mathcal{P}_2 -algebras in \mathcal{E} are the same as \mathcal{P}_2 -algebras in \mathcal{P}_1 -algebras in \mathcal{E} .*

PROOF. The precise formulation of the lemma is that there is an isomorphism of symmetric operads $[\mathcal{P}_1, [\mathcal{P}_2, \mathcal{E}]] \cong [\mathcal{P}_2, [\mathcal{P}_1, \mathcal{E}]]$. The proof is trivial from the symmetry of the Boardman-Vogt tensor product. \square

Consider the operad As that classifies monoids: it has just one object and its arrows of arity n is the set Σ_n of permutations on n -symbols. It is not hard to show that $As \otimes As \cong Comm$ which essentially is Eckman-Hilton duality proving that associative monoids in associative monoids are commutative monoids, except that it is done at the level of classifying operads rather than algebras.

We conclude our review of the basics of operad theory by noting that in the same way that categories can be enriched in a symmetric monoidal category \mathcal{E} (see [21]) so can operads be so enriched. With the evident definitions one then obtains the category $Ope(\mathcal{E})$ of all small operads enriched in \mathcal{E} .

Remark 2.28. In the presence of coproducts in \mathcal{E} any non-enriched symmetric operad \mathcal{P} gives rise to an operad $Dis(\mathcal{P})$ enriched in \mathcal{E} in which each hom-object is a coproduct, indexed by the corresponding hom-set in \mathcal{P} , of the unit I of \mathcal{E} . We will usually refer to $Dis(\mathcal{P})$ as the corresponding discrete operad in \mathcal{E} and call it again \mathcal{P} .

Our main interest in symmetric operads is in their use in the theory of homotopy invariant algebraic structures where enrichment plays a vital role. However, before we embark on the subtleties of homotopy invariance we briefly treat the isomorphism invariance property for non-enriched symmetric operads.

2.3. The isomorphism invariance property. It is a triviality that an algebraic structure can be transferred, uniquely, along an isomorphism. To be more precise and to formulate this in the language of operads, let \mathcal{P} and \mathcal{E} be symmetric operads and $F : \mathcal{P} \rightarrow \mathcal{E}$ an algebra structure on $\{F_0(P)\}_{P \in \mathcal{P}_0}$. Assume that we are given a family $\{f_P : F_0(P) \rightarrow G_0(P)\}_{P \in \mathcal{P}_0}$ of isomorphisms in \mathcal{E} . Then there exists a unique \mathcal{P} -algebra structure $G : \mathcal{P} \rightarrow \mathcal{E}$ on $\{G_0(P)\}_{P \in \mathcal{P}_0}$ for which the family $\{f_P\}_{P \in \mathcal{P}_0}$ forms a natural isomorphism from F to G and thus an isomorphism between the algebras. We call this the *isomorphism invariance property* of algebras.

We can reformulate this property diagrammatically as follows. Let 0 be a one-object symmetric operad with the identity arrow only, and $0 \rightarrow (0 \rightrightarrows 1)$ the inclusion $0 \mapsto 0$ into the free-living isomorphism. Then a choice of functor $F : \mathcal{P} \rightarrow \mathcal{E}$ is the same as a functor $0 \rightarrow [\mathcal{P}, \mathcal{E}]$ while a functor $(0 \rightrightarrows 1) \rightarrow [\mathcal{P}, \mathcal{E}]$ can be identified with two functors $\mathcal{P} \rightrightarrows \mathcal{E}$ and a natural isomorphism between them. The set of objects \mathcal{P}_0 seen as a category with only identity arrows can be seen as a symmetric operad. One then has the evident inclusion functor $\mathcal{P}_0 \rightarrow \mathcal{P}$ which induces a functor $[\mathcal{P}, \mathcal{E}] \rightarrow [\mathcal{P}_0, \mathcal{E}]$. The isomorphism invariance property for \mathcal{P} -algebras in \mathcal{E} is then the statement that in the following diagram:

$$\begin{array}{ccc}
 0 & \xrightarrow{\forall F} & [\mathcal{P}, \mathcal{E}] \\
 \downarrow & \nearrow \exists \alpha & \downarrow \\
 0 \rightrightarrows 1 & \xrightarrow{\forall \{F_0(P) \rightarrow G_0(P)\}_{P \in \mathcal{P}_0}} & [\mathcal{P}_0, \mathcal{E}]
 \end{array}$$

the diagonal filler exists (and is unique) for any functor $F : \mathcal{P} \rightarrow \mathcal{E}$ and any family of isomorphisms $\{F_0(P) \rightarrow G_0(P)\}_{P \in \mathcal{P}_0}$.

In the formalism of Quillen model structures there is a conceptual way to see why a lift in the diagram above exists. To present it we recall that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ of categories is an *isofibration* if it has the right lifting property with

respect to the inclusion $0 \rightarrow (0 \rightrightarrows 1)$. Similarly, a functor $F : \mathcal{P} \rightarrow \mathcal{Q}$ of symmetric operads is an *isofibration* of symmetric operads if it has the right lifting property with respect to the same inclusion $0 \rightarrow (0 \rightrightarrows 1)$ with each category seen as an operad. Equivalently, $F : \mathcal{P} \rightarrow \mathcal{Q}$ is an isofibration (of operads) if, and only if, $j^*(F)$ is an isofibration of categories. We now recall the operadic Quillen model structure on symmetric operads.

Theorem 2.29. *The category \mathcal{Ope} of symmetric operads with the Boardman-Vogt tensor product admits a cofibrantly generated closed monoidal model structure in which the weak equivalences are the operadic equivalences, the cofibrations are those functors $F : \mathcal{P} \rightarrow \mathcal{Q}$ such that the object part of F is injective, and the fibrations are the isofibrations. All operads are fibrant and cofibrant. The Quillen model structure induced on $\mathcal{Cat} \cong \mathcal{Ope}/\star$ is the categorical one (also known as the ‘folk’ or ‘natural’ model structure).*

PROOF. A direct verification of the axioms of a model category is not difficult and not too tedious. Further details can be found in [38]. \square

Now, in the diagram above the left vertical arrow is a trivial cofibration and the right vertical arrow is, by the monoidal model structure axiom, a fibration and hence the lift exists. We summarize the above discussion:

Fact 2.30. *The notion of algebras of operads is internalized to the category \mathcal{Ope} by it being closed monoidal with respect to the Boardman-Vogt tensor product. The isomorphism invariance property of algebras is captured by the operadic Quillen model structure and its compatibility with the Boardman-Vogt tensor product.*

2.4. The homotopy invariance property. In the presence of homotopy in \mathcal{E} one can ask if a stronger property than the isomorphism invariance property holds. Namely, if one merely asks for the arrows $F_0(P) \rightarrow G_0(P)$ to be weak equivalences instead of isomorphisms is it still possible to transfer the algebra structure? A simple example is when one considers a topological monoid X and a topological space Y together with continuous mappings $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $f \circ g$ and $g \circ f$ are homotopic to the respective identities. It is evident that if f and g are not actual inverses of each other then the monoid structure on X will not, in general, induce a monoid structure on Y . The question as to what kind of structure is induced goes back to Stasheff’s study of H -spaces and his famous associahedra that are used to describe the kind of structure that arises [37]. The more general problem for algebraic structures on topological spaces can be addressed by using enriched symmetric operads as is done by Boardman and Vogt in [5]. Their techniques and results were generalized by Berger and Moerdijk in a series of three papers [1, 2, 3] and below we present an expository account of the constructions we will need.

First we give a slightly vague definition of the homotopy invariance property. In the context of dendroidal sets below we will give a precise definition that is completely analogous to the definition of the isomorphism invariance property.

Definition 2.31. Let \mathcal{E} be a symmetric monoidal model category and \mathcal{Q} a symmetric operad enriched in \mathcal{E} . We say that \mathcal{Q} -algebras have the *homotopy invariance property* if given an algebra $F : \mathcal{Q} \rightarrow \mathcal{E}$ on $\{F_0(Q)\}_{Q \in \mathcal{Q}_0}$ and a family

$\{f_Q : F_0(Q) \rightarrow G_0(Q)\}_{Q \in \mathcal{Q}_0}$ of weak equivalences in \mathcal{E} (with perhaps some extra conditions) there exists an essentially unique \mathcal{Q} -algebra structure $G : \mathcal{Q} \rightarrow \hat{\mathcal{E}}$ on $\{G_0(Q)\}_{Q \in \mathcal{Q}_0}$.

It is evident that an arbitrary symmetric operad \mathcal{P} need not have the homotopy invariance property and the problem of sensibly replacing \mathcal{P} by another operad \mathcal{Q} that does have this property is referred to as the problem of finding the up-to-homotopy version of the algebraic structure classified by \mathcal{P} . To make this notion precise we recall that in [1, 2, 3] Berger and Moerdijk establish the following result.

Theorem 2.32. *Let \mathcal{E} be a cofibrantly generated symmetric monoidal model category. Under mild conditions the category $\text{Ope}(\mathcal{E})_A$ of symmetric operads enriched in \mathcal{E} with fixed set of objects equal to A and whose functors are the identity on all objects admits a Quillen model structure in which the weak equivalences are hom-wise weak equivalences and the fibrations are hom-wise fibrations.*

We refer to this model structure as the Berger-Moerdijk model structure on $\text{Ope}(\mathcal{E})_A$.

Remark 2.33. The Berger-Moerdijk model structure on symmetric operads over a singleton $A = \{*\}$ (given in [1]) settles one of the open problems listed by Hovey in [18].

Among the consequences of the model structure Berger and Moerdijk prove the following.

Theorem 2.34. *If \mathcal{Q} is cofibrant in the Berger-Moerdijk model structure on $\text{Ope}(\mathcal{E})_A$ then \mathcal{Q} -algebras in \mathcal{E} have, under mild conditions, the homotopy invariance property.*

PROOF. See Theorem 3.5 in [1] for more details. \square

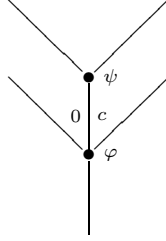
Thus, the problem of finding the up-to-homotopy version of the algebraic structure classified by a symmetric operad \mathcal{P} enriched in \mathcal{E} reduces to finding a cofibrant replacement \mathcal{Q} of \mathcal{P} in the Berger-Moerdijk model structure on $\text{Ope}(\mathcal{E})_{\mathcal{P}_0}$. Of course, a cofibrant replacement always exists just by the presence of the Quillen model structure. However, in order to actually compute with it one needs an efficient construction of it, and this is the aim of the W -construction.

2.4.1. *The original Boardman-Vogt W -construction for topological operads.* The W -construction is a functor $W : \text{Ope}(\text{Top}) \rightarrow \text{Ope}(\text{Top})$ equipped with a natural transformation (an augmentation) $W \rightarrow \text{id}$. A detailed account (albeit in a slightly different language than that of operads) can be found in [5] where it first appeared. We give here an expository presentation aiming at explaining the ideas important to us.

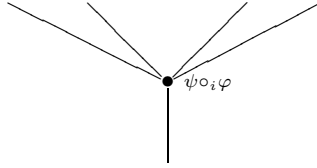
For simplicity let us describe the planar version of the W -construction, that is, we describe a functor taking a planar operad enriched in Top to another such planar operad. We now fix a topological planar operad \mathcal{P} and describe the operad $W\mathcal{P}$. The objects of $W\mathcal{P}$ are the same as those of \mathcal{P} . To describe the arrow spaces we consider standard planar trees (a tree is planar when it comes with an orientation of the edges at each vertex and *standard* means that a choice was made of a single planar tree of each isomorphism class of planar isomorphisms of planar trees) whose edges are labelled by objects of \mathcal{P} and whose vertices are labelled by arrows of \mathcal{P} according to the rule that the objects labelling the input edges of a

vertex are equal (in their natural order) to the input of the operation labelling that vertex. Similarly the object labelling the output of the vertex is the output object of the operation at the vertex. Moreover, each inner edge in such a tree is given a length $0 \leq t \leq 1$. For objects $P_0, \dots, P_n \in (W\mathcal{P})_0$ let $A(P_1, \dots, P_n; P_0)$ be the topological space whose underlying set is the set of all such planar labelled trees \bar{T} for which the leaves of \bar{T} are labelled by P_1, \dots, P_n (in that order) and the root of \bar{T} is labelled by P_0 . The topology on $A(P_1, \dots, P_n; P_0)$ is the evident one induced by the topology of the arrow spaces in \mathcal{P} and the standard topology on the unit interval $[0, 1]$.

The space $W\mathcal{P}(P_1, \dots, P_n; P_0)$ is the quotient of $A(P_1, \dots, P_n; P_0)$ obtained by the following identifications. If $\bar{T} \in A(P_1, \dots, P_n; P_0)$ has an inner edge e whose length is 0 then we identify it with the tree \bar{T}/e obtained from \bar{T} by contracting the edge e and labelling the newly formed vertex by the corresponding \circ_i -composition of the operations labelling the vertices at the two sides of e (the other labels are as in \bar{T}). Thus pictorially we have that locally in the tree a configuration

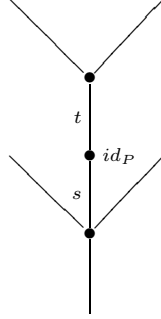


is identified with the configuration

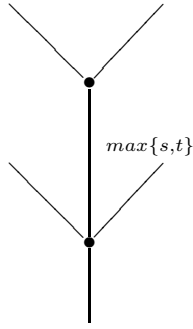


Another identification is in the case of a tree \bar{S} with a unary vertex v labelled by an identity. We identify such a tree with the tree \bar{R} obtained by removing the vertex v and identifying its input edge with its output edge. The length assigned to the new edge is determined as follows. If it is an outer edge then it has no length. If it is an inner edge then it is assigned the maximum of the lengths of s and t (where if either s or t does not have a length, i.e., it is an outer edge, then its length is considered to be 0). The labelling is as in \bar{S} (notice that the label of the newly formed edge is unique since v was labelled by an identity which means that its input and output were labelled by the same object). Pictorially, this identification

identifies the labelled tree



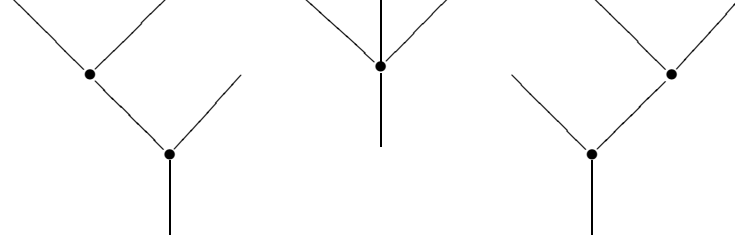
with the tree



The composition in $W\mathcal{P}$ is given by grafting such labelled trees, giving the newly formed inner edge length 1. The augmentation $W\mathcal{P} \rightarrow \mathcal{P}$ is the identity on objects and sends an arrow represented by such a labeled tree to the operation obtained by contracting all lengths of internal edges to 0 and composing in \mathcal{P} . We leave the necessary adaptations needed for obtaining the symmetric version of the W -construction to the reader.

Example 2.35. Let \mathcal{P} be the planar operad with a single object and a single n -ary operation in each arity $n \geq 1$ and no arrows of arity 0. We consider \mathcal{P} to be a discrete operad in Top . It is easily seen that a functor $\mathcal{P} \rightarrow Top$ corresponds to a non-unital topological monoid (we treat this case for simplicity). Let us now calculate the first few arrow spaces in $W\mathcal{P}$. Firstly, $W\mathcal{P}$ too has just one object. We thus use the notation of classical operads, namely $W\mathcal{P}(n)$ for the space of operations of arity n . Clearly $W\mathcal{P}(0)$ is just the empty space. The space $W\mathcal{P}(1)$ consists of labelled trees with one input. Since in such a tree the only possible label at a vertex is the identity, the identification regarding identities implies that $W\mathcal{P}(1)$ is again just a one-point space. In general, since every unary vertex in a labelled tree in $W\mathcal{P}(n)$ can only be labelled by the identity, and those are then identified with trees not containing unary vertices, it suffices to only consider reduced trees, namely trees with no unary vertices. To calculate $W\mathcal{P}(2)$ we need to consider all reduced trees with two inputs, but there is just one such tree, the 2-corolla, and it has no inner edges, thus $W\mathcal{P}(2)$ is also a one-point space. Things become more

interesting when we calculate $W\mathcal{P}(3)$. We need to consider reduced trees with three inputs. There are three such trees, namely



The middle tree contributes a point to the space $W\mathcal{P}(3)$. Each of the other trees has one inner edge and thus contributes the interval $[0, 1]$ to the space. The only identification to be made is when the length of one of those inner edges is 0, in which case it is identified with the point corresponding to the middle tree. The space $W\mathcal{P}(3)$ is thus the gluing of two copies of the interval $[0, 1]$ where we identify both ends named 0 to a single point. The result is then just a closed interval, $[-1, 1]$. However, it is convenient to keep in mind the trees corresponding to each point of this interval. Namely, the tree corresponding to the middle point, 0, is the middle tree. With a point $0 < t \leq 1$ corresponds the tree on the right where the length of the inner edge is t , and with a point $-1 \leq -t < 0$ corresponds the tree on the left where its inner edge is given the length t . In this way one can calculate the entire operad $W\mathcal{P}$. It can then be shown that the spaces $\{W\mathcal{P}(n)\}_{n=0}^{\infty}$, reproduce, up to homeomorphism, the Stasheff associahedra. An A_{∞} -space is then an algebra over $W\mathcal{P}$ and $W\mathcal{P}$ classifies A_{∞} -spaces and their *strong* morphisms.

2.4.2. The Berger-Moerdijk generalization of the W -construction to operads enriched in a homotopy environment. Observe that in the W -construction given above one can construct the space $W\mathcal{P}(P_1, \dots, P_n; P_0)$ as follows. For each labelled planar tree \bar{T} as above let $H^{\bar{T}}$ be H^k where k is the number of inner edges in \bar{T} and $H = [0, 1]$, the unit interval. Further, for each vertex v of \bar{T} let $\mathcal{P}(v) = \mathcal{P}(x_1, \dots, x_n; x_0)$ where x_1, \dots, x_n are (in that order) the inputs of v and x_0 its output. Finally, let $\mathcal{P}(\bar{T})$ be the product of $\mathcal{P}(v)$ where v ranges over the vertices of \bar{T} . Now, The space $A(P_1, \dots, P_n; P_0)$ constructed above is homeomorphic to $\coprod_{\bar{T}} (H^{\bar{T}} \times \mathcal{P}(\bar{T}))$ where \bar{T} varies over all labelled standard planar trees \bar{T} whose leaves are labelled by P_1, \dots, P_n and whose root is labelled by P_0 . The identifications that are then made to construct the space $W\mathcal{P}(P_1, \dots, P_n; P_0)$ are completely determined by the combinatorics of the various trees \bar{T} . This observation is the key to generalizing the W -construction to symmetric operads in monoidal model categories \mathcal{E} other than Top and is carried out in [2, 3]. What is needed is a suitable replacement for the unit interval $[0, 1]$ used above to assign lengths to the inner edges of the trees. Such a replacement is the notion of an interval object in a monoidal model category \mathcal{E} given in [2].

Definition 2.36. Let \mathcal{E} be a symmetric monoidal model category \mathcal{E} with unit I . An *interval* object in \mathcal{E} (see Definition 4.1 in [2]) is a factorization of the codiagonal $I \amalg I \rightarrow I$ into a cofibration $I \amalg I \rightarrow H$ followed by a weak equivalence $\epsilon : H \rightarrow I$ together with an associative operation $\vee : H \otimes H \rightarrow H$ which has a neutral element, an absorbing element, and for which ϵ is a counit. For convenience, when an interval element is chosen we will refer to (\mathcal{E}, H) as a *homotopy environment*.

Relevant examples to our presentation are the ordinary unit interval in Top with the standard model structure (with $x \vee y = \max\{x, y\}$) and the free-living isomorphism $0 \rightrightarrows 1$ in Cat with the categorical model structure.

In such a setting the topological W -construction can be mimicked by gluing together objects $H^{\otimes k}$ instead of cubes $[0, 1]^k$. This is done in detail in [2], to which the interested reader is referred. We thus obtain a functor $W_H : Ope(\mathcal{E}) \rightarrow Ope(\mathcal{E})$ for any homotopy environment \mathcal{E} . Usually we will just write W instead of W_H , which is quite a harmless convention since Proposition 6.5 in [2] guarantees that under mild conditions a different choice of interval object yields essentially equivalent W -constructions.

Example 2.37. Consider the category Cat with the categorical model structure. In this monoidal model category we can choose the category H to be the free-living isomorphism $0 \rightrightarrows 1$ as interval object, with the obvious structure maps. Let us again consider the planar operad \mathcal{P} classifying non-unital associative monoids, this time as a discrete operad in Cat . To calculate $W\mathcal{P}(n)$ we should again consider labelled standard planar trees with lengths. The same argument as above implies that we should only consider reduced trees, and a similar calculation shows that $W\mathcal{P}(n)$ is a one-point category for $n = 1, 2$. Now, to calculate $W\mathcal{P}(3)$ we again consider the three trees as given above. This time the middle tree contributes the category $H^0 = I$. Each of the other trees contributes the category H . The identifications identify the object named 0 in each copy of H with the unique object of I . The result is a contractible category with three objects. In general, the category $W\mathcal{P}(n)$ is a contractible category with $tr(n)$ objects, where $tr(n)$ denotes the number of reduced standard planar trees with n leaves. The composition in $W\mathcal{P}$ is given by grafting of such trees. The operad $W\mathcal{P}$ classifies unbiased monoidal categories and strict monoidal functors (an unbiased monoidal category is a category with an n -ary multiplication functor for each $n \geq 0$ together with some coherence conditions. See [24] for more details as well as a discussion about the equivalence of such categories and ordinary weak monoidal categories).

The generalized Boardman-Vogt W -construction thus provides a computationally tractable way to classify weak algebras for a wide variety of structures in a homotopy environment. However, $W\mathcal{P}$ tends to classify weak \mathcal{P} -algebra with their strong morphisms and not with their weak morphisms. Indeed, for some fixed homotopy environment \mathcal{E} assume that $Ope(\mathcal{E})$ is closed monoidal with respect to a Boardman-Vogt type tensor product. If we now consider for a symmetric operad $\mathcal{P} \in Ope(\mathcal{E})_0$ the internal hom $[W\mathcal{P}, \mathcal{E}]$ then the elements of $[W\mathcal{P}, \mathcal{E}]_0$ are precisely the weak \mathcal{P} -algebras in \mathcal{E} . However, a unary arrow in $[W\mathcal{P}, \mathcal{E}]$ corresponds to a map of symmetric operads $(W\mathcal{P}) \otimes [1] \rightarrow \mathcal{E}$ (where $[1]$ is the operad $0 \rightarrow 1$ considered as a discrete operad in \mathcal{E}). This already shows that the notion one gets is of strong (because W does not act on $[1]$) morphisms between weak (because W does act on \mathcal{P}) \mathcal{P} -algebras.

2.5. Weak maps between weak algebras. Luckily, to arrive at the right notion of weak morphisms between weak algebras no extra work is needed. Following on the observation above we make the following definition.

Definition 2.38. Let \mathcal{P} be an operad in Set and \mathcal{E} a homotopy environment. A *weak \mathcal{P} -algebra* in \mathcal{E} is a functor of symmetric \mathcal{E} -enriched operads $W(\mathcal{P}) \rightarrow \hat{\mathcal{E}}$.

A *weak map* between up-to-homotopy \mathcal{P} -algebras in \mathcal{E} is a functor of symmetric \mathcal{E} -enriched operads $W(\mathcal{P} \otimes [1]) \rightarrow \hat{\mathcal{E}}$.

An obvious question now is whether the collection of all weak \mathcal{P} -algebras and their weak maps forms a category. The answer is that they usually do not. A simple example is provided by A_∞ -spaces where it is known that weak A_∞ -maps do not compose associatively. The theory so far already suggests a solution to that problem. We denote by $[n]$ the operad $0 \rightarrow 1 \rightarrow \cdots \rightarrow n$ seen as a discrete operad in \mathcal{E} . For a symmetric operad \mathcal{P} in \mathbf{Set} consider the symmetric operad $\mathcal{P} \otimes [n]$. An algebra for such a symmetric operad is easily seen to be a sequence X_0, \dots, X_n of \mathcal{P} -algebras together with weak \mathcal{P} -algebra maps:

$$X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n$$

and all their possible compositions.

Proposition 2.39. *Let \mathcal{P} be a symmetric operad in \mathbf{Set} and \mathcal{E} a homotopy environment. For each $n \geq 0$ let X_n be the set of maps*

$$W(\mathcal{P} \otimes [n]) \rightarrow \hat{\mathcal{E}}$$

of symmetric operads enriched in \mathcal{E} . Then the collection $X = \{X_n\}_{n=0}^\infty$ can be canonically made into a simplicial set.

PROOF. The proof follows easily by noting that the sequence $\{\mathcal{P} \otimes [n]\}_{n=0}^\infty$ is a cosimplicial object in \mathbf{Ope} . \square

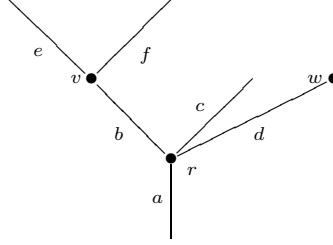
Definition 2.40. We refer to the simplicial set constructed above as the *simplicial set of weak \mathcal{P} -algebras* in \mathcal{E} and denote it by $wAlg[\mathcal{P}, \mathcal{E}]$.

Recall that for strict algebras one could easily iterate structures simply by considering $[\mathcal{P}, [\mathcal{P}, \mathcal{E}]]$ which are classified by $\mathcal{P} \otimes \mathcal{P}$. Our journey into weak algebras in a homotopy environment \mathcal{E} led us to the formation of the simplicial set $wAlg[\mathcal{P}, \mathcal{E}]$ with the immediate drawback that we cannot, at least not in any straightforward manner, iterate. This problem disappears in the dendroidal setting, as we will see below, and is one of the technical advantages of dendroidal sets over enriched operads in the study of weak algebraic structures.

3. Dendroidal sets - a formalism for weak algebras

We now return to non-enriched symmetric operads and introduce the category of dendroidal sets, which is the natural category in which to define nerves of symmetric operads. The category of dendroidal sets is a presheaf category on the dendroidal category Ω and as such one might expect it to be adequate only for the study of non-enriched symmetric operads. However, we will see that it is in fact versatile enough to treat enriched operads quite efficiently by means of the homotopy coherent nerve construction. We do mention that for weak algebraic structures in certain homotopy environments (such as differentially graded vector spaces) dendroidal sets are inappropriate. One might then consider dendroidal objects instead of dendroidal sets as is explained in [34], which also contains all of the results below.

3.1. The dendroidal category Ω . To define the dendroidal category Ω recall the definition of symmetric rooted trees given above. It is evident that any such tree T can be thought of as a picture of a symmetric operad $\Omega(T)$: The objects of $\Omega(T)$ are the edges of T and the arrows are freely generated by the vertices of T . In more detail, consider the tree T given by



then $\Omega(T)$ has six objects, a, b, \dots, f and the following generating operations:

$$r \in \Omega(T)(b, c, d; a),$$

$$w \in \Omega(T)(-; d)$$

and

$$v \in \Omega(T)(e, f; b).$$

The other operations are units (such as $1_b \in \Omega(T)(b; b)$), arrows obtained freely by the Σ_n actions, and formal compositions of such arrows.

Definition 3.1. Fix a countable set X . The *dendroidal category* Ω has as objects all symmetric rooted trees T whose edges $E(T)$ satisfy $E(T) \subseteq X$. The arrows $S \rightarrow T$ in Ω are arrows $\Omega(S) \rightarrow \Omega(T)$ of symmetric operads.

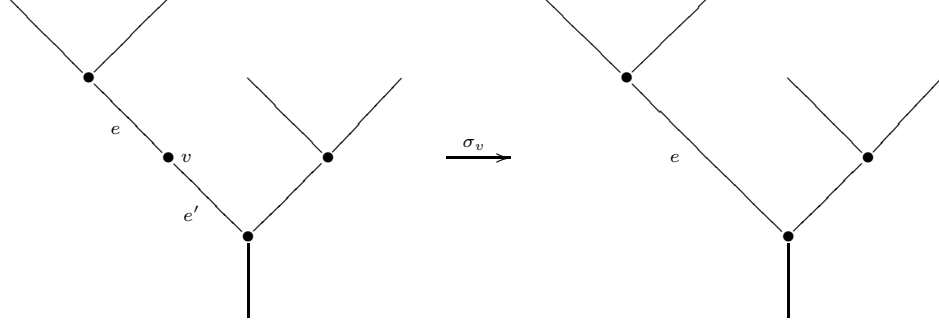
Remark 3.2. The role of the set X above should be thought of as the role variables play in predicate calculus. The edges are only there to be carriers of symbols and countably many such carriers will always be enough. Of course, another choice of X would result in an isomorphic category. Note that the dendroidal category Ω is thus small (in fact is itself countable).

Recall the linear trees L_n and that the simplicial category Δ is a skeleton of the category of finite linearly ordered sets and order preserving maps. The subcategory of Ω spanned by all trees of the form L_n with $n \geq 0$ is easily seen to be equivalent to the simplicial category Δ . In fact, Δ can be recovered from Ω in a more useful way.

Proposition 3.3. (*Slicing lemma for the dendroidal category*) *The simplicial category Δ is obtained (up to equivalence) from the dendroidal category Ω by slicing over the linear tree with one edge and no vertices: $\Delta \cong \Omega/L_0$.*

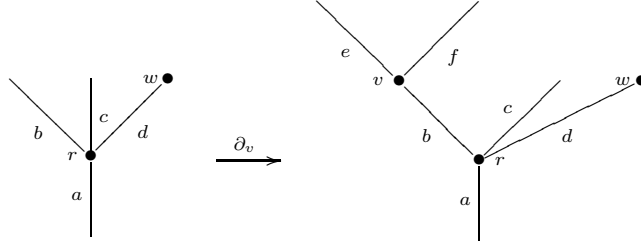
We now describe several types of arrows that generate all of the arrows in Ω . Let T be a tree and v a vertex of valence 1 with $\text{in}(v) = e$ and $\text{out}(v) = e'$. Consider the tree T/v , obtained from T by deleting the vertex v and the edge e' , pictured

locally as

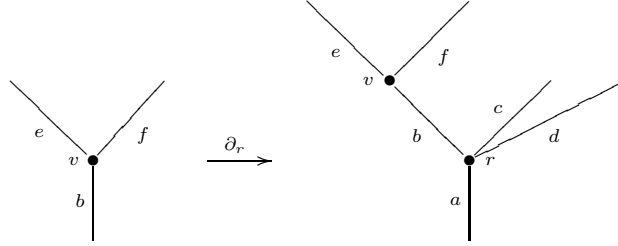


There is then a map in Ω , denoted by $\sigma_v : T \rightarrow T/v$, which sends e and e' in T to e in T/v . An arrow in Ω of this kind is called a *degeneracy*.

Consider now a tree T and a vertex v in T with exactly one inner edge attached to it. One can obtain a new tree T/v by deleting v and all the outer edges attached to it to obtain, by inclusion of edges, the arrow $\partial_v : T/v \rightarrow T$ in Ω called an *outer face*. For example,

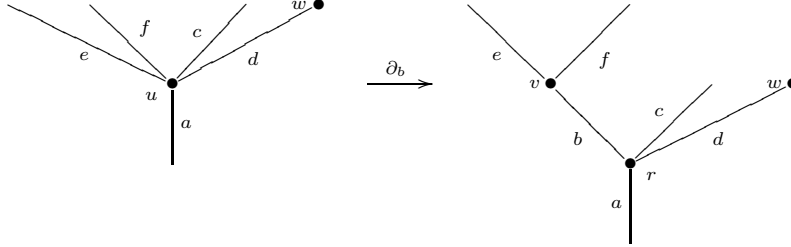


and (to emphasize that it is sometimes possible to remove the root of the tree T)



are both outer faces.

Given a tree T and an inner edge e in T , one can obtain a new tree T/e by contracting the edge e . One then obtains, by inclusion of edges, the map $\partial_e : \Omega(T/e) \rightarrow \Omega(T)$ in Ω called an *inner face*. For example,



Theorem 3.4. Any map $T \xrightarrow{f} T'$ in Ω factors uniquely as $f = \varphi\pi\delta$, where δ is a composition of degeneracy maps, π is an isomorphism, and φ is a composition of (inner and outer) face maps.

This result generalizes the familiar simplicial relations in the definition of a simplicial set.

3.1.1. *The category of dendroidal sets.*

Definition 3.5. The category of *dendroidal sets* is the presheaf category $dSet = Set_\Omega$. Thus a dendroidal set X consists of a collection of sets $\{X_T\}_{T \in \Omega_0}$ together with various maps between them. An element $x \in X_T$ is called a *dendrex* of shape T , or a T -*dendrex*.

For each tree $T \in \Omega_0$ there is associated the *representable dendroidal set* $\Omega[T] = \Omega(-, T)$ which, by the Yoneda Lemma, serves to classify T dendrices in X via the natural bijection $X_T \cong dSet(\Omega[T], X)$. The functor $\Omega \rightarrow Ope$ which sends T to $\Omega(T)$ induces an adjunction $dSet \xrightleftharpoons[N_d]{\tau_d} Ope$, of which N_d , called the *dendroidal nerve functor*, is given explicitly, for a symmetric operad \mathcal{P} , by

$$N_d(\mathcal{P})_T = Ope(\Omega(T), \mathcal{P}).$$

For linear trees L_n we write somewhat ambiguously X_n instead of X_{L_n} . This is a harmless convention since for any two linear trees L_n and L'_n there is a *unique* isomorphism $L_n \rightarrow L'_n$ in Ω . Consider the dendroidal set $\star = \Omega[L_0]$.

Lemma 3.6. (*Slicing lemma for dendroidal sets*) *There is an equivalence of categories $dSet/\star \cong sSet$. If we identify $sSet$ as a subcategory of $dSet$ then the forgetful functor $i_! : sSet \rightarrow dSet$ has a right adjoint i^* which itself has a right adjoint i_* .*

PROOF. We omit the details and just remark that the adjunctions mentioned can be obtained (equivalently) in one of two ways. The first is to consider Δ as a subcategory of Ω via an embedding functor $i : \Delta \rightarrow \Omega$. This functor i then induces a functor $i^* : dSet \rightarrow sSet$ which, from the general theory of presheaf categories (see e.g., [31]), has a left adjoint $i_!$ and a right adjoint i_* . The second way to obtain the adjunctions is to use Remark 2.14, with $\mathcal{C} = dSet$ and $A = \Omega[L_0]$. \square

Proposition 3.7. *Slicing the adjunction $dSet \xrightleftharpoons[N_d]{\tau_d} Ope$ over \star gives the usual*

adjunction $sSet \xrightleftharpoons[N]{\tau} Cat$ with N the nerve functor and τ the fundamental category functor.

PROOF. The precise meaning of the statement is that denoting a one-object operad with just the identity arrow again by \star and for the dendroidal set $\star = \Omega[L_0]$ one has, by slight abuse of notation, that $N_d(\star) = \star$ and $\tau_d(\star) = \star$; thus the functors N_d and τ_d restrict to the respective slices $dSet/\star$ and Ope/\star . Then under the identifications $sSet \cong dSet/\star$ and $Cat \cong Ope/\star$ these restrictions give the nerve functor $N : Cat \rightarrow sSet$ and its left adjoint τ . \square

A general rule of thumb is that any definition or theorem of dendroidal sets will yield, by slicing over $\star = \Omega[L_0]$, a corresponding definition or theorem of simplicial sets. A similar principle is true for operads and categories. We will loosely refer to this process as 'slicing' and say, in the example above for instance, that the usual nerve functor of categories is obtained by slicing the dendroidal nerve functor.

Definition 3.8. Let X and Y be two dendroidal sets. Their tensor product is given by the colimit

$$X \otimes Y = \lim_{\Omega[T] \rightarrow X, \Omega[S] \rightarrow Y} N_d(\Omega(T) \otimes \Omega(S)),$$

Here we use the canonical expression of a presheaf as a colimit of representables.

As an example of our convention about slicing we mention that slicing the tensor product of dendroidal sets yields the cartesian product of simplicial sets. Note however, that the tensor product in $dSet$ is not the cartesian product.

Theorem 3.9. *The category $dSet$ with the tensor product defined above is a closed monoidal category.*

PROOF. This follows by general abstract nonsense. The internal hom is given for two dendroidal sets X and Y by

$$[X, Y]_T = dSet(X \otimes \Omega[T], Y).$$

\square

Slicing this theorem proves that $sSet$ is cartesian closed with the usual formula for the internal hom.

Theorem 3.10. *In the diagram*

$$\begin{array}{ccc} Cat & \xrightleftharpoons[j^*]{j_!} & Ope \\ \tau \uparrow \downarrow N & & \tau_d \uparrow \downarrow N_d \\ sSet & \xrightleftharpoons[i^*]{i_!} & dSet \end{array}$$

all pairs of functors are adjunctions with the left adjoint on top or to the left. Furthermore, the following canonical commutativity relations hold:

$$\begin{aligned} \tau N &\cong id \\ \tau_d N_d &\cong id \\ i^* i_! &\cong id \\ j^* j_! &\cong id \\ j_! \tau &\cong \tau_d i_! \\ N j^* &\cong i^* N_d \\ i_! N &\cong N_d j_!. \end{aligned}$$

If we consider the cartesian structures on Cat and $sSet$, the Boardman-Vogt tensor product on Ope , and the tensor product of dendroidal sets then the four categories are symmetric closed monoidal categories and the functors $i_!$, N , τ , $j_!$ and τ_d are strong monoidal.

Remark 3.11. The dendroidal nerve functor N_d is not monoidal, a fact that plays a vital role in the applicability of dendroidal sets to iterated weak algebraic structures, as we will see below.

We do have the following property.

Proposition 3.12. For symmetric operads \mathcal{P} and \mathcal{Q} there is a natural isomorphism

$$\tau_d(N_d(\mathcal{P}) \otimes N_d(\mathcal{Q})) \cong \mathcal{P} \otimes \mathcal{Q}.$$

Lemma 3.13. The dendroidal nerve functor commutes with internal Homs in the sense that for any two operads \mathcal{P} and \mathcal{Q} we have

$$N_d([\mathcal{P}, \mathcal{Q}]) \cong [N_d(\mathcal{P}), N_d(\mathcal{Q})].$$

Moreover, for simplicial sets X and Y we have

$$[i_!(X), i_!(Y)] \cong i_!([X, Y]).$$

The proofs of these results are not hard.

3.2. Algebras in the category of dendroidal sets. We again introduce a syntactic difference between dendroidal sets thought of as encoding structure and dendroidal sets as environments to interpret structures in.

Definition 3.14. Let E and X be dendroidal sets. The dendroidal set $[X, E]$ is called the dendroidal set of X -algebras in E . An element in $[X, E]_{L_0}$ is called an X -algebra in E . An element of $[X, E]_{L_1}$ is called a map of X -algebras in E .

Let us first note that this definition extends the notion of \mathcal{P} -algebras in \mathcal{E} for symmetric operads in the sense that for symmetric operads \mathcal{P} and \mathcal{E} there is a natural isomorphism

$$[N_d(\mathcal{P}), N_d(\mathcal{E})] \cong N_d([\mathcal{P}, \mathcal{E}]).$$

Indeed, this is just the statement that N_d commutes with internal Homs.

We thus see that the dendroidal nerve functor embeds Ope in $dSet$ in such a way that the notion of algebras is retained and in both cases is internalized in the form of an internal Hom with respect to a suitable tensor product. We now wish to study homotopy invariance of algebra structures in a dendroidal set E . The first step is to specify those arrows along which such algebras are to be invariant.

Recall that for symmetric operads the diagram

$$\begin{array}{ccc} 0 & \rightarrow & 1 \xrightarrow{f} \mathcal{P} \\ & \downarrow & \nearrow (f,g) \\ 0 & \hookrightarrow & 1 \end{array}$$

(where the vertical arrow is the inclusion of the free-living arrow into the free-living isomorphism) admits a lift precisely when the arrow f admits an inverse g . Taking the dendroidal nerve of this diagram and replacing in it $N_d(\mathcal{P})$ by an arbitrary dendroidal set X we arrive at the following definition.

Definition 3.15. Let X be a dendroidal set. An *equivalence* is a dendrex $x : \Omega[L_1] \rightarrow X$ such that in the diagram

$$\begin{array}{ccc} \Omega[L_1] & \xrightarrow{x} & X \\ \downarrow & \nearrow \hat{x} & \\ N_d(0 \hookrightarrow 1) & & \end{array}$$

a lift \hat{x} exists.

Note, that since $\Omega[L_1] \cong i_!(\Delta[1])$ and $N_d(0 \hookrightarrow 1) = i_!(N(0 \hookrightarrow 1))$, by adjunction the dendrex $x : \Omega[L_1] \rightarrow X$ is an equivalence if, and only if, in the corresponding diagram of simplicial sets

$$\begin{array}{ccc} \Delta[1] & \xrightarrow{x} & i^*(X) \\ \downarrow & \nearrow & \\ N(0 \hookrightarrow 1) & & \end{array}$$

a lift exists. Thus, being a weak equivalence in the dendroidal set X is actually a property of the simplicial set $i^*(X)$.

Remark 3.16. Note that $N(0 \hookrightarrow 1)$ is the simplicial infinite dimensional sphere S^∞ and thus a lift $S^\infty \rightarrow i^*(X)$ is a rather complicated object. Intuitively, it is a coherent choice of a homotopy inverse of the simplex $x : \Delta[1] \rightarrow i^*(X)$, together with coherent choices of homotopies, homotopies between homotopies, etc. See [19] for more details. Note moreover, that an equivalence in X is in some sense as weak as X would allow it to be. If $X = N_d(\mathcal{P})$ then a dendrex $\Omega[L_1] \rightarrow X$ is an equivalence if, and only if, the corresponding unary arrow in \mathcal{P} is an isomorphism. We will see below a more refined nerve construction in which equivalences correspond to a notion weaker than isomorphism.

We can now formulate the homotopy invariance property in the language of dendroidal sets. Let X and E be dendroidal sets. We identify, somewhat ambiguously, the set X_η with the dendroidal set $\coprod_{x \in X_\eta} \Omega[\eta]$. Then there is a map of dendroidal sets $X_\eta \rightarrow X$ which induces a mapping $[X, E] \rightarrow [X_\eta, E]$. Consider now a family $\{f_x\}_{x \in X_\eta}$ where each f_x is an equivalence in X . Then this family can be extended (usually in many different ways) to give a map $\hat{f} : N_d(0 \hookrightarrow 1) \rightarrow [X_\eta, E]$.

Definition 3.17. Let X and E be a dendroidal sets. We say that X -algebras in E have the *homotopy invariance property* if for every X -algebra in E , given by $F : X \rightarrow E$, and any family $\{f_x\}_{x \in X_\eta}$ and any extension of it to \hat{f} as above that fit into the commutative diagram

$$\begin{array}{ccc} \Omega[\eta] & \xrightarrow{\forall F} & [X, E] \\ \downarrow & \nearrow \exists \alpha & \downarrow \\ N_d(0 \hookrightarrow 1) & \xrightarrow{\hat{f}} & [X_\eta, E] \end{array}$$

a lift α exists.

Intuitively, the lift α consists of two X -algebras in E , the first being F and the second one being obtained by transferring the X -algebra structure given by F along the equivalences $\{f_x\}_{x \in X_\eta}$.

3.3. The homotopy coherent nerve and weak algebras. We now show how dendroidal sets enter the picture in the context of operads enriched in a symmetric closed monoidal model category \mathcal{E} with a chosen interval object (which we call a homotopy environment). Recall then that the Berger-Moerdijk generalization of the Boardman-Vogt W -construction sends a symmetric operad \mathcal{P} enriched in \mathcal{E} to a cofibrant replacement $W\mathcal{P}$. Recall as well that any non-enriched symmetric operad can be seen as a discrete symmetric operad enriched in \mathcal{E} .

Definition 3.18. Fix a homotopy environment \mathcal{E} . Given a symmetric operad \mathcal{P} enriched in \mathcal{E} its *homotopy coherent dendroidal nerve* is the dendroidal set whose set of T -dendrices is

$$hcN_d(\mathcal{P})_T = \text{Ope}(\mathcal{E})(W(\Omega(T)), \mathcal{P})$$

of \mathcal{E} -enriched functors between \mathcal{E} -enriched operads, where $\Omega(T)$ is seen as a discrete operad enriched in \mathcal{E} .

The homotopy coherent dendroidal nerve construction, together with the closed monoidal structure on $dSet$ given above, allows for the internalization of the notion of weak algebras. We illustrate this:

Definition 3.19. Let \mathcal{P} be a non-enriched symmetric operad and \mathcal{E} a homotopy environment. The dendroidal set $[N_d(\mathcal{P}), hcN_d(\hat{\mathcal{E}})]$ is called the dendroidal set of *weak \mathcal{P} -algebras in \mathcal{E}* . Here we view \mathcal{E} as an operad enriched in itself (since \mathcal{E} is assumed closed) and thus $\hat{\mathcal{E}}$ as a symmetric operad enriched in \mathcal{E} is well-defined.

It can be shown that the L_0 dendrices in $[N_d(\mathcal{P}), hcN_d(\hat{\mathcal{E}})]$ correspond to symmetric operad maps $W(\mathcal{P}) \rightarrow \hat{\mathcal{E}}$ and thus are weak \mathcal{P} -algebras. Moreover, the L_1 dendrices can be seen to correspond to symmetric operad maps $W(\mathcal{P} \otimes [1]) \rightarrow \hat{\mathcal{E}}$ and thus are weak maps of weak \mathcal{P} -algebras. We have thus recovered an internalization of weak algebras and their weak maps and can now consider iterated weak algebraic structures completely analogously to the way this can be done in the context of non-enriched symmetric operads. We illustrate how this works in two examples below.

3.4. Application to the study of A_∞ -spaces and weak n -categories.

Recall that an A_∞ -space is an algebra for the topologically enriched operad $W(As)$, where As is the non-enriched symmetric operad that classifies monoids. Let $A = N_d(As)$; then, by definition, $[A, hcN_d(Top)]$ is the dendroidal set of A_∞ -spaces and their weak (multivariable) mappings. In the classical definition of A_∞ -spaces it is not at all clear how to define n -fold A_∞ -spaces. However, we now have a perfectly natural such definition.

Definition 3.20. The dendroidal set nA_∞ of n -fold A_∞ -spaces is defined recursively as follows. For $n = 1$ we set $1A_\infty = [A, hcN_d(Top)]$ and for $n \geq 1$: $(n+1)A_\infty = [A, nA_\infty]$.

Thus, we obtain at once notions of weak multivariable mappings of n -fold A_∞ -spaces. And, since $dSet$ is closed monoidal, we can immediately classify n -fold A_∞ -spaces.

Proposition 3.21. *For any $n \geq 1$ the dendroidal set $A^{\otimes n}$ classifies n -fold A_∞ -spaces.*

It is at this point not known exactly how n -fold A_∞ -spaces relate to n -fold loop spaces. However, the recent work [11] of Fiedorowicz and Vogt on interchanging A_∞ and E_n structures is a first step towards a full comparison of the dendroidal and classical approaches.

Remark 3.22. Note that were the dendroidal nerve functor monoidal our definition of n -fold A_∞ -spaces would stabilize at $n = 2$. Indeed, we would then have $A^{\otimes n} = N_d(As)^{\otimes n} = N_d(As^{\otimes n}) = N_d(Comm)$.

A similar application, but technically slightly more complicated, is to obtain an iterative definition of weak n -categories. First notice that the fact that categories, as well as symmetric operads, can be enriched in a symmetric monoidal category \mathcal{E} is a consequence of the ability to in fact enrich in an arbitrary symmetric operad. We leave the details of defining what a category (or operad) enriched in a symmetric operad \mathcal{E} is to the reader and only mention that this is related to the idea of enriching in an fc -multicategory (see [26]). We now show how in fact categories (and operads) can be enriched in a dendroidal set. Recall from Example 2.25 that for any set A there is a symmetric operad C_A that classifies categories over A .

Once more, the ability to easily iterate within the category of dendroidal sets naturally leads to a definition of weak n -categories enriched in a dendroidal set X as follows.

Definition 3.23. Let X be a dendroidal set. The dendroidal set $[N_d(C_A), X]$ is called the dendroidal set of *categories over A enriched in X* and is denoted by $Cat(X)_A$.

It can easily be verified that enriching in the dendroidal nerve of $\hat{\mathcal{E}}$ for \mathcal{E} a symmetric monoidal category agrees with the notion of enrichment in the usual sense.

At this point we would like to collate the various dendroidal sets $Cat(X)_A$ into a single dendroidal sets. There is here a technical difficulty and so as not to interrupt the flow of the presentation we refer the reader to Section 4.1 of [34] for the details of the construction. One then obtains the dendroidal set $Cat(X)$ of categories enriched in X . Similarly, using the operad O_A classifying symmetric operads over A , we can obtain the dendroidal set $Ope(X)$ of symmetric operads enriched in X .

Definition 3.24. Let X be a dendroidal set. Let ${}_0Cat(X) = X$ and define recursively ${}_{n+1}Cat(X) = Cat({}_nCat(X))$ for each $n \geq 1$. We call ${}_nCat(X)$ the dendroidal set of *n -categories enriched in X* .

In particular, considering the category Cat with its categorical model structure and taking $X = hcN_d(Cat)$ we obtain for each $n \geq 0$ the dendroidal set ${}_nCat = {}_nCat(X)$, which we call the dendroidal set of weak n -categories. In [27] the dendrices in ${}_2Cat(X)_\eta$ and ${}_3Cat(X)_\eta$ are compared with other definitions of weak 2-categories and weak 3-categories to show that the notions are in fact equivalent. The complexity of such comparisons increases rapidly with n and is currently, as is the case with many definitions of weak n -categories (see [25] for a survey of such), not settled. We mention that we can also consider categories weakly enriched in other dendroidal sets such as $hcN_d(Top)$ or $hcN_d(sSet)$, where again a

full comparison with existing structures is yet to be completed. Of course, we can also consider weak n -operads of various sorts as well.

We conclude this section by considering the Baez-Breen-Dolan stabilization hypothesis for weak n -categories as defined above. With every reasonable definition of weak n -categories there is usually associated a notion of k -monoidal n -categories for every $k \geq 0$. These are weak $(n + k)$ -categories having trivial information in all dimensions up to and including k . The stabilization hypothesis is that for fixed n the complexity of these structures stabilizes at $k = n + 2$. Given a concrete definition of weak n -categories this hypothesis can be made exact and it becomes a conjecture. In our case we proceed as follows.

Definition 3.25. Let $n \geq 0$ be fixed. For $k \geq 0$ we define recursively the dendroidal set $wCat_k^n$ of *weak k -monoidal n -categories* as follows. For $k = 0$ we set $wCat_0^n = {}_nCat$ and for $k > 0$ we define $wCat_k^n = [A, wCat_{k-1}^n]$, where $A = N_d(As)$. A dendrex of shape η in $wCat_k^n$ is called a *k -monoidal n -category*.

Conjecture 3.26. (*The Baez-Breen-Dolan stabilization hypothesis for our notion of n -categories*) For a fixed $n \geq 0$, there is an isomorphism of dendroidal sets between $wCat_k^n$ and $wCat_{n+2}^n$ for any $k \geq n + 2$.

4. Dendroidal sets - models for ∞ -operads

In this section we show how dendroidal sets are used to model ∞ -operads. As very brief motivation for the concepts to follow we first discuss ∞ -categories, then we present that part of the theory of dendroidal sets needed to define the Cisinski-Moerdijk model structure on dendroidal sets which establishes dendroidal sets as models for homotopy operads and illustrate some of its consequences. The proofs of the results below can be found in [8, 35].

4.1. ∞ -categories briefly. We have seen above that in general, weak \mathcal{P} -algebras in a homotopy environment \mathcal{E} and their weak maps fail to form a category and that in fact one is immediately led to define the simplicial set of weak algebras $wAlg[\mathcal{P}, \mathcal{E}]$. The failure of this simplicial set to be the nerve of a category is a reflection of composition not being associative. However, the composition of weak maps is associative up to coherent homotopies, a fact which induces some extra structure on the simplicial set $wAlg[\mathcal{P}, \mathcal{E}]$. Boardman and Vogt in [5] formulated this extra structure by means of a condition called the restricted Kan condition. To define it recall that a horn in a simplicial set X is a mapping $\Lambda^k[n] \rightarrow X$, where $\Lambda^k[n]$ is the union in $\Delta[n]$ of all faces except the one opposite the vertex k . A horn is called *inner* when $0 < k < n$. Boardman and Vogt in [5], page 102, define

A simplicial set X is said to satisfy the restricted Kan condition
if every inner horn $\Lambda^k[n] \rightarrow X$ has a filler.

and so ∞ -categories were born. They consequently prove that in the context of topological operads the simplicial set of weak algebras satisfies the restricted Kan condition. Simplicial sets satisfying the restricted Kan condition are extensively studied by Joyal (in [19, 20] under the name 'quasicategories') and by Lurie (in e.g., [28] under the name ' $(\infty, 1)$ -categories' or more simply ' ∞ -categories').

There are several ways to model ∞ -categories, of which the above restricted Kan condition is one. Three other models are complete Segal spaces, Segal categories, and simplicial categories. For each of these models there is an appropriate

Quillen model structure rendering the four different models Quillen equivalent (see [4] for a detailed survey). By considering dendroidal sets instead of simplicial sets Cisinski and Moerdijk in [9, 10] introduce the analogous dendroidal notions: complete dendroidal Segal spaces, Segal operads, and simplicial operads. Moreover, they establish Quillen model structures for each of these notions, proving they are all Quillen equivalent to a Quillen model structure on dendroidal sets they establish in [8]. All of these model structures and equivalences, upon slicing over a suitable object, reduce to the equivalence of the simplicial based structures mentioned above.

There is yet another approach to ∞ -operads, taken by Lurie [29], which defines an ∞ -operad to be a simplicial set with extra structure. In Lurie's approach the highly developed theory of simplicial sets and quasicategories is readily available to provide a rich theory of ∞ -operads. However, the extra structure that makes a simplicial set into an ∞ -operad is quite complicated, rendering working with explicit examples of ∞ -operads difficult. The approach via dendroidal sets replaces the relative simplicity of the combinatorics of linear trees by the complexity of the combinatorics of trees which renders existing simplicial theory unusable but offers very many explicit examples of ∞ -operads. We believe that this trade-off in complexity will result in these two approaches mutually enriching each other as a future comparisons unfold.

Below, following [8] we give a short presentation of the approach to ∞ -operads embodied in the Cisinski-Moerdijk model structure on $dSet$ that slices to the Joyal model structure on $sSet$ and use this model structure to prove a homotopy invariance property for algebras in $dSet$. From this point on ∞ -category means a quasicategory.

4.2. Horns in $dSet$. We first introduce some concepts needed for the definition referring the reader to [34, 35] for more details.

Definition 4.1. Let T be a tree and $\alpha : S \rightarrow T$ a face map in Ω . The α -face of $\Omega[T]$, denoted by $\partial_\alpha \Omega[T]$, is the dendroidal subset of $\Omega[T]$ which is the image of the map $\Omega[\alpha] : \Omega[S] \rightarrow \Omega[T]$. Thus we have that

$$\partial_\alpha \Omega[T]_R = \{ R \longrightarrow S \xrightarrow{\alpha} T \mid R \rightarrow S \in \Omega[S]_R \}.$$

When α is obtained by contracting an inner edge e in T we denote ∂_α by ∂_e .

Let T be a tree. The *boundary* of $\Omega[T]$ is the dendroidal subset $\partial \Omega[T]$ of $\Omega[T]$ obtained as the union of all the faces of $\Omega[T]$:

$$\partial \Omega[T] = \bigcup_{\alpha \in \Phi_1(T)} \partial_\alpha \Omega[T].$$

where $\Phi_1(T)$, is the set of all faces of T .

Definition 4.2. Let T be a tree and $\alpha \in \Phi_1(T)$ a face of T . The α -horn in $\Omega[T]$ is the dendroidal subset $\Lambda^\alpha[T]$ of $\Omega[T]$ which is the union of all the faces of T except $\partial_\alpha \Omega[T]$:

$$\Lambda^\alpha[T] = \bigcup_{\beta \neq \alpha \in \Phi_1(T)} \partial_\beta \Omega[T].$$

The horn is called an *inner horn* if α is an inner face, otherwise it is called an *outer horn*. We will denote an inner horn $\Lambda^\alpha[T]$ by $\Lambda^e[T]$, where e is the contracted inner edge in T that defines the inner face $\alpha = \partial_e : \Omega[T/e] \rightarrow \Omega[T]$. A horn in a

dendroidal set X is a map of dendroidal sets $\Lambda^\alpha[T] \rightarrow X$. It is inner (respectively outer) if the horn $\Lambda^\alpha[T]$ is inner (respectively outer).

Remark 4.3. It is trivial to verify that these notions for dendroidal sets extend the common ones for simplicial sets in the sense, for example, that for the simplicial horn $\Lambda^k[n] \subseteq \Delta[n]$, the dendroidal set

$$i_!(\Lambda^k[n]) \subseteq i_!(\Delta[n]) = \Omega[L_n]$$

is a horn in the dendroidal sense. Furthermore, the horn $\Lambda^k[n]$ is inner (i.e., $0 < k < n$) if, and only if, the horn $i_!(\Lambda^k[n])$ is inner.

Both the boundary $\partial\Omega[T]$ and the horns $\Lambda^\alpha[T]$ in $\Omega[T]$ can be described as colimits as follows.

Definition 4.4. Let $T_1 \rightarrow T_2 \rightarrow \cdots \rightarrow T_n$ be a sequence of n face maps in Ω . We call the composition of these maps a *subface* of T_n of *codimension* n .

Proposition 4.5. Let $S \rightarrow T$ be a subface of T of codimension 2. The map $S \rightarrow T$ decomposes in precisely two different ways as a composition of faces.

Let $\Phi_i(T)$ be the set of all subfaces of T of codimension i . The proposition implies that for each $\beta : S \rightarrow T \in \Phi_2(T)$ there are precisely two face maps $\beta_1 : S \rightarrow T_1$ and $\beta_2 : S \rightarrow T_2$ that factor β as a composition of face maps. Using these maps we can form two maps γ_1 and γ_2

$$\coprod_{S \rightarrow T \in \Phi_2(T)} \Omega[S] \rightrightarrows \coprod_{R \rightarrow T \in \Phi_1(T)} \Omega[R]$$

where γ_i ($i = 1, 2$) has component $\Omega[S] \xrightarrow{\Omega[\beta_i]} \Omega[T_i] \longrightarrow \coprod \Omega[R]$ for each $\beta : S \rightarrow T \in \Phi_2(T)$.

Lemma 4.6. Let T be a tree in Ω . With notation as above we have that the boundary $\partial\Omega[T]$ is a coequalizer

$$\coprod_{S \rightarrow T \in \Phi_2(T)} \Omega[S] \rightrightarrows \coprod_{R \rightarrow T \in \Phi_1(T)} \Omega[R] \rightarrow \partial\Omega[T]$$

of the two maps γ_1, γ_2 constructed above.

Corollary 4.7. A map of dendroidal sets $\partial\Omega[T] \rightarrow X$ corresponds exactly to a sequence $\{x_R\}_{R \rightarrow T \in \Phi_1(T)}$ of dendrices whose faces match, in the sense that for each subface $\beta : S \rightarrow T$ of codimension 2 we have $\beta_1^*(x_{T_1}) = \beta_2^*(x_{T_2})$.

A similar presentation for horns holds as well. For a fixed face $\alpha : S \rightarrow T \in \Phi_1(T)$ consider the parallel arrows defined by making the following diagram commute

$$\begin{array}{ccc} \Omega[S] & \xrightarrow{\beta_1} & \Omega[T_1] \\ \downarrow & & \downarrow \\ \coprod_{\beta : S \rightarrow T \in \Phi_2(T)} \Omega[S] & \rightrightarrows & \coprod_{R \rightarrow T \neq \alpha \in \Phi_1(T)} \Omega[R] \\ \uparrow & & \uparrow \\ \Omega[S] & \xrightarrow{\beta_2} & \Omega[T_2] \end{array}$$

where the vertical arrows are the canonical injections into the coproduct and where we use the same notation as above.

Lemma 4.8. *Let T be a tree in Ω and α a face of T . In the diagram*

$$\coprod_{S \rightarrow T \in \Phi_2(T)} \Omega[S] \rightrightarrows \coprod_{R \rightarrow T \neq \alpha \in \Phi_1(T)} \Omega[R] \rightarrow \Lambda^\alpha[T]$$

the dendroidal set $\Lambda^\alpha[T]$ is the coequalizers of the two maps constructed above.

Corollary 4.9. *A horn $\Lambda^\alpha[T] \rightarrow X$ in X corresponds exactly to a sequence $\{x_R\}_{R \rightarrow T \neq \alpha \in \Phi_1(T)}$ of dendrices that agree on common faces in the sense that if $\beta : S \rightarrow T$ is a subface of codimension 2 which factors as*

$$\begin{array}{ccc} & R_1 & \\ \beta_1 \nearrow & & \searrow \alpha_1 \\ S & \xrightarrow{\beta} & T \\ \beta_2 \searrow & & \nearrow \alpha_2 \\ & R_2 & \end{array}$$

then $\beta_1^(x_{R_1}) = \beta_2^*(x_{R_2})$.*

Remark 4.10. In the special case where the tree T is linear we obtain the equivalent results for simplicial sets. Namely, the presentation of the boundary $\partial\Delta[n]$ and of the horn $\Lambda^k[n]$ as colimits of standard simplices, and the description of a horn $\Lambda^k[n] \rightarrow X$ in a simplicial set X (see [14]).

We are now able to define the dendroidal sets that model ∞ -operads.

Definition 4.11. A dendroidal set X is an ∞ -operad if every inner horn $h : \Lambda^e[T] \rightarrow X$ has a filler $\hat{h} : \Omega[T] \rightarrow X$ making the diagram

$$\begin{array}{ccc} \Lambda^e[T] & \xrightarrow{h} & X \\ \downarrow & \nearrow \hat{h} & \\ \Omega[T] & & \end{array}$$

commute.

The following relation between ∞ -categories and ∞ -operads is trivial to prove:

Proposition 4.12. *If X is an ∞ -category then $i_!(X)$ is an ∞ -operad. If Y is an ∞ -operad then $i^*(Y)$ is an ∞ -category.*

It is not hard to see that given any symmetric operad \mathcal{P} its dendroidal nerve $N_d(\mathcal{P})$ is an ∞ -operad. In fact we can characterize those dendroidal sets occurring as nerves of operads as follows.

Definition 4.13. An ∞ -operad X is called *strict* if any inner horn in X as above has a unique filler.

Lemma 4.14. *A dendroidal set X is a strict ∞ -operad if, and only if, there is an operad \mathcal{P} such that $N_d(\mathcal{P}) \cong X$.*

A family of examples of paramount importance of ∞ -operads are given by the following. Recall that when \mathcal{E} is a symmetric monoidal model category a symmetric operad \mathcal{P} enriched in \mathcal{E} is called *locally fibrant* if each hom-object in \mathcal{P} is fibrant in \mathcal{E} .

Theorem 4.15. *Let \mathcal{P} be a locally fibrant symmetric operad in \mathcal{E} , where \mathcal{E} is a homotopy environment. The homotopy coherent nerve $hcN_d(\mathcal{P})$ is an ∞ -operad.*

4.3. The Cisinski-Moerdijk model category structure on $dSet$. The objective of this section is to present the Cisinski-Moerdijk model structure on $dSet$. All of the material in this section is taken from [8], to which the reader is referred to for more information and the proofs. In this model structure ∞ -operads are the fibrant objects and it is closely related to the operadic model structure on Ope and to the Joyal model structure on $sSet$.

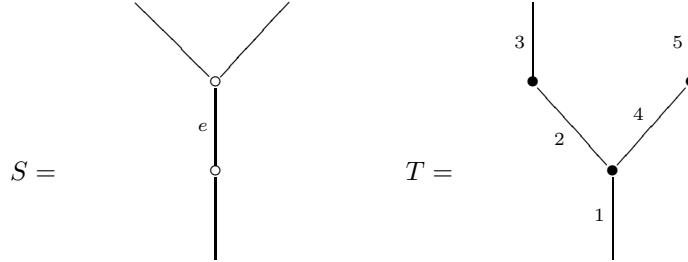
We note immediately that the Cisinski-Moerdijk model structure is not a Cisinski model structure (i.e., a model structure on a presheaf category such that the cofibrations are precisely the monomorphisms) due to a technical complication that prevents the direct application of the techniques developed in [7]. Indeed, the cofibrations in the model structure are the so-called *normal monomorphisms*.

Definition 4.16. A monomorphism of dendroidal sets $f : X \rightarrow Y$ is *normal* if for every dendrex $t \in Y_T$ that does not factor through f the only isomorphism of T that fixes t is the identity.

An important property of dendroidal sets, proved in [35], is the following.

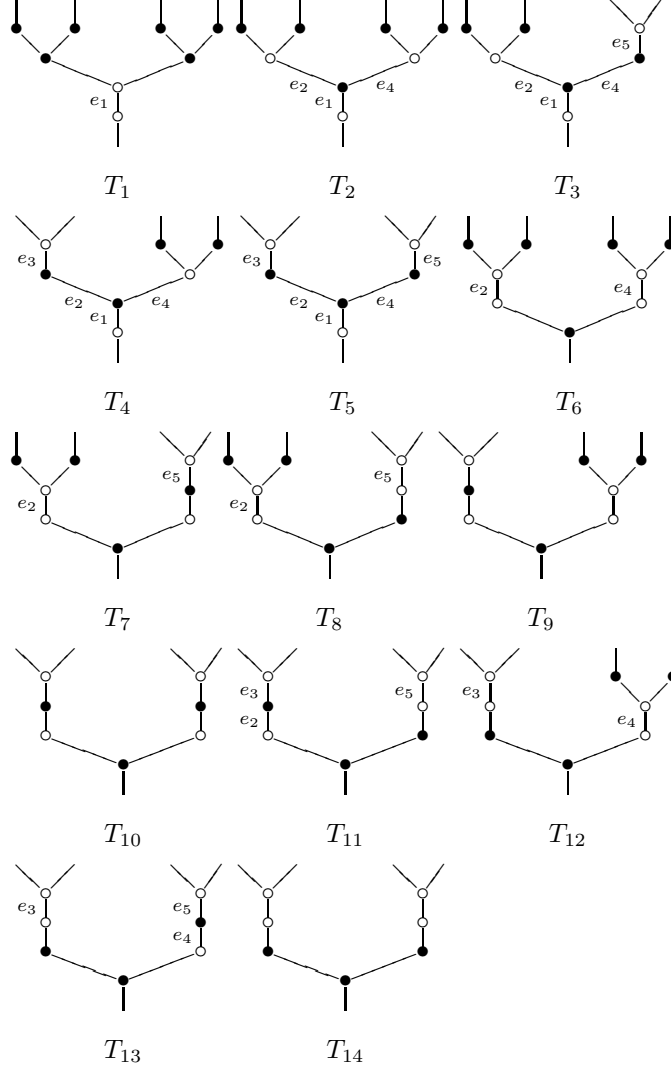
Theorem 4.17. *Let X be a normal dendroidal set (i.e., $\emptyset \rightarrow X$ is normal) and Y an ∞ -operad. The dendroidal set $[X, Y]$ is again an ∞ -operad.*

PROOF. The proof uses the technique of anodyne extensions, as commonly used in the theory of simplicial sets (e.g., [13, 14]), suitably adapted to dendroidal sets. Technically though, the dendroidal case is much more difficult. For simplicial sets there is a rather simple description of the non-degenerate simplices of $\Delta[n] \times \Delta[k]$. But for trees S and T a similar description of the non-degenerate dendrices of $\Omega[S] \otimes \Omega[T]$ is given by the so called poset of percolation trees associated with S and T . Complete details can be found in [35] and we just briefly illustrate the construction for the trees S and T :

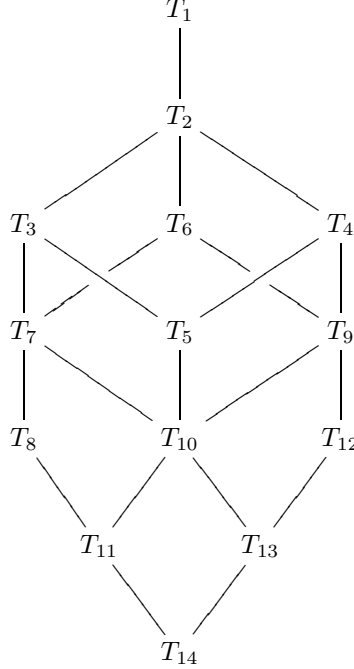


on the following page.

The presentation of $\Omega[T] \otimes \Omega[S]$ is given by the 14 trees T_1, \dots, T_{14} :



The poset structure on these trees is



□

As a special case of this result we may now recover Boardman and Vogt's result that the simplicial set $wAlg[\mathcal{P}, \mathcal{E}]$ of weak \mathcal{P} -algebras in \mathcal{E} is an ∞ -category, as follows.

Theorem 4.18. *Let \mathcal{P} be a symmetric operad and \mathcal{E} a homotopy environment. If the dendroidal set $N_d(\mathcal{P})$ is normal and $hcN_d(\hat{\mathcal{E}})$ is an ∞ -operad then $wAlg[\mathcal{P}, \mathcal{E}]$ is an ∞ -category.*

PROOF. The proof follows by noticing that $i^*([N_d(\mathcal{P}), hcN_d(\hat{\mathcal{E}})]) \cong wAlg[\mathcal{P}, \mathcal{E}]$.

□

As we have seen above, local fibrancy of $\hat{\mathcal{E}}$ assures that $hcN_d(\hat{\mathcal{E}})$ is an ∞ -operad. See below for a condition on \mathcal{P} sufficient to assure $N_d(\mathcal{P})$ is normal. We now turn to the Cisinski-Moerdijk model structure.

Theorem 4.19. *The category $dSet$ of dendroidal sets admits a Quillen model structure where the cofibrations are the normal monomorphisms, the fibrant objects are the ∞ -operads, and the fibrations between ∞ -operads are the inner Kan fibrations whose image under τ_d is an operadic fibration. The class \mathcal{W} of weak equivalences can be characterized as the smallest class of arrows which contains all inner anodyne extensions, all trivial fibrations between ∞ -operads and satisfies the 2 out of 3 property. Furthermore, with the tensor product of dendroidal sets, this model structure is a monoidal model category. Slicing this model structure recovers the*

Joyal model structure on $sSet$ and in the diagram

$$\begin{array}{ccc}
 Cat & \xrightleftharpoons[j^*]{j_!} & Ope \\
 \tau \updownarrow N & & \tau_d \updownarrow N_d \\
 sSet & \xrightleftharpoons[i^*]{i_!} & dSet
 \end{array}$$

where the categories are endowed (respectively starting from the top-left going clockwise) with the categorical, operadic, Cisinski-Moerdijk, and Joyal model structures all adjunctions are Quillen adjunction (and none is a Quillen equivalence).

PROOF. The proof of the model structure is quite intricate and is established in [8]. \square

4.4. Homotopy invariance property for algebras in $dSet$. As a consequence of the Cisinski-Moerdijk model structure on dendroidal sets we obtain the following.

Theorem 4.20. *Let X be a normal dendroidal set and E an ∞ -operad. Then X -algebras in E have the homotopy invariance property.*

PROOF. In the diagram defining the homotopy invariance property in Definition 3.17 above the left vertical arrow is a trivial cofibration and the right vertical arrow is a fibration in the Cisinski-Moerdijk model structure and thus the required lift exists. \square

We now recall Fact 2.30 regarding the internalization of strict algebras by the closed monoidal structure on Ope given by the Boardman-Vogt tensor product and the isomorphism invariance property for such algebras as captured by the operadic monoidal model structure on Ope . We recall that a similar such correspondence for weak algebras and their homotopy invariance property does not seem possible within the confines of enriched operads. We are now in a position to summarize the results recounted above in a form completely analogous to the situation of strict algebras.

Fact 4.21. *The notion of algebras of dendroidal sets is internalized to the category $dSet$ by it being closed monoidal with respect to the tensor product of dendroidal sets. The homotopy invariance property of X -algebras in an ∞ -operad, for a normal X , holds and is captured by the fact that $dSet$ supports the Cisinski-Moerdijk model structure which is compatible with the tensor product. The notion of a weak \mathcal{P} -algebra in a homotopy environment \mathcal{E} , where \mathcal{P} is discrete, is subsumed by the notion of algebras in $dSet$ by means of the dendroidal set $[N_d(\mathcal{P}), hcN_d(\mathcal{E})]$ of weak \mathcal{P} -algebras in \mathcal{E} .*

4.5. Revisiting applications. Recall the iterative construction of the dendroidal set nA_∞ of n -fold A_∞ -spaces and $nCat$ of weak n -categories as well as $wCat_k^n$ of k -monoidal n -categories. It follows from the general theory of dendroidal sets that these are all ∞ -operads. To see that, one uses Theorem 4.17 together with the fact that for a Σ -free symmetric operad \mathcal{P} (for instance if \mathcal{P} is obtained from a planar operad \mathcal{Q} by the symmetrization functor) then $N_d(\mathcal{P})$ is normal. Thus, weak maps of n -fold A_∞ -spaces can be coherently composed and similarly so can

weak functors between weak n -categories. As for k -monoidal n -categories we note that $wCat_k^n$ is, for similar reasons as above, an ∞ -operad as well.

We may now reduce the Baez-Breen-Dolan stabilization conjecture as follows.

Proposition 4.22. *If for any $n \geq 0$ the dendroidal set $wCat_n^n$ is a strict ∞ -operad then the Baez-Breen-Dolan stabilization conjecture is true.*

PROOF. Recall that $wCat_n^n = [A^{\otimes n}, wCat^n]$ and assume it is a strict ∞ -operad. We wish to prove that $[A^{\otimes n+j}, wCat^n] \cong [A^{\otimes n+2}, wCat^n]$ for any fixed $j > 2$. By Lemma 4.14 there is an operad \mathcal{P} such that $[A^{\otimes n}, wCat^n] = N_d(\mathcal{P})$. We now have

$$[A^{\otimes n+j}, wCat^n] = [A^{\otimes j}, [A^{\otimes n}, wCat^n]] = [A^{\otimes j}, N_d(\mathcal{P})]$$

which by adjunction is isomorphic to $N_d([\tau_d(A^{\otimes j}), \mathcal{P}])$. However, A is actually the dendroidal nerve of the symmetric operad As classifying associative monoids. By Proposition 3.12 we have

$$\tau_d(A^{\otimes j}) = \tau_d(N_d(As)^{\otimes j}) \cong As^{\otimes j} \cong Comm.$$

and the result follows. \square

5. Dendroidal sets - combinatorial models of unknown spaces

All of the theory of dendroidal sets that directly or indirectly is concerned with algebras (we include the Cisinski-Moerdijk model structure here as well) is very operadic in nature and is closely related to the theory of ∞ -categories modeled by quasicategories. Simplicial sets are, however, also models for topological homotopy theory. Indeed, simplicial sets were introduced in the context of algebraic topology as combinatorial models of topological spaces. The appropriate equivalence is established in [36] and was the reason to introduce Quillen model categories. To recall the main result recall the singular functor $Sing : Top \rightarrow sSet$ and its left adjoint $|-| : sSet \rightarrow Top$ given by geometric realization.

Theorem 5.1. *The category Top supports a Quillen model structure in which the weak equivalences are the weak homotopy equivalences and the fibrations are the Serre fibrations. The category $sSet$ supports a Quillen model structure in which the weak equivalences are those maps $f : X \rightarrow Y$ for which the geometric realization $|f|$ is a homotopy weak equivalence and the fibrations are the Kan fibrations. With these model structures the adjunction above is a Quillen equivalence.*

Simplicial sets thus support a topologically flavoured Quillen model structure as well as the Joyal model structure which is categorically flavoured and so simplicial sets play two rather different roles. The close connection between dendroidal sets and simplicial sets raises the question as to the existence of a topologically flavoured interpretation of dendroidal sets as well. This problem is open for debate and interpretation and is certainly far from settled.

We remark first that there is some indication that suggests dendroidal sets do carry topological meaning. Recall the Dold-Kan correspondence that establishes an equivalence of categories between the category sAb of simplicial abelian groups and the category Ch of non-negatively graded chain complexes. This correspondence is useful in the calculation of homotopy groups of simplicial sets and in the definition of Eilenberg-Mac Lane spaces $K(G, n)$ for $n > 1$. In [15] it is shown that there is a planar dendroidal version (where one considers a planar version of Ω whose objects are planar trees) of the Dold-Kan correspondence. The equivalence is between the

category dAb of planar dendroidal abelian groups and the category dCh of planar dendroidal chain complexes. The definition of the latter requires that for each face map ∂_α between planar trees there is associated a sign $sgn(\partial_\alpha) \in \{\pm 1\}$ such that the following holds. In the planar version of Ω it is still true that a face $S \rightarrow T$ of codimension 2 decomposes in precisely two ways as the composition of two faces (see Proposition 4.5 above). Thus we can write $S \rightarrow T$ as $\partial_\alpha \circ \partial_\beta$ as well as $\partial_\gamma \circ \partial_\delta$ and we require that $sgn(\partial_\alpha) \cdot sgn(\partial_\beta) = -sgn(\partial_\gamma) \cdot sgn(\partial_\delta)$. One may now wonder whether these dendroidal chain complexes give rise to some sort of generalized Eilenberg-Mac Lane spaces. A first step towards answering this question should be a clearer specification of goals in a broad context, which is the aim of the rest of this section.

As inspiration we consider the Quillen equivalence between topological spaces and simplicial sets mentioned above. The geometric realization plays there a prominent role and thus a significant aspect of understanding the homotopy behind dendroidal sets is to find a category $dTop$ together with functors $Sing_d : dTop \rightarrow dSet$ and $|-|_d : dSet \rightarrow dTop$. The category $dTop$ of course has to be chosen with care so that it will rightfully be considered to be related to topology. We thus expect that there is a fully faithful functor $h_! : Top \rightarrow dTop$ with a right adjoint $h^* : dTop \rightarrow Top$ that should be defined 'purely topologically'. We thus expect $dTop$ to be a category of some generalized topological spaces in which ordinary topological spaces embed via $h_!$. To allow sufficient flexibility for working with these objects we expect that $dTop$ be small complete and small cocomplete. Moreover, the functor $|-|_d : dSet \rightarrow dTop$ should send a dendroidal set X to some generalized space $|X|_d$ in such a way that the combinatorial information in X is not lost. We thus expect of any such functor $|-|_d$ that if for some $f : X \rightarrow Y$ the map $|f|_d$ is an isomorphism then f was already an isomorphism. In other words we expect $|-|_d$ to be conservative.

The term 'purely topologically' above is of course vague and open to discussion. In an attempt to formalize it recall the various slicing lemmas we have seen above: Slicing symmetric operads over \star gives categories, slicing dendroidal sets over $\Omega[\eta] = N_d(\star)$ gives simplicial sets, and slicing Ω over η gives Δ . We thus expect that there is an object $\star \in dTop$ such that slicing $dTop$ over \star gives a category equivalent to Top and that in fact the embedding $h_! : Top \rightarrow dTop$ is essentially the forgetful functor $dTop/\star \rightarrow dTop$. Moreover, noting that the 'correct' tensor product of dendroidal sets is not the cartesian one we expect $dTop$ to possess a monoidal structure different from the cartesian product. And, just as the tensor product of dendroidal sets slices to the cartesian product of simplicial sets we expect the monoidal structure on $dTop$ to slice to the cartesian product of topological spaces. Lastly, an important property of the ordinary geometric realization functor is that it commutes with finite products. We expect of the dendroidal geometric realization functor $dSet \rightarrow dTop$ to be monoidal with respect to the non-cartesian monoidal structure on each category.

We summarize our expectations in the following formulation.

Problem 5.2. Find a category $dTop$ together with a functor $Sing_d : dTop \rightarrow dSet$, a left adjoint $|-|_d : dSet \rightarrow dTop$, and an object $\star \in dTop_0$ such that:

- (1) $dTop$ is small complete and small cocomplete.
- (2) (Slicing lemma) $dTop/\star$ is equivalent to Top .
- (3) The forgetful functor $h_! : Top \rightarrow dTop$ is an embedding.

- (4) Slicing $Sing_d$ gives $Sing$ and slicing $|-|_d$ gives $|-|$.
- (5) $|-|_d$ is conservative.
- (6) $dTop$ admits a non-cartesian monoidal structure that slices over \star to the cartesian product in Top (along $h_!$).
- (7) The functor $|-|_d$ is to be a monoidal functor with respect to the tensor structures on $dSet$ and $dTop$.

We would thus obtain the diagram

$$\begin{array}{ccc}
 sSet & \xrightleftharpoons[Sing]{|-|} & Top \\
 i_! \updownarrow i^* & & h_! \updownarrow h^* \\
 dSet & \xrightleftharpoons[Sing_d]{|-|_d} & dTop
 \end{array}$$

where both squares commute.

The quest will be complete with the establishment of Quillen model structures on $dSet$ and $dTop$ that slice respectively to the standard (topological) ones on $sSet$ and Top and such that in the square above all adjunctions are Quillen adjunctions with both horizontal ones Quillen equivalences.

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