

Cohen-Moore-Neisendorfer Map in Double EHP-sequences

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p : odd prime, $q = 2(p-1)$, $\nu = \nu_p(n)$ for $n = ap^\nu, a \not\equiv 0 \pmod{p}$.
Every spaces and groups are localized at p .

Equivalent maps (homomorphisms) and induced maps are frequently written by the same symbol such as :

$$X \xrightarrow{f} Y \text{ and } \Omega X \xrightarrow{f} \Omega Y, \quad X \xrightarrow{g} \Omega Y \text{ and } SX \xrightarrow{g} Y, \quad \text{etc.}$$

$\pi_{m+k}^m = \pi_{m+k}(S^m)$: The k -stem homotopy group of m -sphere.

Serre decomposition : $\pi_{2r+k}^{2r} \cong \pi_{2r-1+k}^{2r-1} \oplus \pi_{2r+k}^{4r-1}$ by $\partial \xrightarrow{\oplus} H_2$ and $S \xleftarrow{\oplus} w$

The k -stem groups $\pi_{*+k}^* = \sum_r \pi_{2r+1+k}^{2r+1}$ ($\pi_{1+k}^1 = 0$ for $k \neq 0$) :

$$\pi_{3+k}^3 \xrightarrow{\mathbb{E}} \pi_{5+k}^5 \xrightarrow{\mathbb{E}} \cdots \xrightarrow{\mathbb{E}} \pi_{2r-1+k}^{2r-1} \xrightarrow{\mathbb{E}} \pi_{2r+1+k}^{2r+1} \xrightarrow{\mathbb{E}^\infty} \pi_k^S \text{ (stable } k\text{-stem group)}.$$

Double EHP-sequence

$$\cdots \xrightarrow{\mathbb{H}} \pi Q_{2r+k}^{2r-1} \xrightarrow{\mathbb{P}} \pi Q_{2r-1+k}^{2r-1} \xrightarrow{\mathbb{E}} \pi Q_{2r+1+k}^{2r+1} \xrightarrow{\mathbb{H}} \pi Q_{2r-2+k}^{2r-1} \xrightarrow{\mathbb{P}} \cdots,$$

(Whitehead product) (double suspension) (Hopf invariant)

where $\pi Q_i^{2r-1} = \pi_i(Q_2^{2r-1})$ for the homotopy fibre Q_2^{2r-1} of the double suspension map $\mathbb{E} : S^{2r-1} \rightarrow \Omega^2 S^{2r+1}$, and it is called as the group of *invariants*.

IJ Δ -sequence

$$\begin{array}{ccccccc} \xrightarrow{\Delta} & \pi_{2r+k}^{2rp-1} & \xrightarrow{\mathbb{I}} & \pi Q_{2r-2+k}^{2r-1} & \xrightarrow{\mathbb{J}} & \pi_{2r+1+k}^{2rp+1} & \xrightarrow{\Delta} \pi_{2r-1+k}^{2rp-1} \\ & \uparrow \mathbb{E} & & \uparrow \lambda_2 & & \uparrow \mathbb{E} & \parallel \\ & \pi_{2r-2+k}^{2rp-3} & \xrightarrow{i} & \pi Y_{2r-2+k}^{2rp-2} & \xrightarrow{j \circ S} & \pi_{2r-1+k}^{2rp-1} & \xrightarrow{f_p} \pi_{2r-1+k}^{2rp-1} \end{array}$$

where $S^m \xrightarrow{i} Y^{m+1} \xrightarrow{j} S^{m+1} \xrightarrow{f_p} S^{m+1}$ is Moore space cofibering, λ_2 is a homology injection $Y^{2rp-2} \rightarrow Q_2^{2r-1}$ and $\pi Y_i^{m+1} = \pi_i(Y^{m+1})$.

The composition $H_p = \mathbb{J}\mathbb{H} : \pi_{2r+1+k}^{2r+1} \rightarrow \pi_{2r+1+k}^{2rp+1}$ is James-Hopf invariant.

The case $r = 1$

Let $\rho : \tilde{S}^3 \rightarrow S^3$ be the 3-connective fibering of S^3 , then Q_1^2 is homotopy equivalent to $\Omega^3 \tilde{S}^3$. So, in double EHP-sequence of $r = 1$, $\mathbb{H}$ can be replaced by $\rho^{-1} : \pi_i^3 \cong \pi_i(\tilde{S}^3)$, $i \neq 0$. Let $\tilde{\lambda} : Y^{2p+1} \rightarrow \tilde{S}^3$ be a homology injection.

The $IJ\Delta$ -sequence is replaced by the following sequence : $(\alpha_1 = \tilde{\lambda}|_{S^{2p}})$

$$\begin{array}{ccccccc} \xrightarrow{\Delta} & \pi_{2+k}^{2p-1} & \xrightarrow{\alpha_1 \circ S} & \pi_{3+k}^3 & \xrightarrow{H_p} & \pi_{3+k}^{2p+1} & \xrightarrow{\Delta} \pi_{1+k}^{2p-1} \\ & \downarrow S & & \uparrow \rho \circ \tilde{\lambda} & & \parallel & \downarrow \mathbb{E} \\ & \pi_{3+k}^{2p} & \xrightarrow{i} & \pi Y_{3+k}^{2p+1} & \xrightarrow{j} & \pi_{3+k}^{2p+1} & \xrightarrow{f_p} \pi_{3+k}^{2p+1} \end{array}$$

CMN-map $\pi : \Omega^2 S^{2r+1} \rightarrow S^{2r-1}$, ($r \geq 2$)

$$(1a) \quad \pi \circ \mathbb{E} \simeq f_p : S^{2r-1} \rightarrow \Omega^2 S^{2r+1} \rightarrow S^{2r-1}$$

$$(1b) \quad \mathbb{E} \circ \pi \simeq f_p : \Omega^2 S^{2r+1} \rightarrow S^{2r-1} \rightarrow \Omega^2 S^{2r+1}$$

Corollary 1. Exponents $p^r(\pi_{2r+1+k}^{2r+1}) = 0$ for $k \neq 0$. ($r = 1$ by Selick)

Corollary 2. $p(\pi Q_2^{2r-1}) = 0$.

Corollary 3. Extension problem : If $\mathbb{E}\pi(\xi) \neq 0$ for $\xi \in \pi_{2r+1+k}^{2r+1}$ then the extension $\pi_{2r-1+k}^{2r-1} \xrightarrow{\mathbb{E}} \pi_{2r+1+k}^{2r+1} \xrightarrow{\mathbb{H}} \pi Q_{2r-2+k}^{2r-1}$ is non-trivial.

CMN-map and EHP-sequence

Combining Ω^2 of (1a) with (1b) we have

Theorem 0.1 For $r \geq 2$, there exists a map $\bar{\pi} : \Omega^2 Q_2^{2r+1} \rightarrow Q_2^{2r-1}$ such that the following diagram is commutative.

$$\begin{array}{ccccccccc} \pi_{2r+1+k}^{2r+1} & \xrightarrow{\mathbb{E}} & \pi_{2r+3+k}^{2r+3} & \xrightarrow{\mathbb{H}} & \pi Q_{2r+k}^{2r+1} & \xrightarrow{\mathbb{P}} & \pi_{2r+k}^{2r+1} & \xrightarrow{\mathbb{E}} & \pi_{2r+2+k}^{2r+3} \\ \downarrow \pi & & \downarrow \pi & & \downarrow \bar{\pi} & & \downarrow \pi & & \downarrow \pi \\ \pi_{2r-1+k}^{2r-1} & \xrightarrow{\mathbb{E}} & \pi_{2r+1+k}^{2r+1} & \xrightarrow{\mathbb{H}} & \pi Q_{2r-2+k}^{2r-1} & \xrightarrow{\mathbb{P}} & \pi_{2r-2+k}^{2r-1} & \xrightarrow{\mathbb{E}} & \pi_{2r+k}^{2r+1} \end{array}$$

As corollaries we have

Lemma 0.1 Let $y = \bar{\pi}(x) \in \pi Q_{2r-2+k}^{2r-1}$ for $x \in \pi Q_{2r+k}^{2r+1}$.

(1) If an element $\xi \in \pi_{2r+3+k}^{2r+3}$ satisfies $\mathbb{H}(\xi) = x$, then $\eta = \pi(x) \in \pi_{2r+1+k}^{2r+1}$ satisfies $\mathbb{H}(\eta) = y$ and $\mathbb{E}(\eta) = p\xi$.

(2) If an element $\eta \in \pi_{2r-2+k}^{2r-1}$ satisfies $\mathbb{E}(\eta) = \mathbb{P}(x)$, then it holds $\mathbb{P}(y) = p\eta$.

Adams map and $\bar{\pi}$ Let $\alpha : Y^{m+q} \rightarrow Y^m$ be Adams map.

Theorem 0.2 For $r \geq 2$ the following diagram is commutative.

$$\begin{array}{ccccc} \pi Y_i^{2(r+1)p-4} & \xrightarrow{S^2} & \pi Y_{i+2}^{2(r+1)p-2} & \xrightarrow{\lambda_2} & \pi Q_{i+2}^{2r+1} \\ \alpha \downarrow & & & & \bar{\pi} \downarrow \\ \pi Y_i^{2rp-2} & \xlongequal{\quad} & \pi Y_i^{2rp-2} & \xrightarrow{\lambda_2} & \pi Q_i^{2r-1} \end{array}$$

Theorem 0.3 *There exists a homomorphism $\tilde{\pi} : \pi Q_{2+k}^3 \rightarrow \pi_{3+k}(\tilde{S}^3)$ such that the following diagram commutes for $k \neq 0$:*

$$\begin{array}{ccccccc}
\pi_{3+k}^3 & \xrightarrow{\mathbb{E}} & \pi_{5+k}^5 & \xrightarrow{\mathbb{H}} & \pi Q_{2+k}^3 & \xleftarrow{\lambda_2} & \pi Y_{2+k}^{4p-2} \xlongequal{\quad} \pi Y_{2+k}^{4p-2} \\
\downarrow & & \downarrow \pi & & \downarrow \tilde{\pi} & & \downarrow \alpha \\
0 & \longrightarrow & \pi_{3+k}^3 & \xleftarrow{\rho} & \pi_{3+k}(\tilde{S}^3) & \xleftarrow{\tilde{\lambda}} & \pi Y_{3+k}^{2p+1} \xleftarrow{S} \pi Y_{2+k}^{2p}
\end{array}$$

Alpha invariants : $A_{n,r}, a_{n,r}$

Alpha series $\{\alpha_n \in \pi_{nq-1}^S\}$, $\alpha_n = j\alpha^n i : S^{nq-1} \rightarrow Y^{nq} \rightarrow Y^0 \rightarrow S^0$.

Mod p ImJ-series $\{\bar{\alpha}_n \in \pi_{nq-1}^S\}$, $\bar{\alpha}_{n+1} \in \{\bar{\alpha}_n, \alpha_1, p\}$, $n\bar{\alpha}_n = \alpha_n$

$e(\alpha_n) = 1/p \implies \bar{\alpha}_n$ is of order $p^{\nu_p(n)+1}$.

$A_{0,r} = \lambda_2(i) \in \pi Q_{2rp-3}^{2r-1}$, $A_{0,r} = \mathbb{I}(\iota)$, $\iota \in \pi_{2rp-1}^{2rp-1}$,

$A_{n,r} = \lambda_2(\alpha^n i) \in \pi Q_{2r+(r+n)q-3}^{2r-1}$ for $r \geq 1$, $\mathbb{J}(A_{n,r}) = \alpha_n \in \pi_{2rp+(r+n)q}^{2rp+1}$.

$a_{n,r} = \lambda_2(\alpha^{n-1} \delta \alpha i) \in \pi Q_{2r+(r+n)q-4}^{2r-1}$ for $r \geq 1$, where $\delta = ij : Y^m \rightarrow Y^{m+1}$.

$$(2a) \quad \tilde{\pi}(A_{n-r,r}) = A_{n-r+1,r-1} \text{ for } n \geq r > 1$$

$$(2b) \quad \tilde{\pi}(a_{n-r,r}) = a_{n-r+1,r-1} \text{ for } n > r > 1$$

Put $AQ_{2r+nq-3}^{2r-1} = \langle A_{n-r,r} \rangle$ and $AQ_{2r+nq-4}^{2r-1} = \langle a_{n-r,r} \rangle$.

Alpha families : $A_{*+k}^* = \sum_n (A_{2r+nq}^{2r+1} \oplus A_{2r-1+nq}^{2r+1}) \subset \pi_{*+k}^*$

The k -stem alpha family A_{*+k}^* is required by $A_{2r-1+nq}^{2r+1} = 0$ for $r \geq n$ and the exactness the sequence

$$\begin{array}{ccccccc}
0 \longrightarrow & A_{2r-2+nq}^{2r-1} & \xrightarrow{\mathbb{E}} & A_{2r-nq}^{2r+1} & \xrightarrow{\mathbb{H}} & AQ_{2r-3+nq}^{2r-1} & \xrightarrow{\mathbb{P}} \\
& & & & & A_{2r-3+nq}^{2r-1} & \xrightarrow{\mathbb{E}} A_{2r-1+nq}^{2r+1} \xrightarrow{\mathbb{H}} AQ_{2r-4+nq}^{2r-1} .
\end{array}$$

An example of alpha family was given by Gray.

The last $\mathbb{H}$ is surjective iff $r < n - \nu_p(n)$ by the following Gray's lemma.

Lemma 0.2 *There exists an element $X \in \pi_{2n-2\nu+3}^{2n-2\nu-1}$ ($\nu = \nu_p(n)$) satisfying*

$$\mathbb{H}(X) \doteq a_{\nu+1, n-\nu-1} \quad \text{and} \quad \mathbb{E}^\nu(X) = \mathbb{P}(A_{0,n}).$$

Here " $\doteq$ " means " $=$ up to non-zero constant".

Proposition 0.1 . Let $\nu = \nu_p(n)$.

(1) For $\nu+1 \geq r \geq 1$, $A_{2r+nq}^{2r+1} \cong \mathbb{Z}/p^r$ and $\mathbb{E} : A_{2r-2+nq}^{2r-1} \rightarrow A_{2r+nq}^{2r+1}$ is injective.

For $r \geq \nu+1$, $\mathbb{E} : A_{2r+nq}^{2r+1} \cong A_{2r+2+nq}^{2r+3} (\cong \mathbb{Z}/p^{\nu+1})$.

(2) For $\nu+1 \geq r \geq 1$, $A_{2r-1+nq}^{2r+1} \cong \mathbb{Z}/p^r$.

For $n - \nu > r \geq \nu+1$, $A_{2r-1+nq}^{2r+1} \cong \mathbb{Z}/p^{\nu+1}$.

For $n > r \geq \nu+1$, $A_{2r+1+nq}^{2r-1} \cong \mathbb{Z}/p^{n-\nu+1}$.

For $n - \nu > r \geq 1$, the cokernel of $\mathbb{E} : A_{2r-3+nq}^{2r-1} \rightarrow A_{2r-1+nq}^{2r+1}$ is of order p .

For $n > r > \nu+1$, the kernel of $\mathbb{E} : A_{2r-3+nq}^{2r-1} \rightarrow A_{2r-1+nq}^{2r+1}$ is of order p .

$$k = nq - 1$$

$$\begin{array}{ccccccc}
\pi_{1+k}^3 & \rightarrow & \pi_{3+k}^5 & \rightarrow & \pi_{5+k}^7 & \rightarrow & \cdots \rightarrow \pi_{2np-8}^{2n-5} \rightarrow \pi_{2np-6}^{2n-3} \rightarrow \pi_{2np-4}^{2n-1} \rightarrow \pi_{2np-2}^{2n+1} \\
\swarrow & & \swarrow & & \swarrow & & \swarrow \\
a_{n-1,1} & a_{n-2,2} & a_{n-3,3} & & a_{3,n-3} & a_{2,n-2} & a_{1,n-1} \\
\swarrow & \swarrow & \swarrow & & \swarrow & \swarrow & \swarrow \\
A_{2+k}^3 & \rightarrow & A_{4+k}^5 & \rightarrow & A_{6+k}^7 & \rightarrow & \cdots \rightarrow A_{2np-7}^{2n-5} \rightarrow A_{2np-5}^{2n-3} \rightarrow A_{2np-3}^{2n-1} \rightarrow A_{2np-1}^{2n+1} \\
\swarrow & \swarrow & \swarrow & & \swarrow & \swarrow & \swarrow \\
A_{n-1,1} & A_{n-2,2} & A_{n-3,3} & & A_{3,n-3} & A_{2,n-2} & A_{1,n-1} & A_{0,n} \\
\swarrow & \swarrow & \swarrow & & \swarrow & \swarrow & \swarrow & \swarrow \\
A_{3+k}^3 & \rightarrow & A_{5+k}^5 & \rightarrow & A_{7+k}^7 & \rightarrow & \cdots \rightarrow A_{2np-6}^{2n-5} \rightarrow A_{2np-4}^{2n-3} \rightarrow A_{2np-2}^{2n-1} \rightarrow A_{2np}^{2n+1}
\end{array}$$

Extra alpha family

For $0 < r \leq \nu_p(n)$, the invariant $a_{r,n-r}$ is not an $\mathbb{H}$ -image of any alpha family and we investigate non-triviality of $\mathbb{P}(a_{r,n-r}) \in \pi_{2np-2r-4}^{2n-2r-1}$.

Proposition 0.2 (*The case $n = p$*)

- (1) The group $\pi_{2p^2-3}^{2p-1}$ is cyclic of order p^2 generated by an element $\beta_1(2p-1)$ satisfying $\mathbb{H}(\beta_1(2p-1)) \doteq a_{1,p-1}$ and $p(\beta_1(2p-1)) = \mathbb{P}(A_{0,p})$.
- (2) For $r \geq p$, Put $\beta_1(2r+1) = \mathbb{E}^{r-p}(\beta_1(2p-1))$ is of order p .
- (3) Let $\bar{\beta}: Y^{2p^2} \rightarrow S^{2p+1}$ be an extension of $\beta_1(2p+1) = \bar{\beta}|_{S^{2p^2-1}}$, then $\wp^p \neq 0$ in the mapping cone $S^{2p+1} \cup_{\bar{\beta}} CY^{2p^2} = S^{2p+1} \cup_{\beta_1} e^{2p^2} \cup_p e^{2p^2+1}$ of $\bar{\beta}$.

Proposition 0.3 Let $n \equiv 0 \pmod{p}$ and $n > p$.

Then there exists an element $ua_1 \in \pi_{2(n-p)-2+nq}^{2(n-p)+1}$ satisfying

$$\mathbb{H}(ua_1) \doteq \mathbb{H}(\beta_1(2(n-p)p-1)) \quad \text{and} \quad \mathbb{E}^{p-2}(ua_1) = \mathbb{P}(a_{1,n-1}) \neq 0.$$

By induction on $t > 1$ we put

$$\beta_1^t(2r+1) = \beta_1(2r+1) \circ \beta_1^{t-1}(2r-1+pq) \in \pi_{2r+1+t(pq-2)}^{2r+1} \text{ for } r \geq p-1.$$

Theorem 0.4 Let $n \equiv 0 \pmod{p^2}$.

There exists an element $ua_2 \in \pi_{2(n-p^2)+nq}^{2(n-p^2)+3}$ satisfying

$$\mathbb{H}(ua_2) \doteq \mathbb{H}(\beta_1^p(2(n-p^2+1)p-1)) \quad \text{and} \quad \mathbb{E}^{p^2-4}(ua_2) = \mathbb{P}(a_{2,n-2}) \neq 0.$$

Also, $\mathbb{E}^{p-3}(ua_1)$ is of order p^2 and $p\mathbb{E}^{p-3}(ua_1) = \mathbb{E}^{p^2-4}(ua_2)$.

Proposition 0.4 There exists a cell complex

$$S^{2p^2+1} \cup_{\beta_1^p} e^{2p^3-2p+2} \cup_{\alpha_1} e^{2p^3} \cup_p e^{2p^3+1} \text{ with } \wp^{p^2} \neq 0.$$

Theorem 0.5 Let $n \equiv 0 \pmod{p^3}$, then $p\mathbb{E}^t(ua_2) \neq 0$ for $0 \leq t < p^2-4$.

The order of $\mathbb{E}^{p^2-5}(ua_2)$ is p^2 . If $p > 3$ then the order of $\mathbb{E}^{p-4}(ua_1)$ is p^3 .

Proposition 0.5 If $k \neq pq-2$ and $k \not\equiv -2 \pmod{p^4q}$, then A_{2r+1+k}^{2r+1} is a direct summand of π_{2r+1+k}^{2r+1} for all r .

$\mathbb{E}$ -family

A graded subgroup $F_{*+k}^* = \sum_r F_{2r+1+k}^{2r+1}$ of the k -stem group π_{*+k}^* is called as an $\mathbb{E}$ -family if it satisfies the condition

$$(3) \quad \mathbb{E}(F_{2r-1+k}^{2r-1}) = F_{2r+1+k}^{2r+1} \cap \mathbb{E}(\pi_{2r-1+k}^{2r-1}) \quad (\eta \xrightarrow{\mathbb{E}} \xi \xrightarrow{\mathbb{E}} \xi').$$

Alpha family A_{*+k}^* is an $\mathbb{E}$ -family.

For each $\mathbb{E}$ -family F_{*+k}^* , we choose its *belonging invariants* FQ_{2r-1+k}^{2r-1} and FQ_{2r-2+k}^{2r-1} satisfying the exactness of the sequence

$$0 \longrightarrow FQ_{2r-1+k}^{2r-1} \xrightarrow{\mathbb{P}} \pi Q_{2r-1+k}^{2r-1} \xrightarrow{\mathbb{E}} \pi Q_{2r+1+k}^{2r+1} \xrightarrow{\mathbb{H}} FQ_{2r-2+k}^{2r-1} \longrightarrow 0.$$

For given $\mathbb{E}$ -families F_{*+k}^* , consider the quotients

$$\bar{\pi} Q_{2r+1+k}^{2r+1} = \pi Q_{2r+1+k}^{2r+1} / F_{2r+1+k}^{2r+1} \quad \text{and} \quad \bar{\pi} Q_{2r-1+k}^{2r-1} = \pi Q_{2r-1+k}^{2r-1} / FQ_{2r-1+k}^{2r-1},$$

then we have the following exact sequence instead of double EHP-sequence:

$$\cdots \xrightarrow{\mathbb{H}} \bar{\pi} Q_{2r-1+k}^{2r-1} \xrightarrow{\mathbb{P}} \bar{\pi} Q_{2r-1+k}^{2r-1} \xrightarrow{\mathbb{E}} \bar{\pi} Q_{2r+1+k}^{2r+1} \xrightarrow{\mathbb{H}} \bar{\pi} Q_{2r-2+k}^{2r-1} \xrightarrow{\mathbb{P}} \cdots$$

This makes simplify computations of the k -stem groups.

Stable type family.

For a stable generator $\xi \in \pi_k^S$, we consider a *stable type family* converging to ξ , and denote it by $S_{*+k}^*(\xi) = \sum_r S_{2r+1+k}^{2r+1}(\xi)$.

Usually, non-trivial $S_{2r+1+k}^{2r+1}(\xi)$ is generated by an element $\xi(2r+1)$ such that $\xi = \mathbb{E}^\infty(\xi(2r+1))$ and $\mathbb{E}(\xi(2r+1)) = \xi(2r+3)$. The minimum of the degree $2r+1$ of such $\xi(2r+1)$ is denoted by $u(\xi)$.

We exclude the alpha families A_{*+nq-1}^* from stable type family.

$\xi = \beta_1$ is exceptional. $S_{2r-1+pq}^{2r+1}(\beta_1) = \langle \beta_1(2r+1) \rangle \cong \mathbb{Z}/p$ for $r \geq p$, $\cong \mathbb{Z}/p^2$ for $r = p-1$ and $S_{2r-1+pq}^{2r+1} = A_{2r-1+pq}^{2r+1}$ for $p-1 > r \geq 1$.

The followings are a list of generators ξ of π_k^S/A_k^S for $k < (p^2+p)q-3$, in which x and X indicate invariants given by

$$x = \mathbb{I}(\xi(2tp-1)) \in \pi Q_{2tp-3+k}^{2t-1} \quad \text{and} \quad \mathbb{J}(X) = \xi(2tp+1), \quad X \in \pi Q_{2tp-2+k}^{2t-1}.$$

$$\xi = \beta_s \in \pi_{(sp+s-1)q-2}^S, \quad u(\beta_s) = 2p+3, \quad \text{for } s \geq 2, \quad x = b_s, \quad X = B_s.$$

$$\xi = \alpha_1 \beta_s \in \pi_{(sp+s)q-3}^S, \quad u(\alpha_1 \beta_s) = 5, \quad \text{for } s \geq 2, \quad x = ab_s, \quad X = AB_s.$$

$$\xi = \beta_1^r \in \pi_{rpq-2r}^S, \quad u(\beta_1^r) = 2p-1, \quad \text{for } r \geq 1, \quad x = b^r, \quad X = B^r.$$

$$\xi = \alpha_1 \beta_1^p \in \pi_{(rp+1)q-2r-1}^S, \quad u(\alpha_1 \beta_1^p) = 3, \quad \text{for } p > r \geq 1, \quad x = ab^r, \quad X = AB^r.$$

$$\xi = \beta_1^r \beta_s \in \pi_{(rp+sp+s-1)q-2r-2}^S, \quad u(\beta_1^r \beta_s) = 2p-1, \quad \text{for } p \geq r \geq 1, \quad s \geq 2, \\ x = b^r b_s, \quad X = B^r B_s.$$

$$\xi = \alpha_1 \beta_1^r \beta_s \in \pi_{(rp+sp+s)q-2r-3}^S, \quad u(\alpha_1 \beta_1^r \beta_s) = 2p-1, \quad \text{for } p \geq r \geq 1, \quad s \geq 2, \\ x = ab^r b_s, \quad X = AB^r B_s.$$

$$\xi = \epsilon_i \in \pi_{(p^2+i)q-2}^S, \quad u(\epsilon_i) = 2(p-i)p+3, \quad p-1 > i \geq 1, \quad x = e_i, \quad X = E_i.$$

$$\xi = \epsilon'_i \in \pi_{(p^2+i)q-3}^S, \quad u(\epsilon'_i) = 2(p-i-1)p+1, \quad p-1 > i \geq 1, \quad x = e'_i, \quad X = E'_i,$$

where $\beta_p = \epsilon_{p-1}$ and $\epsilon'_i = \alpha_1 \epsilon_{i-1}$ for $p-1 > i > 1$.

Remark that $\epsilon'_i(2r+1)$ cannot express as $\alpha_1(2r+1) \circ \epsilon_{i-1}(2r+q)$ when $2r+1 < 2(p-i)p+6 = u(\epsilon_{i-1}) - (q-1)$.

Removable family

An $\mathbb{E}$ -family $R_{*+k}^* = \sum_r R_{2r+1+k}^{2r+1}$ is called as a *removable family* if $\Delta : R_{2sp+1+k}^{2sp+1} \rightarrow R_{2sp-1+k}^{2sp-1}$ is isomorphic for all s .

The word "removable" means that in the computation of the k -stem group we may exclude removable families together with their belonging invariants.

Any stable type family cannot be removed since it induces non-trivial invariants by $IJ\Delta$ -sequence. Thus every removable family has finite number of terms.

Example 1. Single family

There are many number of removable family which consists of only one term : $R_{*+k}^* = R_{2t-1+k}^{2t-1}$.

Proposition 0.6 Let $x \in \pi Q_{2tp-3+k}^{2t-1}$ and $X \in \pi Q_{2tp-2+k}^{2t-1}$ be invariants corresponding to $\xi \in \pi_k^S$ and let $ax \in \pi Q_{2tp-6+k}^{2t-3}$ and $AX \in \pi Q_{2tp-5+k}^{2t-3}$ corresponding to $\alpha_1 \xi \in \pi_{k+2p-3}^S$.

- (1) $\mathbb{H}\mathbb{P}(x) = (t+1)ax$ and $\mathbb{H}\mathbb{P}(X) = (-t)AX$.
- (2) Assume $t \not\equiv 0, -1 \pmod{p}$, then $R_{2tp-3+k}^{2t-1} = \langle \mathbb{P}(x) \rangle \neq 0$ and $R_{2tp-2+k}^{2t-1} = \langle \mathbb{P}(X) \rangle \neq 0$ are removable families of single term.

Example 2. Alpha-beta family

Proposition 0.7 Let ax be the invariant of Proposition 0.6 of the case $t \equiv -1 \pmod{p}$ and $bx \in \pi Q_{2(t-1)p-5+k}^{2(t-p)+1}$ an invariant corresponding to $\beta_1 \xi \in \pi_{pq-2+k}^S$.

Then there exists an element $uax \in \pi_{2(t-1)p-2+k}^{2(t-p)+1}$ satisfying

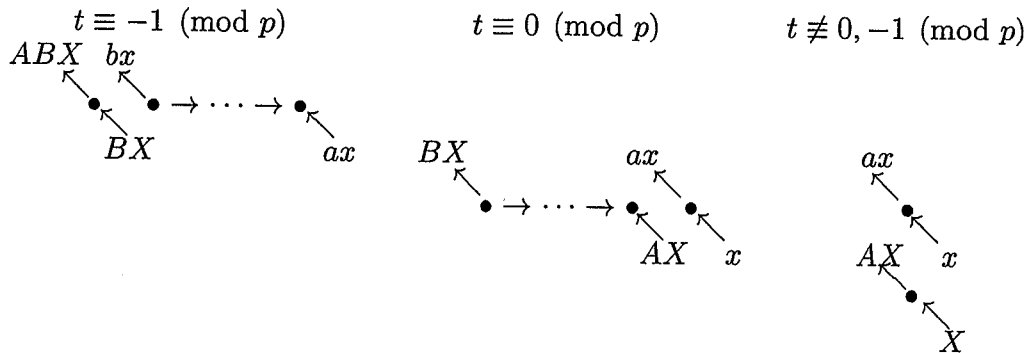
$$\mathbb{H}(uax) = bx \text{ and } \mathbb{E}^{p-2}(uax) = \mathbb{P}(ax).$$

If uax and $\mathbb{P}(ax)$ are of order p , then a removable family $R_{*+tq-3+k}^*$ is given by $R_{2(t-1)p-4+k}^{2(t-p)-1} = \langle \Delta(uax) \rangle$ and $R_{2(t-1)p+2i-2+k}^{2(t-p+i)-1} = \langle \mathbb{E}^i(uax) \rangle$.

Proposition 0.8 Let x and AX be the invariants of Proposition 0.6 of the case $t \equiv 0 \pmod{p}$ and $BX \in \pi Q_{2(t-1)p-4+k}^{2(t-p)-1}$ an invariant corresponding to $\beta_1 \xi \in \pi_{pq-2+k}^S$. Then there exists an element $uAX \in \pi_{2(t-1)p-1+k}^{2(t-p)+1}$ satisfying

$$\mathbb{H}(uAX) = BX \text{ and } \mathbb{E}^{p-2}(uAX) = \mathbb{P}(AX).$$

If uAX and $\mathbb{P}(uAX)$ are of order p , then a removable family $R_{*+tq-2+k}^*$ is given by $R_{2(t-1)p+2i-1+k}^{2(t-p+i)-1} = \langle \mathbb{E}^i(uAX) \rangle$ and $R_{2tp-3+k}^{2t-1} = \langle \mathbb{P}(x) \rangle$.



$$\bullet = 2/p$$

$$\circ = 7/p^2$$

$$\pi_{*+k}^* \bmod A_{*+k}^* \oplus R_{*+k}^*$$

$$\text{Table of } \pi_{2r+1+k}^{2r+1} = \pi_{2r+1+k}^{2r+1} / (A_{2r+1+k}^{2r+1} \oplus R_{2r+1+k}^{2r+1})$$

$k \setminus 2r+1$	3	$2p-1$	
p^2q-3	b^p	$\bullet = \bullet = \dots = \bullet$	$\xrightarrow{AB} \bullet \xrightarrow{b} \circ = \dots = \circ \xrightarrow{a_2} \bullet_a$ $(\alpha_1 \beta_1^p = 0)$
p^2q	$ab^{p-3}b_2$	$\bullet = \bullet = \dots = \bullet$	$\Rightarrow \beta_1^{p-2}\beta_2$
p^2q+q-3		AB_2	$b_2 \bullet = \bullet = \dots = \bullet \Rightarrow \epsilon'$
p^2q+q-2		B_2	$\bullet = \bullet = \dots = \bullet \Rightarrow \epsilon_1$
p^2q+q-1	$b^{p-2}b_2$	$\bullet = \bullet = \dots = \bullet$	$\Rightarrow \alpha_1 \beta_1^{p-2}\beta_2$
$p^2q+iq-q+2i$	$ab^{p-i-2}b_{i+1}$	$\bullet = \bullet = \dots = \bullet$	$\Rightarrow \beta_1^{p-i-1}\beta_{i+1}$
$p^2q+iq-3$		AB_i	$b_{i+1} \bullet = \bullet = \dots = \bullet \Rightarrow \epsilon'_i$
$p^2q+iq-2$		B_{i+1}	$\bullet = \bullet = \dots = \bullet \Rightarrow \epsilon_i$
$p^2q+iq+2i-1$	$b^{p-i-1}b_{i+1}$	$\bullet = \bullet = \dots = \bullet$	$\Rightarrow \alpha_1 \beta_1^{p-i-1}\beta_{i+1}$
$p^2q+pq-q-5$	bb_{p-1}	$\bullet = \bullet = \dots = \bullet$	$\Rightarrow \alpha_1 \beta_1 \beta_{p-1}$
$p^2q+pq-q-4$	ab^p	$\bullet = \bullet = \dots = \bullet$	$\Rightarrow \beta_1^{p+1}$
$p^2q+pq-q-3$		B_{p-1}	$\bullet = \bullet = \dots = \bullet \Rightarrow \epsilon_{p-1} = \beta_p$
$p^2q+pq-5$	b^{p+1}	$\bullet \rightarrow \bullet \rightarrow \dots \rightarrow \bullet \rightarrow \bullet$	
$p^2q+pq-4$	B^{p+1}	$e'_{p-2} \rightarrow e'_{p-3} \rightarrow \dots \rightarrow e'_2 \rightarrow e'$	
$p^2q+pq-3$		$E'_{p-2} \rightarrow E'_{p-2} \rightarrow \dots \rightarrow E'_2 \rightarrow E' \rightarrow w \rightarrow b^p$	$p\varphi = \alpha_1 \epsilon_{p-1}$
$p^2q+pq-2$		$ab^{p-2}b_2$	$\bullet = \bullet = \dots = \bullet \Rightarrow \beta_1^{p-1}\beta_2$
$p^2q+pq+q-5$	uw	$\bullet \rightarrow \bullet = \bullet = \dots = \bullet$	$\Rightarrow \beta_1 \epsilon'$
$p^2q+pq+q-4$	φ	B^{p+1}	

k -stem groups near φ

Lemma 0.3 *Let $e'_1 = e'$ and $E'_1 = E'$.*

- (1) $\bar{\pi}(e'_i) = e'_{i+1}$ and $\bar{\pi}(E'_i) = E'_{i+1}$ for $p-2 > i \geq 1$.
- (2) There exists $ue' \in \pi_{p^2q+pq-2}^3$ satisfying

$$\mathrm{H}^1(ue') = b^{p+1} \quad \text{and} \quad \mathrm{E}^{p-3}(ue') = \mathbb{P}(e').$$

Thus $R_{2r+p^2q+pq-4}^{2r+1} = \langle \mathbb{E}^{r-1}(ue') \rangle$ for $p > r \geq 1$ construct a removable family $R_{*+p^2q+pq-5}^*(e')$.

- (3) *There exists $uE' \in \pi_{p^2q+pq-1}^3$ satisfying*

$$\mathbb{H}(uE') = B^{p+1} \quad \text{and} \quad \mathbb{E}^{p-3}(uE') = \mathbb{P}(E').$$

Thus $R_{2r+p^2q+pq-3}^{2r+1} = \langle \mathbb{E}^{r-1}(uE') \rangle$ for $p > r \geq 1$ construct a removable family $R_{*+p^2q+pq-4}^*(E')$.

Theorem 0.6 *For $p^2q + pq + q - 5 \geq k \geq p^2q + pq - 5$, we have the following decomposition of π_{*+k}^* .*

$$\pi_{*+k}^* = A_{*+k}^* \oplus R_{*+k}^* \oplus S_{*+k}^*(\xi) \oplus E_{*+k}^*,$$

where R_{*+k}^* consists of single family, alpha-beta family and ones in Lemma 0.3, $\xi = \varphi, \beta_1^{p-2}\beta_2, \beta_1\epsilon'$ and E_{*+k}^* occurs in the case $k = p^2q + pq - 3$ with the following structure :

$$E_{2r+p^2q+pq-2}^{2r+1} \cong \mathbb{Z}/p \text{ for } r=1, p \text{ and } \cong \mathbb{Z}/p^2 \text{ for } p>r>1,$$

$\mathbb{E} : E_{2r+p^2q+pq}^{2r+1} \rightarrow E_{2r+p^2q+pq}^{2r+3}$ is bijective for $r = p - 2$ and surjective for $r = p - 1$ and it has $\text{Coker } \mathbb{E} \cong \mathbb{Z}/p$ for $p - 2 > r \geq 1$,

$$\begin{array}{l}
 p^2q + pq - 4 \quad e'_{p-2} \xrightarrow{e'_{p-3}} \dots \rightarrow e'_2 \circ = \circ \rightarrow \bullet \\
 p^2q + pq - 3 \quad E'_{p-2} \xrightarrow{E'_{p-3}} \dots \rightarrow E'_2 \quad w \quad b^p \\
 p^2q + pq - 2 \quad ab^{p-2}b_2 \quad \bullet = \bullet = \dots = \bullet \Rightarrow \varphi \\
 \dots \dots \dots \bullet = \bullet = \dots = \bullet \Rightarrow \beta_1^{p-1} \beta_2 \\
 p^2q + pq + q - 5 \quad \begin{array}{c} uw \quad b^{p+1} \\ \varphi \quad B^{p+1} \end{array} \bullet \bullet = \bullet = \dots = \bullet \Rightarrow \beta_1 \epsilon'
 \end{array}$$

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