Talbot 2024: Topological Cyclic Homology of Ring Spectra

Mentored by Jeremy Hahn and Allen Yuan
Notes compiled by various participants

August 2024

Contents

1 Introduction (Allen Yuan) **3**

2 Definition of THH (Catherine Li) **7**

3 Computation of $\mathrm{THH}(\mathbb{F}_p)$ (Joseph Hlavinka) **13**

4 More computations of THH (Jackson Morris) **16**

5 Basics of synthetic spectra (Max Johnson) **22**

6 The even filtration (Keita Allen) **25**

7 The motivic spectral sequence for $\mathrm{THH}(\mathbb{Z})^\wedge_p$ (Guoqi Yan) **31**

8 The motivic spectral sequence for ℓ via descent from MU (Anish Chedalavada) **34**

9 THH as a cyclotomic spectrum (Maxine Calle) **38**

10 The circle action and redshift (Noah Wisdom) **44**

11 Cyclotomic Frobenius and the Segal conjecture (Isabel Longbottom) **47**

12 Motivic filtrations on TC^- and TP (Avi Zeff) **55**

13 Prismatic cohomology of $\mathbb{Z}$ and ℓ (Shai Keidar) **59**

14 Syntomic cohomology of $\mathbb{Z}$ and ℓ (Preston Cranford) **61**

15 The relationship between TC and algebraic K-theory (Albert Jinghui Yang) **69**

16 The telescope conjecture at a fixed prime p (Lucas Piessevaux) **76**

17 Redshift for $\mathbb{E}_\infty$-ring spectra (Alicia Lima) **87**

18 Conclusion (Jeremy Hahn) **88**

Overview

Topological cyclic homology is a rapidly developing subject sitting between homotopy theory and (via the work of Bhatt–Morrow–Scholze) p-adic arithmetic geometry. There are now several excellent introductions to topological cyclic homology that focus on discrete commutative rings and spherical group rings. We aim to give a computationally focused introduction to the topological cyclic homology of finite height ring spectra. The topic is also closely connected to (even, MU-based) synthetic spectra, and may help familiarize students with their use in computations.

Warning: Many of these talks were live TeX'd and although we have done our best to edit the content, there are possibly still typos, notational inconsistencies, or other mathematical errors. Additionally, notes for some of the talks are missing or incomplete.

Acknowledgements

These notes were written as part of Talbot 2024: Topological Cyclic Homology of Ring Spectra, which took place August 11–17, 2024, and was mentored by Jeremy Hahn and Allen Yuan. We wish to thank the mentors, all the participants and all the note-takers once more! This material is based upon work supported by the K-theory Foundation as well as the National Science Foundation grant DMS-2426392.

1 Introduction (Allen Yuan)

Notes by Eunice Sukarto

In **topology**, we can assign to a space X algebraic invariants $\mathcal{H}^*(X, \mathbb{Z})$ and $\mathrm{KU}^*(X)$. These are related by the Atiyah-Hirzebruch spectral sequence

$$\mathcal{H}^s(X; \pi_t\mathrm{KU}) \Rightarrow \pi_{t-s}\mathrm{KU}^X$$

induced by the Postnikov filtration $\tau_{\geq *}\mathrm{KU}$ on KU.

In **algebraic geometry/rings**, are there similar invariants?

Grothendieck: For a ring $R \rightsquigarrow$ étale cohomology $\mathcal{H}^*_{\text{ét}}(R)$.
e.g. For $R = \mathbb{C}$, $\mathcal{H}^*_{\text{ét}}(\mathbb{C}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$.

Quillen: For a (discrete) commutative ring $R \rightsquigarrow$ algebraic K-theory spectrum $\mathrm{K}(R)$.
$K_0(R) = \{\text{finite dimensional projective } R\text{-modules}, \oplus, \otimes\}$.
e.g. $\mathrm{K}(\mathbb{C})^\wedge_p = ku^\wedge_p$, where ku is connective K-theory.

Lichtenbaum-Quillen conjecture: For R a nice ring, p a prime invertible in R, there exists a spectral sequence

$$E_2^{s,t} = \mathcal{H}^s_{\text{ét}}\left(R; \mathbb{Z}/p\left(-\frac{t}{2}\right)\right) \Rightarrow \pi_{t-s}\mathrm{K}(R)/p$$

where t is a certain twist. Informally, this is saying that $\mathrm{K}(R)_{(p)}$, which has no étale descent, has a filtration by $\mathcal{H}^*_{\text{ét}}(R)$, which has étale descent.

Thomason: K-theory mod p with the Bott element β inverted

$$R \mapsto \mathrm{K}(R)/p[\beta^{-1}]$$

has étale descent.

Waldhausen: Lichtenbaum-Quillen is equivalent to the map

$$\mathrm{K}(R) \to L_1^f\mathrm{K}(R)$$

being an isomorphism on homotopy groups in large enough degrees.

A Detour: Reminders on the Chromatic Filtration

Recall that the complex cobordism MU has homotopy groups

$$\pi_*\mathrm{MU} \cong \mathbb{Z}[x_1, x_2, \ldots]$$

where x_i has degree $2i$. The Lubin-Tate parameters

$$(p, v_1, v_2, \ldots)$$

sit inside $\pi_*\mathrm{MU}$ and can be chosen such that $v_i = x_{p^i-1}$, so v_i has degree $2(p^i-1)$. For each prime p, the p-localization $\mathrm{MU}_{(p)}$ of MU splits as a direct sum of suspensions of the Brown-Peterson spectrum BP, whose homotopy groups are given by

$$\pi_*\mathrm{BP} \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots].$$

Let $\mathrm{BP}\langle n\rangle$ denote the truncated BP-spectrum. It is a BP-algebra spectrum with homotopy groups

$$\pi_*\mathrm{BP}\langle n\rangle \cong \mathbb{Z}_{(p)}[v_1, \ldots, v_n].$$

For example,

$$\begin{aligned} \mathrm{BP}\langle -1\rangle &= \mathbb{F}_p \\ \mathrm{BP}\langle 0\rangle &= \mathbb{Z}_{(p)} \\ \mathrm{BP}\langle 1\rangle &= \ell, \end{aligned}$$

where ℓ injects into p-local connective K-theory as the Adams summand, sending v_1 to β^{p-1}. If $p=2$, then $\ell = ku_{(2)}$. There are ring maps

$$\mathbb{S} \to \mathrm{MU} \to \mathrm{BP} \to \mathrm{BP}\langle n\rangle$$

where $\mathbb{S}$ and MU are $\mathbb{E}_\infty$ rings, while BP and $\mathrm{BP}\langle n\rangle$ are not.

We can mostly pretend that $p, v_1, v_2, \ldots$ lift to $\pi_*\mathbb{S}$. More precisely, we have $p \in \pi_*\mathbb{S}$, a type 0 spectrum. For any e_0, $\mathbb{S}/p^{e_0}$ is a type 1 spectrum, so $\pi_*(\mathbb{S}/p^{e_0})$ contains some power of v_1. Similarly, for any e_1, $\mathbb{S}/p^{e_0}, v_1^{e_1}$ is a type 2 spectrum, so $\pi_*(\mathbb{S}/p^{e_0}, v_1^{e_1})$ contains some power of v_2, and so on.

Let

$$T(n) = \mathbb{S}/p^{e_0}, \ldots, v_{n-1}^{e_{n-1}}[v_n^{-1}]$$

for some $e_0, \ldots, e_{n-1}$. For any $e_0, \ldots, e_{n-1}$, $\mathbb{S}/p^{e_0}, \ldots, v_{n-1}^{e_{n-1}}$ is a type n spectrum, so the Bousfield class of $T(n)$ does not depend on the choice of $e_0, \ldots, e_{n-1}$. Alternatively, we could have picked any type n spectrum in place of $\mathbb{S}/p^{e_0}, \ldots, v_{n-1}^{e_{n-1}}$.

We denote by $L_{T(n)}$ the Bousfield localization with respect to $T(n)$. This is closely related to $L_{\mathrm{K}(n)}$, where $\mathrm{K}(n)$ is the nth Morava K-theory. We define finite localization as

$$L_n^f = L_{T(0)\oplus\ldots\oplus T(n)},$$

which is closely related to

$$L_n = L_{\mathrm{K}(0)\oplus\ldots\oplus\mathrm{K}(n)}.$$

End of Detour

Ausoni-Rognes: computed $\mathrm{K}(\ell)/p, v_1$ for $p \geq 5$ and showed that it is a finitely generated $\mathbb{F}_p[v_p]$-module. If M is a finitely generated graded $\mathbb{F}_2[v_2]$-module, then $M \to M[v_2^{-1}]$ is an isomorphism in high enough grading. This implies that

$$\mathrm{K}(\ell) \to L_2^f K(\ell)$$

is an isomorphism on $\pi_{*>>0}$.

Lichtenbaum-Quillen for ring spectra: For $n \geq 0$,

$$K(BP\langle n\rangle)_{(p)} \to L_{n+1}^{f} K(BP\langle n\rangle)$$

is an isomorphism on $\pi_{*>>0}$ iff $K(BP\langle n\rangle)_{(p)}$ has fp type $n+1$. This was proved by Hahn-Wilson and extended in Hahn-Raksit-Wilson.

The goal of this workshop is to understand the key computations in Hahn-Wilson and Hahn-Raksit-Wilson, focusing on $n = 0, 1$ i.e. $BP\langle n\rangle) = \mathbb{Z}, \ell$.

$$\begin{array}{ccc} K(R) & \longrightarrow & TC(R) \\ & \searrow & \downarrow \\ & & TC^{-}(R) \\ & & \downarrow \\ & & THH(R) \end{array}$$

Day 1: Properties + computations of $THH(R)$

- $THH(R)$ for $R = \mathbb{S}[x]$, MU, $\mathbb{F}_p$
- $THH(\mathbb{Z})/p$, $THH(\ell)/p, v_1$

Day 2: Motivic filtration/motivic spectral sequence

Filtration disambiguation

A spectrum X is the limit of its Postnikov tower

$$X = \lim_{\leftarrow}(\cdots \to \tau_{\leq 1} X \to \tau_{\leq 0} X \to \cdots).$$

The Postnikov/Whitehead filtration on X is

$$\tau_{\geq *} X : \cdots \to \tau_{\geq 1} X \to \tau_{\geq 0} X \to \cdots \to \operatorname{colim}_{\rightarrow} = X.$$

If $X = KU$, every other degree is constant since KU is concentrated in even degrees. This allows us to consider the double-speed Postnikov filtration

$$\tau_{\geq 2*} X : \cdots \to \tau_{\geq 2} X \to \tau_{\geq 0} X \to \cdots .$$

e.g. $\mathrm{Fil}^{*}_{\mathrm{mot}} THH(\mathbb{F}_p) = \tau_{\geq 2*} THH(\mathbb{F}_p)$ is a synthetic spectrum. Synthetic spectra is a nice home for motivic filtrations. We can compute $THH(\mathbb{Z})/p$, $THH(\ell)/p, v_1$ using motivic spectral sequences.

Days ≥ 3: More structures on THH

- circle action:

$$TC^{-}(R) = THH(R)^{hS^1}$$
$$TP(R) = THH(R)^{tS^1}$$

- Frobenius:

$$TC(R) = \mathrm{eq}\left(TC^-(R) \underset{\varphi}{\overset{\mathrm{can}}{\rightrightarrows}} TP(R) \right)$$

- redshift

$X = \operatorname{colim} \mathrm{Fil}^*_{\mathrm{mot}} X$	$\mathrm{gr}^* X$
$K(R)$	$\sim H^*_{\acute{e}t}(R)$
$HH(R)$(HKR filtration)	$\Omega^* R$
$HP(R)$	$H^*_{\mathrm{dR}}(R)$
$TP(R)$	$H^*_{\Delta}(R)$ prismatic cohomology of R
$TC^-(R)$	Nygard filtration on H^*_{Δ}
$TC(R)$	$H^*_{\mathrm{syn}}(R)$ syntomic cohomology

LQ says that $K(R) \sim H^*_{\acute{e}t}(R)$ i.e. $K(R)$ is built out of $H^*_{\acute{e}t}(R)$. If R is algebraically closed/strictly Henselian, then $ku \sim \mathbb{Z}$.

2 Definition of THH (Catherine Li)

Notes by Maxine Calle

In addition to giving a definition of THH, we will also talk about some properties of THH and do some example computations. These example computations will be relatively short, compared to other computations which require a whole talk to cover (see, for instance, Section 3).

We will only consider $\mathbb{E}_\infty$-ring spectra unless explicitly stated otherwise, which allows us to use the following definition of THH.

Definition 2.1. Let $R \to A$ be a map of $\mathbb{E}_\infty$-ring spectra. Then the *topological Hochschild homology of A relative to R* is $\mathrm{THH}(A/R) \coloneqq A \underset{A \otimes_R A}{\otimes} A$.

There's no op in this definition because we're working with $\mathbb{E}_\infty$-rings.

Notation. If we take $R = \mathbb{S}$, then we will simply write $\mathrm{THH}(A)$. If A is an ordinary (discrete) ring, then $\mathrm{THH}(A) \coloneqq \mathrm{THH}(HA)$.

Remark 2.2. If A and R are ordinary (commutative) rings, then $\mathrm{HH}(A/R) = A \otimes^{\mathbb{L}}_{A \otimes^{\mathbb{L}}_R A} A$ is sometimes called *Shukla homology*. It's also sometimes referred to as Hochschild homology, although it is double-derived unlike classical Hochschild homology.

We can give another definition of THH by universal properties (see the next theorem). The idea is that $\mathrm{THH}(A)$ is like "the free $\mathbb{E}_\infty$ A-algebra with S^1-action." But before we state this formulation, we recall that CAlg is tensored over spaces, meaning there is a functor

$$(-)^{\otimes(-)} \colon \mathrm{CAlg} \times \mathrm{Spc} \to \mathrm{CAlg}$$

which preserves colimits in each variable so that there is an adjunction

$$\mathrm{Spc} \underset{\mathrm{CAlg}(A,-)}{\overset{A^{\otimes(-)}}{\rightleftarrows}} \mathrm{CAlg} \qquad (\perp)$$

for $A \in \mathrm{CAlg}$. In particular, if X is a space, then $A^{\otimes X} = \mathrm{colim}_X A$ where A is viewed as a constant diagram, i.e. the colimit of $X \to * \xrightarrow{A} \mathrm{CAlg}$. There are different notations out there for this colimit (such as $A \otimes X, X \otimes A$) but we will stick with $A^{\otimes X}$. The following result appears in [MSV97].

Theorem 2.3 (McClure–Schwänzel–Vogt)**.** *Let A be an $\mathbb{E}_\infty$-ring. Then*

$$\mathrm{THH}(A) \simeq A^{\otimes S^1}.$$

In particular, the unit map $A \to \mathrm{THH}(A)$ induced by $* \to S^1$ is initial among $\mathbb{E}_\infty$-maps $A \to B$ for B with S^1-action, i.e. for any such map there is a factorization

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{THH}(A) \\ & \searrow & \downarrow \\ & & B \end{array}$$

where the dotted arrow is S^1-equivariant.

Proof. Write S^1 as pushout in spaces

$$\begin{array}{ccc} * \amalg * & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^1 \end{array}$$

and apply $A^{\otimes(-)}$ (which preserves pushouts) to get

$$\begin{array}{ccc} A \otimes A & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & \operatorname{colim}_{S^1} A \end{array} ,$$

using the fact that $A \otimes A$ is the coproduct in CAlg(Sp). Observe that this is the same pushout formula for the relative tensor product $A \underset{A\otimes A}{\otimes} A$, and hence we have

$$\mathrm{THH}(A) = A \underset{A\otimes A}{\otimes} A \simeq A^{\otimes S^1}$$

as claimed. The universal property follows from the equivalences

$$\mathrm{Map}^{S^1}_{\mathbb{E}_\infty}(\mathrm{THH}(A), B) \simeq \mathrm{Map}^{S^1}_{\mathrm{Spc}}(S^1, \mathrm{Map}_{\mathbb{E}_\infty}(A, B)) \simeq \mathrm{Map}_{\mathbb{E}_\infty}(A, B).$$

□

We now state some properties of THH, particular base-change formulas, which can also be found in [KN18]. We will also make use of the following result, although we will not prove it here.

Proposition 2.4. *The functor* $\mathrm{THH}\colon \mathrm{CAlg} \to \mathrm{CAlg}$ *is symmetric monoidal.*

Proposition 2.5. *Let* $R \to R'$ *and* $R \to A$ *be* $\mathbb{E}_\infty$*-maps. Then there is an equivalence*

$$\mathrm{THH}(A/R) \otimes_R R' \simeq \mathrm{THH}(A \otimes_R R'/R').$$

Proof. Let B be an $\mathbb{E}_\infty$ R'-algebra with S^1-action. We will show that $\mathrm{THH}(A/R) \otimes_R R'$ has the same universal property as $\mathrm{THH}(A \otimes_R R'/R')$. In particular, we have

$$\begin{aligned} \mathrm{Map}^{S^1}_{R'}(\mathrm{THH}(A/R) \otimes_R R', B) &\simeq \mathrm{Map}^{S^1}_{R}(\mathrm{THH}(A/R), B) \\ &\simeq \mathrm{Map}_R(A, B) \\ &\simeq \mathrm{Map}_{R'}(A \otimes_R R', B) \\ &\simeq \mathrm{Map}^{S^1}_{R'}(\mathrm{THH}(A \otimes_R R'), B), \end{aligned}$$

where the first and third equivalences use the adjunction between restriction and extension of scalars $- \otimes_R R'$, and the second and fourth equivalences use the universal property of THH. □

Theorem 2.6. *Given a commutative diagram of $\mathbb{E}_\infty$-ring maps,*

$$\begin{array}{ccccc} A & \longleftarrow & C & \longrightarrow & B \\ \uparrow & & \uparrow & & \uparrow \\ R & \longleftarrow & T & \longrightarrow & S \end{array},$$

then there is an equivalence

$$\mathrm{THH}(A/R) \otimes_{\mathrm{THH}(C/T)} \mathrm{THH}(B/S) \simeq \mathrm{THH}((A \otimes_C B)/(R \otimes_T S)).$$

We will not prove this theorem in general, but instead in the following special case.

Corollary 2.7. *Let $R \to A$ be a map of $\mathbb{E}_\infty$-rings. Then*

$$\mathrm{THH}(A/R) \simeq \mathrm{THH}(A) \otimes_{\mathrm{THH}(R)} R.$$

Proof. We will show that $\mathrm{THH}(A) \otimes_{\mathrm{THH}(R)} R$ has the correct universal property. Given a $\mathbb{E}_\infty$ R-algebra B with S^1-action, we first claim that

$$\mathrm{Map}_R^{S^1}(\mathrm{THH}(A) \otimes_{\mathrm{THH}(R)} R, B) \simeq \mathrm{Map}^{S^1}_{\mathrm{THH}(R)}(\mathrm{THH}(A), B).$$

In particular, there is an augmentation map $r\colon \mathrm{THH}(R) \to R$ induced by collapsing $S^1 \to *$,

$$\mathrm{THH}(R) \simeq \operatorname*{colim}_{S^1} R \to \operatorname*{colim}_{*} R \simeq R.$$

This is a retraction onto the 0^{th} level, meaning that r is equivariant with respect to the trivial S^1-action on R and precomposing with the unit

$$R \to \mathrm{THH}(R) \xrightarrow{r} R$$

is the identity on R. Now, we use the adjunction

$$\mathrm{CAlg}^{S^1}_{\mathrm{THH}(R)} \underset{r^*}{\overset{-\otimes_{\mathrm{THH}(R)} R}{\rightleftarrows}} \mathrm{CAlg}^{S^1}_{R} \quad (\perp)$$

to derive the claimed equivalence (where, by an abuse of notation, we have written B instead of r^*B).

We now claim that

$$\mathrm{Map}^{S^1}_{\mathrm{THH}(R)}(\mathrm{THH}(A), B) \simeq \mathrm{Map}_R(A, B),$$

which will complete the proof. To see this, observe that the data of a triangle of (non-equivariant) maps

$$\begin{array}{ccc} R & \longrightarrow & A \\ & \searrow & \downarrow \\ & & B \end{array}$$

is equivalent to the data of a triangle of S^1-equivariant maps

$$\begin{array}{ccc} \mathrm{THH}(R) & \longrightarrow & \mathrm{THH}(A) \\ & \searrow & \downarrow \\ & & B \end{array}$$

by using the universal property of both THH(A) and THH(R) to obtain the dashed arrows in the diagram below

$$\begin{array}{ccc} R & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathrm{THH}(R) & \dashrightarrow & \mathrm{THH}(A) \\ & \searrow & \downarrow \\ & & B \end{array} .$$

□

The proof is the same for the general case (i.e. manipulations of universal properties) but requires a bit more book-keeping. Next, we will discuss some example computations.

Example 2.8 (HH of polynomial algebras)**.** Let R be a discrete commutative ring and consider $\mathrm{HH}(R[x_1, \dots, x_n]/R)$. Since THH is symmetric monoidal and $R[x_1, \dots, x_n] \cong R[x]^{\otimes n}$, it suffices to compute $\mathrm{HH}_*(R[x]/R)$. We claim that

$$\mathrm{HH}_*(R[x]/R) \cong R[x] \otimes \Lambda(dx),$$

i.e. it is $R[x]$ concentrated in degrees 0 and 1. (This is part of the HKR theorem.)

To prove the claim, observe that

$$\mathrm{HH}(R[x]/R) = R[x] \overset{\mathbb{L}}{\underset{R[x]\otimes^{\mathbb{L}}_R R[x]}{\otimes}} R[x] \cong R[x] \overset{\mathbb{L}}{\underset{R[a,b]}{\otimes}} R[x]$$

where $R[a, b]$ acts on $R[x]$ by sending $a, b \mapsto x$ and using the multiplication in $R[x]$. We therefore want to compute $\mathrm{Tor}_*^{R[a,b]}(R[x], R[x])$. We resolve $R[x]$ as a $R[a, b]$-module,

$$0 \to R[a, b] \xrightarrow{\cdot(a-b)} R[a, b] \to R[a, b]/(a - b) \cong R[x] \to 0.$$

Then, we obtain the complex

$$0 \to R[x] \otimes_{R[a,b]} R[a, b] \xrightarrow{\mathrm{id} \otimes \cdot(a-b)} R[x] \otimes_{R[a,b]} R[a, b] \to 0$$

but then $\cdot(a - b)$ kills $R[x]$ as an $R[a, b]$-module and so the complex above becomes

$$0 \to R[x] \xrightarrow{0} R[x] \to 0$$

as claimed.

Example 2.9 (THH(MU))**.** There is a formula for THH of a Thom spectrum that will allow us to do this computation; see [Blu10],[BCS10], and [Sch11] for the general formula, as well as [RSV22] for a more recent interpretation. We will list a few variants of this formula (but will not prove them).

Theorem 2.10. *Let R be an $\mathbb{E}_\infty$-ring spectrum and X a connected $\mathbb{E}_{n+1}$-space. If $f\colon X \to B\mathrm{GL}_1(R)$ is an $\mathbb{E}_{n+1}$-map such that the $\mathbb{E}_{n+1}$-structure on the Thom spectrum Mf extends to an $\mathbb{E}_{n+2}$ R-algebra structure, then there is an equivalence of $\mathbb{E}_n$ R-algebras*

$$\mathrm{THH}(Mf/R) \simeq Mf \otimes \mathbb{S}[BX].$$

There is an analogous statement for the $\mathbb{E}_\infty$-setting, replacing X with an $\mathbb{E}_\infty$-group G. Given an $\mathbb{E}_\infty$-map $f\colon G \to B\mathrm{GL}_1(R)$, there is an equivalence

$$\mathrm{THH}(Mf/R) \simeq Mf \otimes \mathbb{S}[BG].$$

Now, to compute $\mathrm{THH}(MU)$, we plug $MU \simeq \mathrm{colim}(BU \to B\mathrm{Gl}_1(\mathbb{S}) \to \mathrm{Sp})$ into this formula to obtain

$$\mathrm{THH}(MU) \simeq MU \otimes \mathbb{S}[BBU].$$

In particular, we can compute

$$\mathrm{THH}_*(MU) = MU_* \otimes \Lambda(x_1, x_2, \dots)$$

where the generators are in odd degrees.

Example 2.11 ($\mathrm{THH}(\mathbb{S}[t])$)**.** Consider $\mathbb{S}[t] \coloneqq \mathbb{S}[\mathbb{N}] = \Sigma^\infty_+ \mathbb{N}$; we expect to obtain

$$\mathrm{THH}(\mathbb{S}[t]) \simeq \mathbb{S} \oplus \bigoplus_{n \geq 1} \Sigma^\infty_+ (S^1/C_n).$$

We will start with a slightly easier computation: $\mathbb{S}[\mathbb{Z}]$. We can get this from a more general computation for spherical group rings, with

$$\mathrm{THH}(\mathbb{S}[G]) \simeq \Sigma^\infty_+ LBG$$

where L denotes the free loop space and the circle action rotates loops. This equivalence follows from the previous example: the group ring $\mathbb{S}[G]$ arises as the Thom spectrum of the trivial map $G \to B\mathrm{GL}_1(\mathbb{S})$, i.e.

$$\mathrm{colim}(G \to B\mathrm{GL}_1(\mathbb{S} \to \mathrm{Sp}) \simeq \underset{G}{\mathrm{colim}}\, \mathbb{S} \simeq \Sigma^\infty_+ G = \mathbb{S}[G].$$

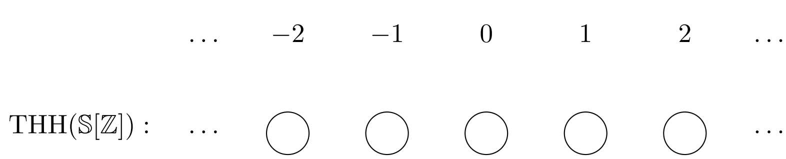
So by the previous example,

$$\mathrm{THH}(\mathbb{S}[G]) \simeq \mathbb{S}[G] \otimes \mathbb{S}[BG] \simeq \mathbb{S}[G \times BG] \simeq \mathbb{S}[LBG].$$

We therefore have

$$\mathrm{THH}(\mathbb{S}[\mathbb{Z}]) \simeq \Sigma^\infty_+ LS^1 \simeq \Sigma^\infty_+ \mathrm{Map}_{\mathrm{Spc}}(S^1, S^1) \simeq \Sigma^\infty_+ (S^1 \times \mathbb{Z}).$$

The visual intuition is that we have a circle for each element of $\mathbb{Z}$, and S^1 acts by rotating the n^{th} loop at n-times speed:



so we might expect THH($\mathbb{S}[\mathbb{N}]$) to look like the suspension spectrum on something like:

$$\begin{array}{lccccccc} & \dots & -2 & -1 & 0 & 1 & 2 & \dots \\ \mathrm{THH}(\mathbb{S}[\mathbb{N}]): & \dots & \varnothing & \varnothing & \bullet & \bigcirc & \bigcirc & \dots \end{array}.$$

Then, as claimed previously, we have

$$\mathrm{THH}(\mathbb{S}[t]) \simeq \mathbb{S} \oplus \bigoplus_{n \geq 1} \Sigma^{\infty}_{+}(S^1/C_n)$$

i.e. a wedge of a point and a bunch of circles, where the quotient $S^1/C_n \simeq S^1$ encodes the circle action (rotating the n^{th} circle at n-times speed).

Here is a sketch of the actual calculation. From the definition, we have

$$\mathrm{THH}(\mathbb{S}[\mathbb{N}]) \simeq \mathbb{S}[\mathbb{N}] \otimes_{\mathbb{S}[\mathbb{N}\times\mathbb{N}]} \mathbb{S}[\mathbb{N}] \simeq \mathbb{S}[\mathbb{N} \otimes_{\mathbb{N}\times\mathbb{N}} \mathbb{N}].$$

The geometric realization that defines the relative tensor product $\mathbb{N} \otimes_{\mathbb{N}\times\mathbb{N}} \mathbb{N}$ looks like

$$|\ \mathbb{N}\times\mathbb{N} \leftleftarrows \mathbb{N}\times(\mathbb{N}\times\mathbb{N})\times\mathbb{N} \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow}} \dots\ |.$$

Let $X_k = \mathbb{N}\times(\mathbb{N}\times\mathbb{N})^{\times k}\times\mathbb{N}$. Since the arrows preserve sums, we can decompose the realization into a disjoint union

$$\coprod_{n\in\mathbb{N}} |\ X_0^n \leftleftarrows X_1^n \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow}} \dots\ |$$

where X_k^n consists of k-simplices which sum to n. For example, taking $n = 0$, there is only one way to sum to 0 for each k (taking all zeros), and so

$$|\ X_0^0 \leftleftarrows X_1^0 \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow}} \dots\ | = |* \leftarrow * \leftarrow \dots| = *.$$

Whereas, for $n = 1$, there are four non-degenerate simplices: two in degree 0 and two in degree 1. So we get a picture that looks like

$$|\ ** \leftleftarrows ** ** \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow}} \dots\ |$$

and the attaching maps imply that we get S^1 after realization. The other choices of $n \geq 2$ similarly produce circles (where all simplices higher than degree n are degenerate). The circle action follows from considering the map $\mathrm{THH}(\mathbb{S}[\mathbb{N}]) \to \mathrm{THH}(\mathbb{S}[\mathbb{Z}])$.

3 Computation of $\mathrm{THH}(\mathbb{F}_p)$ (Joseph Hlavinka)

Notes by Akira Tominaga

The goal of this talk is to explain the following computation.

Theorem 3.1 (Bökstedt)**.** *We have an $\mathbb{E}_1$-algebra equivalence*

$$\mathrm{THH}(\mathbb{F}_p) \simeq \mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3.$$

In particular, using the Serre spectral sequence, one sees that

$$\pi_* \mathrm{THH}(\mathbb{F}_p) \simeq \mathbb{F}_p[x],\ |x| = 2.$$

Remark 3.2. This theorem does not reveal the circle action on $\mathrm{THH}(\mathbb{F}_p)$. The circle action on S^3 is not compatible with the circle action on THH.

Remark 3.3. Recall that there is a trace map $K(\mathbb{F}_p) \to \mathrm{THH}(\mathbb{F}_p)$. Quillen computed that

$$K_*(\mathbb{F}_p) \simeq \begin{cases} 0 & * = 2i, \\ \mathbb{Z}/(p^i - 1) & * = 2i-1. \end{cases}$$

Hence the original trace map is not a good approximation in this case. However, after p-completion, we obtain that $K(\mathbb{F}_p)^\wedge_p \simeq \mathbb{Z}_p$, so the trace map give a better approximation.

One way to deduce Bökstedt's computation is to use following result.

Theorem 3.4 (Hopkins-Mahowald)**.** *There is an $\mathbb{E}_2$-algebra equivalence*

$$\mathbb{F}_p \otimes \mathbb{F}_p \simeq \mathbb{F}_p \otimes \Sigma^\infty_+ \Omega^2 S^3.$$

Proof of Bökstedt's periodicty.

$$\begin{aligned} \mathrm{THH}(\mathbb{F}_p) &\simeq \mathbb{F}_p \otimes_{\mathbb{F}_p \otimes \mathbb{F}_p} \mathbb{F}_p \\ &\simeq \mathbb{F}_p \otimes_{\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega^2 S^3} \mathbb{F}_p \\ &\simeq \mathbb{F}_p \otimes \mathbb{S} \otimes_{\Sigma^\infty_+ \Omega^2 S^3} \mathbb{S} \\ &\simeq \mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3. \end{aligned}$$

□

We will see how to prove the Hopkins-Mahowald's result in the rest of this talk. We restrict to the $p = 2$ case below, but the idea is the same when $p > 2$.

Definition 3.5. Let X be a compact pointed space. The free $\mathbb{E}_n$-algebra on X is the $\mathbb{E}_n$-algebra $\mathrm{Free}_{\mathbb{E}_n}(X) := \Sigma^\infty_+ \Omega^n \Sigma^n X$. This construction is meant to satisfy the universal property

$$\mathrm{Map}_{\mathbb{E}_n}(\mathrm{Free}_{\mathbb{E}_n}(X), Y) \simeq \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty_+ X, Y).$$

The free $\mathbb{E}_n$-$\mathbb{F}_p$-algebra on X is the spectrum $\mathbb{F}_p \otimes \mathrm{Free}_{\mathbb{E}_n}(X)$.

Remark 3.6. We have an equivalence

$$\begin{aligned}\mathrm{Map}_{\mathbb{F}_p-\mathbb{E}_n}(\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega^n S^{n+k}, Y) &\simeq \mathrm{Map}_{\mathbb{E}_n}(\Sigma^\infty_+ \Omega^n S^{n+k}, Y)\\ &\simeq \mathrm{Map}_{\mathrm{Sp}}(S^k, Y).\end{aligned}$$

Construction 3.1. Let A be an $\mathbb{E}_n$-algebra and $x \in \pi_k A$. Take the $\mathbb{F}_p$-$\mathbb{E}_n$-algebra map $\tilde{x} \colon \mathbb{F}_p \otimes \Sigma^\infty_+ \Omega^n S^{n+k} \to A$ using the equivalence above. For an element $Q^j \colon S^j \to \mathbb{F}_p \otimes \Sigma^\infty_+ \Omega^n S^{n+k}$, we denote the composition $\tilde{x} \circ Q^j$ as $Q^j x$.

Theorem 3.7. *Let A be an $\mathbb{E}_n$-algebra. Then there is a suitable choice of Q^r so that the resulting operations $Q^r \colon H_*(A; \mathbb{F}_2) \to H_{*+r}(A; \mathbb{F}_2)$ satisfy the following properties:*

(a) if $x \in H_i(A; \mathbb{F}_2)$, then $Q^r(x) = 0$ for $r < i$ or $i > i + n$,

(b) $Q^r(x) = x^2$ when $|x| = r$, and

(c) Adem, Cartan, and Nishida relations hold.

Let's take $A = \mathbb{F}_2 \otimes \mathbb{F}_2$. For a map $x \in \pi_1(\mathbb{F}_2 \otimes \mathbb{F}_2)$, we associate an $\mathbb{E}_2$-$\mathbb{F}_2$-algebra map $\mathbb{F}_2 \otimes \Sigma^\infty_+ \Omega^2 S^3 \to \mathbb{F}_2 \otimes \mathbb{F}_2$.

Theorem 3.8 (Araki-Kudo)**.** *We have*

$$\pi_* \mathbb{F}_2 \otimes \Sigma^\infty_+ \Omega^2 S^3 \simeq \mathbb{F}_2[x_1, x_2, \dots], \ |x_i| = 2^i - 1.$$

satisfying $Q^{2^i} x_i = x_{i+1}$.

Proof of Hopkins-Mahowald. Recall that $\pi_* \mathbb{F}_2 \otimes \mathbb{F}_2 \simeq \mathbb{F}_2[\xi_1, \xi_2, \dots]$ with $Q^{2^i} \xi_i = \xi_{i+1}$. Consider the map $\tilde{\xi_1} \colon \mathbb{F}_2 \otimes \Sigma^\infty_+ \Omega^2 S^3 \to \mathbb{F}_2 \otimes \mathbb{F}_2$ associated to $\xi_1 \in \pi_1 \mathbb{F}_2 \otimes \mathbb{F}_2$. Then one can check that $\pi_1 \tilde{\xi_1}$ sends ξ_1 to x_1. [1] Because $\tilde{\xi_1}$ is an $\mathbb{E}_2$-map, it commutes with the Dyer-Lashof operations. In particular, $\tilde{\xi_1}$ sends ξ_i to x_i. □

Now we are done on the $\mathrm{THH}(\mathbb{F}_p)$. Let us comment a variant of this computation.

Theorem 3.9. *Let R be a discrete perfect $\mathbb{F}_p$-algebra. Then*

$$\pi_* \mathrm{THH}(A) \simeq A[x], \ |x| = 2.$$

To prove this, we recall the sperical Witt vector spectrum $\mathbb{S}_{W(A)}$. It is a p-complete, flat $\mathbb{S}$-algebra satisfying $\mathbb{S}_{W(A)} \otimes \mathbb{F}_p \simeq A$.

Proof.

$$\begin{aligned}\mathrm{THH}(A) &\simeq \mathrm{THH}(\mathbb{S}_{W(A)} \otimes \mathbb{F}_p)\\ &\simeq \mathrm{THH}(\mathbb{S}_{W(A)}) \otimes \mathrm{THH}(\mathbb{F}_p)\\ &\simeq \mathrm{THH}(\mathbb{S}_{W(A)}) \otimes \mathbb{F}_p \otimes_{\mathbb{F}_p} \mathrm{THH}(\mathbb{F}_p)\\ &\simeq \mathrm{THH}(A/\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{THH}(\mathbb{F}_p)\\ &\simeq A \otimes_{\mathbb{F}_p} \mathrm{THH}(\mathbb{F}_p)\end{aligned}$$

where the second equality uses the fact that THH is symmetric monoidal, and the fourth equality follows from a formula from the last talk. □

[1] We choice the element in $\pi_* \mathbb{F}_2 \otimes \mathbb{F}_2$ so that this will happen. This claim is due to Steinberger.

Lastly, we briefly mention the polynomial generator in $\mathrm{THH}_*(\mathbb{F}_p)$ and the suspension construction. Let A be an $\mathbb{E}_\infty$-ring with the unit $u\colon \mathbb{S} \to A$. Then the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{S} & \longrightarrow & 0 \\
{\scriptstyle u}\downarrow & & \downarrow \\
A & \xrightarrow[u\otimes \mathrm{id} - \mathrm{id}\otimes u]{} & A \otimes A.
\end{array}
$$

Therefore we can take a map $\sigma\colon \mathrm{cofib}(u) \simeq \Sigma\,\mathrm{fib}(u) \to A \otimes A$. Similarly, the following diagram

$$
\begin{array}{ccccc}
\Sigma\,\mathrm{fib}(u) & & \longrightarrow & & 0 \\
\downarrow & \searrow & & & \downarrow \\
 & & A \otimes A & \longrightarrow & A \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & A & \longrightarrow & \mathrm{THH}(A)
\end{array}
$$

commutes and we have a map $\sigma^2\colon \Sigma^2\,\mathrm{fib}(u) \to \mathrm{THH}(A)$.

Proposition 3.10. *Take $A = \mathbb{F}_p$. Then $\pi_0\,\mathrm{fib}(u) = p\mathbb{Z}$ and we have $\sigma^2(p) = x \in \pi_2\,\mathrm{THH}(\mathbb{F}_p)$.*

4 More computations of THH (Jackson Morris)

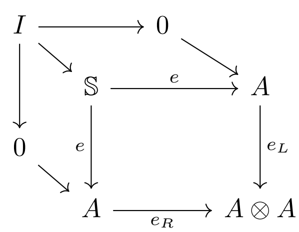
Notes by Atticus Wang

Fix an odd prime p throughout. In this talk we'll review some background on suspension operations and filtered spectra, and then use some of this technology to compute the homotopy groups of $\mathrm{THH}(\mathbb{Z}_p)/p$ and $\mathrm{THH}(\ell_p)/(p, v_1)$.

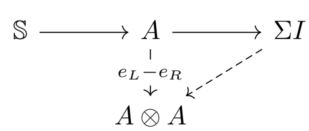
4.1 Suspension operations

Let us first elaborate on the suspension operations mentioned near the end of the last talk. This will be useful for later talks. The main references for this section are [HW22] appendix A and [LL23] section 3.1.

Without aiming for full generality, let's work in the familiar land of spectra. Let A be a homotopy commutative $\mathbb{E}_1$-algebra, and let $I = \mathrm{fib}(\mathbb{S} \to A)$. There are two maps $A \to A \otimes_{\mathbb{S}} A$, given by the left and right units. We can precompose both maps with $I \to \mathbb{S} \to A$. Then any point in $I \to A \otimes_{\mathbb{S}} A$ is nullhomotopic "for two reasons", so we obtain a loop in $A \otimes_{\mathbb{S}} A$. In other words we get a map $\Sigma I \to A \otimes A$. This is how you could think about it. Formally, there are natural homotopies filling all the squares

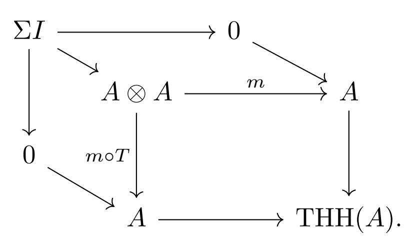


so you automatically get a map $\Sigma I \to A \otimes A$ by pushing out from the back face. Another way to say the same thing is that there is a natural nullhomotopy of the composition $\mathbb{S} \to A \xrightarrow{e_L - e_R} A \otimes A$, so we get a dotted map



This is the suspension map σ.

Doing this once more gives a map $\sigma^2 \colon \Sigma^2 I \to \mathrm{THH}(A)$. In the intuitive picture, the loop you got before in $A \otimes A$ is zero in $\mathrm{THH}(A)$ again for two reasons, so we get a 2-sphere in $\mathrm{THH}(A)$. Formally:



Here, we need the natural map $e_L - e_R : A \to \mathrm{fib}(m\colon A \otimes A \to A)$ to factor through ΣI. It suffices to show that $\mathbb{S} \to A \to \mathrm{fib}(m)$ is zero, i.e. that the map $\pi_0(A) \to \pi_0(\mathrm{fib}(m))$ maps the unit to 0. But because m is surjective on homotopy groups, $\pi_0(\mathrm{fib}(m)) \subset \pi_0(A \otimes A)$, so the map $\pi_0(A) \to \pi_0(A \otimes A)$ induced by $e_L - e_R$ does indeed map the unit to zero.

Exercise 4.1. Show that the Bott element $u \in \mathrm{THH}_2(\mathbb{F}_p)$ arising from the identification $\mathrm{THH}(\mathbb{F}_p) \simeq \mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3$ is the same element as $\sigma^2 p$. (Hint: first show that σp is the degree 1 exterior power generator of the Steenrod algebra by comparing with $\mathbb{F}_p \otimes^L_{\mathbb{Z}} \mathbb{F}_p$.)

Precomposing σ^2 with the connecting homomorphism gives the map

$$d\colon \Sigma A \to \Sigma^2 I \xrightarrow{\sigma^2} \mathrm{THH}(A).$$

The geometric picture is that this is supposed to take a point in A, take its image in $\mathrm{THH}(A)$, and use the S^1 action on $\mathrm{THH}(A)$ to form a circle. In other words:

Exercise 4.2. The map d is the same map as

$$\Sigma A \to A \otimes \Sigma A = (\Sigma^\infty_+ S^1) \otimes A \to \mathrm{THH}(A)$$

where the last map is induced by the circle action on THH.

The last map we'll define is a map $t\colon \Sigma^{-2}\,\mathrm{THH}(A) \to \lim_{\mathbb{CP}^1} \mathrm{THH}(A)$, where the limit is taken over the diagram $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty = BS^1 \to \mathrm{Sp}$ encoding the S^1-action on THH. To construct this, write $\mathbb{CP}^1$ as a colimit of spaces

$$\mathbb{CP}^1 = \mathrm{colim}(S^1 \rightrightarrows S^0).$$

When taking the limit over an indexing category which can be written as a colimit, one can break it up as a limit of the limits over each individual part[2]. Applying this to the above cofiber sequence we get

$$\begin{aligned}\lim_{\mathbb{CP}^1} \mathrm{THH}(A) &\simeq \lim(\mathrm{THH}(A) \oplus \mathrm{THH}(A) \rightrightarrows \mathrm{THH}(A) \oplus \Sigma^{-1}\,\mathrm{THH}(A)) \\ &\simeq \lim(\mathrm{THH}(A) \to \Sigma^{-1}\,\mathrm{THH}(A))\end{aligned}$$

and t is just the connecting homomorphism for this fiber sequence.

The geometric picture is that if we take a 2-sphere in $\mathrm{THH}(A)$ and apply t, it recovers the fixed points of the S^1-action on the sphere, namely the two poles; in other words it is a kind of inverse to the suspension operation. The geometric picture suggests the following proposition (whose proof we skip):

Proposition 4.3 ([HW22], lemma A.4.1)**.** *The following square commutes:*

$$\begin{array}{ccc} I & \longrightarrow & \mathbb{S} \\ {\scriptstyle\sigma^2}\downarrow & & \downarrow \\ \Sigma^{-2}\,\mathrm{THH}(A) & \xrightarrow[t]{} & \lim_{\mathbb{CP}^1} \mathrm{THH}(A). \end{array}$$

[2] This is explained e.g. here.

4.2 Filtered objects

In this section we discuss filtered and graded objects. References include [BHS20] appendix B and [LL23] section 2.

Definition 4.4. Let $\mathcal{C}$ be a stable ∞-category. A *t-structure* on $\mathcal{C}$ is the data of a pair of two subcategories $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq -1}$ such that:

- $\mathcal{C}_{\geq 0}$ is closed under Σ, and $\mathcal{C}_{\leq -1}$ is closed under Σ^{-1}.
- For $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq -1}$, $\mathrm{Map}(X, Y)$ is contractible.
- For any $X \in \mathcal{C}$ there exist $Y \in \mathcal{C}_{\geq 0}$ and $Z \in \mathcal{C}_{\leq -1}$ such that $Y \to X \to Z$ is a fiber sequence.

Exercise 4.5. Show that:

(a) $\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq -1} = \varnothing$.

(b) There exists a right adjoint $\tau_{\geq 0} : \mathcal{C} \to \mathcal{C}_{\geq 0}$ to the inclusion $\mathcal{C}_{\geq 0} \hookrightarrow \mathcal{C}$. Similarly there exist a left adjoint $\tau_{\leq -1}$.

(c) $\mathcal{C}^{\heartsuit} := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ is an abelian 1-category.

(d) Given a stable ∞-category $\mathcal{C}$, consider its homotopy category $h\mathcal{C}$ which is a triangulated category. It is clear that a t-structure on $\mathcal{C}$ gives rise to a t-structure on $h\mathcal{C}$. Show that this is a bijection.

Our filtrations are decreasing by convention:

Definition 4.6. Define the category of filtered objects $\mathrm{fil}(\mathcal{C}) = \mathrm{Fun}(\mathbb{Z}_{\geq}, \mathcal{C})$, and the category of graded objects $\mathrm{gr}(\mathcal{C}) = \mathrm{Fun}(\mathbb{Z}, \mathcal{C})$. Both are symmetric monoidal categories by Day convolution. Here $\mathbb{Z}_{\geq}$ is the poset $\cdots 2 \to 1 \to 0 \to -1 \cdots$ and $\mathbb{Z}$ is discrete.

Now suppose $\mathcal{C}$ is a presentably symmetric monoidal stable ∞-category with a t-structure which is compatible with the monoidal structure: this means that the tensor unit is connective (i.e. in $\mathcal{C}_{\geq 0}$), and $\mathcal{C}_{\geq 0}$ is closed under tensor product. The example we'll use is of course Sp. Between the categories $\mathcal{C}, \mathrm{fil}(\mathcal{C}), \mathrm{gr}(\mathcal{C})$, there are the following functors:

- $-_n \colon \mathrm{fil}(\mathcal{C}) \to \mathcal{C}$, taking the nth object.
- the left adjoint to the above, $c^{0,n} \colon \mathcal{C} \to \mathrm{fil}(\mathcal{C})$. This takes X to the filtered object $\cdots \to 0 \to X \to X \to \cdots$ with the first X at the nth position. This is symmetric monoidal. Let $c^{k,n} := \Sigma^k c^{0,n+k}$.
- the colimit (or "realization") $\mathrm{colim} \colon \mathrm{fil}(\mathcal{C}) \to \mathcal{C}$. This is symmetric monoidal.
- the right adjoint to the above, the constant functor $Y \colon \mathcal{C} \to \mathrm{fil}(\mathcal{C})$. This is symmetric monoidal.
- the right adjoint to the above $\lim \colon \mathrm{fil}(\mathcal{C}) \to \mathcal{C}$.
- the associated graded, $\mathrm{gr} \colon \mathrm{fil}(\mathcal{C}) \to \mathrm{gr}(\mathcal{C})$. This is symmetric monoidal.
- the right adjoint to the above, taking zeros for all the filtration maps: $\mathrm{gr}(\mathcal{C}) \to \mathrm{fil}(\mathcal{C})$.

- the Whitehead tower $\tau_{\geq *}\colon \mathcal{C} \to \mathrm{fil}(\mathcal{C})$, which is lax symmetric monoidal.
- the bigraded homotopy groups $\pi^{\heartsuit}_{k,n}X = \pi^{\heartsuit}_{k}X_{n+k} = \Sigma^{-k}\tau_{\geq k}\tau_{\leq k}X_{n+k} \in \mathcal{C}^{\heartsuit}$ and $\pi_{k,n}X = \pi_k X_{n+k} = [\mathbb{1}^{k,n}, X]$.

Exercise 4.7. Let the *filtration parameter* $\tau \in \pi_{0,-1}(\mathbb{1}^{0,0})$ be defined by the map

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{1} & \longrightarrow & \mathbb{1} & \longrightarrow & \cdots
\end{array}$$

and let $C\tau \in \mathrm{fil}(\mathcal{C})$.

(a) For any $X \in \mathrm{fil}(\mathcal{C})$, what is $X \otimes \tau$?

(b) Show that $\mathrm{gr}(\mathcal{C})$ can be identified with $\mathrm{Mod}_{C\tau}(\mathrm{fil}(\mathcal{C}))$, and taking associated graded amounts to base changing $- \otimes C\tau$.

(c) On the other hand show that $\mathrm{fil}(\mathcal{C})[\tau^{-1}] \simeq \mathcal{C}$ by $X \mapsto \operatorname{colim} X$.

Proposition 4.8. *Let $X \in \mathrm{fil}(\mathcal{C})$. There is a spectral sequence associated to X:*

$$E^1_{s,t} = \pi^{\heartsuit}_{t-s,s}(\mathrm{gr}\, X) \implies \pi^{\heartsuit}_{t-s}(\operatorname{colim} X).$$

The differentials d_r are of bidegree $(r+1, r)$. Intuition check: d_1 goes from $\pi_{t-s}(\mathrm{gr}_t\, X) \to \pi_{t-s-1}X_{t+1} \to \pi_{t-s-1}(\mathrm{gr}_{t+1}\, X)$ which is correct.

Remark 4.9. Many people prefer for this spectral sequence to be graded according to the Adams convention, where the x-axis represents total degree $t-s$ and the y-axis represents Adams filtration degree s.

We now specialize to $\mathcal{C} = \mathrm{Sp}$ where $\pi^{\heartsuit}$ and π agree. Given a spectrum, we often study it by putting a filtration on it and studying its associated spectral sequence. The suspension operations allow us to formally determine some differentials in the spectral sequence. To start with, given any spectrum X and an element $x \in \pi_*(X)$ with a lift $\widetilde{x} \in \pi_*(X \otimes I)$, we can tensor the maps σ, σ^2 with X to obtain

$$\sigma x \in \pi_{*+1}(X \otimes A \otimes A),\ \sigma^2 x \in \pi_{*+2}(X \otimes \mathrm{THH}(A)).$$

In general they depend on the lift $\widetilde{x}$. Similarly given any $y \in \pi_*(X \otimes A \otimes A)$ with a lift $\widetilde{y} \in \pi_*(X \otimes \mathrm{fib}(m))$, we can define $\sigma y \in \pi_{*+1}(X \otimes \mathrm{THH}(A))$. Then for any element $x \in \pi_*(X \otimes A)$ we can define $dx = \sigma((e_L - e_R)x)$.

Now let R be an $\mathbb{E}_1$-algebra in $\mathrm{fil}(\mathrm{Sp})$, $X \in \mathrm{Sp}$. Suppose $x \in \pi_k(X)$ whose image in $\pi_k(X \otimes R_0) = \pi_{k,-k}(X \otimes R)$ is equal to $\tau^r y$ for some $y \in \pi_k(X \otimes R_r)$, $r \geq 1$. This means that x maps to zero in $\pi_{k,-k}(X \otimes \mathrm{gr}(R))$. The following is the key fact we'll use to determine the differentials:

Proposition 4.10 ([LL23], lemma 3.6)**.** *There is a lift $\widetilde{x} \in \pi_{k,-k}(X \otimes \mathrm{fib}(\mathbb{S}^{0,0} \to \mathrm{gr}\, R))$ such that in the spectral sequence associated to $X \otimes \mathrm{THH}(R)$, $\sigma^2 x \in \pi_{k+2,-k-2}(X \otimes \mathrm{THH}(\mathrm{gr}\, R))$ survives to the E_r-page, and there is a differential $d_r(\sigma^2 x) = \pm dy$.*

Sketch of proof. The idea is to map into this spectral sequence from a simpler one where we know the differentials. Namely, a nullhomotopy of x in the associated graded is the same as a homotopy $x \sim \tau^r y$, which is the same as a nullhomotopy ν of $\tau^r y$ in $X \otimes \mathrm{cofib}(\mathbb{S}^{0,0} \to R)$. Then we get a map of filtered objects

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow 0 & \longrightarrow & \mathbb{S}^k & \longrightarrow & \cdots & \longrightarrow & \mathbb{S}^k & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & {\scriptstyle y}\downarrow & & & & \downarrow & \overset{\nu}{\swarrow\!\!\swarrow} & \downarrow & & \downarrow & & \\ \cdots \longrightarrow & & X \otimes R_r & \longrightarrow & \cdots & \longrightarrow & X \otimes R_1 & \longrightarrow & X \otimes (R_0/\mathbb{S}) & \longrightarrow & X \otimes (R_{-1}/\mathbb{S}) & \longrightarrow & \cdots \end{array}$$

we know the d_r differentials of the spectral sequence of the top object because everything has to be killed. That differential maps to a nontrivial differential in the bottom object, which in turn maps to a nontrivial differential in $X \otimes \mathrm{THH}(R)$ by applying σ^2. □

4.3 Main calculation

We will now do the calculation of $\mathrm{THH}(\mathbb{Z}_p)/p$ and $\mathrm{THH}(\ell_p)/(p, v_1)$ for p an odd prime, following the streamlined method of [LL23], example 4.2 and 4.3. Note that this computation goes back to Bökstedt '86 (for $\mathbb{Z}_p$) and McClure–Staffeldt '93 (for ℓ_p).

Our strategy is to put filtrations on $\mathbb{Z}_p$ and ℓ_p, which induce filtrations on their THH and quotients. Moreover, gr commutes with THH (because of monoidality) and quotients (because of exactness), so the associated graded are just THH of $\mathrm{gr}\,\mathbb{Z}_p$ and $\mathrm{gr}\,\ell_p$, which are easy to compute.

Recall that ℓ_p is the connective Adams summand: it is a p-complete spectrum which can be defined as the connective cover of $\mathrm{KU}_{(p)}^{h\mu_{p-1}}$ where μ_{p-1} acts via Adams operations. It satisfies

$$\mathrm{ku}_{(p)} = \ell_p \oplus \Sigma^2 \ell_p \oplus \cdots \oplus \Sigma^{2p-4} \ell_p,$$

and it has homotopy groups $\pi_* \ell_p = \mathbb{Z}_p[v_1]$, $|v_1| = 2p - 2$.

Recall also that $\mathbb{Z}_p$ and ℓ_p are the first two examples in a series of spectra called truncated Brown–Peterson spectra, $\mathrm{BP}\langle n \rangle$. They are obtained from BP by quotienting by the elements $v_{n+1}, v_{n+2}, \dots \in \pi_* \mathrm{BP}$. But $\mathbb{Z}_p$ and ℓ_p are highly structured: they are $\mathbb{E}_\infty$-rings, whereas $\mathrm{BP}\langle n \rangle$ in general do not admit such structures.

Let's begin. We filter $\mathbb{Z}_p$ and ℓ_p by:

- $\mathbb{Z}_p^{\mathrm{fil}}$ is given the filtration $\cdots \to \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p = \mathbb{Z}_p = \dots$ and let $\widetilde{v_0}$ be $1 \in \mathbb{Z}_p$ in filtration degree 1. It is an $\mathbb{E}_\infty$-ring in fil(Sp), whose associated graded is $\mathbb{F}_p[v_0]$. (By the way, this is also the Adams filtration of $\mathbb{Z}_p$ along $\mathbb{S} \to \mathbb{F}_p$.)
- ℓ^{fil} is given the Whitehead filtration $\cdots \to \tau_{\geq 2}\ell \to \tau_{\geq 1}\ell \to \ell = \ell = \dots$ and let p be in filtration degree 0 and let $\widetilde{v_1}$ be in filtration degree $2p-2$. It is an $\mathbb{E}_\infty$-ring in fil(Sp) because $\tau_{\geq *}$ is lax symmetric monoidal.

We can take $\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})$ in the symmetric monoidal category $\mathrm{fil}(\mathcal{C})$. The image of $\widetilde{v_0}$ in $\mathbb{Z}_p^{\mathrm{fil}} \to \mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})$ is also denoted $\widetilde{v_0}$. We'll compute $\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})/\widetilde{v_0}$. Note that colimits and taking associated graded both commute with THH and cofibers, so $\mathrm{colim}\,\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})/\widetilde{v_0} = \mathrm{THH}(\mathbb{Z}_p)/p$ and

$$\mathrm{gr}(\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})/\widetilde{v_0}) = \mathrm{THH}(\mathbb{F}_p[v_0])/v_0 = \mathbb{F}_p[\sigma^2 p] \otimes \wedge[dv_0].$$

Here $\sigma^2 p$ has bidegree $(-2,0)$ and dv_0 has bidegree $(0,1)$. By Proposition 4.10 applied to $R=\mathbb{Z}_p^{\mathrm{fil}}$, we get $d_1(\sigma^2 p)=dv_0$ in the spectral sequence for $\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})$. This maps to the spectral sequence for $\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})/\widetilde{v_0}$, so the relation holds for us as well. By Leibniz rule, this determines all differentials. There are no extension problems for degree reasons ($(\sigma t_1)^2=0$) and we get

$$\pi_*(\mathrm{THH}(\mathbb{Z}_p)/p)\simeq \mathbb{F}_p[\sigma^2 v_1]\otimes \wedge[\sigma t_1]$$

with $|\sigma^2 v_1|=2p$ and $|\sigma t_1|=2p-1$. These elements can be named this way by [HW22], proposition 6.1.6.

Similarly, let's compute $\mathrm{THH}(\ell)/(p,v_1)$. As above, we compute $\mathrm{THH}(\ell^{\mathrm{fil}})/(p,\widetilde{v_1})$, whose colimit is what we want, and whose associated graded is

$$\mathrm{THH}(\mathrm{gr}(\ell^{\mathrm{fil}}))/(p,v_1)=\mathrm{THH}(\mathbb{Z}_p[v_1])/(p,v_1)=\mathbb{F}_p[\sigma^2 v_1]\otimes\wedge[\lambda_1,dv_1],$$

where $|\sigma^2 v_1|=(-2p,0)$, $|\lambda_1|=(-2p-1,0)$, and $|dv_1|=(-1,2p-2)$. By degree reasons only one page can carry differentials, namely the E^{2p-2}-page; and it does have a differential by Proposition 4.10, with $d^{2p-2}(dv_1)=\sigma^2 v_1$. Thus the E^∞ page looks like $\mathbb{F}_p[\mu_1]\otimes\wedge[\lambda_1,\lambda_2]$. Again, there are no extension problems for degree reasons, so we conclude that

$$\pi_*(\mathrm{THH}(\ell)/p,v_1)=\mathbb{F}_p[\mu]\otimes\wedge[\lambda_1,\lambda_2]$$

with $|\mu|=2p^2$, $|\lambda_1|=2p-1$, $|\lambda_2|=2p^2-1$.

5 Basics of synthetic spectra (Max Johnson)

Notes by Akira Tominaga

Let's recall some terminology on filtered and graded objects. Let $\mathcal{C}$ be a presentable stable category (namely, $\mathcal{C} = \mathrm{Sp}$).

Definition 5.1. The category of filtered objects in $\mathcal{C}$ is the functor category

$$\mathrm{Fil}(\mathcal{C}) \coloneqq \mathrm{Fun}(\mathbb{Z}_{\geq}, \mathcal{C}).$$

The category of graded objects is the functor category

$$\mathrm{Gr}(\mathcal{C}) \coloneqq \mathrm{Fun}(\mathbb{Z}_{\leq}, \mathcal{C}).$$

The category of filtered objects and graded objects have several associated functors.

Definition 5.2. Let the associated graded functor

$$\mathrm{gr}_* \colon \mathrm{Fil}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$$

be the functor determined by

$$\mathrm{gr}_n X \coloneqq \mathrm{cofib}(X_{n+1} \to X_n).$$

Definition 5.3. One can define a functor $\iota \colon \mathrm{Gr}(\mathcal{C}) \to \mathrm{Fil}(\mathcal{C})$ by zero map.

Definition 5.4. Let $\tau \colon \mathrm{Fil}(\mathcal{C}) \to \mathrm{Fil}(\mathcal{C})$ be the functor given by the structure map in filtered objects.

Definition 5.5. Recall that we defined a functor

$$c^{i,j} \colon \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$$

by

$$(c^{i,j}X)_n \coloneqq \begin{cases} \Sigma^i X & n \leq i+j, \\ 0 & n > i+j. \end{cases}$$

Definition 5.6. Let the realization functor $\mathrm{Re} \colon \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$ be the functor given by

$$\mathrm{Re}\, X \coloneqq \operatorname*{colim}_n X_n.$$

Remark 5.7. When $\mathcal{C}$ is symmetric monoidal, then $\mathrm{Fil}(\mathcal{C})$ and $\mathrm{Gr}(\mathcal{C})$ are also symmetric monoidal via Day convolution. The idea for the Day convolution is, for $X, Y \in \mathcal{C}$ and $n \in \mathbb{Z}$, we define $X \otimes Y$ by the formula

$$(X \otimes Y)_n \coloneqq \operatorname*{colim}_{i+j \geq n} X_i \otimes Y_j.$$

Let $\mathbb{1}$ be the unit in $\mathcal{C}$. Then the unit $\mathbb{1}_{\mathrm{Fil}}$ in $\mathrm{Fil}(\mathcal{C})$ is $c^{0,0}(\mathbb{1})$, and the unit $\mathbb{1}_{\mathrm{Gr}}$ in $\mathrm{Gr}(\mathcal{C})$ is the graded object whose image of $0 \in \mathbb{Z}$ is $\mathbb{1}$ and $n \neq 0$ is the zero object.

Remark 5.8. Note that gr_* is symmetric monoidal and left adjoint to ι.

Lemma 5.9. *Denote* $\mathcal{C}\tau \in \mathrm{Fil}(\mathcal{C})$ *the filtered object* $\mathrm{cofib}(\tau\colon c^{0,-1}(\mathbb{1}) \to c^{0,0}(\mathbb{1}))$. *Then the map* $c^{0,0}(\mathbb{1}) \to C\tau$ *is* $\mathbb{E}_\infty$*-algebra in* $\mathrm{Fil}(\mathcal{C})$.

Proof. Note that $\mathcal{C}\tau$ equivalent to the image of the unit in $\mathrm{Gr}(\mathcal{C})$ via ι. The claim follows because ι is lax symmetric monoidal. □

Proposition 5.10. *There is a categorycal equivalence* $\mathrm{Mod}(C\tau) \simeq \mathrm{Gr}(\mathcal{C})$. *Under this equivalence, we have a natural equivalence of functors* $- \otimes C\tau \simeq \mathrm{gr}_*$.

For filtered objects, we can define the bigraded homotopy group

$$\pi_{s,t}X \coloneqq \pi_s X_{s+t} \simeq [\Sigma^s c^{0,s+t}(\mathbb{1}), X].$$

Associated to this bigraded homotopy group, for each filtered object $X \in \mathrm{Fil}(\mathcal{C})$, one can construct a spectral sequence

$$E_1^{s,t} = \pi_{t-s}(\mathrm{gr}_t X) \Rightarrow \pi_{t-s} \operatorname{Re} X.$$

In addition to this, one can set up τ-Bockstein spectral sequence

$$E_1^{s,t} = (\pi_{t-s}\,\mathrm{gr}_t X)[\tau] \Rightarrow \pi_{s,t}X.$$

For any filtered object X, these two spectral sequences are closely related each other. If we understand the differentials in one spectral sequence, then we can deduce the differential of the other. In fact, one can show $d_r(x) = y \Leftrightarrow d_r(x) = \tau^r y$.

Remark 5.11. Tate spectral sequence and homotopy fixed point spectral sequence.

Definition 5.12. A tower functor is a lax monoidal functor

$$T\colon \mathcal{C} \to \mathrm{Fil}(\mathcal{C}).$$

The deformation of $\mathcal{C}$ by a tower functor T is the category $\mathrm{Mod}(T(\mathbb{1}))$.

Note the endomophism $\tau\colon \mathrm{Fil}(\mathcal{C}) \to \mathrm{Fil}(\mathcal{C})$ restricts to $\mathrm{Mod}(T(\mathbb{1}))$. Denote $\mathrm{Mod}(T(\mathbb{1}))[\tau^{-1}]$ be the full subcategory spanned by τ-invertible objects.

Theorem 5.13. *Denote* $C\tau$ *to be the cofiber* $\mathrm{cofib}(\tau\colon T(\mathbb{1}) \to T(\mathbb{1}))$. *There are symmetric monoidal equivalences of categories*

$$\mathrm{Mod}(T(\mathbb{1}))[\tau^{-1}] \simeq \mathrm{Mod}(\mathrm{Re}(T(\mathbb{1}))),$$
$$\mathrm{Mod}(C\tau) \simeq \mathrm{Mod}(\mathrm{gr}_* T(\mathbb{1})).$$

The first category $\mathrm{Mod}(T(\mathbb{1}))[\tau^{-1}]$ can be thought of the generic fiber with respect to the deformation by τ. The latter category $\mathrm{Mod}(C\tau)$ is the special fiber when $\tau = 0$.

Now we specialize the case $\mathcal{C} = \mathrm{Sp}$.

Proposition 5.14. *The functor*

$$\nu\colon \mathrm{Sp} \to \mathrm{Fil}(\mathrm{Sp}),\ X \mapsto \mathrm{Tot}\,\tau_{\geq 2*}(\mathrm{MU}^{\otimes \bullet+1} \otimes X)$$

is a tower functor.

Definition 5.15. We call the module category $\mathrm{Mod}(\nu\mathbb{S})$ to be the category of MU-synthetic spectra $\mathrm{Syn}_{\mathrm{MU}}^{\mathrm{ev}}$.

Proposition 5.16. *The generic fiber* $\mathrm{Mod}(\nu\mathbb{S})[\tau^{-1}]$ *is equivalent to* Sp.

Proof. This follows from the observation that

$$\begin{aligned}\operatorname*{colim}_n \mathrm{Tot}\,\tau_{\geq 2n}(\mathrm{MU}^{\otimes \bullet+1}) &\simeq \mathrm{Tot}\operatorname*{colim}_n \tau_{\geq 2n}(\mathrm{MU}^{\otimes \bullet+1}) \\ &\simeq \mathrm{Tot}\,\mathrm{MU}^{\otimes \bullet+1} \\ &\simeq \mathbb{S}.\end{aligned}$$

Note that the sequential colimit and Tot commutes because, Tot is a sequential limit of finite limit Tot^n, and the map $\mathrm{Tot} \to \mathrm{Tot}^n$ induces an isomorphism on homotopy groups through a range increasing in n. ☐

Also, we can analyze the special fiber. The special fiber becomes more algebraic, namely, it is related to the E_2-term of the Adams Novikov spectral sequence.

Proposition 5.17. *There is a symmetric monoidal equivalence*

$$\mathrm{Mod}(C\tau) \simeq \mathrm{Stable}_{\mathrm{MU}_*\mathrm{MU}}^{\mathrm{ev}}$$

where the latter is the Hovey's stable category of even $\mathrm{MU}_*\mathrm{MU}$*-comodules.*

Hovey's stable category $\mathrm{Stable}_{\mathrm{MU}_*\mathrm{MU}}^{\mathrm{ev}}$ has a feature that its heart $\mathrm{Stable}_{\mathrm{MU}_*\mathrm{MU}}^{\mathrm{ev},\heartsuit}$ is isomorphic to the category of even comodules $\mathrm{Comod}_{\mathrm{MU}_*\mathrm{MU}}^{\mathrm{ev}}$. In particular, we have an isomorphism of abelian groups

$$\pi_s\,\mathrm{gr}_{s+t}\,C\tau \simeq \mathrm{Ext}_{\mathrm{MU}_*\mathrm{MU}}^{s,t}(\mathrm{MU}_*, \mathrm{MU}_*).$$

Remark 5.18. Even $\mathrm{MU}_*\mathrm{MU}$-comodule algebra gives an algebra in $C\tau$-modules. For example, $\mathrm{MU}_*/(p, v_1, \cdots, v_n)$ is an even $\mathrm{MU}_*\mathrm{MU}$-comodule algebra. Therefore $C\tau/(p, v_1, \cdots, v_n)$ is an $\mathbb{E}_\infty$-ring in the category of $C\tau$-modules, although the Smith-Toda complex does not always exist in Sp.

As a concluding remark, we note the following. This result will not be used in the rest of this workshop.

Theorem 5.19. *After p-completion, there is an equivalence*

$$\mathrm{Syn}_{\mathrm{MU}}^{\mathrm{ev}} \simeq \mathrm{SH}(\mathbb{C})^{\mathrm{cell}}$$

where $\mathrm{SH}(\mathbb{C})^{\mathrm{cell}}$ *is the smallest subcategory of all complex motivic spectra containing spheres and closed under colimits.*

6 The even filtration (Keita Allen)

Notes by Keita Allen

6.1 Introduction

Yesterday we introduced topological Hochschild homology, a spectral invariant which, to an $\mathbb{E}_\infty$ ring A, associates another $\mathbb{E}_\infty$ ring $\mathrm{THH}(A)$. This is an invariant which is related to algebraic K-theory; the homotopy groups $\mathrm{THH}_*(A)$ serve as a first approximation to $K_*(A)$. We know algebraic K-groups are hard to compute directly, so the following question is then of interest to us.

Question 6.1. Can we compute $\mathrm{THH}_*(A)$?

We saw some fundamental computations yesterday ($\mathrm{THH}_*(\mathbb{F}_p)$, $\mathrm{THH}_*(\mathbb{Z})/p$, $\mathrm{THH}_*(\ell)/(p, v_1)$) but the way we computed these was somewhat ad-hoc. We can then ask the following slight modification of Question 6.1:

Question 6.2. Can we compute $\mathrm{THH}_*(A)$ in a principled way?

Max provided us a first approach to an answer to this question; we saw that if we have a filtration on a spectrum A, then we get a spectral sequence converging to the homotopy groups of A. So as a further refinement, we can ask the following question.

Question 6.3. Can we provide a functorial filtration on $\mathrm{THH}(A)$ which yields an interesting spectral sequence?

The goal of this talk is to provide an answer to this question in the affirmative; this will be the *even filtration*. We will see that this filtration recovers various other interesting filtrations on objects of interest, and later in the day we will see that it can be used to advance our understanding of the THH of ring spectra.

Notation

In a slight departure from [HRW22], we will denote the ∞-category of spectra by Sp and the category of $\mathbb{E}_\infty$ rings by CAlg. Given an ∞-category $\mathcal{C}$, we will denote the category of filtered objects in $\mathcal{C}$ (the category of functors from the poset $\mathbb{Z}$ into $\mathcal{C}$) by $\mathrm{Fil}\mathcal{C}$.

6.2 The even filtration and evenly faithfully flat maps

Definition 6.4. An $\mathbb{E}_\infty$ ring A is said to be *even* if $\pi_* X$ is concentrated only in even degrees.

Even $\mathbb{E}_\infty$ rings are nice; for example, their homotopy groups form a commutative ring, and they are always complex-orientable. The even filtration can be thought of as a means of approximating an arbitrary E_∞ ring by even ones.

Definition 6.5 (The even filtration)**.** Let A be an $\mathbb{E}_\infty$ ring. For each integer n, we define

$$\mathrm{fil}^n_{\mathrm{ev}} A = \lim_{\substack{A \to B \\ B \text{ even}}} \tau_{\geq 2n} B \in \mathrm{CAlg},$$

where $\tau_{\geq *}B$ denotes the Postnikov truncation of B. By a universal property argument, we have maps

$$\cdots \to \mathrm{fil}^{n+1}_{\mathrm{ev}} A \to \mathrm{fil}^{n}_{\mathrm{ev}} A \to \mathrm{fil}^{n-1}_{\mathrm{ev}} A \to \cdots ;$$

these assemble to define a filtered object

$$\mathrm{fil}^{*}_{\mathrm{ev}} A \in \mathrm{Fil}\,\mathrm{CAlg}\,.$$

This assignment promotes to a functor

$$\mathrm{fil}^{*}_{\mathrm{ev}} : \mathrm{CAlg} \to \mathrm{Fil}\,\mathrm{CAlg},$$

which we will refer to as the *even filtration*.

Those more well-versed than I am in the language of ∞-categories may be unsatisfied by my assertion that this assignment can be made functorial. For those of you, the following simple categorical description may be appealing.

Definition 6.6 (The even filtration, again)**.** The even filtration is the right Kan extension

$$\begin{array}{ccccc} \{\text{even } \mathbb{E}_\infty \text{ rings}\} & = & \mathrm{CAlg}_{\mathrm{ev}} & \xrightarrow{\tau_{\geq 2*}} & \mathrm{Fil}\,\mathrm{CAlg} \\ & & \big\downarrow & \nearrow_{\mathrm{fil}^{*}_{\mathrm{ev}}} & \\ & & \mathrm{CAlg} & & \end{array}$$

In addition to actually guaranteeing that we have a functor, some other satifying things fall out of the above definition. First, we will note that on even $\mathbb{E}_\infty$ rings, the even filtration is just the double-speed Postnikov filtration. Further, since $\tau_{\geq 2*}$ is lax symmetric monoidal, $\mathrm{fil}^{*}_{\mathrm{ev}}$ is also canonically lax symmetric monoidal, and so takes $\mathbb{E}_\infty$ rings to $\mathbb{E}_\infty$ algebras in filtered spectra. In particular, for any $\mathbb{E}_\infty$ ring A, $\mathrm{fil}^{*}_{\mathrm{ev}} A$ will be a module over the commutative algebra $\mathrm{fil}^{*}_{\mathrm{ev}}\, \mathbb{S}$. We'll see later that this even filtered sphere is something familiar.

Now, having this definition is great, but as it stands it is a little scary; it is easy enough to understand when the ring we are filtering is even, but otherwise this gigantic limit does not seem very computable. The following definition is our first step towards understanding this filtration.

Definition 6.7 (Evenly faithfully flat maps)**.** A map $A \to B$ of $\mathbb{E}_\infty$ rings is *evenly faithfully flat*, or *eff*, if for any even $\mathbb{E}_\infty$ ring C and a map of $\mathbb{E}_\infty$ ring $A \to C$, the pushout $B \otimes_A C$ is even and $\pi_*(B \otimes_A C)$ is faithfully flat over $\pi_*(C)$.

The even filtration enjoys a certain "descent" property along evenly faithfully flat maps.

Proposition 6.8. *Let $A \to B$ be an eff map. Then*

$$\mathrm{fil}^{*}_{\mathrm{ev}} A = \mathrm{Tot}_\bullet \left(\mathrm{fil}^{*}_{\mathrm{ev}} B^{\otimes_A \bullet+1}\right) := \lim_{\Delta,\bullet} \left(\mathrm{fil}^{*}_{\mathrm{ev}} B^{\otimes_A \bullet+1}\right)$$

We won't prove this here, but the proof of this is actually kind of formal, using general stuff about Grothendieck topologies.

6.3 Some evenly faithfully flat maps

Now, we'll move to discussing some examples of eff maps, and what this can tell us about the even filtration.

Proposition 6.9. *The map $\mathbb{S} \to MU$ is evenly faithfully flat.*

Proof. Let $\mathbb{S} \to C$ be a map into an even $\mathbb{E}_\infty$ ring. Then we have $MU \otimes_{\mathbb{S}} C = MU \otimes C$, where the latter is the tensor product of underlying spectra. So

$$\begin{aligned} \pi_*(MU \otimes_{\mathbb{S}} C) &= \pi_*(MU \otimes C) \\ &= MU_* C \\ &= (\pi_* C)[b_1, b_2, \ldots] \quad |b_i| = 2i, \end{aligned}$$

where the second equality is the definition of homology, and the third equality follows from the fact that even ring spectra are complex orientable and a computation with the Atiyah-Hirzebruch spectral sequence. (See [Lur10], Lecture 7 for more details.) In particular, $MU \otimes_{\mathbb{S}} C$ is even, and as a free $\pi_* C$-module it is also faithfully flat. □

This, in conjunction with the eff descent result 6.8, show us that the even filtration on the sphere is actually something familiar.

Corollary 6.10. *The even filtration on $\mathbb{S}$ is the (double-speed Postnikov)* décalage *of the Adams-Novikov filtration on $\mathbb{S}$:*

$$\operatorname{fil}^*_{\mathrm{ev}} \mathbb{S} \simeq \operatorname{Déc} \operatorname{fil}^*_{AN} \mathbb{S} = \operatorname{Tot}_\bullet \left(\tau_{\geq 2*} MU^{\otimes \bullet + 1} \right).$$

In particular, the even filtration on the sphere converges; $\operatorname{colim}_*(\operatorname{fil}^*_{\mathrm{ev}} \mathbb{S}) = \mathbb{S}$ *and* $\lim_*(\operatorname{fil}^*_{\mathrm{ev}} \mathbb{S}) = 0$.

Proof. By the proposition above and Proposition 6.8, we have

$$\operatorname{fil}^*_{\mathrm{ev}} \mathbb{S} \simeq \operatorname{Tot}_\bullet \left(\operatorname{fil}^*_{\mathrm{ev}} MU^{\otimes \bullet + 1} \right).$$

Since MU is even, the even filtration on MU is the double-speed Postnikov filtration, yielding the corollary. □

Remark 6.11. For those unfamiliar, the Adams-Novikov filtration on $\mathbb{S}$ comes from the cosimplicial "descent" resolution

$$\mathbb{S} \simeq \operatorname{Tot}_\bullet MU^{\bullet+1},$$

which gives rise to a filtered object $\operatorname{fil}^{AN}_* \mathbb{S} := \operatorname{fil}_* MU^{\bullet+1}$ with $\operatorname{fil}_n MU^{\bullet+1} = \operatorname{fib}(\operatorname{Tot} MU^{\bullet+1} \to \operatorname{Tot}_{n-1} MU^{\bullet+1})$. The spectral sequence associated to this filtration is the Adams-Novikov spectral sequence. Décalage is way of turning a filtered object into another filtered object, which has the effect of "turning the page" on associated spectral sequences; the E_1 page of the spectral sequence associated to $\operatorname{Déc} \operatorname{fil}_* MU^{\bullet+1}$ is the E_2 page of the spectral sequence associated to $\operatorname{fil}_* MU^{\bullet+1}$. In particular, the two filtrations/spectral sequences have the same convergence properties. There is a nice discussion of décalage in Section 2 of [Ang+23], and more details in [Hed21].

Remark 6.12. Recall that for any $\mathbb{E}_\infty$ ring A, $\mathrm{fil}^*_{\mathrm{ev}} A$ is a module over $\mathrm{fil}^*_{\mathrm{ev}} \mathbb{S}$. Thus, the corollary above tells us that the even filtration actually lands in $\mathrm{Mod}_{\mathrm{FilSp}}(\mathrm{Tot}_\bullet\left(\tau_{\geq 2*} MU^{\otimes \bullet + 1}\right))$, which is the category of (MU-)*synthetic spectra* as discussed by Max; that is, $\mathrm{fil}^*_{\mathrm{ev}}$ is a functor

$$\mathrm{fil}^*_{\mathrm{ev}} : \mathrm{CAlg} \to \mathrm{Syn}_{MU}\,.$$

We follow this up with another class of eff maps, this time involving something we have heard a lot of about yesterday: THH.

Proposition 6.13. *The maps*

$$\mathrm{THH}(MU) \to MU$$
$$\mathrm{THH}(\mathbb{S}[t]) \to \mathbb{S}[t]$$

are evenly faithfully flat.

Proof. We will prove that $\mathrm{THH}(\mathbb{S}[t]) \to \mathbb{S}[t]$ is eff; the proof proceeds essentially the same way when we replace $\mathbb{S}[t]$ with MU[3]. Let $\mathrm{THH}(\mathbb{S}[t]) \to C$ be a ring map into an even $\mathbb{E}_\infty$ ring C; we need to show that $\pi_*(\mathbb{S}[t] \otimes_{\mathrm{THH}(\mathbb{S}[t])} C)$ is even and faithfully flat. To compute the homotopy groups of a pushout

$$A \otimes_B C,$$

we use the *Tor spectral sequence*; this has signature

$$\mathrm{Tor}_{B_*}(\pi_* A, \pi_* C) = \pi_* A \otimes^{\mathbb{L}}_{\pi_* B} \pi_* C \Longrightarrow \pi_*(A \otimes_B C).$$

However, it is difficult to use this spectral sequence directly to $\mathbb{S}[t] \otimes_{\mathrm{THH}(\mathbb{S}[t])} C$, written this way; $\mathrm{THH}_*(\mathbb{S}[t])$ is not a nice ring to try to compute Tor over. However, we can massage this a bit; we have

$$\mathbb{S}[t] \otimes_{\mathrm{THH}(\mathbb{S}[t])} C \simeq C[t] \otimes_{C \otimes \mathrm{THH}(\mathbb{S}[t])} C.$$

Now our base is a bit nicer; we get

$$\begin{aligned} \pi_*\left(C \otimes \mathrm{THH}(\mathbb{S}[t])\right) &= \pi_*\left(C \otimes \left(\mathbb{S}[t] \otimes S^1\right)\right) \\ &= C_*[t] \otimes_{C_*} C_*(S^1) \\ &= C_*[t] \otimes_{C_*} \Lambda_{C_*}(dt) \\ &= \Lambda_{C_*[t]}(dt) : \end{aligned}$$

an exterior algebra. Tor over this is a free module generated in even degrees, and further the spectral sequence will collapse for degree reasons. This shows us that $\pi_*(C \otimes_{\mathrm{THH}(\mathbb{S}[t])} \mathbb{S}[t])$ is even and free over $\pi_* C$, and is in particular faithfully flat. □

6.4 The even filtration and motivic filtrations

One of the remarkable features of the even filtration is its very close relationship to some invariants of interest in arithmetic geometry.

Theorem 6.14 ([BMS19]). *Let R be a (discrete) quasisyntomic commutative ring. Then there is a* motivic filtration *on* $\mathrm{THH}(R)$*:*

$$\mathrm{fil}^*_{\mathrm{mot}} \mathrm{THH}(R),$$

where the associated graded is known as Hodge-Tate cohomology.

[3]See `https://www.youtube.com/watch?v=y92tka3bwLc` for more details on the MU case.

Theorem 6.15 ([HRW22], Theorem 5.0.3)**.** *Let R be a quasisyntomic commutative ring. Then there is an equivalence*

$$\mathrm{fil}^*_{\mathrm{ev}}\,\mathrm{THH}(R) \simeq \mathrm{fil}^*_{\mathrm{mot}}\,\mathrm{THH}(R).$$

Remark 6.16. Bhatt-Morrow-Scholze also define motivic filtrations on the richer invariants of TC^-, TP, and TC; these have associated graded given by things such as syntomic and prismatic cohomology. There are also variations of the even filtration on these objects which recover the BMS motivic filtrations; these will come up in discussion later in the week.

Computing the motivic filtration on $\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p)$

To close, we'll take a look at the motivic filtration on $\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p)$, and see how this shows up in its homotopy. We begin with the following lemma, which we will omit proof of; the argument is similar to our proof of Proposition 6.13.

Proposition 6.17 (Special case of [HRW22], Lemma 4.2.5)**.** *The map $\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p) \to \mathbb{F}_p[t]$ is evenly faithfully flat.*

Using this, eff descent tells us that

$$\mathrm{fil}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p) = \mathrm{Tot}_\bullet\left(\mathrm{fil}^*_{\mathrm{ev}}\,\mathbb{F}_p[t]^{\otimes_{\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p)}\bullet+1}\right).$$

The fact that the map $\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p) \to \mathbb{F}_p[t]$ is eff shows us that the tensor powers $\mathbb{F}_p[t]^{\otimes_{\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p)}\bullet+1}$ are even:

- For $\bullet = 0$, this is automatic, and
- for $\bullet > 0$, this follows inductively; since

  $$\mathbb{F}_p[t]^{\otimes_{\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p)}\bullet+1} = \mathbb{F}_p[t] \otimes_{\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p)} \left(\mathbb{F}_p[t]^{\otimes_{\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p)}\bullet}\right),$$

  assuming $\mathbb{F}_p[t]^{\otimes_{\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p)}\bullet}$ is even, the fact that $\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p) \to \mathbb{F}_p[t]$ is eff tells us that $\mathbb{F}_p[t]^{\otimes_{\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p)}\bullet+1}$ is also even.

Thus we can make the further identification

$$\mathrm{fil}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p) = \mathrm{Tot}_\bullet\left(\tau_{\geq 2*}\mathbb{F}_p[t]^{\otimes_{\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p)}\bullet+1}\right),$$ [4]

that is, the motivic filtration is again the (décalage of the) descent filtration.

Now, Catherine showed us that

$$\mathrm{THH}_*(\mathbb{F}_p[t]/\mathbb{F}_p) \cong \mathbb{F}_p[t] \otimes \Lambda(dt). \tag{6.1}$$

Let's see how these generators are incarnated in the motivic filtration. We have seen that

$$\mathrm{THH}(\mathbb{S}[t]/\mathbb{S}) \cong \mathbb{S}[t] \otimes \Sigma^\infty_+ S^1,$$

which tells us that

$$\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p) \simeq \mathrm{THH}(\mathbb{S}[t]/\mathbb{S}) \otimes \mathbb{F}_p \simeq \left(\mathbb{F}_p \otimes \Sigma^\infty_+ \mathbb{N}\right) \otimes_{\mathbb{F}_p} \left(\mathbb{F}_p \otimes \Sigma^\infty_+ S^1\right) = \mathbb{F}_p[t] \otimes_{\mathbb{F}_p} \mathbb{F}_p[S^1].$$

[4] Note that the same argument shows us that if $A \to B$ is an eff map into an even target B, then $\mathrm{fil}^*_{\mathrm{mot}}\,A = \mathrm{Tot}_\bullet\left(\tau_{\geq 2*}B^{\otimes_A \bullet+1}\right)$

It turns out we can understand the descent filtration on $\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p)$ in terms of that of these factors; we have an equivalence of maps

$$(\mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p) \longrightarrow \mathbb{F}_p[t]) \simeq \mathbb{F}_p[t] \otimes_{\mathbb{F}_p} \left(\mathbb{F}_p[S^1] \longrightarrow \mathbb{F}_p\right),$$

and so descent along this map tells us that

$$\mathrm{fil}^*_{\mathrm{mot}} \mathrm{THH}(\mathbb{F}_p[t]/\mathbb{F}_p) = \mathbb{F}_p[t] \otimes_{\mathbb{F}_p} \mathrm{Tot}_\bullet \left(\tau_{\geq 2*} \mathbb{F}_p^{\otimes_{\mathbb{F}_p[S^1]} \bullet+1}\right).$$

Comparing this with our known answer for $\mathrm{THH}_*(\mathbb{F}_p[t]/\mathbb{F}_p)$ in 6.1, we can do some fiddling to identify where each part comes from; the $\mathbb{F}_p[t]$ part comes from filtration degree zero, and the exterior generator dt comes from filtration degree 1.[5]

[5] I think the idea is supposed to be that $\mathbb{F}_p \otimes_{\mathbb{F}_p[S^1]} \mathbb{F}_p = \mathbb{F}_p[\mathbb{S} \otimes_{\mathbb{S}[S^1]} \mathbb{S}] = \mathbb{F}_p[BS^1] = \mathbb{F}_p[\mathbb{CP}^\infty] = \mathbb{F}_p[dt]$.

7 The motivic spectral sequence for $\mathrm{THH}(\mathbb{Z})_p^\wedge$ (Guoqi Yan)

Notes by Jackson Morris

A quick note is that we will really be computing $\mathrm{THH}(\mathbb{Z}_p)_p^\wedge$. Jeremy gave a reason why this is equivalent to the above computation related to how p-completion interacts with the tensor product.

We saw in the previous talk that the map $\mathrm{THH}(\mathbb{S}[z]) = \mathbb{S}[z] \otimes S^1 \to \mathbb{S}[z]$ coming from collapsing the circle to a point is eff. The previous descent result gives us an isomorphism

$$\mathrm{THH}(\mathbb{Z}) \cong \mathrm{Tot}(\mathrm{THH}(\mathbb{Z})/\mathbb{S}[z]^{\otimes \bullet+1}),$$

which, after p-completion, is an isomorphism

$$\mathrm{THH}(\mathbb{Z}_p)_p^\wedge \cong \mathrm{Tot}(\mathrm{THH}(\mathbb{Z}_p)/\mathbb{S}_{\mathbb{Z}_p}[z]^{\otimes \bullet+1}),$$

where $\mathbb{S}_{\mathbb{Z}_p} = \mathbb{S}_p^\wedge$ are the spherical Witt vectors. A key point here is that since the map $\mathrm{THH}(\mathbb{S}[z]) \to \mathbb{S}[z]$ is eff, the motivic filtration agrees with the Adams tilration, and so the motivic spectral sequence ends up being the same as the Adams spectral sequence.

Now, consider the map $\mathbb{S}_{\mathbb{Z}_p} \to \mathrm{THH}(\mathbb{S}_{\mathbb{Z}_p})^{\otimes \bullet+1}$. When we apply $\mathrm{THH}(\mathbb{Z}_p/-)$, we get the map

$$\mathrm{THH}(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}) \to \mathrm{THH}(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z]^{\otimes \bullet+1}) \cong \mathrm{THH}(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z])^{\otimes_{\mathrm{THH}(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p})} \bullet+1}.$$

Further, we have that

$$\mathrm{THH}(\mathbb{Z}_p)_p^\wedge \cong \mathrm{THH}(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}) \xrightarrow{\simeq} \mathrm{Tot}(\mathrm{THH}(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z]^{\otimes \bullet+1}).$$

Later, we will see that we have a Hopf algebroid

$$(\mathrm{THH}_*(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z]), \mathrm{THH}_*(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z_1, z_2]))) = (A, \Gamma).$$

To try to alleviate some notational annoyances, we will use A and Γ as indicated above. The following theorem is due to Krause-Nikolaus.

Theorem 7.1. *$\pi_* A = \mathbb{Z}_p[u]$, where $|u| = 2$.*

Now, we want to understand Γ. First, we observe that there is a map $\mathbb{Z}_p[z_1, z_2] \to \mathbb{Z}_p$ given by sending $z_1, z_2 \mapsto p$. By construction, we have that $z_1 - z_2 \in \text{kernel} =: I$. Further, we can see that

$$I/I^2 = \mathrm{HH}_2(\mathbb{Z}_p/\mathbb{Z}_p[z_1, z_2]) \cong \mathrm{THH}_2(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z_1, z_2]).$$

Thus some quotient representative, which we will dubiously refer to as $\sigma^2(z_1 - z_2)$, lives in $\mathrm{THH}_2(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z_1, z_2])$. The idea, pointed out by Allen, is that elements of the kernel are precisely the elements we can apply σ^2 to. We can define the left unit as being induced by the map on THH:

$$\eta_L \colon \mathrm{THH}(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z]) \to \mathrm{THH}(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z_1, z_2]), z \mapsto z_1.$$

Our theorem tells us that $\mathrm{THH}_*(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z]) = \mathbb{Z}_p[u_1]$, and we can identify this polynomial generator as $\sigma^2(p - z)$. By taking powers of u_1 in the image, we get a filtration on Γ.

Lemma 7.2. $\operatorname{gr}_* \Gamma = \mathbb{Z}_p[u_1] \otimes \mathbb{Z}_p\langle \sigma^2(z_1 - z_2)\rangle$

Here, we are using the angle brackets to denote the divided power algebra.

Proof. We have a string of isomorphisms which we can chase:

$$\begin{aligned} \mathrm{THH}(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z_1, z_2])/u_1 &\cong \mathrm{THH}(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z_1, z_2]) \otimes_{\mathrm{THH}(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z])} \mathrm{THH}(\mathbb{Z}_p/\mathbb{Z}_p) \\ &\cong \mathrm{THH}(\mathbb{Z}_p/\mathbb{Z}_p[z_2]). \end{aligned}$$

Thus $\mathrm{THH}_*(\mathbb{Z}_p/\mathbb{Z}_p[z_2]) \cong \mathbb{Z}_p\langle\sigma^2(z_1 - z_2)\rangle$. Notice that thing is even, and so the u_1-Bockstein spectral sequence coming from the filtration collapses. Thus:

$$\operatorname{gr}_* \Gamma = \mathbb{Z}_p[u_1] \otimes \mathbb{Z}_p\langle \sigma^2(z_1 - z_2)\rangle$$

□

We can now proceed to describe the Hopf algebroid (A, Γ). Consider the following puhsout diagram:

$$\begin{array}{ccc} \mathbb{S}_{\mathbb{Z}_p}[z] & \xrightarrow{\eta_R} & \mathbb{S}_{\mathbb{Z}_p}[z_1, z_2] \\ {\scriptstyle \eta_L}\downarrow & & \downarrow \\ \mathbb{S}_{\mathbb{Z}_p}[z_1, z_2] & \longrightarrow & \mathbb{S}_{\mathbb{Z}_p}[z_1, z_2, z_3] \end{array}$$

where $\eta_L(z) = z_1, \eta_R(z) = z_2$ induce the left and right unit maps for the Hopf algebroid, the rightmost vertical map is $z_i \mapsto z_i$, and the bottom horizontal map is $z_i \mapsto z_{i+1}$. Once we apply $\mathrm{THH}(\mathbb{Z}_p/-)$, we can apply base change to get the following isomorphism:

$$\begin{aligned} \mathrm{THH}(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z_1, z_2, z_3]) &\cong \mathrm{THH}(\mathbb{Z}_p, \mathbb{S}_{\mathbb{Z}_p}[z_1, z_2]) \otimes_{\mathrm{THH}(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z_2])} \mathrm{THH}(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z_2, z_3]) \\ &\cong \Gamma \otimes_A \Gamma \end{aligned}$$

The diagonal $\Delta \colon \Gamma \to \Gamma \otimes_A \Gamma$ is induced by

$$\mathbb{S}_{\mathbb{Z}_p}[z_1, z_2] \to \mathbb{S}_{\mathbb{Z}_p}[z_1, z_2, z_3], z_1 \mapsto z_1, z_2 \mapsto z_3.$$

The conjugation $c \colon \Gamma \to \Gamma$ is induced by

$$\mathbb{S}_{\mathbb{Z}_p}[z_1, z_2] \to \mathbb{S}_{\mathbb{Z}_p}[z_1, z_2], z_1 \leftrightarrow z_2.$$

The conunit $\varepsilon \colon \Gamma \to A$ is induced by

$$\mathbb{S}_{\mathbb{Z}_p}[z_1, z_2] \to \mathbb{S}_{\mathbb{Z}_p}[z], z_i \mapsto z.$$

Recall that we have an Adams resolution

$$\mathrm{THH}(\mathbb{Z}_p, \mathbb{S}_{\mathbb{Z}_p}) \cong \mathrm{Tot}(\mathrm{THH}(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}[z]^{\otimes \bullet + 1}))$$

and so we can look at the associated Adams spectral sequence with E_2-page

$$E_2^{n,s} = \mathrm{Ext}_\Gamma^{n,s}(A) \implies \pi_{n-s,s} \mathrm{THH}(\mathbb{Z}_p/\mathbb{S}_{\mathbb{Z}_p}).$$

We can actually compute this E_2-page in a nice manner by building a resolution of A by relative injective extended Γ-comodules. Recall that *extended* Γ-comodules are those of the form $M \otimes_A \Gamma$ for $M \in Mod_A$, and a Γ-comodule is a *relative injective* if it is a summand of $\Gamma \otimes_A M$ for some $M \in Mod_A$. Our description of is

$$\Gamma = \mathbb{Z}_p[u_1] \otimes \mathbb{Z}_p\langle \sigma(z_1 - z_2) \rangle.$$

We can choose a free $\mathbb{Z}_p$-basis for $\mathbb{Z}_p\langle \sigma(z_1 - z_2) \rangle$ as $\{\langle \sigma(z_1 - z_2)^{[i]} : i \geq 0\}$, where

$$\sigma(z_1 - z_2)^{[i]} \sigma(z_1 - z_2)^{[j]} = \binom{i+j}{i} \sigma(z_1 - z_2)^{[i+j]}.$$

Magically, we can use this basis to construct our resolution. If we let

$$D : \sigma(z_1 - z_2)^{[i]} \to \sigma(z_1 - z_2)^{[i-1]}$$

and extend by $\mathbb{Z}[u_1]$-linearity to a map $D \colon \Gamma \to \Gamma\{Dz\}$, we can not only observe that this is surjective, but that $\ker(D) = A$. Thus, our resolution is very short.

$$0 \longrightarrow A \longrightarrow \Gamma \xrightarrow{D} \Gamma \longrightarrow 0$$

Applying $\mathrm{Hom}_\Gamma(A, -)$ gives us the following:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_\Gamma(A, \Gamma) & \xrightarrow{d} & \mathrm{Hom}_\Gamma(A, \Gamma) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & A & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Since A is a polynomial on one generator we can conclude that $d(u^n) = -nu^n dz$. This completely determines the differentials of the spectral sequence. If I could code up spectral sequences, it would look nice.

8 The motivic spectral sequence for ℓ via descent from MU (Anish Chedalavada)

Notes by Juan Moreno

The goal for the this talk is to compute $\mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\ell)/(p, v_1)$. This will be used later in Shai's talk (Section 13) to compute some stuff about TC^- and TP. There are three main ingredients for this computation and we'll record them here as the first three propositions.

Proposition 8.1. *The map* $\mathrm{THH}(\ell) \to \mathrm{THH}(\ell/\mathrm{MU})$ *is eff.*

Proof. This one is not so bad so we can go ahead and prove it. From Keita's talk we know that $\mathrm{THH}(\mathrm{MU}) \to \mathrm{MU}$ is eff. The result then follows from the pushout square below

$$\begin{array}{ccc} \mathrm{THH}(\mathrm{MU}) & \longrightarrow & \mathrm{MU} \\ \downarrow & & \downarrow \\ \mathrm{THH}(\ell) & \longrightarrow & \mathrm{THH}(\ell/\mathrm{MU}). \end{array}$$

□

The next ingredient is a computation of $\mathrm{THH}(\ell/\mathrm{MU})_*$. We will postpone the proof until the end as it is a bit involved.

Proposition 8.2. $\mathrm{THH}(\ell/\mathrm{MU})_*$ *is even and given by*

$$\ell_*[w_{2,i} | i \geq 0] \otimes_{\ell_*} \ell_*[y_{j,i} | j \not\equiv -1 \mod p, j \geq 1, i \geq 0].$$

Furthermore $w_{2,0}$ *can be taken to be* $\sigma^2 v_2$.

The last ingredient is to recall the abudment of this motivic spectral sequence which was computed in Jackson's talk.

Proposition 8.3. *There is an isomorphism*

$$\mathrm{THH}_*(\ell)/(p, v_1) \cong \mathbb{F}_p[\sigma^2 v_2] \otimes \Lambda(\sigma t_1, \sigma t_2),$$

where Λ *denotes an exterior algebra.*

The upshot of Proposition 8.1 is that [HRW22] then tells us that

$$\mathrm{fil}^*_{\mathrm{mot}} \mathrm{THH}(\ell) = \mathrm{fil}^*_{\mathrm{ev}} \mathrm{THH}(\ell) = \mathrm{Tot}(\mathrm{fil}^*_{\mathrm{ev}} \mathrm{THH}(\ell/\mathrm{MU})^{\otimes_{\mathrm{THH}(\ell)} \bullet+1}).$$

Proposition 8.2 tells us that $\mathrm{THH}(\ell/\mathrm{MU})_*$ is even so $\mathrm{fil}^*_{\mathrm{ev}}$ can be replaced with the Postnikov filtration

$$\mathrm{fil}^*_{\mathrm{mot}} \mathrm{THH}(\ell) \simeq \mathrm{Tot}(\tau_{\geq 2*} \mathrm{THH}(\ell/\mathrm{MU})^{\otimes_{\mathrm{THH}(\ell)} \bullet+1})$$

We can rewrite this further by remembering the some formulas from Catherine's talk:

$$\mathrm{THH}(\ell/\mathrm{MU}) \otimes_{\mathrm{THH}(\ell)} \mathrm{THH}(\ell/\mathrm{MU}) \simeq \mathrm{THH}(\ell/\mathrm{MU}^{\otimes \bullet+1}).$$

The cobar complex for $\mathrm{THH}(\ell) \to \mathrm{THH}(\ell/\mathrm{MU})$ then has the form $\mathrm{THH}(\ell/\mathrm{MU}^{\otimes \bullet+1})$, so we have

$$\mathrm{fil}^*_{\mathrm{mot}} \mathrm{THH}(\ell) \simeq \mathrm{Tot}(\tau_{\geq 2*} \mathrm{THH}(\ell/\mathrm{MU}^{\otimes \bullet+1})).$$

Now we start modding out by (p, v_1). From Proposition 8.2 we know that for each $\bullet$, $\mathrm{THH}_*(\ell/\mathrm{MU}^{\otimes \bullet+1})_*$ is a free ℓ_*-module. This implies that we can just mod out by (p, v_1) levelwise in the cobar complex and the resulting spectral sequence will compute $\pi_* \mathrm{THH}(\ell)/(p, v_1)$. Thus

$$\pi_s \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\ell)/(p, v_1) \cong H^s(C(*)),$$

where $C(*)$ is the cobar complex computing $\mathrm{Tot}(H\pi_{\geq 2*} \mathrm{THH}(\ell/\mathrm{MU}^{\otimes \bullet+1})/(p, v_1))$. To summarize, the motivic spectral sequence has the form

$$H^s(C(2t)) \implies \mathbb{F}_p[\sigma^2 v_2] \otimes \Lambda(\sigma t_1, \sigma t_2).$$

Remember, our goal is to give an explicit description of this E_2-page. Our approach will be to construct a May-Ravenel spectral sequence converging to the E_2-page of the motivic spectral sequence such that the E_2-page of the May-Ravenel spectral sequence has the same rank as the answer for the motivic spectral sequence from Proposition 8.3. By a homological squeeze theorem, we then deduce that the motivic spectral sequence collapses allowing us to identify $\operatorname{gr}^*_{\mathrm{mot}}$.

Notation. For the sake of legibility, let's establish some notation before proceeding with the argument. We denote

- $\mathrm{THH}(\ell)/(p, v_1)$ by A,
- $\mathrm{THH}(\ell/\mathrm{MU})/(p, v_1)$ by B,
- $(B \otimes_A B)_*$ by Σ_* ,
- and $B_* \otimes^{\mathbb{L}}_{A_*} B_*$ by $\overline{\Sigma}$.

Note that B_* is even so Proposition 8.1 tells us that

$$B_* \to \Sigma_*$$

is flat. The cohomology of the cobar complex can then be identified as

$$H^*(C(*)) \cong \mathrm{Ext}_{\Sigma_*}(B_*, B_*)$$

for the Hopf algebroid (B_*, Σ_*). To get the May-Ravenel spectral sequence, we first construct some other spectral sequences obtained by filtering Σ_* by the Postnikov filtrations for A and B.

Explicitly, view A and B as filtered by $\{\tau_{\geq *} A\}$ and $\{\tau_{\geq *} B\}$. Then for each n, we can filter $B^{\otimes_A n+1}$ by $\{\tau_{\geq *} B^{\otimes_{\tau_{\geq *} A} n+1}\}$. This yields a spectral sequence of the form

$$\pi_*(B_*^{\otimes^{\mathbb{L}}_{A_*} n+1}) \implies \pi_*(B^{\otimes_A n+1}) \tag{8.1}$$

To describe the E_2-page, notice that the map $A_* \to B_*$ sends $\sigma^2 v_2 \mapsto w_{2,0}$ and, since B_* is even, must send the exterior classes to zero. Thus

$$\overline{\Sigma}_* \cong B_* \otimes_{\mathbb{F}_p} P \otimes_{\mathbb{F}_p} \Gamma\{\sigma^2 t_1, \sigma^2 t_2\},$$

where P is a polynomial algebra on classes in even deegrees and $\Gamma\{-\}$ denotes a divided power algebra on the given classes. Since $\overline{\Sigma}_*$ is concentrated in even degrees, we have that for each n, the spectral sequence of Equation (8.1) collapses.

It follows that for each n, $\pi_*(B^{\otimes_A n+1})$ has a filtration with associated graded given by $\pi_*(B_*^{\otimes^{\mathbb{L}}_A n+1})$. One can check that these filtrations for varying n are suitably compatible with the maps in the cobar complex so that this filtration to gives rise to the *May-Ravenel spectral sequence*

$$\mathrm{Ext}_{\overline{\Sigma}_*}(B_*, B_*) \implies \mathrm{Ext}_{\Sigma}(B_*, B_*).$$

The last step is to analyze the E_2-page of this spectral sequence. For this, we use the evident tensor product decomposition of the Hopf algebroid $(B_*, \overline{\Sigma}_*)$ as

$$(\mathbb{F}_p[\sigma^2 v_2], \mathbb{F}_p[w_{2,0}]) \otimes (P, P \otimes_{\mathbb{F}_p} P) \otimes (\mathbb{F}_p, \Gamma(\sigma^2 t_1, \sigma^2 t_2))$$

to decompose $\mathrm{Ext}_{\overline{\Sigma}}(B_*, B_*)$ as

$$\mathbb{F}_p[\sigma^2 v_2] \otimes_{\mathbb{F}_p} \mathbb{F}_p \otimes \Lambda_{\mathbb{F}_p}(\sigma t_1, \sigma t_2) = A_*.$$

Thus, writting degrees as a pair (stem, weight), we have $\pi_* \operatorname{gr}^*_{\mathrm{mot}} A \cong A_*$ with $|\sigma^2 v_2| = (2p^2, p^2)$ and $|\sigma t_i| = (2p-1, p)$. This completes the computation.

As promised, we now return to Proposition 8.2.

proof of Proposition 8.2. We compute

$$\mathrm{THH}(\ell/\mathrm{MU}) \simeq \ell \otimes_{\ell \otimes_{\mathrm{MU}} \ell} \ell$$

using first a Tor spectral sequence followed by a bar spectral sequence. As one might hope, getting to the E_∞-page will be painless. The hard work will be resolving the extension problems.

The exposition will be cleaner if we introduce the following notation first. For an MU-algebra R, write $E_r(R)$ for the E_r-page of the Tor spectral sequence

$$\mathrm{Tor}^{p,q}_{\mathrm{MU}_*}(R_*, R_*) \implies \pi_*(R \otimes_{\mathrm{MU}_*} R),$$

and write $E'_r(R)$ for the E_r-page of the bar spectral sequence computing

$$\pi_* \operatorname{Tot}_\bullet(R \otimes_{\mathrm{MU}} (R \otimes_{\mathrm{MU}} R)^{\otimes_{\mathrm{MU}} \bullet} \otimes_{\mathrm{MU}} R).$$

Start with

$$\mathrm{Tor}^{p,q}_{\mathrm{MU}_*}(\ell_*, \ell_*) \implies \pi_*(\ell \otimes_{\mathrm{MU}_*} \ell).$$

The E_2-page is Tor over a polynomial algebra on infinitely many generators and ℓ_* is a quotient of this polynomial algebra by some of those generators. Thus, the E_2-page is given by an exterior algebra over ℓ_* with a generator for every generator in the kernel of $\mathrm{MU}_* \to \ell_*$. The spectral sequence collapses for degree reasons and we get

$$E_2(\ell) = E_\infty(\ell) \cong \Lambda_{\ell_*}(\sigma v_2, \sigma v_3, ...) \otimes_{\ell_*} \Lambda_{\ell_*}(\sigma x_j | j \neq p^k - 1).$$

Now consider the bar spectral sequence for

$$\operatorname{Tot}_\bullet(\ell \otimes_{\mathrm{MU}} (\ell \otimes_{\mathrm{MU}} \ell)^{\otimes_{\mathrm{MU}} \bullet} \otimes_{\mathrm{MU}} \ell).$$

Again, this collapses for degree reasons and we get

$$E'_2(\ell) = E'_\infty(\ell) \cong \Gamma_{\ell_*}(\sigma^2 v_2, ...) \otimes_{\ell_*} \Gamma_{\ell_*}(\sigma^2 x_j | j \neq p^k - 1).$$

Now, to resolve multiplicative extensions.

Running the same argument with ℓ replaced by $H\mathbb{F}_p$, we obtain

$$E'_2(H\mathbb{F}_p) = E'_\infty(H\mathbb{F}_p) \cong \Gamma_{\mathbb{F}_p}(\sigma^2 v_1, \sigma^2 v_2, ...) \otimes_{\mathbb{F}_p} \Gamma_{\mathbb{F}_p}(\sigma^2 x_j | j \neq p^k - 1).$$

We see that the map

$$E'_\infty(\ell) \to E'_\infty(H\mathbb{F}_p)$$

induced by $\ell \to H\mathbb{F}_p$ is injective. It therefore, suffices to resolve multiplicative extensions in $E'_\infty(H\mathbb{F}_p)$. We turn to this now.

Viewing $H\mathbb{F}_p \otimes_{\mathrm{MU}} H\mathbb{F}_p$ as an $\mathbb{E}_\infty$-$H\mathbb{F}_p$ algebra, we have the following actions of the Dyer-Lashof operations:

$$Q^{e_p(v_i)} \sigma v_i = \sigma v_{i+1},$$

and

$$Q^{e_p(x_j)} \sigma x_j = \sigma x_{jp+p-1} \text{ modulo decomposables},$$

where

$$e_p(v_i) = \begin{cases} p^i & \text{if } p > 2 \\ 2^{i+1} & \text{if } p = 2 \end{cases},$$

and

$$e_p(x_j) = \begin{cases} j+1 & \text{if } p > 2 \\ 2(j+1) & \text{if } p = 2 \end{cases}.$$

The reader is referred to Lemma 2.4.1 in [HW22] for a proof of these. The proof then follows from Theorem 3.6 in [BM10], which exhibits operations on the E_1-page of the Künneth spectral sequences used here that are compatible with the Dyer-Lashof operations, together with compatibility of σ and σ^2 with the Dyer-Lashof operations. □

9 THH as a cyclotomic spectrum (Maxine Calle)

Notes by Sanjana Agarwal

In this talk, we have the following goals:

- To define what a cyclotomic/ p-cyclotomic spectrum is
- To understand why THH is a cyclotomic spectrum
- To see, as a result, how one can define other spectra like TP, TC, etc.

Notation. We will write S^1 for the circle group; this is also written as $\mathbb{T}$ in the literature. For p a prime, we write $C_p \subseteq S^1$ for the finite cyclic group of order p given by the p^{th} roots of unity. A key observation that we will make repeated use of is that $S^1/C_p \cong S^1$ as topological groups.

A p-cyclotomic spectrum is a spectrum X that has a *circle action* by S^1 and an equivariant *Frobenius map*

$$\phi_p \colon X \to X^{tC_p}$$

where X^{tC_p} is a *Tate construction*. The talk will explain these concepts and make this idea more precise; some references for this material are [NS18], [KN18], [HS18].

9.1 Equivariant preliminaries

Definition 9.1. Let G denote any finite or compact Lie group. A **(Borel) G-spectrum** or a **spectrum with a G-action** is a functor $X \colon \mathrm{B}G \to \mathrm{Sp}$.

If G is a finite group, a model for $\mathrm{B}G$ is the nerve of the category with a single object, $*$, and one morphism each for each element of G. An element $g \in G$ 'acts' on $X(*) := X$ (abuse of notation) via the endomorphisms on X given by $X(g)$. For $G = S^1$, we have $\mathrm{B}G \simeq \mathbb{CP}^\infty$.

Definition 9.2. The category of (Borel) G-spectra is $\mathrm{Sp}^{\mathrm{B}G} := \mathrm{Fun}(\mathrm{B}G, \mathrm{Sp})$.

Here are some first examples of spectra with G-action.

- Any spectrum X can be viewed as a G-spectrum with a trivial G-action via the functor $X(*) = X$ and $X(g) = \mathrm{id}$. More abstractly, there is a map $\mathrm{Sp} = \mathrm{Fun}(*, \mathrm{Sp}) \to \mathrm{Fun}(\mathrm{B}G, \mathrm{Sp})$ induced by the pullback along $\mathrm{B}G \to *$.
- If Y is a G-space, viewed as a functor $Y \colon \mathrm{B}G \to \mathrm{Spc}$, then $\Sigma^\infty_+ Y$ is a G-spectrum via the composition $\mathrm{B}G \xrightarrow{Y} \mathrm{Spc} \xrightarrow{\Sigma^\infty_+} \mathrm{Sp}$.

We now focus on our main example, $\mathrm{THH}(A)$: We want to show that $\mathrm{THH}(A)$ is an S^1-spectrum. Intuitively, one way to think about about why and how $\mathrm{THH}(A)$ should inherit a circle action is by looking at another definition of THH that we haven't seen this week. Given any ring object A in a nice category, one can do the cyclic bar construction and define THH as the realization of this cyclic object:

$$\mathrm{THH}(A) := \left|\ A \leftleftarrows A \otimes A \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\; A \otimes A \otimes A \ \cdots \right|.$$

On the ith term of the cyclic object, there is an action of C_{i+1} which cyclically permutes the A terms. Geometrically realizing the cyclic object gives you an object with a S^1-action. Formalizing this idea in the setting of ∞-categories takes some work, and can be found in [NS18, §III.2].

When A is an $\mathbb{E}_\infty$-ring, we can make use of the universal property of THH (Section 2, Theorem 2.3) to prove that $\mathrm{THH}(A)$ has an S^1-action. Recall that, given any $\mathbb{E}_\infty$-ring B with S^1 action which receives a map from A, we have

$$\mathrm{Map}^{S^1}_{\mathrm{CAlg}}(\mathrm{THH}(A), B) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{CAlg}}(A, B)$$

and $\mathrm{THH}(A) \simeq A^{\otimes S^1}$. That gives us

$$S^1 \to \mathrm{Map}(S^1, S^1) \to \mathrm{Map}(\mathrm{THH}(A), \mathrm{THH}(A))$$

by taking the functor $(-)^{\otimes S^1}$ on both sides of $\mathrm{Map}(S^1, S^1)$ (noting that the first map exists since S^1 acts on itself). Hence we have an action of S^1 on $\mathrm{THH}(A)$ via the maps in the image of the above composition.

We now return to a general equivariant notion. Given any spectrum with G-action X, there are two ways to obtain an ordinary (i.e. non-equivariant) spectrum.

Definition 9.3. Homotopy orbits and homotopy fixed points: Given a G-spectrum $X\colon \mathrm{B}G \to \mathrm{Sp}$, the *homotopy orbits* are $X_{hG} \coloneqq \mathrm{colim}_{\mathrm{B}G} X$ and the *homotopy fixed points* are $X^{hG} \coloneqq \lim_{\mathrm{B}G} X$.

Definition 9.4. Negative topological cyclic homology: The *negative cyclic homology* of A is the homotopy fixed point spectrum

$$\mathrm{TC}^-(A) \coloneqq \mathrm{THH}(A)^{hS^1}.$$

When working one prime at a time, we may instead want to consider the p-completion

$$\mathrm{TC}^-(A;p) \coloneqq (\mathrm{THH}(A)^{hS^1})^\wedge_p.$$

We note that $\mathrm{TC}^-(A;p)$ may not be standard notation.

9.2 The Tate construction

The Tate construction is a generalization of Tate cohomology. Recall that for a finite group G and a G-module M, there is a norm map from the G-orbits of M to the G-fixed points given by

$$\begin{aligned} \mathrm{Nm}\colon M_G &\to M^G \\ [m] &\mapsto \sum_{g\in G} g\cdot m. \end{aligned}$$

This map crucially allows us to define long exact sequences between group homology and group cohomology of G with coefficients in M, which gives us Tate cohomology $\hat{H}^*(G;M)$.

Tate construction for finite groups: If G is a finite group, then by abstract nonsense there is always a *norm map* $\mathrm{Nm}\colon X_{hG} \to X^{hG}$ of spectra. The *Tate construction* is the cofiber

$$X^{tG} \coloneqq \mathrm{cofib}(X_{hG} \xrightarrow{\mathrm{Nm}} X^{hG}).$$

Note that since X^{tG} is the cofiber, there is a canonical map $\mathrm{can}\colon X^{hG} \to X^{tG}$. We now list some properties of Tate construction:

(a) The functor $(-)^{tG}\colon \mathrm{Sp}^{\mathrm{B}G} \to \mathrm{Sp}$ is an exact functor of stable ∞-categories.

(b) $(-)^{tG}$ is a lax symmetric monoidal functor such that $(-)^{hG} \xRightarrow{can} (-)^{tG}$ is also lax symmertic monoidal.

(c) If M is a G-module, then the Tate construction recovers Tate cohomology in the sense that $\pi_*(HM^{tG}) \cong \hat{H}^{-*}(G; M)$.

Tate construction for $G = S^1$: For S^1, there is a norm map $\Sigma X_{hS^1} \xrightarrow{\mathrm{Nm}} X^{hS^1}$ and the Tate construction is defined to be

$$X^{tS^1} \coloneqq \mathrm{cofib}(\Sigma X_{hS^1} \xrightarrow{\mathrm{Nm}} X^{hS^1}).$$

Remark 9.5. The suspension appears in the source of the norm because S^1 is not a finite group. There is a more general formula for the norm when G is compact Lie, which specializes to the case above for $G = S^1$.

Definition 9.6. The *topological periodic homology* of A is the Tate construction

$$\mathrm{TP}(A) \coloneqq \mathrm{THH}(A)^{tS^1}.$$

We let $\mathrm{TP}(A; p)$ denote the p-completion of the spectrum $\mathrm{TP}(A)$.

9.3 p-cyclotomic spectra

If X is a spectrum with S^1-action, we can always restrict to an action of $C_p \leq S^1$ for any fixed prime p. Again by abstract nonsense, there is a norm map $X_{hC_p} \to X^{hC_p}$ and we map consider the Tate construction

$$X^{tC_p} \coloneqq \mathrm{cofib}(X_{hC_p} \to X^{hC_p})$$

for this C_p-action. In this case, X^{tC_p} inherits a residual action by $S^1/C_p \cong S^1$.

Definition 9.7. A ***p*-cyclotomic spectrum** is $X \in \mathrm{Sp}^{\mathrm{B}S^1}$ with an S^1-equivariant map $\Phi_p : X \to X^{tC_p}$ called the *Frobenius.*

More generally, a *cyclotomic spectrum* is a spectrum with S^1-action along with Frobenii maps ϕ_p for all primes p. However, for our purposes, we can get away with working one prime at a time so we will mostly focus on the p-cyclotomic setting.

Remark 9.8. Our definition differs slightly from others in the literature, where a p-cyclotomic spectrum may be required to have an action by C_{p^∞} rather than S^1. We note that C_{p^∞} arises as the "p-power torsion" subgroup within S^1; in particular, $(\mathrm{B}S^1)^\wedge_p \simeq (\mathrm{B}C_{p^\infty})^\wedge_p$, so in the p-complete setting there is no meaningful difference.

The (infinity) **category of p-cyclotomic spectra** CycSp_p has as objects all p-cyclotomic spectra as above, and a 1-morphism $f\colon X \to Y$ is an S^1-equivariant map of spectra such that

$$\begin{array}{ccc} X & \xrightarrow{\phi_X} & X^{tC_p} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{tC_p}} \\ Y & \xrightarrow{\phi_Y} & Y^{tC_p}. \end{array} .$$

This infinity category can be formalized as a *lax equalizer* of the functors

$$\mathrm{Sp}^{\mathrm{B}S^1} \underset{(\ )^{tC_p}}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{Sp}^{\mathrm{B}S^1}.$$

In general, a lax equalizer of two functors $F, G\colon \mathcal{C} \to \mathcal{D}$ is computed as a pullback

$$\begin{array}{ccc} \mathrm{LEq}(F,G) & \longrightarrow & \mathcal{D}^{\Delta^1} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ \mathcal{C} & \xrightarrow[(F,G)]{} & \mathcal{D} \times \mathcal{D} \end{array} ;$$

note that an object is $(c, Fc \xrightarrow{f} Gc)$ and a 1-morphism is the data of a 1-morphism in $\mathcal{C}$ that makes the relevant square commute. One may similarly define the ∞-category of cyclotomic spectra where the maps must be compatible with the Frobenii across all primes p, i.e. the lax equalizer of

$$\mathrm{Sp}^{\mathrm{B}S^1} \underset{\prod_p (\ )^{tC_p}}{\overset{\prod_p \mathrm{id}}{\rightrightarrows}} \prod_p \mathrm{Sp}^{\mathrm{B}S^1}.$$

Defining these ∞-categories as lax equalizers immediately implies some nice properties:

(a) Equivalences of p-cyclotomic spectra are given by underlying equivalences.[6]

(b) CycSp_p is a stable ∞-category.

(c) The forgetful functor $\mathrm{CycSp}_p \to \mathrm{Sp}^{\mathrm{B}S^1}$ is exact and preserves all (small) colimits.

(d) For all primes p, there is a restriction map $\mathrm{CycSp} \to \mathrm{CycSp}_p$.

(e) CycSp_p inherits a symmetric monoidal structure from Sp.

We now claim that $\mathrm{THH}(A)$ is a p-cyclotomic spectrum. We have already discussed how $\mathrm{THH}(A)$ is a spectrum with S^1-action, so it suffices to construct a Frobenius map $\mathrm{THH}(A) \to (\mathrm{THH}(A))^{tC_p}$. To do so, we use something called the *Tate diagonal*.

[6] In particular, we are not considering p-cyclotomic spectra as a subcategory of *genuine S^1 spectra*, although there is a comparison between the Borel notion we have described here and a more genuine equivariant notion.

First, observe that we have a natural way to map any spectrum X into a spectrum with C_p-action by taking the diagonal $X \to X^{\otimes p}$, where C_p permutes the factors of $X^{\otimes p}$. However, $(-)^{\otimes p}$ is not an exact functor (it is not even additive), but post-composing with the Tate construction fixes this issue.

Theorem 9.9. *For every $X \in \mathrm{Sp}$, there is a map $X \xrightarrow{\Delta_p} (X^{\otimes p})^{tC_p}$ called the Tate diagonal which extends to a natural transformation*

$$\mathrm{Sp} \underset{((-)^{\otimes p})^{tC_p}}{\overset{\mathrm{id}}{\Downarrow \Delta_p}} \mathrm{Sp} \ .$$

Moreover, there is a unique such natural transformation Δ_p which is also symmetric monoidal.

The Tate diagonal has the nice property that it takes $\mathbb{E}_n$-algebras to $\mathbb{E}_n$-algebra maps. Moreover, if X is bounded below, then Δ_p is a p-completion. We can use the Tate diagonal to construct Frobenius maps. Before examining $\mathrm{THH}(A)$, we first discuss an easier example.

Example 9.10. The sphere spectrum $\mathbb{S}$ with the trivial S^1-action is a p-cyclotomic spectrum. The S^1-equivariant Frobenius map $\phi_p \colon \mathbb{S} \to \mathbb{S}^{tC_p}$ can be constructed using the Tate diagonal:

$$\phi_p \colon \mathbb{S} \xrightarrow{\Delta_p} (\mathbb{S}^{\otimes p})^{tC_p} \xrightarrow{\cong} \mathbb{S}^{tC_p}.$$

More generally, note that this composition factors as

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\quad \Delta_p \quad} & (\mathbb{S}^{\otimes p})^{tC_p} \\ {\scriptstyle p^*}\big\downarrow & & \big\downarrow{\scriptstyle =} \\ \mathrm{map}(\mathrm{B}C_p, \mathbb{S}) \simeq \mathbb{S}^{hC_p} & \xrightarrow[\mathrm{can}]{} & \mathbb{S}^{tC_p} \end{array}$$

where $p \colon BC_p \to *$ is the collapse map. This composition can be made equivariant, using the fact that $\mathbb{S}$ has trivial action and recalling various adjunctions involving mapping into/out of objects with trivial action. In particular, p^* and the equivalence $\mathrm{map}(\mathrm{B}C_p, \mathbb{S}) \simeq \mathbb{S}^{hC_p}$ are S^1-equivariant, so ϕ_p is as well (since can is equivariant by construction).

Remark 9.11. This construction (using p^* and can) works more generally to give any spectrum X with trivial S^1-action the extra structure of a p-cyclotomic spectrum.

We are finally ready to construct the Frobenius map for $\mathrm{THH}(A)$. We only sketch the proof for when A is an $\mathbb{E}_\infty$-ring (where we can use the universal property of THH), although the statement holds for $\mathbb{E}_1$-algebras; the proof is more involved in that case, see [NS18, §III.2].

Proposition 9.12. *There is a Frobenius map $\phi_p \colon \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$.*

Proof. First, observe that $A \to A^{\otimes p}$ is initial in $\mathbb{E}_\infty$-algebra maps $A \to B$ where B has a C_p-action. Thus we have a C_p-equivariant map $A^{\otimes p} \to \mathrm{THH}(A)$ of $\mathbb{E}_\infty$-algebras. We apply the Tate construction to obtain a map

$$(A^{\otimes p})^{tC_p} \to \mathrm{THH}(A)^{tC_p}.$$

Recall that the Tate construction $\mathrm{THH}(A)^{tC_p}$ has a residual action by $S^1/C_p \cong S^1$. We then have a diagram of $\mathbb{E}_\infty$-maps

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{THH}(A) \\ {\scriptstyle \Delta_p}\downarrow & & \downarrow \\ (A^{\otimes p})^{tC_p} & \longrightarrow & \mathrm{THH}(A)^{tC_p} \end{array}$$

and the universal property of $\mathrm{THH}(A)$ implies that there is an S^1-equivariant factorization via the dashed arrow. This is the desired map of $\mathbb{E}_\infty$-algebras $\phi_p\colon \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$. □

9.4 Topological Cyclic Homology, briefly

We finally define $\mathrm{TC}(A)$. We will assume that A is a connective $\mathbb{E}_\infty$-ring (so in particular $\mathrm{THH}(A)$ is bounded below), as this allows us to make use of the Tate orbit lemma in the definition below.

Definition 9.13. The *topological cyclic homology* of A at a prime p is the equalizer

$$\mathrm{TC}(A,p) \coloneqq \mathrm{eq}\left(\mathrm{TC}^-(A,p) \underset{\phi}{\overset{\mathrm{can}}{\rightrightarrows}} \mathrm{TP}(A,p) \right)$$

Recall that $(\mathrm{THH}(A)^{hS^1})^\wedge_p =: \mathrm{TC}^-(A,p)$ and $\mathrm{TP}(A,p) \coloneqq (\mathrm{THH}(A)^{tS^1})^\wedge_p$. By a slight abuse of notation, the map can in the diagram above is the p-completion of the canonical map $\mathrm{THH}(A)^{hS^1} \to \mathrm{THH}(A)^{tS^1}$. The map ϕ is a bit more complicated: first, since the Frobenius map ϕ_p is S^1-equivariant, we may consider

$$(\phi_p)^{hS^1}\colon \mathrm{THH}(A)^{hS^1} \to (\mathrm{THH}(A)^{tC_p})^{hS^1} \simeq (\mathrm{THH}(A)^{tC_p})^{h(S^1/C_p)},$$

where the equivalence is merely using our favorite fact $S^1/C_p \cong S^1$. Now, the Tate orbit lemma implies that we may miraculously "cancel fractions" and obtain

$$(\mathrm{THH}(A)^{tC_p})^{h(S^1/C_p)} \simeq \mathrm{THH}(A)^{tS^1}.$$

The p-completion of the composition

$$\mathrm{THH}(A)^{hS^1} \xrightarrow{(\phi_p)^{hS^1}} (\mathrm{THH}(A)^{tC_p})^{hS^1} \simeq (\mathrm{THH}(A)^{tC_p})^{h(S^1/C_p)} \simeq \mathrm{THH}(A)^{tS^1}$$

is the map ϕ in the equalizer above.

There is a different but equivalent formulation of TC, which makes use of a previous example that $\mathbb{S}$ with trivial action can be extended to a p-cyclotomic spectrum via the Tate diagonal.

Theorem 9.14. *There is an equivalence* $\mathrm{TC}(A,p) \simeq \mathit{map}_{\mathrm{CycSp}_p}(\mathbb{S}, \mathrm{THH}(A))^\wedge_p$.

10 The circle action and redshift (Noah Wisdom)

Notes by Sanjana Agarwal

10.1 Talk Outline

In this talk we will

(a) Introduce S^1-homotopy fixed point spectral sequence (HFPSS) and S^1-Tate fixed point spectral sequence (TFPSS).

(b) Relate the σ^2 construction (from talk 3) with the circle action.

(c) Calculate $\pi_*(\mathrm{TC}^-(\mathbb{F}_p))$ and $\pi_*(\mathrm{TP}(\mathbb{F}_p))$.

(d) Prove redshift for $\mathrm{TC}^-(\mathbb{Z}_p/\mathrm{MU})$ and $\mathrm{TC}^-(\ell/\mathrm{MU})$.

(e) And deduce analogous statements for absolute TC and K-theory.

10.2 S^1-HFPSS and TFPSS

We want to be able to compute the spectra we have defined earlier, like TC^-, TP, etc. The main tool towards that are the fixed point spectral sequences, which we define in this section. But to be able to define that, we need the notion of filtered spectra.

Definition 10.1. A filtered spectrum is a diagram $... \to X_i \to X_{i+1} \to ...$ of spectra indexed on $\mathbb{Z}$.

Any filtered spectrum gives rise to a spectral sequence with signature

$$E_1^{p,q} := \pi_p(\mathrm{cofib}(X_q \to X_{q+1})) \implies \pi_{p+q}(\mathrm{colim} X_\bullet).$$

In the case when $X \in Fun(\mathrm{B}G, \mathrm{Sp})$ (example of interest for us is when $G = S^1$, and $X = \mathrm{THH}(-)$) consider $\tau_{\geq p} : \mathrm{Sp} \to \mathrm{Sp}$ and compose. Doing this for each p, gives a filtration, $\tau_{\geq \bullet} X \in \mathrm{Fil}(\mathrm{Fun}(\mathrm{B}G, \mathrm{Sp}))$. On this filtered spectra, one can compatibly apply the functors $(-)^{hG}$ and $(-)^{tG}$ (this is not immediate but can be shown - one has to check that $(-)^{hG}$ and $(-)^{tG}$ are compatible with Whitehead filtrations [NS18, p. I.2.6]) and the resulting spectral sequences are called the homotopy fixed spectral sequence and Tate fixed point spectral sequence, respectively.

To write down the E_2-page of the spectral sequence we identify the cofibers or the graded pieces and use the fact that $(-)^{hG}$ is exact. Turns out that the relevant cofiber is $\mathrm{K}(\pi_l X, k)^{hG}$ giving the E_2-page of HFPSS as the group cohomology $H^*(G; \pi_* X)$ and the E_2-page of the TFPSS as the Tate cohomology $\widehat{H}^*(G; \pi_* X)$.

Theorem 10.2. *There exists the following spectral sequences called the homotopy fixed point spectral sequence and the Tate fixed point spectral sequence respectively*

$$E^2_{p,q} \cong H^{-p}(G; \pi_q(X)) \implies \pi_{p+q}(E^{hG})$$

$$E^2_{p,q} \cong \widehat{H}^{-p}(G; \pi_q(X)) \implies \pi_{p+q}(E^{tG})$$

These two spectral sequences will be our two essential tools in this talk.

10.3 The σ^2 construction and the circle action

Recall

$$\sigma^2 \colon \Sigma^{-1}(A/1) \to \Sigma^{-2}\,\mathrm{THH}(A)$$

from the third talk. We want to create a way of "undoing" this map. We have an action map

$$\Sigma\,\mathrm{THH}(A) \to S^1_+ \otimes \mathrm{THH}(A) \to \mathrm{THH}(A)$$

with adjoint $\mathrm{THH}(A) \to \Sigma^{-1}\,\mathrm{THH}(A)$.

This gives a fiber sequence

$$\lim_{\mathcal{C}P^1} \mathrm{THH}(A) \to \mathrm{THH}(A) \to \Sigma^{-1}\,\mathrm{THH}(A)$$

with connecting morphism $t \colon \Sigma^{-2}\,\mathrm{THH}(A) \to \lim_{\mathcal{C}P^1} \mathrm{THH}(A)$. Using the σ^2 map above, we get the following diagram which gives us a notion of 'undoing' σ :

Theorem 10.3. *There's a functorial diagram*

$$\begin{array}{ccc} \Sigma^{-1}(A/1) & \longrightarrow & 1 \\ \big\downarrow{\scriptstyle \sigma^2} & & \big\downarrow \\ \Sigma^{-2}\,\mathrm{THH}(A) & \xrightarrow{t} & \lim_{\mathcal{C}P^1} \mathrm{THH}(A). \end{array}$$

10.4 Calculating $\pi_*\,\mathrm{TC}^-(\mathbb{F}_p)$ and $\pi_*\,\mathrm{TP}(\mathbb{F}_p)$

In this subsection, we are finally ready to use the tools above for our first example computation - we would like to compute the homotopy groups of $\mathrm{TC}^-(\mathbb{F}_p)$ and $\mathrm{TP}(\mathbb{F}_p)$.

Theorem 10.4. *$\pi_*\,\mathrm{TC}^-(\mathbb{F}_p) \cong \mathbb{Z}_p[t, \sigma^2 p]/(t\sigma^2 p - p)$ and $\pi_*\,\mathrm{TP}(\mathbb{F}_p) \cong \mathbb{Z}_p[t^{\pm 1}]$.*

Proof. Recall that $\mathrm{THH}_*(\mathbb{F}_p) \cong \mathbb{F}_p[\sigma^2 p]$. The HFPSS computing $\pi_*\,\mathrm{TC}^-(\mathbb{F}_p)$ has E_2-page $H^*(\mathrm{B}S^1; \mathbb{F}_p[\sigma^2 p])$. Since this is concentrated in even degrees, it collapses. Thus the E_∞-page is $\mathbb{F}_p[\sigma^2 p, t]$ with $t \in H^2(CP^\infty; \mathbb{F}_p)$. But now this is a bad extension problem with $\mathbb{F}_p$ as all the associated graded pieces. Thinking of t as the complex orientation of $\mathrm{TC}^-(\mathbb{F}_p)$, we obtain the equation $t \cdot \sigma^2 p - p = 0$ in $\pi_*\,\mathrm{TC}^-(\mathbb{F}_p)$

$$\begin{array}{ccc} p & \longrightarrow & p \\ \big\downarrow{\scriptstyle \sigma^2} & & \big\downarrow \\ \sigma^2 p & \xrightarrow{t} & t\sigma^2 p = p \end{array}$$

which resolves the extension problem, giving $\pi_*\,\mathrm{TC}^-(\mathbb{F}_p) \cong \mathbb{Z}_p[t, \sigma^2 p]/(t\sigma^2 p - p)$.

Similarly the TFPSS computing $\pi_*\,\mathrm{TP}(\mathbb{F}_p)$ has E_2-page $\widehat{H}^*(\mathrm{B}S^1, \mathbb{F}_p[\sigma^2 p])$ which collapses as well. Comparing it with HFPSS gives that $\pi_*\,\mathrm{TP}(\mathbb{F}_p) \to \pi_*\,\mathrm{TC}^-(\mathbb{F}_p)$ is an isomorphism for $* \leq 0$. Since the map is multiplicative, we have the image of this map $p^k\mathbb{Z}_p$ for $* = 2k \geq 0$. Thus, $\pi_*\,\mathrm{TP}(\mathbb{F}_p) \cong \mathbb{Z}_p[t^{\pm 1}]$. □

10.5 Towards Redshift for $\mathrm{TC}^{-}(\mathbb{Z}_p/\mathrm{MU})$ and $\mathrm{TC}^{-}(\ell/\mathrm{MU})$

Our next example computation is computation for $\mathrm{TC}^{-}(\mathbb{Z}_p/\mathrm{MU})$ and $\mathrm{TC}^{-}(\ell/\mathrm{MU})$. In the process we see red-shift for these spectra.

Theorem 10.5. *$\pi_* \mathrm{TC}^{-}(\mathbb{Z}_p/\mathrm{MU}) \cong \mathrm{THH}_*(\mathbb{Z}_p/\mathrm{MU})[[t]]$ and $\pi_* \mathrm{TC}^{-}(\ell/\mathrm{MU}) \cong \mathrm{THH}_*(\ell/\mathrm{MU})[[t]]$.*

Proof. Recall that $\mathrm{THH}_*(\mathbb{Z}_p/\mathrm{MU})$ is a polynomial $\mathbb{Z}_p$-algebra and $\mathrm{THH}_*(\ell/\mathrm{MU})$ is a polynomial $\mathbb{Z}_p[v_1]$-algebra. In both of these cases, the HFPSS is concentrated in even degrees, and there is no extension problem.
Thus $\pi_* \mathrm{TC}^{-}(\mathbb{Z}_p/\mathrm{MU}) \cong \mathrm{THH}_*(\mathbb{Z}_p/\mathrm{MU})[[t]]$ and $\pi_* \mathrm{TC}^{-}(\ell/\mathrm{MU}) \cong \mathrm{THH}_*(\ell/\mathrm{MU})[[t]]$. Here $t \in H^2(\mathbb{C}P^\infty; \mathrm{THH}_0(-/\mathrm{MU}))$ is the standard generator. ☐

Redshift for $\mathbb{Z}_p/\mathrm{MU}$ and ℓ/MU : Recall from previous talks that one polynomial generator for $\mathrm{THH}_*(\mathbb{Z}_p/\mathrm{MU})$ is $\sigma^2 v_1$, and a generator for $\mathrm{THH}_*(\ell/\mathrm{MU})$ is $\sigma^2 v_2$.

Choose $x \in \mathrm{TC}^{-}(-/\mathrm{MU})$ corresponding to $\sigma^2 v_i$ in the HFPSS ($i = 1, 2$ respectively). The $\lim_{\mathbb{C}P^1}$ term in our diagram from theorem 10.3 controls mod t^2 behavior in the spectral sequence (by a different choice of filtration for the HFPSS). Hence by the same diagram, tx is v_i modulo t^2, ie. $tx - v_i = t^2 y$.

Write $x' = x - ty$. Then $tx' = v_i$, and we may replace the polynomial generator x in $\mathrm{TC}^{-}(-/\mathrm{MU})$ with x'. The unit map $\pi_*(\mathrm{MU}_{(p)}^{hS^1}) \to \mathrm{TC}^{-}(-/\mathrm{MU})$ thus sends the complex orientation to t and v_i to $t(\sigma^2 v_i)$.

By work of Hovey, since $\mathrm{TC}^{-}(\mathbb{Z}_p/\mathrm{MU})$ is an MU-module,

$$L_{K(1)} \mathrm{TC}^{-}(\mathbb{Z}_p/\mathrm{MU}) = (\mathrm{TC}^{-}(-/\mathrm{MU}))[v_1^{-1}]^\wedge_{(p)}$$

which by our computation may be directly checked to be nonzero. Additionally,

$$L_{K(2)} \mathrm{TC}^{-}(\ell/\mathrm{MU}) = (\mathrm{TC}^{-}(\ell/\mathrm{MU}))[v_2^{-1}]^\wedge_{(p,v_1)}$$

which is again directly seen to be nonzero.

Consider the composition of ring maps

$$\mathrm{K}(\mathbb{Z}_p) \to \mathrm{TC}(\mathbb{Z}_p) \to \mathrm{TC}^{-}(\mathbb{Z}_p) \to \mathrm{TC}^{-}(\mathbb{Z}_p/\mathrm{MU})$$

. Taking $K(1)$ localization gives a sequence of ring maps for which the last ring is nonzero. Hence all rings are nonzero. Similarly, consider

$$\mathrm{K}(\ell) \to \mathrm{TC}(\ell) \to \mathrm{TC}^{-}(\ell) \to \mathrm{TC}^{-}(\ell/\mathrm{MU})$$

and take $K(2)$-localizations. We've proven the following rings are nonzero: $L_{K(1)}\mathrm{K}(\mathbb{Z}_p)$, $L_{K(1)} \mathrm{TC}(\mathbb{Z}_p)$, $L_{K(1)} \mathrm{TC}^{-}(\mathbb{Z}_p)$, $L_{K(1)} \mathrm{TC}^{-}(\mathbb{Z}_p/\mathrm{MU})$, $L_{K(2)}\mathrm{K}(\ell)$, $L_{K(2)} \mathrm{TC}(\ell)$, $L_{K(2)} \mathrm{TC}^{-}(\ell)$, and $L_{K(2)} \mathrm{TC}^{-}(\ell/\mathrm{MU})$!

11 Cyclotomic Frobenius and the Segal conjecture (Isabel Longbottom)

Notes by Atticus Wang

Fix a prime p throughout. The goal of this talk is to prove the following theorem:

Theorem 11.1 ([HW22], 4.3.1)**.** *The Frobenius*

$$\varphi\colon \mathrm{THH}(\mathrm{BP}\langle n\rangle)/(v_0,\dots,v_n) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p}/(v_0,\dots,v_n)$$

is an isomorphism in π_* for all $*$ sufficiently large.

To explain the context for this theorem, recall that the classical Segal's conjecture for C_p is equivalent to the fact that the Frobenius for $\mathrm{THH}(\mathbb{S}) = \mathbb{S}^{\mathrm{triv}}$,

$$\varphi\colon \mathbb{S} \to \mathbb{S}^{tC_p},$$

exhibits the right hand side as the p-completion of $\mathbb{S}$.

We'll make the following informal definition: For a bounded below, p-complete, p-cyclotomic spectrum X, we say it satisfies Segal's conjecture if the Frobenius $\varphi\colon X \to X^{tC_p}$ is an isomorphism in π_* for all $*$ sufficiently large.

It follows by the thick subcategory theorem in BP-modules that in Theorem 11.1 we can replace each v_i by any arbitrarily large power $v_i^{e_i}$, so that $\mathbb{S}/(v_0^{e_0},\dots,v_n^{e_n})$ exists as a type $n+1$ complex. By the thick subcategory theorem in spectra, we conclude that for any type $n+1$ spectrum E, $E\otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)$ satisfies Segal's conjecture in the above sense.

Remark 11.2. The significance of Segal's conjecture is not only historical, but it is also one of the key inputs for proving the Quillen–Lichtenbaum conjecture for $\mathrm{BP}\langle n\rangle$. We sketch this deduction given the necessary inputs. Suppose that X satisfies Segal's conjecture, then

$$\varphi\colon X^{hS^1} \to (X^{tC_p})^{hS^1} = X^{tS^1}$$

is also an isomorphism in large degrees. Suppose in addition that the canonical map $\mathrm{can}\colon X^{hS^1} \to X^{tS^1}$ vanishes in all sufficiently large degrees, a property which turns out also to be true for $X = E\otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)$ where E is a type $n+2$ complex ([HW22], Section 6). Then we conclude that $E\otimes \mathrm{TC}(\mathrm{BP}\langle n\rangle)$ is bounded above for any type $n+2$ complex E. With some more work, one can in fact show that $\pi_*(E\otimes \mathrm{TC}(\mathrm{BP}\langle n\rangle))$ is finite. By results of Mahowald–Rezk [MR99] this implies that

$$\mathrm{TC}(\mathrm{BP}\langle n\rangle) \to L_{n+1}^f\,\mathrm{TC}(\mathrm{BP}\langle n\rangle)$$

is an isomorphism in sufficiently large degrees. The Quillen–Lichtenbaum conjecture is the analogous statement with TC replaced by algebraic K-theory.

The proof strategy for Theorem 11.1 is the following. We'll filter $\mathrm{BP}\langle n\rangle$ by the $\mathbb{F}_p$-Adams filtration, and after taking care of convergence issues we only need to prove the theorem for the associated graded, which is a graded $\mathbb{E}_2$-polynomial algebra over $\mathbb{F}_p$. Such a case can be further reduced to the case of a spherical polynomial ring in a single variable, where the result follows from Segal's conjecture for $\mathbb{S}$.

11.1 Descent towers

One of the oldest techniques to understand a spectrum X is to approximate it with a well-understood ring spectrum A. For example, suppose $X = \mathbb{S}$, then the Adams (resp. Adams–Novikov) spectral sequence tries to approximates it by $A = \mathbb{F}_p$ (resp. $A = \mathrm{MU}$).

The way this works is by writing down the cosimplicial spectrum

$$X \otimes A \rightrightarrows X \otimes A \otimes A \substack{\rightarrow\\\rightarrow\\\rightarrow} X \otimes A \otimes A \otimes A \dots$$

whose limit we denote by $X^\wedge_A$. Then we can filter $X^\wedge_A$ by the *descent tower*

$$\operatorname{desc}^{\geq *}_A X := \lim(\tau_{\geq *}(X \otimes A) \rightrightarrows \tau_{\geq *}(X \otimes A \otimes A) \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots).$$

The functor assigning X to $\operatorname{desc}^{\geq *}_A X$ is a lax symmetric monoidal functor $\mathrm{Sp} \to \mathrm{fil}(\mathrm{Sp})$. This means for example that if our input X is an $\mathbb{E}_2$-ring such as $\mathrm{BP}\langle n \rangle$, its descent tower is a $\mathbb{E}_2$-ring in the category of filtered spectra. See talk 4 and references therein for more on filtered spectra.

The associated graded of $\operatorname{desc}^{\geq *}_A X$ is

$$\operatorname{gr}^*_A X = \Sigma^* \lim(H\pi_*(X \otimes A) \rightrightarrows H\pi_*(X \otimes A \otimes A) \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots).$$

Totalization of a cosimplicial abelian group like this is computed by the homology of the associated chain complex (by taking alternating sums of the cosimplicial maps). In other words,

$$\pi_{-\bullet} \operatorname{gr}^*_A X = H^{\bullet+*}(\pi_*(X \otimes A) \to \pi_*(X \otimes A \otimes A) \to \dots)).$$

Now, suppose in addition that the map $\pi_*(A) \to \pi_*(A \otimes A)$ induced by $1 \otimes \mathrm{unit}$ (equivalently, $\mathrm{unit} \otimes 1$) is flat. Then a standard trick on cohomology theories shows that

$$\pi_*(X \otimes A \otimes A) = \pi_*((X \otimes A) \otimes_A (A \otimes A)) = A_*X \otimes_{A_*} A_*A$$

and similarly for the other terms. This means that the chain complex computing $\operatorname{gr}^*_A X$ can be rewritten as

$$A_*X \to A_*X \otimes_{A_*} A_*A \to A_*X \otimes_{A_*} A_*A \otimes_{A_*} A_*A \to \dots$$

which is the bar complex which computes the Ext groups $\operatorname{Ext}_{A_*A}(A_*A, A_*X)$ of comodules over the Hopf algebroid (A_*, A_*A) (see [Rav86], appendix A1).

Putting this all together, we have the following:

Proposition 11.3. *There is a natural spectral sequence*

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{A_*A}(A_*, A_*X) = \pi_{t-s} \operatorname{gr}^t_A X \implies \pi_{t-s} X^\wedge_A.$$

When X is additionally a ring spectrum, this spectral sequence is multiplicative.

We say this spectral sequence converges *conditionally* if the limit

$$\lim_*(\operatorname{desc}^{\geq *}_A X) = 0.$$

For example if X and A are connective, this is clearly satisfied.

Remark 11.4. There is a natural map $X \to X^\wedge_A$. When the fiber of the unit map $\mathbb{S} \to A$ is 1-connective, this map is an equivalence. For example $A = \mathrm{MU}$ satisfies this. In other cases, we can sometimes still determine $X^\wedge_A$. Since $X^\wedge_A$ is A-local, the map $X \to X^\wedge_A$ factors through the Bousfield localization $L_A X$. The map $L_A X \to X^\wedge_A$ is an equivalence in many situations (see [Bou79], theorems 6.5–6.7). For example, when $A = \mathbb{F}_p$, this is satisfied and we get the p-completion $X^\wedge_p$.

Let's see what happens in the case of $X = \mathrm{BP}\langle n\rangle$ and $A = \mathbb{F}_p$. Since $\mathrm{BP}\langle n\rangle$ is p-complete, we get the spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbb{F}_p, H_*(\mathrm{BP}\langle n\rangle; \mathbb{F}_p)) \implies \pi_{t-s}\mathrm{BP}\langle n\rangle,$$

where $\mathcal{A}^*$ is the mod p Steenrod algebra and $\mathcal{A}_*$ its dual. By Steve Wilson's result ([Wil75], proposition 1.7) we know the $\mathbb{F}_p$-cohomology

$$H^*(\mathrm{BP}\langle n\rangle; \mathbb{F}_p) = \mathcal{A}^*/(Q_0, \dots, Q_n)$$

where $Q_0, \dots, Q_n$ are defined as in Milnor [Mil58]. Taking its dual we get

$$H_*(\mathrm{BP}\langle n\rangle; \mathbb{F}_p) = \mathcal{A}_* \square_{\wedge(\tau_0,\dots,\tau_n)} \mathbb{F}_p = \mathrm{eq}(\mathcal{A}_* \rightrightarrows \mathcal{A}_* \otimes \wedge(\tau_0, \dots, \tau_n))$$

where the two maps are inclusion into the first factor and the diagonal $\mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ composed with the quotient map by ξ's and $\tau_{n+1}, \tau_{n+2}, \dots$. The fact that $H_*(\mathrm{BP}\langle n\rangle; \mathbb{F}_p)$ is extended means that we can apply the change-of-rings theorem ([Rav86], A1.3.13) to get

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbb{F}_p, H_*(\mathrm{BP}\langle n\rangle; \mathbb{F}_p)) = \mathrm{Ext}^{s,t}_{\wedge(\tau_0,\dots,\tau_n)}(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p[v_0, \dots, v_n]$$

with the last step by Koszul duality. These classes v_i live in degree $(s,t) = (1, 2p^i - 1)$ and detect the classes $p, v_1, \dots, v_n \in \pi_*(\mathrm{BP}\langle n\rangle) = \mathbb{Z}_p[v_1, \dots, v_n]$ respectively. We can choose lifts $\widetilde{v_i} \in \pi_* \operatorname{desc}^{\geq 2p^i-1}_{\mathbb{F}_p} \mathrm{BP}\langle n\rangle$. Note that v_0 can be uniquely lifted and so we denote the lift just by v_0 as well. It in fact exists as an element $v_0 \in \pi_0 \operatorname{desc}^{\geq 1}_{\mathbb{F}_p} \mathbb{S}$ detecting p.

11.2 Reduction to the associated graded

From now on we'll just write desc for $\operatorname{desc}^{\geq *}_{\mathbb{F}_p}$. Consider the descent tower $\operatorname{desc}\mathrm{BP}\langle n\rangle$. As we remarked above, it is an $\mathbb{E}_2$-algebra in $\mathrm{fil}(\mathrm{Sp})$, therefore we can apply THH to it in $\mathrm{fil}(\mathrm{Sp})$, and the resulting filtered spectrum will have underlying object (i.e. colimit) $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$. Furthermore, there is also a cyclotomic Frobenius map in the filtered context, and it is compatible with taking the colimit: the only caveat is that the Frobenius here is twisted by a shear in filtration degree:

Definition 11.5. For a filtered spectrum $X\colon \mathbb{Z}_{\geq} \to \mathrm{Sp}$, let $L_p X$ denote the filtered spectrum

$$\dots X_1 \to X_0 \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} X_0 \to X_{-1} \dots$$

where $(L_p X)_{ip} = \dots = (L_p X)_{ip-p+1} = X_i$. More concisely, it is the left Kan extension of X along $p\colon \mathbb{Z} \to \mathbb{Z}$.

Proposition 11.6 ([Ant+22], example A.11)**.** *For any $\mathbb{E}_1$-algebra R in $\mathrm{fil}(\mathrm{Sp})$ (resp. $\mathrm{gr}(\mathrm{Sp})$), there is a natural S^1-equivariant map of filtered (resp. graded) spectra*

$$\varphi : L_p\,\mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

compatible with taking the colimit and taking the associated graded.

Recall in the beginning we wanted to show that the Frobenius

$$\varphi\colon\ \mathrm{THH}(\mathrm{BP}\langle n\rangle)/(v_0,\dots,v_n) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p}/(v_0,\dots,v_n)$$

induces an isomorphism in π_* for $* \gg 0$. We can upgrade this to a map of filtered spectra by Proposition 11.6, which recovers the original map after taking colimits:

$$\varphi : L_p\,\mathrm{THH}(\operatorname{desc}\mathrm{BP}\langle n\rangle)/(v_0,\dots,\widetilde{v}_n) \to \mathrm{THH}(\operatorname{desc}\mathrm{BP}\langle n\rangle)^{tC_p}/(v_0,\dots,\widetilde{v}_n) \tag{11.1}$$

so we have broken the problem down to two separate parts:

(a) Show that the filtered objects on both sides have limit 0 (i.e. conditional convergence);

(b) Show that the induced map on the associated graded is an equivalence in large enough degrees.

We'll first deal with (a) and leave (b) for the next section. Recall that because desc is symmetric monoidal, $\operatorname{desc} X$ is a $\operatorname{desc}\mathbb{S}$-module for any spectrum X, in particular there is a map

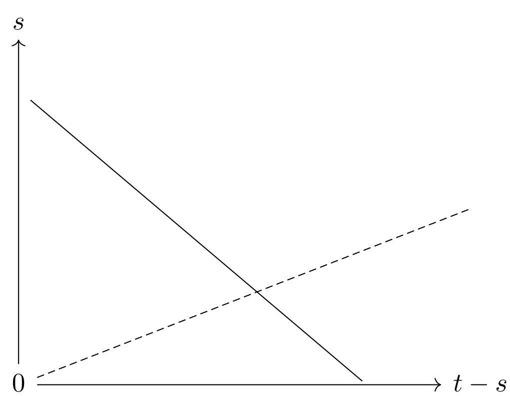
$$v_0 : \operatorname{desc} X[1] \to \operatorname{desc} X$$

where $\operatorname{desc}^{\geq j} X[1] = \operatorname{desc}^{\geq j-1} X$.

Proposition 11.7. *Let A be a connective $\mathbb{E}_1$-algebra, then the tower $\mathrm{THH}(\operatorname{desc} A)^\wedge_{v_0}$ converges conditionally.*

Proof. It suffices to show that $\mathrm{THH}(\operatorname{desc} A)/v_0$ converges conditionally. This is computed by the colimit of a simplicial filtered spectrum, with terms $(\operatorname{desc} A)^{\otimes n}/v_0 = \operatorname{desc}(A^{\otimes n})/v_0$.

Let us first show that $\lim(\operatorname{desc}(A^{\otimes n})/v_0) = 0$. In fact it is true for any bounded below spectrum X that $\operatorname{desc}^{\geq *}(X)/v_0$ becomes increasingly connective as $*$ grows. It is computed by a spectral sequence with E_2 page



where the only nonzero entries need to be both above the solid line $t \geq *$ and below the dashed line, which is the classical vanishing line in the Adams spectral sequence (see [HW22] C.3.4 and references therein). From this it is easy to see that $\operatorname{desc}^{\geq *}(X)/v_0$ has to be increasingly connective.

We now have $U = \mathrm{THH}(\operatorname{desc} A)/v_0$ as a geometric realization of a simplicial filtered spectrum $U(\bullet)$ whose each individual term is conditionally convergent. Consider its skeletal filtration

$$0 \to \mathrm{sk}_0 \to \cdots \to \mathrm{sk}_{n-1} \to \mathrm{sk}_n \to \cdots \to U$$

where sk_n is the realization of the left Kan extension

$$\begin{array}{ccc} \Delta^{op}_{\leq n} & \xrightarrow{U(\leq n)} & \mathrm{fil(Sp)} \\ \downarrow & \nearrow & \\ \Delta^{op} & & \end{array}$$

Each sk_n is a finite colimit, which is equivalently a finite limit, which commute with limits. So sk_n are all conditionally convergent. The cofiber $\mathrm{sk}_n / \mathrm{sk}_{n-1}$ is the *latching object* $L_{[n]}U$ (see [Lur09]), which is a retract of $\Sigma^n U([n])$. In particular, because the filtered object $U([n]) = \operatorname{desc} A^{\otimes n+1}/v_0$ is uniformly connective in each term, $\mathrm{sk}_n / \mathrm{sk}_{n-1}$ is uniformly n-connective.

Now, fix an index k, and look at Milnor's sequence

$$0 \to \lim^1_s \pi_{k-1}(U_s) \to \pi_k(\lim_s U_s) \to \lim_s \pi_k(U_s) \to 0$$

where U_s denotes the filtration on U. For any fixed index s, $\pi_k(U_s) = \pi_k(\operatorname{colim}_n(\mathrm{sk}_n)_s) = \operatorname{colim}_n \pi_k(\mathrm{sk}_n)_s = \pi_k(\mathrm{sk}_{k+1})_s$ by the above. So $\lim_s \pi_k(U_s) = \lim_s \pi_k(\mathrm{sk}_{k+1})_s$. Similarly $\lim^1_s \pi_{k-1}(U_s) = \lim^1_s \pi_{k-1}(\mathrm{sk}_{k+1})_s$. But Milnor's sequence for sk_{k+1} implies that both of these are zero. So we conclude that $\lim U = 0$ as desired. ☐

Proposition 11.8. *Let A be a connective $\mathbb{E}_1$-algebra, then the tower $\mathrm{THH}(\operatorname{desc} A)^{tC_p}$ converges conditionally.*

Proof. We know that $(\lim \mathrm{THH}(\operatorname{desc} A))^{tC_p} = ((\lim \mathrm{THH}(\operatorname{desc} A))^\wedge_p)^{tC_p}$ by [NS18] I.2.9, and the latter is zero because $(\lim \mathrm{THH}(\operatorname{desc} A))/p = \lim(\mathrm{THH}(\operatorname{desc} A)/v_0) = 0$ by Proposition 11.7. So it suffices to show that in general for a filtered G-spectrum X which is uniformly bounded below, the natural map $(\lim X)^{tG} \to \lim(X^{tG})$ is an isomorphism.

First, this is clear if we replace tG by hG, so it suffices to prove that the natural map $(\lim X)_{hG} \to \lim(X_{hG})$ is an equivalence. The object X_{hG} is computed by the colimit of the simplicial object

$$\cdots \Sigma^\infty_+ G \otimes \Sigma^\infty_+ G \otimes X \mathrel{\substack{\to\\\to\\\to}} \Sigma^\infty_+ G \otimes X \rightrightarrows X.$$

Denote its skeleta by $\mathrm{sk}_n(X)$. Then since it is a finite colimit, the natural map $\mathrm{sk}_n(\lim X) \to \lim(\mathrm{sk}_n X)$ is an equivalence. Moreover, since X is uniformly bounded below, the connectivity of the map $\mathrm{sk}_n X \to X_{hG}$ uniformly increases with n. Taking the limit and using Milnor's sequence, we see that the maps $\lim(\mathrm{sk}_n X) \to \lim(X_{hG})$ and $\mathrm{sk}_n(\lim X) \to (\lim X)_{hG}$ both become increasingly connective with n. Thus, the natural map $(\lim X)_{hG} \to \lim(X_{hG})$ is an equivalence. ☐

Combining Propositions 11.7 and 11.8 shows (a).

11.3 The Segal conjecture for polynomial $\mathbb{F}_p$-algebras

For part (b), suppose the associated graded object of $\operatorname{desc} \mathrm{BP}\langle n\rangle$ is the $\mathbb{E}_2$-algebra R, then the associated graded of Equation (11.1) is given by

$$\varphi : L_p \,\mathrm{THH}(R)/(v_0, \ldots, v_n) \to \mathrm{THH}(R)^{tC_p}/(v_0, \ldots, v_n).$$

Recall from the discussion above that $\pi_* R = \mathbb{F}_p[v_0, \dots, v_n]$ as a commutative $\mathbb{F}_p$-algebra, with v_i in total degree $t - s = 2p^i - 2$ and weight[7] $t = 2p^i - 1$. This fact alone determines the $\mathbb{E}_2$-structure of R completely:

Definition 11.9 (spherical polynomial rings)**.** For integers r, w, let $\mathbb{S}^{2r}(w)$ denote the graded spectrum with $\mathbb{S}^{2r}$ in weight w and zero elsewhere. By $\mathbb{S}[x]$ (where x has degree $2r$ and weight w), we will mean the free graded $\mathbb{E}_1$-algebra on $\mathbb{S}^{2r}(w)$, with structural map $x \colon \mathbb{S}^{2r}(w) \to \mathbb{S}[x]$. Recall that as a spectrum it is given by

$$\mathbb{S}^0(0) \oplus \mathbb{S}^{2r}(w) \oplus \mathbb{S}^{4r}(2w) \oplus \dots$$

It in fact admits the structure of a graded $\mathbb{E}_2$-ring ([HW22], 4.1.1). This $\mathbb{E}_2$ structure originates from the fact that there exists an $\mathbb{E}_2$-map $\mathbb{Z} \to \mathrm{Pic}(\mathbb{S})$ sending $1 \mapsto \mathbb{S}^2$, which comes from the one-point compactification of complex vector spaces (the J-homomorphism).

Proposition 11.10 ([HW22], proposition 4.2.1)**.** *Let R be a graded $\mathbb{E}_2$-$\mathbb{F}_p$-algebra with $\pi_* R = \mathbb{F}_p[a_1, \dots, a_n]$ with $|a_i|$ in nonnegative even degrees and positive weights. Then R must be equivalent to $\mathbb{F}_p \otimes \mathbb{S}[a_1] \otimes \dots \otimes \mathbb{S}[a_n]$ as graded $\mathbb{E}_2$-$\mathbb{F}_p$-algebras.*

We skip the proof but the idea is to show that $\mathbb{F}_p \otimes \mathbb{S}[a_1] \otimes \dots \otimes \mathbb{S}[a_n]$ has an $\mathbb{E}_2$-$\mathbb{F}_p$-algebras cell structure concentrated in even degrees, which then implies that there are no obsctructions to writing down an $\mathbb{E}_2$ map from it to R, which then must be an equivalence. Granted this, it suffices to prove the following:

Proposition 11.11. *For R as above, the cyclotomic Frobenius*

$$\varphi : L_p \operatorname{THH}(R) \to \operatorname{THH}(R)^{tC_p}$$

induces on homotopy groups the obvious inclusion of rings

$$\mathbb{F}_p[x, a_1, \dots, a_n] \otimes \wedge(\sigma a_1, \dots, \sigma a_n) \to \mathbb{F}_p[x^{\pm 1}, a_1, \dots, a_n] \otimes \wedge(\sigma a_1, \dots, \sigma a_n)$$

where x has degree 2 and weight 0, and σ raises degree by 1 and does not change the weight.

This clearly implies (b) since the exterior algebra has bounded degree. See talk 4 for background on the suspension operation σ.

To prove this proposition, we can write φ as the composition

$$\operatorname{THH}(\mathbb{F}_p) \otimes_i L_p \operatorname{THH}(\mathbb{S}[a_i]) \to \operatorname{THH}(\mathbb{F}_p)^{tC_p} \otimes_i \operatorname{THH}(\mathbb{S}[a_i])^{tC_p} \to \operatorname{THH}(R)^{tC_p} \tag{11.2}$$

where the first map is the tensor product of each individual Frobenii and the second is from the lax monoidal structure of $(-)^{tC_p}$. Thus, it suffices to show that the first map is of the desired form on homotopy groups, and the second map is an equivalence. To do this, we need to better understand the cyclotomic Frobenius on spherical polynomial rings.

11.4 Segal's conjecture for $\mathbb{S}[x]$

Suppose x has degree $2r$ and weight w.

Proposition 11.12 ([Mat21], theorem 3.8)**.** *As a S^1-spectrum,*

$$\operatorname{THH}(\mathbb{S}[x]) = \bigoplus_{k \geq 0} \operatorname{Ind}_{C_k}^{S^1} \mathbb{S}^{2kr}(kw).$$

[7] Here, "weight" means the grading from the descent filtration.

Here C_k acts on $\mathbb{S}^{2kr}(kw) = (\mathbb{S}^{2r}(w))^{\otimes k}$ by permuting the tensor factors, $\mathrm{Ind}_{C_k}^{S^1}$ is the functor which is left adjoint to the functor $\mathrm{Sp}^{BS^1} \to \mathrm{Sp}^{BC_k}$ restricting along $BC_k \to BS^1$, and the $k = 0$ summand is just $\mathbb{S}$.

As an illustrative example, taking $r = w = 0$ recovers the classical computation

$$\mathrm{THH}(\mathbb{S}[x]) = \mathbb{S}^{\mathrm{triv}} \bigoplus_{k\geq 1} \Sigma_+^\infty (S^1/C_k).$$

Assume from now on that $w > 0$, so that all the different summands of $\mathrm{THH}(\mathbb{S}[x])$ live in distinct weights, and we can look at each k separately.

Definition 11.13. For a topological group G, a G-spectrum X is *finite* if it lies in the thick subcategory of G-spectra generated by $\Sigma_+^\infty(G/H)$ for $H \leq G$ closed.

Proposition 11.14. *If $w \neq 0$, then $\mathrm{THH}(\mathbb{S}[x])$ is weight-wise finite as a C_p-spectrum.*

Proof. Clearly it suffices to look at a fixed k. First, for $r \geq 0$, $(\mathbb{S}^{2r})^{\otimes k}$ is a retract of $\Sigma_+^\infty S^{2r\rho}$ where ρ is the regular representation of C_k. The space $S^{2r\rho}$ admits a finite C_k-CW-structure, so $\mathbb{S}^{2kr}$ is finite. Now, induction preserves finiteness, because $\mathrm{Ind}_G^F(G/H) = F/H$. So $\mathrm{Ind}_{C_k}^{S^1} \mathbb{S}^{2kr}$ is finite as a S^1-spectrum. We now need to show that any finite S^1-spectrum is finite as a C_p-spectrum. The proper closed subgroups of S^1 are C_n for $n \geq 1$. If $p \mid n$ then C_p fixes S^1/C_n so it's just $\Sigma(\Sigma_+^\infty *)$. If $p \nmid n$, then C_p acts on S^1/C_n freely, and it also clearly admits a finite C_p-CW-structure. □

Let's now see what $\mathrm{THH}(\mathbb{S}[x])^{tC_p}$ looks like. We can look at each k independently. By the proof above, when $p \nmid k$, $\mathrm{Res}_{C_p}^{S_1} \mathrm{Ind}_{C_k}^{S_1} \mathbb{S}^{2kr}$ can be built only using suspensions of $\Sigma_+^\infty C_p = \mathbb{S}^{\oplus p}$, whose Tate fixed point is 0 since C_p acts freely. When $k = mp$, we have

$$\mathrm{Res}_{C_p}^{S_1} \mathrm{Ind}_{C_k}^{S_1} \mathbb{S}^{2kr} = \mathbb{S}^{2kr} \otimes \Sigma_+^\infty S^1$$

where C_p acts on the second factor trivially and on the first factor by permuting the tensor factors in $\mathbb{S}^{2kr} = (\mathbb{S}^{2kr})^{\otimes p}$. Therefore,

$$\mathrm{THH}(\mathbb{S}[x])^{tC_p} \simeq \bigoplus_{m\geq 0} \mathbb{S}^{2rmp}(wmp)^{tC_p} \otimes \Sigma_+^\infty S^1.$$

Proposition 11.15. *The cyclotomic Frobenius*

$$\varphi : L_p \,\mathrm{THH}(\mathbb{S}[x]) \to \mathrm{THH}(\mathbb{S}[x])^{tC_p}$$

exhibits the right side as the p-completion of the left side.

Proof. By definition of φ we have the following square whose arrows are all S^1-equivariant:

$$\begin{array}{ccc} L_p\mathbb{S}[x]^{\otimes m} \otimes \Sigma_+^\infty S^1 & \longrightarrow & (\mathbb{S}[x]^{\otimes mp})^{tC_p} \otimes \Sigma_+^\infty S^1 \\ \downarrow & & \downarrow \\ L_p \,\mathrm{THH}(\mathbb{S}[x]) & \xrightarrow[\varphi]{} & \mathrm{THH}(\mathbb{S}[x])^{tC_p}. \end{array}$$

Looking at the summand corrsponding to x^m in the top map, it is by definition the Tate-valued Frobenius

$$\mathbb{S}^{2rm} \to (\mathbb{S}^{2rmp})^{tC_p}$$

tensored with S^1_+, which is the p-completion map by the classical Segal's conjecture for spheres. On the bottom, this map corresponds to φ in weight wpm. This proves that φ is p-completion in each weight and thus we are done. □

We can finally prove Proposition 11.11, which finishes the proof of the main Theorem 11.1.

Proof of Proposition 11.11. Recall that we need to show the first map in Equation (11.2) is the desired map on homotopy groups and the second map there is an equivalence. It is a classical fact ([NS18], IV.4) that the Frobenius $\mathrm{THH}(\mathbb{F}_p) \to \mathrm{THH}(\mathbb{F}_p)^{tC_p}$ is $\mathbb{F}_p[x] \to \mathbb{F}_p[x^{\pm}]$ on homotopy, and we have just shown that $L_p \,\mathrm{THH}(\mathbb{S}[x]) \to \mathrm{THH}(\mathbb{S}[x])^{tC_p}$ is p-completion, so the first map is indeed as claimed on homotopy groups.

To show the second map is an equivalence, using Proposition 11.14, it suffices to show that for X, Y nonnegatively graded C_p-spectra where Y is weight-wise finite, then the natural map $X^{tC_p} \otimes Y^{tC_p} \to (X \otimes Y)^{tC_p}$ is an equivalence. Because they are nonnegatively graded and $(-)^{tC_p}$ is taken degreewise, it suffices to show the same statement for X, Y (ungraded) C_p-spectra. In this case, the subcategory of C_p-spectra Y such that the map is an equivalence is thick and contains $\Sigma^\infty_+ C_p$ (because both sides are 0) and $\mathbb{S}$ (by the classical Segal conjecture), so we are done. □

12 Motivic filtrations on TC^- and TP (Avi Zeff)

Notes by Roger Murray

12.1 More even filtrations

In Talk 6 we saw that, for A an $\mathbb{E}_\infty$-algebra, we could define the even filtration as a right Kan extension, and that it had a description as the limit

$$\lim_{\substack{A\to B\\ B\in\mathrm{CAlg}^{\mathrm{ev}}}} \tau_{\geq 2*}B.$$

If we make the further assumption that A comes equipped with an S^1-action, then we can make a suite of similarly-flavoured definitions. For example, we define

$$\mathrm{fil}^*_{\mathrm{ev},\mathrm{h}S^1} A := \lim_{\substack{A\to B \ S^1\text{-equivariant}\\ B\in(\mathrm{CAlg}^{\mathrm{ev}})^{\mathrm{B}S^1}}} \tau_{\geq 2*}(B^{\mathrm{h}S^1}),$$

$$\mathrm{fil}^*_{\mathrm{ev},\mathrm{t}S^1} A := \lim_{\substack{A\to B \ S^1\text{-equivariant}\\ B\in(\mathrm{CAlg}^{\mathrm{ev}})^{\mathrm{B}S^1}}} \tau_{\geq 2*}(B^{\mathrm{t}S^1}).$$

In particular $\mathrm{fil}^*_{\mathrm{ev},\mathrm{h}S^1}(-)$ is the right Kan extensions of

$$\tau_{\geq 2*}((-)^{\mathrm{h}S^1}) : (\mathrm{CAlg}^{\mathrm{ev}})^{\mathrm{B}S^1} \to \mathrm{FilSp}$$

along the inclusion $(\mathrm{CAlg}^{\mathrm{ev}})^{\mathrm{B}S^1} \hookrightarrow \mathrm{CAlg}^{\mathrm{B}S^1}$, and likewise for $\mathrm{fil}^*_{\mathrm{ev},\mathrm{t}S^1}(-)$ and $\tau_{\geq 2*}((-)^{\mathrm{t}S^1})$.

Although we won't mention it again, another definition we could make for $B \in (\mathrm{CAlg}^{\mathrm{ev}})^{\mathrm{B}S^1}$ is

$$\mathrm{fil}^*_+\,\mathrm{fil}^*_{\mathrm{ev},\mathrm{h}S^1} := \lim_{\substack{A\to B\\ B \text{ even}}} \tau_{\geq 2*}\left(\tau_{\geq *}B\right)^{\mathrm{h}S^1}.$$

This is a bifiltered spectrum.

Yet another place we can come up with an even filtration is for cyclotomic spectra. Earlier, in Talk 9, we observed that, for A a (p-complete, bounded below) cyclotomic spectrum, there is an equivalence

$$\mathrm{TC}(A) \simeq \mathrm{fib}(\varphi - \mathrm{can} : A^{\mathrm{h}S^1} \to A^{\mathrm{t}S^1}).$$

Now suppose A is an *even* cyclotomic $\mathbb{E}_\infty$-ring, meaning that A is a (bounded below, p-typical) cyclotomic $\mathbb{E}_\infty$-ring such that the underlying $\mathbb{E}_\infty$-ring is even. Then we have a functor $\mathrm{CycSp}_p^{\mathrm{ev}} \to \mathrm{FilSp}$ defined by

$$A \mapsto \mathrm{fib}\left(\varphi - \mathrm{can} : \tau_{\geq 2*}(A^{\mathrm{h}S^1}) \to \tau_{\geq 2*}(A^{\mathrm{t}S^1})\right).$$

We define

$$\mathrm{fil}^*_{\mathrm{ev},\mathrm{TC}} : \mathrm{CycSp} \to \mathrm{FilSp}$$

to be the right Kan extension of this functor along the inclusion

$$\mathrm{CycSp}^{\mathrm{ev}} \hookrightarrow \mathrm{CycSp}\,.$$

12.2 Time to Descend

Our newly defined filtrations play nicely with eff maps. In particular, just like Proposition 6.8, if we have an eff map $A \to B$ between two $\mathbb{E}_\infty$-algebras with S^1-action, then the following canonical map is an equivalence:

$$\mathrm{fil}^*_{\mathrm{ev},\mathrm{h}S^1}(A) \xrightarrow{\sim} \lim_{\Delta} \mathrm{fil}^*_{\mathrm{ev},\mathrm{h}S^1}\left(B^{\otimes_A \bullet+1}\right). \tag{12.1}$$

One of the most important examples of an eff map we've seen is the unit $\mathbb{S} \to \mathrm{MU}$. It turns out that for A any $\mathbb{E}_\infty$-ring with S^1-action, the map $A \to A \otimes \mathrm{MU}$ is eff and therefore, by above, we have an equivalence

$$\mathrm{fil}^*_{\mathrm{ev},\mathrm{h}S^1}(A) \xrightarrow{\sim} \lim_{\Delta} \mathrm{fil}^*_{\mathrm{ev},\mathrm{h}S^1}\left(A \otimes \mathrm{MU}^{\otimes_A \bullet+1}\right).$$

This is a particularly nice example of descent which we call *Novikov descent.*

We would like to study even filtrations of THH. It turns out that, in order to have nice forms of descent, it will be useful to consider a certain class of maps. We now take some time to set things up.

Fact 12.1.

(1) If $k \to S$ is a map of even, connective $\mathbb{E}_\infty$-algebras such that $\pi_*(S)$ is polynomial over $\pi_*(k)$ then the map $\mathrm{THH}(S/k) \to S$ is eff.

(2) If $S \to R$ is a *quasiregular quotient* (meaning it is a map of even $\mathbb{E}_\infty$-rings such that the cotangent complex $\mathbb{L}_{\pi_*(R)/\pi_*(S)}$ has Tor-amplitude concentrated in degree 1), then $\mathrm{THH}(R/S)$ is even.

Suppose we have maps

$$k \xrightarrow{f} S \xrightarrow{g} R$$

where f is as in (1) and g is as in (2) of 12.1, i.e. such that $\mathrm{THH}(R/S)$ is even and $\mathrm{THH}(S/K) \to S$ is eff. We have the pushout diagram

$$\begin{array}{ccc} \mathrm{THH}(S/k) & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathrm{THH}(R/k) & \longrightarrow & \mathrm{THH}(R/S), \end{array}$$

whose bottom map $\mathrm{THH}(R/k) \to \mathrm{THH}(R/S)$, under our assumptions, is eff with even target. This is especially nice since, for example, then

$$\begin{aligned} \mathrm{fil}^*_{\mathrm{ev}}(\mathrm{THH}(R/k)) &\xrightarrow{\sim} \lim_{\Delta} \mathrm{fil}^*_{\mathrm{ev}}\left(\mathrm{THH}(R/S)^{\otimes_{\mathrm{THH}(R/k)} \bullet+1}\right) \\ &\simeq \lim_{\Delta} \tau_{\geq 2*}\left(\mathrm{THH}(R/S^{\otimes_k \bullet+1})\right) \end{aligned} \tag{12.2}$$

where the last equivalence follows from the fact that $\mathrm{THH}(R/S)$ is even. This descent will be used, for example, in the next talk. We give this particularly nice situation a name:

Definition 12.2. A map $k \to R$ of even, connective $\mathbb{E}_\infty$-rings is *quasi-lci* if it has a factorisation

$$k \xrightarrow{f} S \xrightarrow{g} R$$

where f and g are as in (1) and (2) of Fact 12.1 respectively.

Remark 12.3.

(1) We could alternately define it to be a map $k \to R$ of $\mathbb{E}_\infty$-rings such that $\mathbb{L}_{\pi_*(R)/\pi_*(k)}$ has Tor-amplitude concetrated in degrees $[0, 1]$.

(2) Note that this doesn't quite match up with the actual definition of [HRW22] but for our purposes it will suffice.

Definition 12.4. A map of $k \to R$ of $\mathbb{E}_\infty$-rings is *chromatically quasi-lci* if

(1) $k \otimes MU$ and $R \otimes MU$ are even,

(2) and $k \otimes MU \to R \otimes MU$ is quasi-lci.

Definition 12.5. An $\mathbb{E}_\infty$-algebra R is *quasisyntomic* if $\mathbb{Z} \to R$ is quasi-lci and $\pi_*(R)$ has bounded p-power-torsion.

Definition 12.6. An $\mathbb{E}_\infty$-algebra R is *chromatically quasisyntomic* if $R \otimes \mathrm{MU}$ is quasisyntomic.

Remark 12.7. Being both even and chromatically quasi-lci (resp., even and chromatically quasisyntomic) is equivalent to being quasi-lci (resp., quasisyntomic).

We are currently building up towards prismatic cohomology. Perfectoid rings are of particularly interest here as they have the property that taking THH, TC^- or TP spits out something even. The class of *quasiregular semiperfectoid rings* (or qsrp rings), which we will discuss below, also satisfy this property but generally lead to higher computational complexity. These rings are of significance in number theory since, very roughly and vaguely speaking, quasisyntomic rings are covered by qrsp rings and these are the largest/coarsest covers that are still tractable via THH techniques.

Definition 12.8. If $k \to R$ is chromatically quasi-lci, we define

$$\mathrm{fil}^*_{\mathrm{mot}}(\mathrm{THH}(R/k)) = \mathrm{fil}^*_{\mathrm{ev}}(\mathrm{THH}(R/k)),$$

$$\mathrm{fil}^*_{\mathrm{mot}}(\mathrm{TC}^-(R/k)) = \mathrm{fil}^*_{\mathrm{ev},\mathrm{h}S^1}(\mathrm{THH}(R/k)),$$
$$\mathrm{fil}^*_{\mathrm{mot}}(\mathrm{TP}(R/k)) = \mathrm{fil}^*_{\mathrm{ev},\mathrm{t}S^1}(\mathrm{THH}(R/k)).$$

We again have descent statements similar to those of 12.2.

We give the the corresponding associated graded spectra names, which specialise to those of the same name made in [BMS19] in the case R is, for example, a discrete ring which is perfectoid or qsrp.

Definition 12.9.

(1) $\mathrm{gr}^*_{\mathrm{mot}}(\mathrm{TC}^-(R))$ is the *Nygaard-filtered prismatic cohomology* of R denoted $\mathcal{N}^{\geq *}\widehat{\mathbb{\Delta}}_R\{*\}$,

(2) $\mathrm{gr}^*_{\mathrm{mot}}(\mathrm{TP}(R))$ is the *Prismatic cohomology* of R denoted $\widehat{\mathbb{\Delta}}_R\{*\}$ (where $\{*\}$ denotes the *Breuil-Kisin twist* and $\widehat{(-)}$ denotes completion with respect to Nygaard filtration),

(3) $\mathrm{gr}^*_{\mathrm{mot}}(\mathrm{THH}(R))$ is (related to) the *Hodge-Tate cohomology* of R denoted $\overline{\widehat{\mathbb{\Delta}}}_R\{*\}$.

To prove that these definitions specialise to those of [BMS19], one argues that, for Grothendieck topology reasons, one can restrict to looking at qrsp rings. In this case, Bhatt–Morrow–Scholze showed that their filtrations were just double-speed Postnikov filtrations and that THH, TP, and TC^- were all even, and therefore agree.

If k is something called a *cyclotomic-base*, for example k is $\mathbb{S}$, we can also define:

Definition 12.10.

$$\mathrm{fil}^*_{\mathrm{mot}}\,\mathrm{TC}(R/k) := \mathrm{fil}^*_{\mathrm{ev,TC}}\,\mathrm{THH}(R/k).$$

The associated graded for $\mathrm{fil}^*_{\mathrm{mot}}\,\mathrm{TC}(R)$ is called the *syntomic cohomology* of R.

12.3 Understanding $\mathrm{gr}^*_{\mathrm{mot}}(\mathrm{TC}^-(\mathbb{S}_{(p)}))$

Next we begin to study $\mathrm{gr}^*_{\mathrm{mot}}(\mathrm{TC}^-(\mathbb{S}_{(p)}))$. Using our descent for $\mathrm{fil}_{\mathrm{ev,h}S^1}$ plus Novikov descent, there is an equivalence

$$\pi_*\left(\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\mathbb{S}_{(p)})\right) \xrightarrow{\sim} \lim_{\Delta}\,\pi_*\left(\left(\mathrm{THH}(\mathbb{S}_{(p)}/\mathrm{MU}_{(p)}^{\otimes\bullet+1})\right)^{\mathrm{h}S^1}\right)$$

We refer to the right hand side as the cobar complex for $\mathrm{TC}^-(\mathbb{S}_{(p)})$.

The complex orientation for BP can be represented as a class $t \in \pi_{-2}\mathrm{BP}^{\mathrm{h}S^1}$ by identifying the E_2-page of the homotopy fixed point spectral sequence with $\mathrm{BP}^*\mathbb{CP}^\infty \simeq \pi_{-*}\mathrm{BP}[\![t]\!]$. Thus by considering the composition

$$\mathrm{BP}^{\mathrm{h}S^1} \to \mathrm{MU}_{(p)}^{\mathrm{h}S^1} \longrightarrow \mathrm{TC}^-(\mathbb{S}_{(p)}/\mathrm{MU}_{(p)}),$$

where the second map is the homotopy fixed points of the natural map

$$\mathrm{MU}_{(p)} \to \mathrm{THH}(\mathbb{S}_{(p)}/\mathrm{MU}_{(p)}) \simeq \mathrm{THH}(\mathbb{S}_{(p)}) \otimes_{\mathrm{THH}(\mathrm{MU}_{(p)})} \mathrm{MU}_{(p)},$$

we obtain a class, which we also call t, in the cobar complex for $\mathrm{TC}^-(\mathbb{S}_{(p)})$. We saw this yesterday in Talk 10. More generally we have a map of cobar complexes

$$\pi_*\left(\mathrm{BP}^{\otimes\bullet+1}\right)^{\mathrm{h}S^1} \to \pi_*\left(\left(\mathrm{THH}(\mathbb{S}_{(p)}/\mathrm{MU}_{(p)}^{\otimes\bullet+1})\right)^{\mathrm{h}S^1}\right).$$

To understand $\mathrm{gr}^*_{\mathrm{mot}}(\mathrm{TC}^-(\mathbb{S}_{(p)}))$ it will be important to understand more about this class t.

The cobar complex $\pi_*\left(\mathrm{BP}^{\otimes\bullet+1}\right)^{\mathrm{h}S^1}$ is determined, in part, by the hopf algebroid structure for $(\mathrm{BP}^*\mathbb{CP}^\infty, (\mathrm{BP}\otimes\mathrm{BP})^*(\mathbb{CP}^\infty))$. In particular the structure maps

$$\mathrm{BP}_*[\![t]\!] \underset{\eta_R}{\overset{\eta_L}{\rightrightarrows}} \mathrm{BP}_*\mathrm{BP}[\![t]\!] \simeq \mathrm{BP}_*[t_1, t_2, \ldots][\![t]\!]$$

extending the left and right unit for the Hopf algebroid $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$. We extend the conjugation self map c on $\mathrm{BP}_*\mathrm{BP}$ to $\mathrm{BP}_*\mathrm{BP}[\![t]\!]$ via $c(t) = t$. We would like to describe $\eta_R(t)$ as this will help us to be able to calculate differential for the cobar complex for, e.g. $\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-$. We will see this next up in the talk 13.

We have

$$\eta_R(t) = c(t +_{\mathbb{G}} t_1 t^p +_{\mathbb{G}} +t_2 t^{p^2} +_{\mathbb{G}} \cdots)$$

where $\mathbb{G}$ is the universal p-typical formal group law of BP_* [Wil82, Lemma 3.14].

13 Prismatic cohomology of $\mathbb{Z}$ and ℓ (Shai Keidar)

Notes by Anish Chedalavada

In this talk we compute the prismatic cohomology of $\mathbb{Z}$ and ℓ mod (p, v_1) or (p, v_1, v_2) respectively, and eventually TC^- and TP of $\mathbb{Z}$ and ℓ. Recall that we have computed

$$\mathrm{THH}_*(\mathbb{Z}/\mathrm{MU}) \simeq \mathbb{Z}[w_{1,i} \mid i \geq 0] \otimes \mathbb{Z}[y_{j,i} \mid j \neq 1 \bmod p,\ i \geq 0]$$

and

$$\mathrm{THH}_*(\ell/\mathrm{MU}) \simeq \ell_*[w_{2,i} \mid i \geq 0] \otimes_{\ell_*} \ell_*[y_{j,i} \mid j \neq 1 \bmod p,\ i \geq 0]$$

in the previous talk. The even filtrations

$$\begin{aligned}
&\mathrm{fil}^*_{\mathrm{mot}} \mathrm{THH}(\mathbb{Z}) \simeq \mathrm{Tot}(\tau_{\geq 2*} \mathrm{THH}(\mathbb{Z}/\mathrm{MU}^{\otimes \bullet+1})),\\
&\mathrm{fil}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathbb{Z}) \simeq \mathrm{Tot}(\tau_{\geq 2*} \mathrm{THH}(\mathbb{Z}/\mathrm{MU}^{\otimes \bullet+1})^{hS^1}),\\
&\mathrm{fil}^{\square}_{+} \mathrm{fil}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathbb{Z}) \simeq \mathrm{Tot}(\tau_{\geq 2\square}(\tau_{\geq 2*} \mathrm{THH}(\mathbb{Z}/\mathrm{MU}^{\otimes \bullet+1})^{hS^1})),
\end{aligned}$$

give rise to *algebraic t-Böckstein spectral sequences* for $A = \mathbb{Z}, \ell$ as below.

$$\pi_*(\mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(A))[t] \Rightarrow \mathrm{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(A)$$

and

$$\pi_*(\mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(A))[t^{\pm 1}] \Rightarrow \mathrm{gr}^*_{\mathrm{mot}} \mathrm{TP}(A).$$

13.1 Case $A = \mathbb{Z}$

The E_1 terms are explicitly given as:

$$\mathrm{THH}_*(\mathbb{Z})/p \simeq \mathbb{F}_p[\sigma^2 v_1] \otimes \Lambda(\lambda)$$

with degrees $|\sigma^2 v_1| = 2p$ and $|\lambda| = 2p - 1$. Owing to the regularity of the action of t on the cobar complex computing $\mathrm{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathbb{Z})$[8], the t-adic filtration on $(\mathrm{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathbb{Z}))/(p, v_1)$ may be computed by quotienting (p, v_1) in each graded component[9]. Since $t\sigma^2 v_1$ detects v_1, the E_1-pages of these spectral sequences become

$$\mathbb{F}_p[\sigma^2 v_1, t]/(t\sigma^2 v_1) \otimes \Lambda(\lambda) \Rightarrow \mathrm{gr}^{\square}_{\mathrm{mot}} \mathrm{TC}^-(\mathbb{Z})/p \tag{13.1}$$

and

$$\mathbb{F}_p[t^{\pm 1}] \otimes \Lambda(\lambda) \Rightarrow \mathrm{gr}^{\square}_{\mathrm{mot}} \mathrm{TP}(\mathbb{Z}). \tag{13.2}$$

We note that when displayed in Adams grading, the differentials above change bidegree $d_r : \pi_{a,b} \to \pi_{a-1,b+1}$.

Theorem 13.1. *The algebraic t-Boc̆kstein spectral sequence of 13.2 is determined by multiplicative structure along with the following facts:*

(a) The classes t^p, μ, λ_1 are permanent cycles.

[8] resp. $\mathrm{gr}^*_{\mathrm{mot}} \mathrm{TP}(\ell)$
[9] resp. for $\mathrm{gr}^*_{\mathrm{mot}} \mathrm{TP}(\ell)/(p, v_1, v_2)$

(b) There is a d_p differential $d_p(t) \doteq t^{p+1}\lambda_i$.

The E_∞ page is $\mathbb{F}_p[t^{\pm p}] \otimes \Lambda(\lambda_1)$.

Remark 13.2. The unit map $\mathrm{BP} \to \mathrm{MU}_{(p)}$ via the Quillen idempotent supplies the composite

$$\pi_{2*}(\mathrm{BP}^{\otimes \bullet+1})^{hS^1} \to \pi_{2*}(\mathrm{MU}^{\otimes \bullet+1})^{hS^1} \to \pi_{2*}\,\mathrm{TP}(\ell/\mathrm{MU}^{\otimes \bullet+1})$$

whose totalization yields a map from the cobar complex for $(\mathrm{BP}^*\mathbb{CP}^\infty, (\mathrm{BP}\otimes\mathrm{BP})^*\mathbb{CP}^\infty)$ to the cobar complex computing $\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TP}(\ell)$. Using that this Hopf algebroid is associated to a flat cover of the universal p-typical formal group law, [Wil82, p. 3.14] deduces the following formula:

$$\eta_R(t) = c(t +_{\mathbb{G}} t_1 t^p +_{\mathbb{G}} t_2 t^{p^2} + \dots)$$

where $+_{\mathbb{G}}$ denotes the addition on the formal group law BP^{hS^1} and c denotes the conjugation action on $\mathrm{BP}_*\mathrm{BP}$. In *loc. cit.* this is used to derive the relation

$$\eta_R(t) = t - t^p t_1 \mod p, v_1, t^{p+2}. \tag{13.3}$$

Proof of Theorem 13.1. We follow the proof in [HRW22, p. 6.5.1]. We first note that the relation of 13.3 implies that $\eta_R(t^{p^k}) = t^{p^k} - t^{p^{k+1}} t_1^{p^k} \mod (I_n, t^{p^k+2})$. The claimed differentials of part (2) follow from the fact that $t^p t_1 = t^{p+1}\sigma^2 t_1$ and that $(\sigma^2 t)^{p^i}$ is detected by λ_i for $1 \le i \le n$. Furthermore, we have that

$$\eta_R(t^{p^{n+1}}) = t^{p^{n+1}} + t^{p^{n+2}} t_1^{p^2} = t^{p^{n+1}} + t^{p^{n+2}+p^{n+1}}(\sigma^2 t_1)^{p^{n+1}} \mod (I_n, t^{p^{n+2}+2p^{n+1}}).$$

In particular, $\eta_R(t^{p^{n+1}}) = t^{p^{n+1}} \mod t^{p^{n+2}+p^{n+1}}$, so $t^{p^{n+1}}$ must survive to the $E_{p^{n+1}}$ page.

For the first claimed cycle in (1), note that any recipient of a differential from $t^{p^{n+1}}$ must live in Adams weight 1: the only candidates are of the form d_{p^i} □

Notes from the rest of this talk are unfortunately not available. Here is the description from the preliminary syllabus:

- Calculate, via the algebraic Tate spectral sequence [HRW22, §6.5], the mod (p, v_1) prismatic cohomology of $\mathbb{Z}$ and the mod (p, v_1, v_2) prismatic cohomology of ℓ (the two cases are analogous and we leave the speaker to decide how to handle them; for instance, it would be fine to only do one of the cases).
- Conclude also the bigraded homotopy groups of $\pi_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\mathbb{Z})/(p, v_1)$ and $\pi_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\ell)/(p, v_1, v_2)$.

14 Syntomic cohomology of $\mathbb{Z}$ and ℓ (Preston Cranford)

Notes by Preston Cranford, edited by Joe Hlavinka and Atticus W.

In this talk we calculate the (p, v_1, v_2) syntomic cohomology of the p-completed connective Adams summand $\ell_p^\wedge$. Then we discuss three features of the calculation: Tate duality, collapsing of v_2-Bockstein spectral sequence, and satisfaction of the Lichtenbaum–Quillen conjecture.

Fix a prime p. Let ℓ denote the p-completed connective Adam's summand $\ell_p^\wedge$, the ring-spectrum with it's usual $\mathbb{E}_\infty$-structure.

Before we calculate the (p, v_1, v_2) syntomic cohomology of ℓ, we start by defining syntomic cohomology of suitable $\mathbb{E}_\infty$-rings, chromatically quasisyntomic $\mathbb{E}_\infty$ rings. For the statement of the definition we recall that

Definition 14.1. [HRW22, Section 1] For R a chromatically p-quasisyntomic $\mathbb{E}_\infty$-ring, the syntomic cohomology of R is the associated graded of $\mathrm{fil}_{\mathrm{mot}}\mathrm{TC}(R)_p^\wedge = \mathrm{fil}_{\mathrm{ev},p,\mathrm{TC}}\mathrm{THH}(R)_p^\wedge$.

Now that we have defined syntomic cohomology of chromatically p-quasisyntomic $\mathbb{E}_\infty$-rings, we define mod $(p, v_1, \ldots, v_n)$ syntomic cohomology to be the mod $(p, v_1, \ldots, v_n)$ reduction of syntomic cohomology.

Recalling from a previous talk that ℓ is chromatically p-quasisyntomic, we now begin calculating it's mod (p, v_1, v_2) syntomic cohomology. Since we've already calculated what the canonical map

$$\left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}^-(\ell)\right)/(p, v_1, v_2) \xrightarrow{''can"} \left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell)\right)/(p, v_1, v_2)$$

looks like on the level of homotopy groups in Talk 4.3, it only remains to understand what the Frobenius map is on the level of homotopy groups. The main idea of the proof is to show that the mod (p, v_1, v_2) Frobenius

$$\left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}^-(\ell)\right)/(p, v_1, v_2) \xrightarrow{\varphi} \left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell)\right)/(p, v_1, v_2)$$

is given by the mod (p, v_1) Frobenius

$$\left(\mathrm{THH}(\ell)\right)/(p, v_1) \xrightarrow{\varphi} \left(\mathrm{THH}(\ell)^{\mathrm{tC}_p}\right)/(p, v_1)$$

as we recall now.

Lemma 14.2. *[Talk 4.1] The mod $(p, v_1, \ldots, v_n)$ Frobenius map $\pi_*\left(\mathrm{THH}(BP\langle n\rangle) \otimes_{BP\langle n\rangle} \mathbb{F}_p\right) \xrightarrow{\varphi} \pi_*\left(\mathrm{THH}(BP\langle n\rangle)^{\mathrm{tC}_p} \otimes_{BP\langle n\rangle} \mathbb{F}_p\right)$ is identified with the ring map*

$$\Lambda(\lambda_1, \lambda_2, \ldots, \lambda_n) \otimes \mathbb{F}_p[\mu] \to \Lambda(\lambda_1, \lambda_2, \ldots, \lambda_n) \otimes \mathbb{F}_p[\mu^{\pm 1}]$$

that inverts the class μ.

Now that we've recalled Lemma 14.2 just for the case $n = 1$ so that $BP\langle n\rangle = \ell$ we are ready to compute the mod (p, v_1, v_2) Frobenius in Lemma 14.3.

Lemma 14.3. *[HRW22, Corollary 6.6.1] In terms of the isomorphisms*

$$\left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}^-(\ell)\right)/(p,v_1,v_2) \cong \mathbb{F}_p[t^{p^2},\mu]/(t^{p^2}\mu)\otimes\Lambda(\lambda_1,\lambda_2)$$
$$\oplus\,\mathbb{F}_p\{t^d\lambda_1, t^{pd}\lambda_2, t^d\lambda_1\lambda_2, t^{pd}\lambda_1\lambda_2 \mid 0<d<p\},$$

$$\left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell)\right)/(p,v_1,v_2) \cong \mathbb{F}_p[t^{\pm p^2}]\otimes\Lambda(\lambda_1,\lambda_2)$$

calculated in a previous talk, the Frobenius is trivial on classes not of the form $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^k$ where $k\geq 0$ and $\epsilon_1,\epsilon_2\in\{0,1\}$ and sends each class of the form $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^k$ to an $\mathbb{F}_p^\times$ multiple of the class named $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}t^{-p^2k}$.

Proof. Consider the diagram

$$\begin{array}{ccc} \mathrm{TC}^-(\ell) & \xrightarrow{\varphi} & \mathrm{TP}(\ell) \\ \downarrow & & \downarrow \\ \mathrm{THH}(\ell) & \xrightarrow{\varphi} & \mathrm{THH}(\ell)^{\mathrm{tC}_p} \end{array}$$

with the left vertical arrow given by the unit map. The diagram commutes by the Tate orbit lemma. Modding out by (p,v_1) gives the commuting diagram

$$\begin{array}{ccc} \left(\mathrm{TC}^-(\ell)\right)/(p,v_1) & \xrightarrow{\varphi} & \left(\mathrm{TP}(\ell)\right)/(p,v_1) \\ \downarrow & & \downarrow \\ \left(\mathrm{THH}(\ell)\right)/(p,v_1) & \xrightarrow{\varphi} & \left(\mathrm{THH}(\ell)^{\mathrm{tC}_p}\right)/(p,v_1). \end{array}$$

Since we understand the Frobenius bottom horizontal map, we have an oppurtunity to understand the Frobenius top horizontal map. We will do this using the fact that the Frobenius preserves the motivic filtration, as stated in the following black-boxed lemma:

Lemma 14.4. *[HRW22, Theorem 1.3.6] Let R be a chromatically quasisyntomic $\mathbb{E}_\infty$-ring. Then, for each prime number p, the Nikolaus–Scholze Frobenius*

$$\varphi : \mathrm{TC}^-(R)^\wedge_p \to \mathrm{TP}(R)^\wedge_p$$

refines to a natural map

$$\varphi : \mathrm{fil}^\star_{\mathrm{mot}}\mathrm{TC}^-(R)^\wedge_p \to \mathrm{fil}^\star_{\mathrm{mot}}\mathrm{TP}(R)^\wedge_p.$$

The same is true of the canonical map between the same objects, and $\mathrm{fil}^\star_{\mathrm{mot}}\mathrm{C}_p l\mathrm{TC}(R)_p$ can be computed as the equalizer of the filtered Frobenius and canonical maps.

Taking mod (p,v_1) reductions, we get the following commutative diagram

$$\begin{array}{ccc} \left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}^-(\ell)\right)/(p,v_1) & \xrightarrow{\varphi} & \left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell)\right)/(p,v_1) \\ \downarrow & & \downarrow \\ \left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{THH}(\ell)\right)/(p,v_1) & \xrightarrow{\varphi} & \left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{THH}(\ell)^{\mathrm{tC}_p}\right)/(p,v_1). \end{array}$$

We argue that the above diagram factors through one of the form

$$\begin{array}{ccc} \left(\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{TC}^{-}(\ell)\right)/(p,v_1,v_2) & \xrightarrow{\varphi} & \left(\mathrm{gr}^{*}_{\mathrm{mot}}\mathrm{TP}(\ell)\right)/(p,v_1,v_2) \\ \downarrow f & & \downarrow g \\ \left(\mathrm{gr}^{*}_{\mathrm{mot}}\mathrm{THH}(\ell)\right)/(p,v_1) & \xrightarrow{\varphi} & \left(\mathrm{gr}^{*}_{\mathrm{mot}}\mathrm{THH}(\ell)^{\mathrm{tC}_p}\right)/(p,v_1). \end{array}$$

Note that $v_2 = 0$ in $(\mathrm{gr}^{\star}_{\mathrm{ev}}\,\ell)/(p,v_1)$, so the sequence of algebra maps

$$\mathrm{gr}^{\star}_{\mathrm{ev}}\,\ell \to \mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{THH}(\ell) \xrightarrow{\varphi} \mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{THH}(\ell)^{\mathrm{tC}_p}$$

imply that $v_2 = 0$ in $\left(\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{THH}(\ell)^{\mathrm{tC}_p}\right)/(p,v_1)$. Thus, the natural map

$$\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{TP}(\ell)/(p,v_1) \to \mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{THH}(\ell)^{\mathrm{tC}_p}/(p,v_1)$$

factors over a map

$$g : \mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{TP}(\ell)/(p,v_1,v_2) \to \mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{THH}(\ell)^{\mathrm{tC}_p}/(p,v_1).$$

Now that we have the Frobenius map which we wish to understand in a commuting diagram with a Frobenius map that we do understand, we can start studying the former. We leave the proof of g being an isomorphism to the next Lemma 14.5. The map f is trivial on every class not of the form $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^k$. Lemma 14.2 and g being an isomorphism (Lemma 14.5) together imply that each class of the form $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^k$ has non-trivial Frobenius image. The only non-trivial classes in the codomain in the same degree as $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^k$, are $\mathbb{F}_p^{\times}$ multiples of the class named $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}t^{-p^2k}$. □

The above proof is complete except for the justification of the map g being isomorphism. We justify this now.

Lemma 14.5. *[HRW22, Theorem 6.4.1] The map g above is an isomorphism*

$$\left(\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{TP}(\ell)\right)/(p,v_1,v_2) \xrightarrow{\cong} \left(\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{THH}(\ell)^{\mathrm{tC}_p}\right)/(p,v_1)$$

Proof. We can compute the motivic associated graded for $\mathrm{TP}(\ell)$ via the cobar complex associated to the descent

$$TP(\ell) \to \mathrm{TP}(\ell/\mathrm{MU})$$

with sth term given by $\pi_*\,\mathrm{TP}(\ell/MU^{s+1})$. The domain of g is calculated by the complex obtained from modding out each term $\pi_*\,\mathrm{TP}(\ell/\mathrm{MU}^{\otimes s+1})$ by p, v_1 and v_2. The codomain is obtained from the complex $\pi_*\mathrm{THH}(\ell/\mathrm{MU}^{\otimes \bullet+1})^{\mathrm{tC}_p}$ by levelwise killing p and v_1. We first note that, for each value of $s \geq 0$, $v_2 = 0$ in $\left(\pi_*\mathrm{THH}(\ell/\mathrm{MU}^{\otimes s+1})^{\mathrm{tC}_p}\right)/(p,v_1)$. This can be seen from the existence of the relative cyclotomic Frobenius map

$$\pi_*\mathrm{THH}(\ell/\mathrm{MU}^{\otimes s+1})/(p,v_1) \to \pi_*\mathrm{THH}(\ell/\mathrm{MU}^{\otimes s+1})^{\mathrm{tC}_p}/(p,v_1),$$

because $v_2 \equiv 0$ modulo (p,v_1) in any $\pi_*\ell$-algebra such as $\pi_*\mathrm{THH}(\ell/\mathrm{MU}^{\otimes s+1})$. It follows that g extends to a map of cobar complexes which levelwise is of the form

$$g_s : \left(\pi_*\,\mathrm{TP}(\ell/\mathrm{MU}^{\otimes s+1})\right)/(p,v_1,v_2) \to \left(\pi_*\mathrm{THH}(\ell/\mathrm{MU}^{\otimes s+1})^{\mathrm{tC}_p}\right)/(p,v_1).$$

To prove that g is an isomorphism, we will prove the stronger claim that g_s is an isomorphism for each $s \geq 0$. To see that each g_s is an isomorphism we first the Lemma 14.6 below that the group $\left(\pi_*\mathrm{THH}(\ell/\mathrm{MU}^{\otimes s+1})^{\mathrm{tC}_p}\right)/(p,v_1,v_2)$ can be computed from $\left(\pi_*\,\mathrm{TP}(\ell/\mathrm{MU}^{\otimes s+1})\right)/(p,v_1)$ by killing $[p](t)$ for any complex orientation t.

Lemma 14.6. *[HRW22, Lemma 6.4.2] Let* $M \in \mathrm{Mod}_{\mathrm{MU}}^{\mathrm{BS}^1}$ *be an* S^1*-equivariant* MU*-module. Then the map*

$$M^{\mathrm{tS}^1}/[p](t) = M^{\mathrm{tS}^1} \otimes_{\mathrm{MU}^{\mathrm{tS}^1}} \mathrm{MU}^{\mathrm{tC}_p} \to M^{\mathrm{tC}_p}$$

is an equivalence. In paticular for $\left(\pi_* \mathrm{THH}(\ell/\mathrm{MU}^{\otimes s+1})^{\mathrm{tC}_p}\right)/(p, v_1) \in \mathrm{Mod}_{\mathrm{MU}}^{\mathrm{BS}^1}$*, we have an equivalence*

$$\left(\pi_* \mathrm{THH}(\ell/\mathrm{MU}^{\otimes s+1})^{\mathrm{tC}_p}\right)/(p, v_1) \cong \left(\left(\pi_* \mathrm{THH}(\ell/\mathrm{MU}^{\otimes s+1})^{\mathrm{t}S^1}\right)/(p, v_1)\right)/([p](t))$$

Proof. This argument largely follows [NS18, p. IV.4.12] Since $\mathrm{MU}^{\mathrm{tC}_p} = \mathrm{MU}^{\mathrm{tS}^1}/[p](t)$ is a perfect $\mathrm{MU}^{\mathrm{tS}^1}$-module as seen in the following sequence

$$\mathrm{MU}^{\mathrm{tC}_p} \to \mathrm{MU}^{\mathrm{tS}^1} \xrightarrow{[p](t)} \mathrm{MU}^{\mathrm{tS}^1}.$$

Therefore, the functor $(-) \otimes_{\mathrm{MU}^{\mathrm{tS}^1}} \mathrm{MU}^{\mathrm{tC}_p}$ commutes with all limits and colimits. Using the equivalences $M^{tG} = \mathrm{colim}(\tau_{\geq n} M)^{tG}$ and $M^{tG} = \lim(\tau_{\leq n} M)^{tG}$ for $G = C_p$ it is sufficient to assume that M is bounded. By filtering M by it's Postnikov filtration it suffices to assume that M is discrete. For M is discrete then it's S^1-action is trivial because BS^1 is connected and

$$\mathrm{Fun}(BS^1, \mathrm{Aut}(M)) = \mathrm{Fun}(BS^1, \mathrm{Aut}(\pi_0(M)))$$

By a similar argument one can see that its induced C_p-action is trivial too. Now that we've reduced to discrete S^1-action case, we show for M with a trivial S^1-action that

$$M^{\mathrm{tS}^1}/[p](t) = M^{\mathrm{tS}^1} \otimes_{\mathrm{MU}^{\mathrm{tS}^1}} \mathrm{MU}^{\mathrm{tC}_p} \to M^{\mathrm{tC}_p}$$

as desired. Since M has a trivial S^1-action we have that $A^{hS^1} = A^{BS^1}$. But since M is discrete it also has trivial C_p-action, so likewise $A^{hC_p} = A^{BC_p}$. Now consider the fiber sequence

$$S^1 \to BC_p \to BS^1 \xrightarrow{d \to d^{\otimes p}} BS^1.$$

of spaces. Dualizing and lifting to spectra gives a cofiber (=fiber) sequence

$$\Sigma^\infty BC_p \to \Sigma^\infty BS^1 \to (BS^1)^\wedge.$$

Homming from M then determines a fiber (=cofiber) sequence

$$M^{BC_p} \leftarrow M^{BS^1} \leftarrow M^{(BS^1)^\wedge}.$$

Because M is a MU-module, there is a Thom isomoprhism $M^{(BS^1)^\wedge} \cong \Sigma^{-2} M^{BS^1}$. Note that

$$(BS^1)^\wedge = \mathrm{cofib}(S(V) \to D(V))$$

and $D(V) \cong \mathrm{CP}^\infty$. Finally, the map

$$\Sigma^{-2} M^{BS^1} \to M^{BC_p}$$

can be identified with the map $M[[t]] \to M[[t]]$ sending 1 to the Euler characteristic $p[t]$. This proves

$$M^{h\mathrm{S}^1}/[p](t) = M^{h\mathrm{S}^1} \otimes_{\mathrm{MU}^{h\mathrm{S}^1}} \mathrm{MU}^{h\mathrm{C}_p} \to M^{h\mathrm{C}_p}$$

is an equivalence, and the analogous statement with $\mathrm{t}S^1$ and $\mathrm{t}C_p$ in place of $h\mathrm{S}^1$ and $\mathrm{t}C_p$, respectively, follows from inverting t.

□

We finish by using the black-boxed Lemma 14.7

Lemma 14.7. *[HRW22, Section 6] We have that v_2 is a unit multiple of $[p](t)$.*

Finally knowing that $v_2 = cp[t](t)$ for a unit c allows us to conclude that

$$\begin{aligned}\left(\left(\pi_*\mathrm{THH}(\ell/\mathrm{MU}^{\otimes s+1})^{\mathrm{t}S^1}\right)/(p,v_1)\right)/([p](t)) &= \left(\left(\pi_*\mathrm{THH}(\ell/\mathrm{MU}^{\otimes s+1})^{\mathrm{t}S^1}\right)/(p,v_1)\right)/(cv_2)\\ &= \left(\left(\pi_*\mathrm{THH}(\ell/\mathrm{MU}^{\otimes s+1})^{\mathrm{t}S^1}\right)/(p,v_1)\right)/(v_2)\\ &= \left(\left(\pi_*\mathrm{THH}(\ell/\mathrm{MU}^{\otimes s+1})^{\mathrm{t}S^1}\right)/(p,v_1,v_2)\right)\end{aligned}$$

as desired. □

Now that we know the Frobenius, we are ready for the syntomic cohomology calculation.

Theorem 14.8. *[HRW22, Theorem 6.0.4] The mod (p, v_1, v_2) syntomic cohomology of ℓ is a finite $\mathbb{F}_p$-vector space. As a vector space, it is isomorphic to*

(a) $\mathbb{F}_p\{1\}$*, in Adams weight* 0 *and degree* 0*.*

(b) $\mathbb{F}_p\{\partial, t^d\lambda_1, t^{dp}\lambda_2 \mid 0 \le d < p\}$*, in Adams weight* 1*. Here,* $|\partial| = -1$*,* $|t^d\lambda_1| = 2p-2d-1$*, and* $|t^{dp}\lambda_2| = 2p^2 - 2dp - 1$*.*

(c) $\mathbb{F}_p\{t^d\lambda_1\lambda_2, t^{dp}\lambda_1\lambda_2, \partial\lambda_1, \partial\lambda_2 \mid 0 \le d < p\}$*, in Adams weight* 2*. Here,* $|t^d\lambda_1\lambda_2| = 2p^2 - 2p - 2d - 2$*,* $|t^{dp}\lambda_1\lambda_2| = 2p^2 - 2p - 2dp - 2$*,* $|\partial\lambda_1| = 2p - 2$*, and* $|\partial\lambda_2| = 2p^2 - 2$*.*

(d) $\mathbb{F}_p\{\partial\lambda_1\lambda_2\}$*, in Adams weight* 3 *and degree* $2p^2 + 2p - 3$*.*

Proof. By definition of TC, we have the fiber sequence

$$\mathrm{TC}(\ell) \to \mathrm{TC}^-(\ell) \xrightarrow{\varphi - \mathrm{can}} \mathrm{TP}(\ell),$$

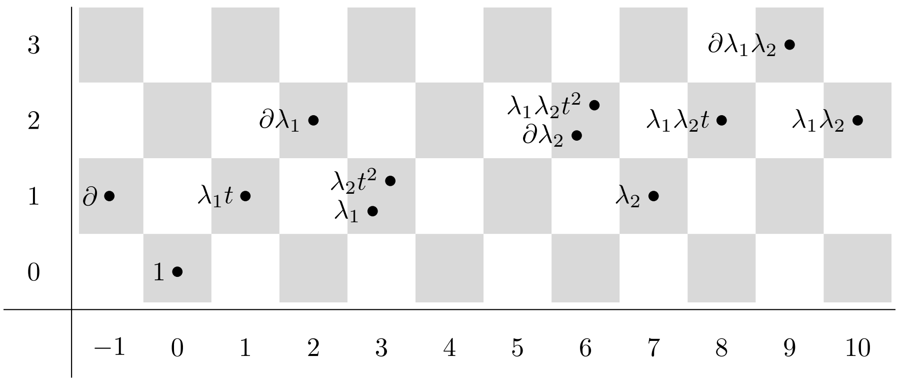
and since the motivic filtration preserving the fiber sequence, we have a fiber sequence

$$\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(\ell) \to \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\ell) \xrightarrow{\varphi - \mathrm{can}} \mathrm{gr}_{\mathrm{mot}} * \mathrm{TP}(\ell).$$

By taking appropiate reductions, we have the induced fiber sequence

$$\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(\ell)/(p,v_1,v_2) \to \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\ell)/(p,v_1,v_2) \xrightarrow{\varphi - \mathrm{can}} \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TP}(\ell)/(p,v_1,v_2).$$

The mod $(2, v_1, v_2)$-syntomic cohomology of $\ell = \mathrm{ku}$



which induces a long exact sequence

$$\begin{aligned}\cdots \to \operatorname{gr}^{*-1}_{\mathrm{mot}} \mathrm{TP}(\ell)/(p, v_1, v_2) &\to \operatorname{gr}^{*}_{\mathrm{mot}} \mathrm{TC}(\ell)/(p, v_1, v_2) \\ \to \operatorname{gr}^{*}_{\mathrm{mot}} \mathrm{TC}^{-}(\ell)/(p, v_1, v_2) &\xrightarrow{\varphi - \mathrm{can}} \operatorname{gr}^{*}_{\mathrm{mot}} \mathrm{TP}(\ell)/(p, v_1, v_2) \to \cdots\end{aligned}$$

of groups which we will use to deduce the result. The map

$$\operatorname{gr}^{*}_{\mathrm{mot}} \mathrm{TC}^{-}(\ell)/(p, v_1, v_2) \xrightarrow{\varphi - \mathrm{can}} \operatorname{gr}^{*}_{\mathrm{mot}} \mathrm{TP}(\ell)/(p, v_1, v_2)$$

is given by the rule that classes of the form $\lambda_1^{e_1}\lambda_2^{e_2}t^{kp^2} \in \operatorname{gr}^{*}_{\mathrm{mot}} \mathrm{TC}^{-}(\ell)/(p, v_1, v_2)$ go to $-\lambda_1^{e_1}\lambda_2^{e_2}t^{kp^2} \in \operatorname{gr}^{*}_{\mathrm{mot}} \mathrm{TP}(\ell)/(p, v_1, v_2)$, classes of the form $\lambda_1^{e_1}\lambda_2^{e_2}t\mu^{k} \in \operatorname{gr}^{*}_{\mathrm{mot}} \mathrm{TC}^{-}(\ell)/(p, v_1, v_2)$ go to a $\mathbb{F}_p^{\times}$-multiple of $\lambda_1^{e_1}\lambda_2^{e_2}t^{-kp^2} \in \operatorname{gr}^{*}_{\mathrm{mot}} \mathrm{TP}(\ell)/(p, v_1, v_2)$, and all other classes go to zero. The kernel of this map gives the terms in the syntomic cohomology calculation that don't have λ_i's in them. The cokernel is given by

$$\frac{\operatorname{gr}^{*}_{\mathrm{mot}} \mathrm{TP}(\ell)/(p, v_1, v_2)}{\operatorname{im}(\varphi - \mathrm{can})} \cong \frac{\mathbb{F}_p[t^{p^2}] \otimes \Lambda(\lambda_1, \lambda_2)}{F_p[t^{p^2}]} \cong \Lambda(\lambda_1, \lambda_2).$$

This contributetes the basis elements with a ∂ coming from the degree shift.

□

Now we discuss three consequences of the syntomic cohomology calculation. The first is the following.

Proposition 14.9. *[HRW22, Theorem 6.0.4] The v_2-Bockstein spectral sequence (converging to the mod (p, v_1) syntomic cohomology of ℓ as an $\mathbb{F}_p[v_2]$-module) collapses with no differentials. As a consequence, we've calculated $\operatorname{gr}^{*}_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p, v_1)$ as a $\mathbb{F}_p[v_2]$-vector space.*

Proof. Recall that the v_2-Bockstein spectral sequence (converging to the mod (p, v_1) syntomic cohomology of ℓ as an $\mathbb{F}_p[v_2]$-module) has signature

$$\pi_{*,*}\left(\left(gr^{*}_{\mathrm{mot}}\left(\mathrm{TC}(\ell)\right)/(p, v_1)\right)/(v_2)\right)[v_2] \to \left(gr^{*}_{\mathrm{mot}}\left(\mathrm{TC}(\ell)\right)/(p, v_1)\right)$$

which can be rewritten

$$\pi_{*,*}\left(gr^*_{\text{mot}}\left(\text{TC}(\ell)\right)/(p, v_1, v_2)\right)[v_2] \to \left(gr^*_{\text{mot}}\left(\text{TC}(\ell)\right)/(p, v_1)\right)$$

due to regularity. Since we know $gr^*_{\text{mot}}\left(\text{TC}(\ell)\right)/(p, v_1, v_2)$ and that $|v_2| = (2p^2 - 2, 0)$ we have the following description of the E_1 page of the spectral sequence.

We have $\pi_{*,*}\left(gr^*_{\text{mot}}\left(\text{TC}(\ell)\right)/(p, v_1, v_2)\right)[v_2]$ as an $\mathbb{F}_p$ vector space isomorphic to

(a) $\mathbb{F}_p\{v_2^e \mid 0 \le d < p, e \ge 0\}$, in Adams weight 0 and degree $(2p^2 - p)e$.

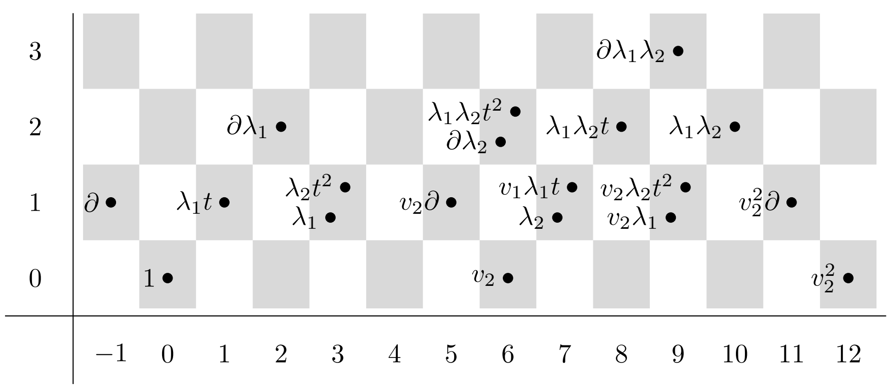
(b) $\mathbb{F}_p\{v_2^e\partial, v_2^e t^d\lambda_1, v_2^e t^{dp}\lambda_2 \mid 0 \le d < p, e \ge 0\}$, in Adams weight 1. Here, $|\partial| = -1$, $|t^d\lambda_1| = 2p - 2d - 1$, and $|t^{dp}\lambda_2| = 2p^2 - 2dp - 1$.

(c) $\mathbb{F}_p\{v_2^e t^d\lambda_1\lambda_2, v_2^e t^{dp}\lambda_1\lambda_2, v_2^e\partial\lambda_1, v_2^e\partial\lambda_2 \mid 0 \le d < p, e \ge 0\}$, in Adams weight 2. Here, $|t^d\lambda_1\lambda_2| = 2p^2 - 2p - 2d - 2$, $|t^{dp}\lambda_1\lambda_2| = 2p^2 - 2p - 2dp - 2$, $|\partial\lambda_1| = 2p - 2$, and $|\partial\lambda_2| = 2p^2 - 2$.

(d) $\mathbb{F}_p\{v_2^e\partial\lambda_1\lambda_2, e \ge 0\}$, in Adams weight 3 and degree $2p^2 + 2p - 3$.

We argue that there are no non-trivial differentials. In a Bockstein spectral sequence, the differentials are trigraded. The differentials increase Adams weight by 1 and must increase v_2-degree by a positive amount. Differentials increases v_2-degree by at least 1, Adams weight by 1, and degree by -1. Any differential increasing v_2-degree by $b \ge 1$ map $v_2^a x_1$ to $v_2^{a+2}x_2$ for some $a \ge 0$ and x_1, x_2 having no factor of v_2 where $\deg(v_2^{a+b}x_2) - \deg(v_2^a x_1) = -1$. This implies $\deg(x_2) - \deg(x_1) = b(2p^2 - 2) - 1$. Thus $b < 2$ since the maximum $\deg(x_2) - \deg(x_1)$ that can occur is $2p^2 - 2$. Since $\deg(x_2) - \deg(x_1) = 2p^2 - 1$ cannot occur, b cannot be 1, so we conclude that all differentials collapse. ☐

The E_1-page of the v_2-Bockstein spectral sequence for $p = 2$



Now that we've stated the first observation, we state the second.

Proposition 14.10. *The syntomic cohomology calculation exhibits something that looks like Tate duality.*

Now that we've stated the first observation, we state the third.

Proposition 14.11. *[HRW22, Corollary 6.6.3] For any prime $p \ge 2$ and type 3 p-local finite complex F, $F_*\text{TC}(\ell)$ is finite.*

Proof. Let $\mathcal{C}$ denote the category of p-complete finite spectra V such that $V_*\mathrm{TC}(\ell)$ is finite. See that C is a thick subcategory of p-complete finite spectra: it contains 0 because $0_*\mathrm{TC}(\ell)$ is finite, it's closed under fibers and cofibers by observing the long exact sequence associated to a fiber or cofiber sequence, and retraction preserves tensoring with $\mathrm{TC}(\ell)$ and so will preserve finiteness.

By the Thick Subcategory Theorem, C must be $\mathcal{C}_{\geq n}$ for some $\geq n$, the category of finite p-local spectra of type $\geq n$. Fix (i, j, k) so that the generalized Moore spectrum $V := \mathbb{S}/(p^i, v_1^j, v_2^k)$ exists and thus is type 3. If we show that $V \in \mathcal{C}_{\geq n}$, it follows that $k \leq 3$ in which case $\mathcal{C}_{\geq 3} \subset \mathcal{C}$, that is, the proposition would be proven.

We show that $V_*\mathrm{TC}(\ell)$ is finite for V the type 3 complex $\mathbb{S}/(p^i, v_1^j, v_2^k)$. Note that (i, j, k) have been picked so that there is a motivic spectral sequence with signature

$$\mathrm{gr}^\star_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p^i, v_1^j, v_2^k) \Rightarrow V_*\mathrm{TC}(\ell)$$

.

Now we see $\mathrm{gr}^\star_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p^i, v_1^j, v_2^k)$ may be resolved by finitely many copies of $\mathrm{gr}^\star_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p, v_1, v_2)$ in the following way. We inductively for $1 \leq m < i$ exhibit the cofiber sequence

$$\mathrm{gr}^\star_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p^m, v_1, v_2) \xrightarrow{p} \mathrm{gr}^\star_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p^m, v_1, v_2) \to \mathrm{gr}^\star_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p^{m+1}, v_1, v_2)$$

giving a cofiber sequence

$$\mathbb{S}/(p^m, v_1, v_2)_*\mathrm{TC}(\ell) \to \mathbb{S}/(p^m, v_1, v_2)_*\mathrm{TC}(\ell) \to \mathbb{S}/(p^{m+1}, v_1, v_2)_*\mathrm{TC}(\ell)$$

where the first two terms are finite by inductive hypothesis so that the final term is also finite. Repeating this process for the powers of v_1 and v_2 completes the proof.

□

Finally we discuss the following corollary of the Proposition 14.11.

Corollary 14.12. *[HRW22, Theorem 6.6.4] The Litchenbaum-Quillen conjecture holds for* $\mathrm{TC}(\ell)$*, that is,*

$$\mathrm{TC}(\ell)_{(p)} \to L_2^f\mathrm{TC}(\ell)_{(p)}$$

is a π_**-iso for* $* \gg 0$.

Proof. Recall that the L_2^f-localization map $\mathbb{S} \to L_2^f\mathbb{S}$ fits into a cofiber sequence

$$C \to \mathbb{S} \to L_2^f\mathbb{S}$$

Such that C is a filtered colimit of objects of $C_{\geq 3}$. In the last Proposition 14.11, we proved that the objects $\mathcal{C}_{\geq 3}$ coincides with the spectra v such that $V_*\mathrm{TC}(\ell)$ is finite, thus C itself as the property that $C_*\mathrm{TC}(\ell)$ is finite. Applying $\mathrm{TC}(\ell)_*$ gives the fiber sequence

$$\mathrm{TC}(\ell) \to L_2^f\mathrm{TC}(\ell) \to \mathrm{TC}(\ell)_*V.$$

whose long exact sequence has $\mathrm{TC}(\ell) \to L_2^f\mathrm{TC}(\ell)$ must be an equivalence in degrees the highest degree appearing in $\mathrm{TC}(\ell)_*V$ plus one and onward. □

15 The relationship between TC and algebraic K-theory (Albert Jinghui Yang)

Notes by Kai Shaikh

15.1 The Cyclotomic Trace

We begin with the Waldhausen S-construction of K. Let $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{st}}$; we will define a stable infinity category $S_n\mathcal{C}$ for each $n \in \mathbb{N}$. Let $S_n\mathcal{C}$ have the objects sequences of maps:

$$* \to X_{0,1} \to X_{0,2} \to \ldots \to X_{0,n}$$

and choices of cofibers for each of the maps $X_{0,i} \hookrightarrow X_{0,j} \to X_{i,j}$, which are assembled into a diagram:

$$\begin{array}{ccccccccc}
* & \longrightarrow & X_{0,1} & \longrightarrow & X_{0,2} & \longrightarrow & \ldots & \longrightarrow & X_{0,n} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & * & \longrightarrow & X_{1,2} & \longrightarrow & \ldots & \longrightarrow & \ldots \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & * & \longrightarrow & \ldots & \longrightarrow & X_{n-1,n} \\
 & & & & & & & & \downarrow \\
 & & & & & & & & *
\end{array}$$

where each square is a pushout. The nerve of the category of such diagrams with compatible maps is then a stable infinity category, which we denote $S_n\mathcal{C}$. The $S_n\mathcal{C}$'s are merged into a bisimplicial set, whose diagonal corresponds to a stable infinity category, which we denote $S_\bullet\mathcal{C}$; we can now iterate this construction: Let $S_\bullet^{(n)}$ be the stable infinity category obtained by iterating this procedure n times and then taking the diagonal. Finally, we get the K-theory spectrum as

$$\mathrm{K}(\mathcal{C})_n = \left| \left(S_\bullet^{(n)}(\mathcal{C}) \right)^{\simeq} \right|$$

where $\simeq$ denotes the core of an infinity category. Then we have:

$$\Omega^\infty \mathrm{K}(\mathcal{C}) \cong \Omega \left| S_\bullet(\mathcal{C})^{\simeq} \right|.$$

We note that this recovers Quillen's Q-construction of algebraic K-theory. K as defined above upgrade to a lax symmetric monoidal functor

$$\mathrm{K}\colon \mathrm{Cat}^{\mathrm{st}}(\mathcal{C}) \to \mathrm{Sp}.$$

At this point we briefly digress to discuss idempotent completion.

We observe that idempotent complete stable infinity categories are closed under colimits, so the forgetful functor:

$$F\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Cat}^{\mathrm{st}}$$

admits a right adjoint, which we denote $\mathrm{Idem}(-)$. We now make the following definition.

Definition 15.1. $F \in \left(\mathrm{Cat}^{\mathrm{st}}\right)^{\Delta^1}$ is called a **Morita equivalence** if $\mathrm{Idem}(F)$ is an equivalence.

Definition 15.2. In $\mathrm{Cat}^{\mathrm{perf}}$, we call a sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{g} \mathcal{D}$ **exact** (**Karoubi**) if

(a) $g \circ f \simeq 0$,

(b) f is fully faithful, and

(c) $\mathcal{E} \simeq \mathcal{D}/\mathcal{C}$.

Definition 15.3. $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$ is called an **additive invariant** if:

(a) F inverts Morita equivalences,

(b) F preserves filtered colimits, and

(c) F sends split exact sequences to exact sequences.

Example 15.4. THH and K are additive invariants; TC does not preserve filtered colimits, so it is not.

Notation. The forgetful functor $\mathrm{PSh}^{\mathrm{add}}_{\mathrm{Sp}}(\mathrm{Cat}^{\mathrm{st}}) \hookrightarrow \mathrm{PSh}_{\mathrm{Sp}}(\mathrm{Cat}^{\mathrm{st}})$ has a left adjoint which we denote L^{add}. Denote $\mathcal{M}_{\mathrm{add}}$ by the composition of L^{add} with the Yoneda embedding $\mathrm{Cat}^{\mathrm{st}} \hookrightarrow \mathrm{PSh}_{\mathrm{Sp}}(\mathrm{Cat}^{\mathrm{st}})$. This is also known as the (**non-commutative**) **additive motive**. The following theorem is critical:

Theorem 15.5 (Blumberg-Gepner-Tabuada)**.**

$$\mathrm{K}(\mathcal{C}) \simeq \mathrm{Map}(\mathcal{M}_{\mathrm{add}}(\mathrm{Sp}), \mathcal{M}_{\mathrm{add}}(\mathcal{C}))$$

for all $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$. *For* F *an additive invariant, we have*

$$\mathrm{Map}(K, F) \simeq F(\mathrm{Sp}^{\omega}).$$

We are now ready to define the (abstract) Dennis trace via the theorem. Note that

$$\mathrm{Map}(\mathrm{K}, \mathrm{THH}) \simeq \mathrm{THH}(\mathrm{Sp}^{\omega}) \simeq \mathbb{S},$$

we have

$$\pi_0 \,\mathrm{Map}(\mathrm{K}, \mathrm{THH}) \simeq \pi_0 \,\mathrm{THH}(\mathbb{S}) \simeq \pi_0\mathbb{S} \simeq \mathbb{Z}.$$

The **Dennis trace** is defined to be the unique lax symmetric monoidal transformation $\mathrm{K} \to \mathrm{THH}$ corresponding to $1 \in \mathbb{Z}$. The definition of cyclotomic trace is slightly more involved, since, as we noted earlier, TC is not an additive invariant. However, it can be approximated by additive invariants TC^n as follows.

Construction 15.1. We recover TC as $\lim_n \mathrm{TC}^n$ where TC^n is given by an equalizer

$$\mathrm{TC}^n \coloneqq \mathrm{eq}\left(\mathrm{TR}^n \underset{\varphi}{\overset{incl}{\rightrightarrows}} \mathrm{TR}^{n-1} \right)$$

where $\mathrm{TR}^n(-) = \mathrm{THH}(-)^{C_{p^n}}$, *incl* is the inclusion map, and φ is given by

$$\mathrm{THH}(\mathcal{C})^{C_{p^n}} = \left(\mathrm{THH}(\mathcal{C})^{C_p}\right)^{C_{p^{n-1}}} \to \left(\Phi^{C_p}\,\mathrm{THH}(\mathcal{C})\right)^{C_{p^{n-1}}} \simeq \mathrm{THH}(\mathcal{C})^{C_{p^{n-1}}}.$$

Noting that each TC^n is an additive invariant (and even a localizing invariant), we are ready to (sketchically) define the cyclotomic trace tr_c:

Theorem 15.6 (Blumberg-Gepner-Tabuada)**.** *The maps* $K \to \mathrm{TC}$ *correspond to compatible systems of maps* $\mathrm{K} \to \mathrm{TC}^n$*; thus, we have*

$$\mathrm{Map}(\mathrm{K}, \mathrm{TC}) \simeq \lim_n \mathrm{Map}(\mathrm{K}, \mathrm{TC}^n) = \lim_n \mathrm{TC}^n(\mathbb{S}) = \mathrm{TC}(\mathbb{S}) \simeq \mathbb{S} \oplus \Sigma\,\mathrm{fib}(\mathbb{S}_{hS^1} \to \Sigma^{-1}\mathbb{S}).$$

After p-completion, $\mathrm{tr}_c \colon \mathrm{K} \to \mathrm{TC}$ *corresponds to* $1 \in \mathbb{Z}_p = \pi_0\left(\mathrm{TC}(\mathbb{S})_p^\wedge\right)$.

15.2 The theorem of Dundas-Goodwillie-McCarthy

Here, we simply state the ultimate result of Dundas-Goodwillie-McCarthy.

Theorem 15.7 (Dundas-Goodwillie-McCarthy)**.** *For* $f \colon B \to A$ *a map of connective* $\mathbb{E}_\infty$*-rings with* $\pi_0(f)$ *being surjective with a nilpotent kernel, we have a cartesian square:*

$$\begin{array}{ccc} \mathrm{K}(B) & \xrightarrow{\mathrm{tr}_c} & \mathrm{TC}(B) \\ \downarrow & & \downarrow \\ \mathrm{K}(A) & \xrightarrow{\mathrm{tr}_c} & \mathrm{TC}(A) \end{array}$$

15.3 Goodwillie Calculus

In order to prove the theorem of Dundas-Goodwillie-McCarthy, we need the concept of the Goodwillie Calculus. For $F \colon \mathcal{C} \to \mathcal{D}$ a functor preserving filtered colimits, we can approximate it by n-excisive functors[10] P_nF, which perform similar to a polynomial of degree n in a certain sense. These can be assembled into the following (also known as a **Taylor tower**):

$$F \to \dots \to P_nF \to P_{n-1}F \to \dots \to P_0F.$$

Let D_nF denote

$$D_nF \coloneqq \mathrm{fib}\left(P_nF \to P_{n-1}F\right).$$

D_nF behaves like the homogeneous piece of F of degree n (and is determined by the derivative ∂_nF). To know what is happening here, we need the following definition.

Definition 15.8. Let I be a finite set of cardinality n, and $\mathcal{P}(I)$ be the category of subsets of I. An n**-cube** is a functor $\chi : N(\mathcal{P}(I)) \to \mathcal{C}$. It is

(a) **cartesian** if $\chi(\varnothing) \simeq \mathrm{holim}_{\varnothing \neq S \subset I}\, \chi(S)$,

(b) **cocartesian** if $\chi(\varnothing) \simeq \mathrm{hocolim}_{\varnothing \neq S \subset I}\, \chi(S)$,

(c) **strongly cocartesian** if any 2-dimensional face is a pushout.

A functor $F : \mathcal{C} \to \mathcal{D}$ is n**-excisive** if it takes strongly cocartesian $(n+1)$-cubes to cartesian $(n+1)$-cubes. The category of n-excisive functors is denoted by $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$.

Example 15.9. By definition, 1-excisive functors are just ordinarily excisive functors, i.e. sends pushouts to pullbacks.

[10] More precisely, P_n is left adjoint to the forgetful functor $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

The nice thing about n-excisive functors is that $\mathrm{Exc}^n(\mathcal{C},\mathcal{D}) \subset \mathrm{Exc}^{n+1}(\mathcal{C},\mathcal{D})$ for any $n \geq 0$. The aforementioned approximations D_nF, P_nF are n-excisive. They are related by the following diagram:

$$\begin{array}{ccc} \mathcal{C}^n & \xrightarrow{\mathrm{cr}_{(n)} D_n F} & \mathcal{D} \\ \big\downarrow {\scriptstyle (\Sigma^\infty_{\mathcal{C}})^n} & & \big\uparrow {\scriptstyle \Omega^\infty_{\mathcal{D}}} \\ \mathrm{Sp}(\mathcal{C})^n & \xrightarrow{\partial_n F} & \mathrm{Sp}(\mathcal{D}) \end{array}$$

where $\mathrm{cr}_{(n)} D_n F$ is the symmetric cross-effect of D_nF. See Construction 6.1.3.20 in [Lur17]. More precisely, $\Omega^\infty_{\mathcal{D}} \circ \partial_n F \simeq \mathrm{cr}_{(n)} D_n F \circ \prod \Omega^\infty_{\mathcal{C}}$.

Example 15.10. For U, V an open cover of $X \in \mathrm{Top}_*$, with $U \cap V \simeq *$, and $F\colon \mathrm{Top}_* \to \mathrm{Sp}$, we have $\partial_1 F(X) \simeq \partial_1(U) \oplus \partial_1(V)$. In other word, Goodwillie derivatives "see" the "linear" part of F.

Theorem 15.11. *If $\mathcal{C}$ has pushouts, $\mathcal{D}$ has sequential colimits and finite limits, and they are commutative, then $F : \mathcal{C} \to \mathcal{D}$ admits n-th approximation P_nF, and this is universal among all natural transformations from F to any n-excisive functors.*

15.4 Sketch of proof of DGM theorem

The following theorem of Goodwillie will be one of main ingredients in the proof:

Theorem 15.12 (Goodwillie)**.** *Let F and G be two functors from $\mathcal{C}$ to $\mathcal{D}$ and satisfy the conditions being "ρ-analytic" functors (see [Goo92]), and both of them preserve filtered colimits. If for all $X \in \mathcal{C}$, we have $\partial_1 F(X) \simeq \partial_1 G(X)$, then we get a cartesian square:*

$$\begin{array}{ccc} F(Y) & \longrightarrow & G(Y) \\ \big\downarrow & & \big\downarrow \\ F(X) & \longrightarrow & G(X) \end{array}$$

for all $(\rho+1)$-connected $Y \to X$.

Take $\mathcal{C} = \mathrm{Cat}^{\mathrm{st}}_\infty$, $\mathcal{D} = \mathrm{Sp}$, $F = \mathrm{K}$, and $G = \mathrm{TC}(= \lim_n \mathrm{TC}^n)$. If we can prove that $\partial_1 \mathrm{K} \simeq \partial_1 \mathrm{TC}$, then by Goodwillie's theorem we are done. To do that, we reduce the case to the square-zero extension.

Step 1: Reduce to the square-zero extension

Theorem 15.13 (Goodwillie)**.** *If the DGM theorem is true in the special case when $Y = A \ltimes M \to A$, $X = A$ (i.e. trivial square-zero extension), then it is true in general.*

Idea of proof. Use the simplicial approximations to X and Y by bar constructions $X_\bullet$ and $Y_\bullet$, respectively. Then we can prove it level-wise, i.e. focus on $\mathrm{K}(Y_r \to X_r)$ and $\mathrm{TC}(Y_r \to X_r)$. Each X_r is free associative, and by assumpion $Y_r \to X_r$ is split surjective, yielding $Y_r = M_r \rtimes A_r$. □

Step 2: Compute the Goodwillie derivatives

We will follow what Dundas and McCarthy did in their original work. A standard reference for this part is the Chapter 3 of [Mad95]. Let M be a A-bimodule, and $\mathcal{P}_A$ be the category of projective A-bimodules. A fact by Dundas and McCarthy is that

$$\mathrm{THH}(A, M) \simeq \operatorname*{colim}_n \Omega^n \mid \bigoplus_{c \in S_\bullet^{(n)} \mathcal{P}_A} \hom_{S_\bullet \mathrm{Mod}_A}(c, c \otimes_A M) \mid .$$

Let $\mathcal{P}(A, M)$ be the category consisting of objects (P, α) for $P \in \mathcal{P}_A$ and $\alpha : P \to P \otimes_A M$ being A-linear. The morphisms in this category are $(P_1, \alpha_1) \xrightarrow{f} (P_2, \alpha_2)$ satisfying

$$\begin{array}{ccc} P_1 & \longrightarrow & P_1 \otimes_A M \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f \otimes 1} \\ P_2 & \longrightarrow & P_2 \otimes_A M \end{array}$$

Define

$$\mathrm{K}(\mathcal{P}(A, M)) \coloneqq \operatorname*{colim}_n \Omega^n \mid \coprod_{c \in S_\bullet^{(n)} \mathcal{P}_A} \hom_{S_\bullet \mathrm{Mod}_A}(c, c \otimes_A M) \mid$$

and

$$\tilde{\mathrm{K}}(A, M) \coloneqq \mathrm{fib}(\mathrm{K}(\mathcal{P}(A, M)) \to \mathrm{K}) = \partial_1 \mathrm{K}(A, M).$$

Dundas and McCarthy proved that $\mathrm{K}(\mathcal{P}(A, M)) = \mathrm{K}(A \ltimes M)$. Now like in Step 1, we can also use the simplicial approximation $M_\bullet \to M$ to reduce the question level-wise. Suppose M is m-connected, i.e. $\pi_*|M_\bullet| = \pi_* M = 0$ for $* \leq m$. Consider the diagram

$$\begin{array}{ccccc} \tilde{\mathrm{K}}(A, M) & & \dashrightarrow & & \mathrm{THH}(A, M) \\ {\scriptstyle \simeq}\downarrow & & & & \downarrow{\scriptstyle \simeq} \\ \Omega^p \,\mathrm{fib}^{(p)} & \xrightarrow{\alpha^{(p)}} & \Omega^p \,\mathrm{cofib}^{(p)} & \xrightarrow{\beta^{(p)}} & \Omega^p \,\mathrm{THH}^{(p)} \end{array}$$

where

$$\mathrm{fib}^{(p)} = \mathrm{fib}\left(\left| \coprod_{c \in S_\bullet^{(p)} \mathcal{P}_A} \hom_{S_\bullet^{(p)} \mathrm{Mod}_A}(c, c \otimes_A M) \right| \to \left| S_\bullet^{(p)} \mathcal{P}_A \right| \right),$$

$$\mathrm{cofib}^{(p)} = \mathrm{cofib}\left(\left| S_\bullet^{(p)} \mathcal{P}_A \right| \to \left| \coprod_{c \in S_\bullet^{(p)} \mathcal{P}_A} \hom_{S_\bullet^{(p)} \mathrm{Mod}_A}(c, c \otimes_A M) \right| \right),$$

$$\mathrm{THH}^{(p)} = \left| \bigoplus_{c \in S_\bullet^{(p)} \mathcal{P}_A} \hom_{S_\bullet^{(p)} \mathrm{Mod}_A}(c, c \otimes_A M) \right|.$$

Lemma 15.14. *$\alpha^{(p)}$ is at least $(p-3)$-connected, $\beta^{(p)}$ is $2m$-connected.*

Idea of proof. Consider the diagram

$$\begin{array}{ccc} \left|\coprod_{c\in S^{(p)}_\bullet\mathcal{P}_A} \hom_{S^{(p)}_\bullet \mathrm{Mod}_A}(c, c\otimes_A M)\right| & \longrightarrow & \left|S^{(p)}_\bullet \mathcal{P}_A\right| \\ \downarrow & & \downarrow \\ \left|\bigvee_{c\in S^{(p)}_\bullet\mathcal{P}_A} \hom_{S^{(p)}_\bullet \mathrm{Mod}_A}(c, c\otimes_A M)\right| & \longrightarrow & * \end{array}$$

and then use the Blakers-Massey theorem. □

Choose $p \gg 2m$ and $p \to \infty$, we have $\partial_1 \mathrm{K}(A, M) = \mathrm{THH}(A, M)$. For the TC part, the story is similar. A standard reference for this part is [Hes94]. Consider

$$\tilde{\mathrm{TC}}(A, M) \coloneqq \mathrm{fib}(\mathrm{TC}(A \ltimes M) \to \mathrm{TC}(A)) = \partial_1 \mathrm{TC}(A, M).$$

Again, we use the simplicial approximation. The underlying space is $\mathrm{TC}(A \oplus M)$. Suppose M is m-connected, and $M_\bullet \to M$. Recall that $\mathrm{THH}(A, M) = \|\mathrm{B}^{cyc}_\bullet(A \oplus M)\|$, where

$$\mathrm{B}^{cyc}_n(A \oplus M) \simeq (A \oplus M)^{\otimes n+1} \simeq \bigvee_{S\subset[n]} A^{\otimes([n]-S)} \otimes M^{\otimes S}$$

is the cyclic bar construction. Write

$$T_{a,n}(A, M) = \bigvee_{S\subset[n], |S|=a} A^{\otimes([n]-S)} \otimes M^{\otimes S}.$$

For example, $T_{0,\bullet}(A, M) = \mathrm{THH}(A)$.

Lemma 15.15. *I* $|T_{1,\bullet}(A, M)| \simeq S^1_+ \wedge \mathrm{THH}(A, M)$.

II The cyclotomic structure map is given by

$$R_p : T_{a,\bullet}(A, M)^{C_{p^r}} \to T_{a/p,\bullet}(A, M)^{C_{p^{r-1}}}$$

which induces

$$\tilde{\mathrm{TC}}(A, M)^\wedge_p \simeq \left(\operatorname*{holim}_{R_p} \left(\bigvee_{s=0}^{\infty} T_{p^s,\bullet}(A, M)\right)^{C_{p^r}}\right)^\wedge_p$$

III By checking the connectivity, and looking at the free S^1-action. The right hand side in II is weak equivalant to $\left(\mathrm{holim}_{R_p} (T_{1,\bullet}(A, M))^{C_{p^r}}\right)^\wedge_p$, which is weak equivalent to

$$\left(\operatorname*{holim}_r (S^1/C_{p^r}{}_+ \wedge \mathrm{THH}(A, M))\right)^\wedge_p,$$

which is a consequence of I.

By the proposition and the fact $S^1/C_{p^r} \simeq S^1$, one has

$$\partial_1 \mathrm{TC}(A, M)^\wedge_p = \tilde{\mathrm{TC}}(A, M)^\wedge_p \simeq (\Sigma\, \mathrm{THH}(A, M))^\wedge_p.$$

Step 3: Analyticity of K and TC

This is to ensure both K and TC satisfy the requirement of "ρ-analytic" functors. We will not give further explanations but rather directly leave the statement.

Theorem 15.16 (Goodwillie, 1992; McCarthy, 1997)**.** K *and* TC *are* (-1)-*analytic.*

Step 4: From p-completed case to usual case

Again, approximate $Y \to X$ by simplicial rings $Y_\bullet$ and $X_\bullet$, respectively. Use the p-completed version of DGM (a.k.a. McCarthy's theorem) in Step 2, and some complicated examination on connectivity yields the desired result (a.k.a. Dundas' theorem). See Chapter 3.4 and 3.5 of [Mad95] for details.

Remark 15.17 (Liam)**.** Nowadays, we can appeal to a surprising result of Clausen-Mathew-Morrow, which says that as a functor $\mathrm{CycSp}_{\geq 0} \to \mathrm{Sp}$, $\mathrm{TC}(-)/p$ commutes with small colimits. Strictly speaking, this result is stronger than we need for the proof of DGM, but it is an important structural property of TC.

16 The telescope conjecture at a fixed prime p (Lucas Piessevaux)

Notes by Lucas Piessevaux

Let $\mathrm{K}(n)$ denote height n Morava K-theory with homotopy groups given by

$$\mathrm{K}(n)_* \cong \mathbb{F}_p[v_n^{\pm}].$$

Let $F(n)$ denote a finite type n complex, and define the mapping telescope

$$T(n) = v_n^{-1}F(n)$$

with respect to the v_n^i-self map on $F(n)$. These spectra define Bousfield localisations of the form

$$\mathrm{Sp}_{\mathrm{K}(n)} \hookrightarrow \mathrm{Sp}_{T(n)} \to \mathrm{Sp}.$$

The $\mathrm{K}(n)$-local category has good behaviour in the Adams–Novikov spectral sequence, while the $T(n)$-local category has good behaviour on homotopy groups, as evidenced by the definition. It is then natural to ask whether the best of both worlds is true.

Conjecture 16.1 (Telescope conjecture, Ravenel)**.** The inclusion

$$\mathrm{Sp}_{\mathrm{K}(n)} \to \mathrm{Sp}_{T(n)}$$

is an equivalence.

In fact, there are many equivalent formulations of the telescope conjecture, and the original formulation was phrased in terms of the Adams–Novikov spectral sequence of a $T(n)$-local spectrum. At $n = 0$ the telescope conjecture is clearly true if we use the convention that $\mathrm{K}(0) = \mathbb{Q}$. One can use computational techniques to prove that this is also true at $n = 1$.

Theorem 16.2 (Mahowald ($p = 2$), Miller ($p > 2$))**.** *The telescope conjecture is true at height $n = 1$.*

Over time, computational evidence suggested that the telecope conjecture fails at heights $n \geq 2$. We now know that this is true by recent results of Burklund–Hahn–Levy–Schlank in [Bur+23], where a counterexample is constructed using K-theory at all heights $n \geq 2$ and all primes. It is still an interesting question to ask for which spectra the telescope conjecture is true, and in this talk we will discuss an example of a spectrum satisfying the height two telescope conjecture, as well a counterexample. In the first case, we will see that the natural map

$$L_2^f\,\mathrm{TC}(\ell) \to L_2\,\mathrm{TC}(\ell)$$

is an equivalence, where we remind the reader that L_2^f is the (smashing) localisation at $T(0) \oplus T(1) \oplus T(2)$, while L_2 is the (equally smashing) localisation at height two Lubin–Tate theory or $\mathrm{K}(0) \oplus \mathrm{K}(1) \oplus \mathrm{K}(2)$.

Remark 16.3. Using this interpretation, one sees that the telescope conjecture should be true for all spectra arising as homotopy MU-modules; indeed their Adams–Novikov spectral sequence collapses so that a localisation on the ANSS E_2-page corresponds to a localisation on homotopy. The precise statement is that for an MU-module X the map

$$L_n^f X \to L_n X$$

is an equivalence. Since both localisations are smashing, it suffices to prove this for $X = \mathrm{MU}$, and this follows from a simple argument as in Theorem 2.7 (iii) of [Rav93].

16.1 An example

Let us now prove the following theorem:

Theorem 16.4 ([HRW22], Theorem 6.6.4)**.** *The height two telescope conjecture is true for* $\mathrm{TC}(\ell)$*, i.e. the map*

$$L_2^f\,\mathrm{TC}(\ell) \to L_2\,\mathrm{TC}(\ell)$$

is an equivalence.

The proof goes through several reductions.

- For bounded below ring spectra such as ℓ, we can compute TC using the Nikolaus–Scholze fibre sequence

  $$\mathrm{TC}(\ell) \simeq \mathrm{fib}(\mathrm{can} - \phi^{h\mathbb{T}}\colon\ \mathrm{TC}^-(\ell) \to \mathrm{TP}(\ell)).$$

  Since both maps involved in the fibre sequence are ring maps, we conclude that $\mathrm{TC}(\ell)$ admits the structure of a $\mathrm{TC}^-(\ell)$-module. Since both L_S^f and L_2 are smashing, it therefore suffices to prove the theorem above for $\mathrm{TC}^-(\ell)$.

- The telescope conjecture is true at height one (and height zero), so it suffices to prove the theorem after smashing with a finite type two complex $F(2)$. Further, it follows from the definition of L_2^f that there is an equivalence

  $$L_2^f F(2) \otimes \mathrm{TC}^-(\ell) \simeq v_2^{-1} F(2) \otimes \mathrm{TC}^-(\ell),$$

  where we are inverting the v_2^i-self map on $F(2)$.

- The motivic filtration on $\mathrm{TC}^-(\ell)$ arises from the motivic cobar complex

  $$\mathrm{TC}^-(\ell) \xrightarrow{\sim} \mathrm{Tot}(\mathrm{TC}^-(\ell/\mathrm{MU}^{\otimes \bullet+1})).$$

  Every term of the form $\mathrm{TC}^-(\ell/\mathrm{MU}^{\otimes \bullet+1})$ admits the structure of a module over $\mathrm{MU}^{h\mathbb{T}}$, which itself is a module over MU by virtue of the fact that the action on MU is trivial. In particular, the telescope conjecture is true for every term in this cosimplicial object

Using the final remark above, we have reduced the problem to analysing the map

$$\begin{aligned} v_2^{-1} F(2) \otimes \mathrm{TC}^-(\ell) &\xrightarrow{\sim} v_2^{-1} F(2) \otimes \mathrm{Tot}(\mathrm{TC}^-(\ell/\mathrm{MU}^{\otimes \bullet+1})),\\ &\xrightarrow{\sim} v_2^{-1}\,\mathrm{Tot}(F(2) \otimes \mathrm{TC}^-(\ell/\mathrm{MU}^{\otimes \bullet+1})),\\ &\longrightarrow \mathrm{Tot}(v_2^{-1} F(2) \otimes \mathrm{TC}^-(\ell/\mathrm{MU}^{\otimes \bullet+1})),\\ &\xrightarrow{\sim} \mathrm{Tot}(L_2\,\mathrm{TC}^-(\ell/\mathrm{MU}^{\otimes \bullet+1})), \end{aligned}$$

where we used the fact that $F(2)$ is finite to commute it past the limit. Indeed, the target is a limit of L_2-local spectra hence is L_2-local itself, so it suffices to show that we can commute v_2-inversion with the totalisation in the motivic cobar complex.

Lemma 16.5. *To commute v_2^{-1} with* Tot *above, it suffices to show that the spectral sequence associated to this cosimplicial object has a horizontal vanishing line on a finite page when displayed in Adams grading.*

Remark 16.6. This type of lemma is part of a larger picture of descent techniques, in which a horizontal vanishing line on a finite page of the spectral sequence associated to a cosimplicial object tells us that the totalisation tower $\{\mathrm{Tot}_n\}_n$ was sufficiently finite to allow us to treat the totalisation as a finite limit hence commute it with a filtered colimit. The result we use here is Lemma 2.34 in [CM21].

Note that this horizontal vanishing line does indeed exist! The spectral sequence associated to this cosimplicial object is the motivic spectral sequence computing $\mathrm{TC}^-(\ell)$, and the explicit description of the prismatic cohomology of ℓ in Shai's talk displays this horizontal vanishing line (note that the Adams grading convention is sheared from the grading convention we were using there). We conclude that $v_2^{-1}F(2)\otimes\mathrm{TC}^-(\ell)$ is L_2-local, whence the comparison map

$$L_2^f\,\mathrm{TC}(\ell)\to L_2\,\mathrm{TC}(\ell)$$

is an equivalence.

16.2 A counterexample

Let us now provide a counterexample to the telescope conjecture at height two; we will construct a spectrum that is $T(2)$-local but not K(2)-local. The goal of this section is to prove that for some action of the group $\mathbb{Z}$ on ℓ, the coassembly map

$$L_{T(2)}\,\mathrm{TC}(\ell^{h\mathbb{Z}})\to L_{T(2)}\,\mathrm{TC}(\ell)^{h\mathbb{Z}}$$

is not an equivalence. Let us restrict to primes $p\geq 7$ to simplify arguments with Smith–Toda complexes and Adams operations. The primary reference for the computational part of the argument is Ishan Levy's Oberwolfach report [GHR23]. In the second part of this section, we will discuss the recent advances in chromatic homotopy theory and trace methods that allow us to leverage this computational result into a disproof of the telescope conjecture at height two. These arguments are discussed in more detail in [Bur+23].

Remark 16.7. The term coassembly refers to a limit comparison map: Let $F\colon\mathcal{C}\to\mathcal{D}$ be a functor between categories with I-indexed limits for some category I. Let $X\simeq\lim_{i\in I}X_i$ be a limit diagram in $\mathcal{C}$, then functoriality of F gives us maps

$$F(X)\xrightarrow{F(\pi_i)}F(X_i)$$

assembling to a map

$$F(\lim_{i\in I}X_i)\to\lim_{i\in I}F(X_i).$$

16.2.1 Coassembly in TC

Note that ℓ already comes equipped with a natural multiplicative action of $\mathbb{Z}$ given by the Adams operations restricted from those on complex K-theory.

Remark 16.8. Recall that to compute $\mathrm{THH}(\ell)$, we used a lift of ℓ to a filtered algebra by means of its $\mathbb{F}_p$-descent filtration or (p, v_1)-adic filtration. Letting Ψ^{1+p} be a generator of the Adams operations on ℓ, we compute that on homotopy

$$\Psi^{1+p}(v_1) = (1+p)^{p-1} v_1.$$

We conclude that ℓ lifts not only to $\ell^{\mathrm{Fil}} \in \mathrm{CAlg}(\mathrm{Sp}^{\mathrm{Fil}})$ but to an object of $\mathrm{CAlg}(\mathrm{Sp}^{\mathrm{Fil}})^{B\mathbb{Z}}$, i.e. the Adams operations respect the multiplicative structure for the Day convolution on the (p, v_1)-adic filtration.

While the Adams action is certainly not trivial, recall that the associated graded of ℓ^{Fil} is given by $\mathbb{F}_p[v_0, v_1] \in \mathrm{CAlg}(\mathrm{Sp}^{\mathrm{Gr}})^{B\mathbb{Z}}$. By the formula established above, it is clear that the Adams operations act trivially on this graded ring spectrum as $p = 0$. As a consequence, we obtain

$$\mathrm{gr}_*(\ell^{\mathrm{Fil}})^{h\mathbb{Z}} \simeq \mathbb{F}_p[v_0, v_1] \otimes_{\mathbb{F}_p} \mathbb{F}_p^{h\mathbb{Z}}.$$

Once again, we obtain a spectral sequence with E_1-page given by the graded THH of $\mathrm{gr}_*(\ell^{\mathrm{Fil}})^{h\mathbb{Z}}$ and converging to the usual THH of the underlying object of $(\ell^{\mathrm{Fil}})^{h\mathbb{Z}}$ which is $\ell^{h\mathbb{Z}}$ by the compatibility of the filtration with the action. This is of the form

$$\begin{aligned} E_1 &= \pi_*(\mathrm{THH}^{\mathrm{Gr}}(\mathbb{F}_p[v_0, v_1] \otimes_{\mathbb{F}_p} \mathbb{F}_p^{h\mathbb{Z}})), \\ &\cong \pi_*(\mathrm{THH}^{\mathrm{Gr}}(\mathbb{F}_p[v_0, v_1]) \otimes_{\mathbb{F}_p} \mathrm{HH}^{\mathrm{Gr}}(\mathbb{F}_p^{h\mathbb{Z}}/\mathbb{F}_p)), \\ &\implies \pi_* \mathrm{THH}(\ell^{h\mathbb{Z}}). \end{aligned}$$

The first tensor factor in the E_1-page is the usual one computing $\mathrm{THH}(\ell)$, while the second one is more interesting: in fact the failure of the coassembly map to be an equivalence is already happening at this level.

Lemma 16.9. *Let $C \coloneqq C^0(\mathbb{Z}_p, \mathbb{F}_p)$ denote the $\mathbb{F}_p$-algebra of continuous $\mathbb{F}_p$-valued functions on $\mathbb{Z}_p$. Then the coassembly map*

$$\pi_* \mathrm{HH}(\mathbb{F}_p^{h\mathbb{Z}}/\mathbb{F}_p) \to \pi_* \mathrm{HH}(\mathbb{F}_p/\mathbb{F}_p)^{h\mathbb{Z}}$$

is given by the map

$$\mathrm{ev}_0\langle\zeta\rangle \colon C\langle\zeta\rangle \to \mathbb{F}_p\langle\zeta\rangle$$

that evaluates a function at $0 \in \mathbb{Z}_p$ and preserves the exterior generator ζ.

Proof. First, note that the action on $\mathbb{F}_p$ by $\mathbb{Z}$ is trivial, so we are really describing the coassembly map for $\mathbb{F}_p^{B\mathbb{Z}}$. Now the $\mathbb{F}_p$-cochains on a space factor through the unstable p-completion of that space, which in this case allows us to replace the circle $B\mathbb{Z}$ by the p-adic algebraic circle $B\mathbb{Z}_p$ which is now a p-profinite space. The upshot of this reduction is that Hochschild homology (over $\mathbb{F}_p$) of cochains on p-profinite spaces recovers continuous cochains on the free loop space. In our case:

$$\mathrm{HH}(\mathbb{F}_p^{B\mathbb{Z}_p}/\mathbb{F}_p) \simeq C^*(LB\mathbb{Z}_p; \mathbb{F}_p).$$

The topological group structure on $B\mathbb{Z}_p$ allows us to split $LB\mathbb{Z}_p$ as a product $\Omega B\mathbb{Z}_p \times B\mathbb{Z}_p$, so by the Künneth formula we obtain

$$\begin{aligned} \mathrm{HH}(\mathbb{F}_p^{B\mathbb{Z}_p}/\mathbb{F}_p) &\simeq C^*(B\mathbb{Z}_p; \mathbb{F}_p) \otimes_{\mathbb{F}_p} C^*(\mathbb{Z}_p; \mathbb{F}_p), \\ &\simeq \mathbb{F}_p\langle\zeta\rangle \otimes_{\mathbb{F}_p} C^0(\mathbb{Z}_p; \mathbb{F}_p) \end{aligned}$$

with the exterior class ζ arising as the fundamental class of the circle in degree -1. The coassembly map then reduces to evaluation at the constant loop, which under the equivalences above corresponds to evaluation at $0 \in \mathbb{Z}_p$. $\square$

This virtuous behaviour on the E_1-page, where the coassembly problem splits off as a separate tensor factor, actually persists all the way to the mod p, v_1 E_∞-page as is proven in [LL23]. We obtain an identification

$$\pi_* \operatorname{THH}(\ell^{h\mathbb{Z}})/p, v_1 \cong \pi_* \operatorname{THH}(\ell)/p, v_1 \otimes_{\mathbb{F}_p} \operatorname{HH}_*(\mathbb{F}_p^{h\mathbb{Z}}/\mathbb{F}_p).$$

One could then proceed to compute $\operatorname{TC}(\ell^{h\mathbb{Z}})/p, v_1$ similarly to the way we computed it for ℓ. However, the group action does make the computation intractable at first sight. The trick is to somehow deform this problem into a continuous family of spectral sequence computations in which the hard equivariant computation degenerates to the classical Ausoni–Rognes style computation at a special fibre. We will now make this precise using some seemingly inoccuous observations

Remark 16.10. In the discussion above about the $\mathbb{Z}$-action on ℓ^{Fil} as well as the resulting computation in Hochschild homology, we could have replaced $\mathbb{Z}$ by $p^k\mathbb{Z}$ for any $k \geq 0$. The only difference is that one then replaces $B\mathbb{Z}$ by its p^k-fold cover.

For every $k \geq 0$ the Hochschild homology computations then assemble into a system

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \operatorname{HH}_*(\mathbb{F}_p^{hp^k\mathbb{Z}}/\mathbb{F}_p) & \longrightarrow & \operatorname{HH}_*(\mathbb{F}_p^{hp^{k+1}\mathbb{Z}}/\mathbb{F}_p) & \longrightarrow & \cdots & \longrightarrow & \operatorname{HH}_*(\mathbb{F}_p/\mathbb{F}_p) \\
& \downarrow{=} & & \downarrow{=} & & & & \downarrow{=} \\
\cdots \longrightarrow & C^0(p^k\mathbb{Z}_p)\langle\zeta_k\rangle & \longrightarrow & C^0(p^{k+1}\mathbb{Z}_p)\langle\zeta_{k+1}\rangle & \longrightarrow & \cdots & \longrightarrow & \mathbb{F}_p
\end{array}$$

induced by the inclusions $p^{k+1}\mathbb{Z} \to p^k\mathbb{Z}$. On Hochschild homology one sees that the maps are given by restriction

$$f \in C^0(p^k\mathbb{Z}_p) \mapsto f\mid_{p^{k+1}\mathbb{Z}_p} \in C^0(p^{k+1}\mathbb{Z}_p),$$

hence eventually evaluation at 0 as k tends to infinity. On the other hand, the fundamental class ζ_k is sent to the fundamental class of its p-fold cover, i.e. $p\zeta_{k+1}$ which vanishes since we are in characteristic p. One should think of the system above as a sort of sheaf on the open subsets $p^k\mathbb{Z}_p$ of $\mathbb{Z}_p$ with stalk at zero corresponding to no action.

Lemma 16.11. *Given $f\colon A \to B$ a map of finite rank free $C = C^0(\mathbb{Z}_p)$-modules, there exist a $k \gg 0$ such that*

$$f \otimes_C C^0(p^k\mathbb{Z}_p) = f \otimes_{C,\mathrm{ev}_0} \mathbb{F}_p \otimes_{\mathbb{F}_p} C^0(p^k\mathbb{Z}_p).$$

The lemma is to be interpreted as stating that for every f–to be thought of as a continuous $\mathbb{Z}_p$-indexed morphism of $\mathbb{F}_p$-vector spaces–there exists a sufficiently large k such that the restriction of f to $p^k\mathbb{Z}_p$ is entirely determined by its effect at $0 \in \mathbb{Z}_p$.

Proof. Since A, B are assumed to be finite rank free modules, we reduce to the case $A = B = C$ so that f is given by multiplication by some function also denoted f. In this case, we just use the fact that f is a continuous function to see that it must be locally constant for the p-adic topology on $\mathbb{Z}_p$, whence there exists a neighbourhood $p^k\mathbb{Z}_p$ of the origin on which it is constant. $\square$

The idea is now to apply this lemma to the case where f is a differential in one of the spectral sequences involved in computing $\mathrm{TC}(\ell^{h\mathbb{Z}})/p, v_1$ or $\mathrm{TC}(\ell)^{h\mathbb{Z}}/p, v_1$. Let us first recall what these spectral sequences were (as in Shai's talk).

I The mod p, v_1, v_2 Nygaard filtered prismatic cohomology of ℓ was computed using the algebraic t-Bockstein spectral sequence

$$E_1 = \mathbb{F}_p[t,\mu]/t\mu \otimes \Lambda(\lambda_1,\lambda_2) \implies \mathrm{gr}^{\mathrm{mot}}_* \mathrm{TC}^-(\ell)/p, v_1, v_2.$$

The nontrivial differentials of interest were

$$\begin{aligned} d_p(t) &= t^{p+1}\lambda_1, \\ d_{p^2}(t^p) &= t^{p^2+p}\lambda_2, \end{aligned}$$

and their propagations along the Leibniz rule. The resulting computation of the mod p, v_1, v_2 Nygaard filtered prismatic cohomology of ℓ is given by

$$\begin{aligned} \mathrm{gr}^{\mathrm{mot}}_* \mathrm{TC}^-(\ell)/p, v_1, v_2 = {} & \mathbb{F}_p[t^{p^2},\mu]/t^{p^2}\mu \otimes \Lambda(\lambda_1,\lambda_2), \\ & \oplus \mathbb{F}_p\{t^d\lambda_1, t^{pd}\lambda_2, t^d\lambda_1\lambda_2, t^{pd}\lambda_1\lambda_2 \mid 0 < d < p\}. \end{aligned}$$

II The mod p, v_1, v_2 prismatic cohomology of ℓ was computed using the algebraic Tate-t-Bockstein spectral sequence

$$E_1 = \mathbb{F}_p[t^{\pm}] \otimes \Lambda(\lambda_1,\lambda_2) \implies \mathrm{gr}^{\mathrm{mot}}_* \mathrm{TP}(\ell)/p, v_1, v_2.$$

The nontrivial differentials of interest were

$$\begin{aligned} d_p(t) &= t^{p+1}\lambda_1, \\ d_{p^2}(t^p) &= t^{p^2+p}\lambda_2, \end{aligned}$$

and their propagations along the Leibniz rule. The resulting computation of the mod p, v_1, v_2 prismatic cohomology of ℓ is given by

$$\mathrm{gr}^{\mathrm{mot}}_* \mathrm{TP}(\ell)/p, v_1, v_2 = \mathbb{F}_p[t^{\pm p^2}] \otimes \Lambda(\lambda_1,\lambda_2).$$

The final step–as in Preston's talk–was now to compute the mod p, v_1, v_2 syntomic cohomology using the Nikolaus–Scholze fibre sequence. Recall in particular that the Frobenius map from Nygaard filtered prismatic cohomology to prismatic cohomology was described by

$$\begin{aligned} \phi(\lambda_i) &= \lambda_i, \\ \phi(\mu) &= t^{-p^2}, \\ \phi(tx) &= 0, \end{aligned}$$

where tx denotes a generic class in $\mathrm{gr}^{\mathrm{mot}}_* \mathrm{TC}^-(\ell)/p, v_1, v_2$ with t in its name. The canonical map sends a class of the form $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}t^{kp^2}$ to the correspondingly named class for $\epsilon_i \in \{0,1\}, k \geq 0$ and is zero otherwise. The results above then glue together to tell us that

- The mod p, v_1, v_2 syntomic cohomology $\mathrm{gr}^{\mathrm{mot}}_* \mathrm{TC}(\ell)/p, v_1, v_2$ is bounded.

- The v_2-Bockstein spectral sequence collapses for degree reasons, so that
$$\mathrm{gr}^{\mathrm{mot}}_* \mathrm{TC}(\ell)/p, v_1$$
is finitely generated as a $\mathbb{F}_p[v_2]$-module.
- We conclude that $v_2^{-1}F(2) \otimes \mathrm{TC}(\ell)$ is finite dimensional over $\mathbb{F}_p[v_2^{\pm}]$.

In Preston's talk, we saw that the conclusions above were precisely what is needed to establish Quillen–Lichtenbaum for ℓ. Of interest to us is the simple observation that all processes above were *finite*: up to Leibniz propagation there were only finitely many differentials in both algebraic t-Bockstein spectral sequences, and the assembly processes–gluing and running the collapsing v_2-Bockstein spectral sequence–were finitary as well. Therefore, when we repeat this process to compute $\mathrm{gr}^{\mathrm{mot}}_* \mathrm{TC}(\ell)^{h\mathbb{Z}}/p, v_1$, we can apply Lemma 16.11 to every differential in sight and take the maximum of all the k's that the Lemma spits out to obtain a neighbourhood of 0 in $\mathbb{Z}_p$ on which the computation reduces to the computation above augmented with a *trivial* action. Letting k be at least this maximum we therefore obtain
$$\mathrm{TC}(\ell)^{hp^k\mathbb{Z}}/p, v_1 \simeq \mathrm{TC}(\ell)/p, v_1\langle\zeta\rangle,$$
i.e. just cochains on the sphere $Bp^k\mathbb{Z}$ with values in $\mathrm{TC}(\ell)/p, v_1$.

Remark 16.12. Recall that historically $\mathrm{TC}(\ell)/p, v_1$ was already computed by Ausoni–Rognes using other techniques. However, this computation is not finitary hence would not have been suitable for application of our Lemma; one of the key innovations of the motivic techniques in [HRW22] is that this computation becomes finite (and in particular that the v_2-Bockstein spectral sequence on syntomic cohomology just collapses entirely).

So we see that the computation of the target of the coassembly map was not too hard by virtue of asymptotic constancy and the finiteness of the computation of mod p, v_1 syntomic cohomology of the Adams summand. In fact, similar asymptotic constancy techniques can be applied to the computation of the source $\mathrm{TC}(\ell^{h\mathbb{Z}})/p, v_1$, but they are far more involved. We refer the reader to Section 7 in [Bur+23] for a discussion. Let us state the consequences of this computation:

Theorem 16.13 (Theorem 7.20 of [Bur+23])**.** *The trivial action coassembly map*
$$\pi_* \mathrm{TC}(\ell^{B\mathbb{Z}})/p, v_1 \to \pi_* \mathrm{TC}(\ell)^{B\mathbb{Z}}/p, v_1$$
is given–as a graded $\Lambda(\lambda_1, \lambda_2)$-module map–by the direct sum of

I $\mathrm{ev}_0 \colon C^0(p\mathbb{Z}_p)\langle\zeta\rangle \otimes_{\mathbb{F}_p} N \to \mathbb{F}_p\langle\zeta\rangle \otimes_{\mathbb{F}_p} N$*,*

II $i \colon \Lambda(\lambda_1, \lambda_2) \hookrightarrow \Lambda(\lambda_1, \lambda_2, \zeta)$*,*

III $(\pi_2 \colon \mathrm{coker}(\mathbb{F}_p \xhookrightarrow{\mathrm{cst}} C^0(\mathbb{Z}_p^\times))\{\partial\zeta\} \oplus \mathbb{F}_p\{\partial\} \to \mathbb{F}_p\{\partial\}) \otimes i$*,*

where N is the direct summand appearing in the prismatic cohomology of ℓ on generators whose name contains t (i.e. those of Nygaard filtration at least one) in the kernel of the canonical map.

It is clear from this description that the associated $\mathbb{F}_p[v_2^{-1}]$-module map
$$\pi_* v_2^{-1} \mathrm{TC}(\ell^{hp^k\mathbb{Z}})/p, v_1 \to \pi_* v_2^{-1} \mathrm{TC}(\ell)^{hp^k\mathbb{Z}}/p, v_1$$
has infinite dimensional source and finite dimensional target! This allows us to conclude that the coassembly map in $T(2)$-local TC of ℓ is *not* an equivalence as desired.

16.2.2 Coassembly in K-theory

Having obtained our computational result on the failure of the coassembly map to be an equivalence, let us now discuss how to turn this into a counterexample of the telescope conjecture at height two (and $p \geq 7$). First–as promised in the title of [Bur+23]–let us turn to K-theory.

Theorem 16.14 (Corollary 6.3 of [Bur+23])**.** *For $n \geq 1$ let R be a $T(n+1)$-acyclic connective $\mathbb{E}_1$-ring spectrum with an action of $\mathbb{Z}$. Then the coassembly maps for this action assemble to a commutative diagram*

$$\begin{array}{ccccc} L_{T(n+1)}\mathrm{K}(L_{T(n)}R^{h\mathbb{Z}}) & \xleftarrow{\sim} & L_{T(n+1)}\mathrm{K}(R^{h\mathbb{Z}}) & \xrightarrow{\sim} & L_{T(n+1)}\,\mathrm{TC}(R^{h\mathbb{Z}}) \\ \downarrow & & \downarrow & & \downarrow \\ L_{T(n+1)}\mathrm{K}(L_{T(n)}R)^{h\mathbb{Z}} & \xleftarrow{\sim} & L_{T(n+1)}\mathrm{K}(R)^{h\mathbb{Z}} & \xrightarrow{\sim} & L_{T(n+1)}\,\mathrm{TC}(R)^{h\mathbb{Z}}, \end{array}$$

where leftward maps are induced by localisation and rightward maps are cyclotomic trace maps.

While we will not prove this theorem, let us state the main players in its proof, most of which are recent results in telescopically localised K-theory and trace methods.

- The equivalence in the bottom left corner follows from applying $\mathbb{Z}$-homotopy fixed points to the purity isomorphism of Land–Mathew–Meier–Tamme ([Lan+24]) combined with a vanishing result of Clausen–Mathew–Naumann–Noel ([Cla+20]). This states that for an $\mathbb{E}_1$-ring R there is an equivalence

  $$L_{T(n+1)}\mathrm{K}(R) \xrightarrow{\sim} L_{T(n+1)}\mathrm{K}(L_{T(n)\oplus T(n+1)}R)$$

  and should be viewed as a preliminary form of redshift. Since R is $T(n+1)$-acyclic by assumption, we obtain the bottom left equivalence. In fact, there are no connectivity assumptions in the purity theorem stated here, so we can apply it to $R^{h\mathbb{Z}}$, which is equally $T(n+1)$-acylic, to obtain the top left equivalence. ‘

- The equivalence in the bottom right corner follows from the Dundas–Goodwillie–McCarthy (DGM) theorem from Albert's talk: this gives us a Cartesian square

  $$\begin{array}{ccc} L_{T(n+1)}\mathrm{K}(R) & \longrightarrow & L_{T(n+1)}\,\mathrm{TC}(R) \\ \downarrow & & \downarrow \\ L_{T(n+1)}\mathrm{K}(\pi_0 R) & \longrightarrow & L_{T(n+1)}\,\mathrm{TC}(\pi_0 R). \end{array}$$

  Since $n+1 \geq 2$ by assumption, we can use Mitchell's theorem which states that $L_{T(n+1)}\mathrm{K}(\mathbb{Z}) = 0$ to see that $L_{T(n+1)}\mathrm{K}$ (and hence also $L_{T(n+1)}\,\mathrm{TC}$ since the cyclotomic trace map is now an $\mathbb{E}_0$-map) vanish on discrete rings, whence the bottom row in the DGM theorem vanishes and the top row is an equivalence.

- All that needs to be shown is now that the top right arrow is an equivalence. Recall that $R^{h\mathbb{Z}}$ is no longer necessarily connective, so we can not just apply the DGM theorem. However, a recent result of Levy extends the DGM theorem to precisely the context we need it in.

Theorem 16.15 (Theorem B in [Lev22])**.** *Let R and S be connective $\mathbb{E}_1$-rings with $\mathbb{Z}$-action with a $\mathbb{Z}$-equivariant ring map $f\colon R \to S$. If f is 1-connective, then for any truncating invariant E the induced map*

$$E(R^{h\mathbb{Z}}) \to E(S^{h\mathbb{Z}})$$

is an equivalence.

A truncating invariant–in the sense of Land–Tamme ([LT19], Definition 3.1) is a localising invariant such that for every connective $\mathbb{E}_1$-ring A the map $A \to \pi_0 A$ induces

$$E(A) \xrightarrow{\sim} E(\pi_0 A).$$

It is clear from the classical DGM theorem that

$$\mathrm{K}^{\mathrm{inv}} := \mathrm{fib}(\mathrm{tr}\colon \mathrm{K} \to \mathrm{TC})$$

is a truncating invariant. Levy's theorem (which is proved using the techniques of [LT19]) then tells us that

$$L_{T(n+1)}\mathrm{K}^{\mathrm{inv}}(R^{h\mathbb{Z}}) \to L_{T(n+1)}\mathrm{K}^{\mathrm{inv}}((\pi_0 R)^{h\mathbb{Z}})$$

is an equivalence, where $f\colon R \to \pi_0 R$ is just the 0-truncation. This further implies that the commutative diagram

$$\begin{array}{ccc} L_{T(n+1)}\mathrm{K}(R^{h\mathbb{Z}}) & \longrightarrow & L_{T(n+1)}\,\mathrm{TC}(R^{h\mathbb{Z}}) \\ \downarrow & & \downarrow \\ L_{T(n+1)}\mathrm{K}((\pi_0 R)^{h\mathbb{Z}}) & \longrightarrow & L_{T(n+1)}\,\mathrm{TC}((\pi_0 R)^{h\mathbb{Z}}) \end{array}$$

induces an equivalence on horizontal fibres, hence is Cartesian! It now suffices to note that the rings on the bottom row admit $\mathbb{Z}^{h\mathbb{Z}}$-algebra structures, and since the group $\mathbb{Z}$ acts trivially on the ring spectrum $\mathbb{Z}$ (indeed, $B\mathbb{Z}$ only has cells in even degrees) we see that $\mathbb{Z}^{h\mathbb{Z}} \simeq \mathbb{Z}^{B\mathbb{Z}}$ is now itself a $\mathbb{Z}$-algebra whence we can apply Mitchell's theorem and conclude that the bottom row vanishes and the top row must be an equivalence.

In conclusion, we see that the coassembly map

$$L_{T(2)}\,\mathrm{TC}(\ell^{h\mathbb{Z}}) \to L_{T(2)}\,\mathrm{TC}(\ell)^{h\mathbb{Z}}$$

not being an equivalence tells us that the coassembly map

$$L_{T(2)}\mathrm{K}(L^{h\mathbb{Z}}) \to L_{T(2)}\mathrm{K}(L)^{h\mathbb{Z}}$$

is not an equivalence either. Indeed, $n = 1$ since ℓ is clearly $T(2)$-acyclic, and $L_{T(1)}\ell = v_1^{-1}\ell$ can be identified with the nonconnective Adams summand L.

16.2.3 Assembly: hyperdescent

Having obtained our result on failure of coassembly in telescopically localised K-theory, let us now finally discuss the relation between these coassembly maps and the telescope conjecture. The key strategy that we will leverage to obtain this disproof is that the $\mathbb{E}_\infty$ rings $L_{\mathrm{K}(n)}\mathbb{S}$ and $L_{T(n)}\mathbb{S}$ have different arithmetic behaviour with respect to Galois extensions.

- By a celebrated result of Devinatz–Hopkins, the map
$$L_{\mathrm{K}(n)}\mathbb{S} \to E_n(\overline{\mathbb{F}}_p)$$
to height n Lubin–Tate theory over $\overline{\mathbb{F}}_p$ is the algebraic closure of the $\mathrm{K}(n)$-local sphere, and is in fact a pro–Galois extension with Galois group $\mathbb{G}_n$.

- On the other hand, it is not clear how many Galois extensions[11] of $L_{\mathrm{K}(n)}\mathbb{S}$ lift to $L_{T(n)}\mathbb{S}$. One important class of such lifts has been constructed by Carmeli–Schlank–Yanovski in [CSY21]; namely the cyclotomic extensions
$$L_{T(n)}\mathbb{S} \to L_{T(n)}\mathbb{S}[\omega^{(n)}_{p^k}]$$
for all $k \geq 0$ with Galois group $(\mathbb{Z}/p^k)^\times$

Remark 16.16. The reason cyclotomic extensions can be lifted is that they depend only on the fact that the $T(n)$-local category (hence also the $\mathrm{K}(n)$-local category) are such that certain p-finite spaces are in a sense *invertible*. In particular, the space $B^n C_{p^k}$ is invertible (in the sense that colimits and limits indexed over this space agree in $L_{T(n)}\mathrm{Sp}$) which allows one to construct these cyclotomic extensions as summands of $L_{T(n)}\mathbb{S}[B^n C_{p^k}]$. This ought to be compared with the $(\mathbb{Z}/p^k)^\times$-Galois extensions
$$\mathbb{Q} \to \mathbb{Q}[\omega_{p^k}]$$
at height zero, which crucially use that p is invertible in $\mathbb{Q}$.

Taking the colimit over all k, we obtain a map
$$L_{T(n)}\mathbb{S} \to R^f_n \coloneqq L_{T(n)}\mathbb{S}[\omega^{(n)}_{p^\infty}].$$
However, it is *not* clear that this is a $\mathbb{Z}_p^\times$-Galois extension, as $(R^f_n)^{h\mathbb{Z}_p^\times}$ need not be equivalent to $L_{T(n)}\mathbb{S}$. The issue is that one has now taken a large colimit and $\mathbb{Z}_p^\times$ is a profinite topological group. For descent to hold, one needs that the finite Galois extensions assemble to a *hypersheaf* on the site of finite subgroups of $\mathbb{Z}_p^\times$.
One can localise at $(R^f_n)^{h\mathbb{Z}_p^\times}$ to obtain a smashing localisation
$$L_{T(n)}\mathrm{Sp}^\wedge_{\mathrm{cyc}} \hookrightarrow L_{T(n)}\mathrm{Sp},$$
i.e. the subcategory in which cyclotomic hyperdescent is true (per construction). In fact, there is a further inclusion
$$L_{\mathrm{K}(n)}\mathrm{Sp} \hookrightarrow L_{T(n)}\mathrm{Sp}^\wedge_{\mathrm{cyc}}.$$
One sees that $L_{\mathrm{K}(n)}R^f_n$ is equivalent to the $L_{\mathrm{K}(n)}\mathbb{S}$-algebra
$$R_n \coloneqq E_n(\overline{\mathbb{F}}_p)^{\ker \det\colon \mathbb{G}_n \to \mathbb{Z}_p^\times}$$
previously considered by Westerland. Indeed, the $\mathrm{K}(n)$-localisation functor sends this to a $\mathbb{Z}_p^\times$-Galois extension of $L_{\mathrm{K}(n)}L_{T(n)}\mathbb{S} = L_{\mathrm{K}(n)}\mathbb{S}$, but by the Devinatz–Hopkins theorem and

[11] Forthcoming work of Burklund Clausen and Levy shows that étale fundamental groups of ring spectra actually cannot tell the difference between $\mathrm{K}(n)$ and $T(n)$-localisation. However, this correspondence need not preserve *faithfulness* of profinite Galois extensions.

the Galois correspondence these extensions can be identified with subgroups of $\mathbb{G}_n$ and it suffices to note that Westerland's R_n corresponds to the right subgroup. In this case it is clear that

$$R_n^{h\mathbb{Z}_p^\times} \simeq (E_n(\overline{\mathbb{F}}_p)^{\ker\det\colon \mathbb{G}_n \to \mathbb{Z}_p^\times})^{h\mathbb{Z}_p^\times} \simeq L_{\mathrm{K}(n)}\mathbb{S}$$

so that the $\mathrm{K}(n)$-local sphere *does* satisfy cyclotomic hyperdescent. We therefore have a chain of inclusions

$$L_{\mathrm{K}(n)}\mathrm{Sp} \hookrightarrow L_{T(n)}\mathrm{Sp}^\wedge_{\mathrm{cyc}} \hookrightarrow L_{T(n)}\mathrm{Sp}.$$

It is not known whether the first inclusion is strict, but the computation we made above shows that the second inclusion is strict. The final crucial ingredient is the cyclotomic redshift theorem of Ben-Mosche–Carmeli–Schlank–Yanovski.

Theorem 16.17 ([Ben+23] Theorem B)**.** *Let R be a $T(n)$-local ring spectrum. Then there is a $\mathbb{Z}_p^\times$-equivariant equivalence*

$$L_{T(n+1)}\mathrm{K}(R[\omega_{p^\infty}^{(n)}]) \simeq L_{T(n+1)}\mathrm{K}(R)[\omega_{p^\infty}^{(n+1)}].$$

The idea is now that when $n+1=2$ and $p \geq 7$ the $p^k\mathbb{Z}$-pro-Galois-extension $L_{T(1)}\mathbb{S} = L_{\mathrm{K}(1)}\mathbb{S} \to L$ is an example of a cyclotomic extension. Taking fixed points tells us that

$$L_{T(2)}\mathrm{K}(L)^{hp^k\mathbb{Z}} \simeq L_{T(2)}\mathrm{K}(L^{hp^k\mathbb{Z}})^\wedge_{\mathrm{cyc}}.$$

Therefore, failure of the coassembly map to be an equivalence implies that the cyclotomic completion map is not an equivalence and we are done.

Remark 16.18. The failure of the telescope conjecture at height two is related to the failure of one part of the Ausoni–Rognes redshift conjectures. They originally conjectured that the map

$$\mathrm{K}(L_{\mathrm{K}(1)}\mathbb{S})/p, v_1 \to \mathrm{K}(\mathrm{KU})^{h\mathbb{Z}_p^\times}/p, v_1$$

should have bounded above fibre, hence be an equivalence after inverting v_2. Now we see that this is too much to ask for: it should only be a cyclotomic completion map. The appropriate replacement for this conjecture is Theorem E in [Bur+23], which states that the map

$$\mathrm{K}(L_{\mathrm{K}(1)}\mathbb{S})/p, v_1 \to v_2^{-1}\mathrm{K}(L_{\mathrm{K}(1)}\mathbb{S})/p, v_1$$

has bounded above fibre.

17 Redshift for $\mathbb{E}_\infty$-ring spectra (Alicia Lima)

Notes for this talk are unfortunately not available. Here is the description from the preliminary syllabus:

- Discuss the telescope conjecture in stable homotopy theory at a fixed prime p.
- Show why what we have done proves that $\mathrm{TC}(\mathbb{Z})$ and $\mathrm{TC}(\ell)$ satisfy the telescope conjecture [HRW22, Theorem 6.6.4].
- Explain how TC and Lichtenbaum–Quillen properties enter into the disproof of the telescope conjecture in general [Bur+23].

18 Conclusion (Jeremy Hahn)

Notes by Albert Jinghui Yang

The study of topological cyclic homology has been quite active recently, with various approaches and applications in arithmetic geometry, homotopy theory, and more. The following are some crucial tasks that the lecturer believes will lead to considerable progress and that readers should consider:

I Make new computations in hope of observing new phenomena and making conjectures.

II Understand both classical and modern results about the K-theory of discrete rings, and try to generalize them to the K-theory of ring spectra.

III Use K-theory to produce powerful examples of spectra, which detect surprising parts of the stable stem $\pi_*\mathbb{S}$.

Let's delve into a few concrete questions arising from each of these tasks, starting with Task III.

18.1 Task III

From the previous talks, we know that $\mathrm{K}(L_{\mathrm{K}(1)}\mathbb{S})$ is simultaneously simple enough to understand and compute, but able to detect the failure of the telescope conjecture (see Lecture 16). This is a significant result, and we are curious whether K-theory calculations could have further implications in classical homotopy theory.

The first related question can then be described as follows:

Question 18.1 (Hahn, Senger)**.** Can $\mathrm{K}(\mathrm{BP}\langle n\rangle)$ detect or prove the existence of Greek letters elements?

Remark 18.2 (hint from Hahn)**.** Note that motivic spectral sequences we have studied receive maps from the Adams–Novikov spectral sequence for $\pi_*\mathbb{S}$. Thus, it should be possible to analyze elements that are easy to name in terms of the Adams–Novikov spectral sequence.

The following questions are related to telescope conjectures:

Question 18.3. Does TC(MU) satisfy the telescope conjecture? The answer to this is almost surely yes, but it would be good to work out. The purity theorem may be helpful here.

Question 18.4. How does the algebraic K-theory redshift the Bousfield classes between $\mathrm{Sp}_{T(n)}$ and $\mathrm{Sp}_{K(n)}$?

There is a partial answer to Question 18.4 by Ben-Moshe, Carmeli, Schlank, and Yanovski [Ben+23]:

Theorem 18.5 (Ben-Moshe, Carmeli, Schlank, Yanovski, 2023)**.** *K-theory redshifts the cyclotomic hyperdescent Bousfield classes.*

A specific follow-up question is described as follows:

Question 18.6. Does K-theory redshift Bousfield classes of $\mathbb{E}_\infty$-rings in the 'Bockstein tower', as [CHY24, Open Question 6]? Note that the cyclotomic redshift above is a special case of this conjecture.

18.2 Task I

Antieau, Krause, and Nikolaus developed a powerful algorithm to compute $\pi_* \operatorname{TC}(\mathbb{Z}/p^n)$ for $n \in \mathbb{Z}$ in their paper [AKN24], via the computation of $\pi_* \operatorname{gr}^*_{\mathrm{mot}} \operatorname{TC}(\mathbb{Z}/p^n)$. Let R be a quasisyntomic ring. The graded piece

$$\operatorname{gr}^i_{\mathrm{mot}} \operatorname{TC}(R; \mathbb{Z}_p) \simeq \mathbb{Z}_p(i)(R)[2i],$$

where $\mathbb{Z}_p(i)(R)$ is a sequence of p-complete complexes in the p-complete derived category $D(\mathbb{Z}_p)^\wedge_p$. When $R = \mathbb{Z}/p^n$, it turns out the associated BMS motivic spectral sequence

$$E_2^{i,j} = H^{i-j}(\mathbb{Z}_p(-j)(R)) \Rightarrow \operatorname{TC}_{-i-j}(R; \mathbb{Z}_p)$$

is simple enough since it degenerates and there are no non-trivial extensions for degree reasons. Currently, using a computing cluster, they are able to compute $\pi_* \operatorname{TC}(\mathbb{Z}/8)$ for $* \leq 41$, after which the computational costs become too high to easily go further. There is still significant room for improved understanding.

In their computations, they observed that many even degree groups vanish. They developed this fact into a theorem, known as *even vanishing*:

Theorem 18.7 (Antieau, Krause, Nikolaus, 2024)**.** $\pi_{2*} \operatorname{TC}(\mathbb{Z}/p^n) = 0$ *for* $* \gg 0$.

We can take this as a model for excellent research following Task I. There are undoubtedly many interesting and conceptual patterns that, like even vanishing, will only become apparent upon carrying out further K-theory computations.

There's another theorem by Hahn, Levy, Senger, that was only conjectured after making lots of computations:

Lemma 18.8. $\pi_* \operatorname{gr}^*_{\mathrm{mot}} \operatorname{TC}(\mathbb{Z}/p^n)/(p, v_1)$ *does not depend on* $n \geq 2$.

Theorem 18.9 (Hahn, Levy, Senger)**.** *For any* $m \geq 0$ *and any animated commutative ring* R*, the functor* $R \mapsto \operatorname{gr}^*_{\mathrm{mot}} \operatorname{TC}(R)/(p, v_1^{p^n})$ *factors through another functor* $R \mapsto R/p^{n+2}$.

Question 18.10. Suppose R is an ℓ-algebra. Does $\operatorname{gr}^*_{\mathrm{mot}} \operatorname{TC}(R)/(p, v_1, v_2)$ depend only on R mod some powers of p and v_1?

Question 18.11. Can you compute some examples in their entirety? For instance, what is $\pi_* \operatorname{gr}^*_{\mathrm{mot}} \operatorname{TC}(\mathbb{Z}/4)/2$?

Question 18.12. For any of the following rings R, can you compute more than is currently known about $\operatorname{TC}(R)$, $\operatorname{TC}^-(R)$ or $\operatorname{TP}(R)$, possibly integrally, mod v_i's, mod powers of v_i's, or mod powers of v_i's with v_j inverted, etc.? In cases where motivic spectral sequences exists, one can ask the same questions about e.g. $\operatorname{gr}^*_{\mathrm{mot}} \operatorname{TC}(R)$ as a $C\tau$-module. What patterns and conjectures can you uncover?

Example rings R: $\mathrm{BP}\langle n \rangle$, BP, MU, $\mathbb{S}$, ko, tmf, $\tau_{\geq 0} E_n$, $\tau_{\geq 0} E_n^{tC_p}$, $\ell/(p^{k_0}, v_1^{k_1})$, $L_{K(1)}\mathbb{S}$, or $\tau_{\geq 0} K_n \coloneqq \tau_{\geq 0}(E_n/p, u_1, \cdots, u_{n-1})$, or $k(n)$ the connective cover of $2p^n - 2$ periodic Morava K-theory.

Although Question 18.12 is broad and largely open, it has recently become approachable in many cases. One recent work is by Angelini-Knoll, Ausoni, Rognes on $\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(\mathrm{ko})/(2, v_1, v_2)$ (see [GAR23]).

18.3 Task II

To begin, let's recall the following classical theorem by Voevodsky and others:

Theorem 18.13 (Voevodsky)**.** *For R a sufficiently nice discrete ring, the map*

$$K(R)_{(p)} \to L_1^f K(R)_{(p)}$$

is a π_*-isomorphism for $* \gg 0$.

We discuss a higher chromatic version of this theorem conjectured by Ausoni and Rognes for $\mathrm{BP}\langle n\rangle$. As a side note, Lichtenbaum-Quillen also holds for ko (see [GAR23]) and for $L_{K(1)}\mathbb{S}$ at $p \geq 5$ (see [Bur+23]).

Proposition 18.14 (Mathew)**.** *Let R be a connective $\mathrm{BP}\langle n\rangle$-algebra satisfying the Segal conjecture, i.e. the map*

$$\mathrm{THH}(R)/(p, v_1, \cdots, v_n) \to \mathrm{THH}(R)^{tC_p}/(p, v_1, \cdots, v_n)$$

is a π_*-isomorphism for $* \gg 0$. Then the map

$$\mathrm{TC}(R)_{(p)} \to L_{n+1}^f\,\mathrm{TC}(R)_{(p)}$$

is also a π_*-isomorphism for $* \gg 0$.

Question 18.15. What is the largest class of rings for which the Segal conjecture implies the Lichtenbaum-Quillen conjecture?

A recent work of Bhatt-Scholze [BS22] on the étale comparison theorem implies that if R is a p-complete discrete ring, then $\pi_*(v_1^{-1}\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(R)/p)$ is the mod p étale cohomology of the generic fiber of R with Tate twists. A follow-up question to both this and the LMMT purity theorem, investigated by Andy Senger:

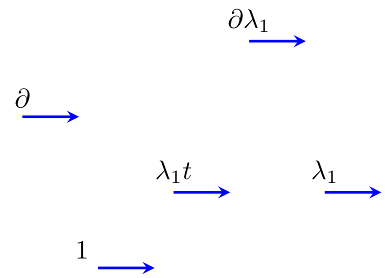
Question 18.16. Let R be a height n ring (e.g. E_n). Does

$$v_{n+1}^{-1}\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(R)/(p, v_1, \cdots, v_n)$$

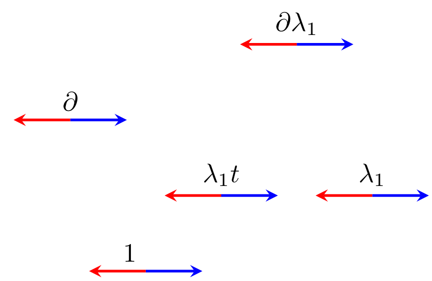
depend only on $v_n^{-1}\,\mathrm{gr}^*_{\mathrm{ev}}\,R/(p, v_1, \cdots, v_{n-1})$?

In Bhatt's lecture on the prismatic F-gauges [Bha22], he introduced a concept called the Lagrangian refinement of Tate duality. Recall that the Tate duality mainly deals with Poincaré duality of Galois cohomology, namely there is a pairing of Galois cohomology groups of some local field in the group scheme coefficient (working in the local case). To illustrate the idea of how this new concept could be applied to our interest case, let's look at the following example.

Example 18.17. Consider $\pi_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(\mathbb{Z}_2)/2$. We can give a quick sketch of the elements in E_2-page of the associated spectral sequence:



Then the corresponding sketch of the elements in E_2-page of the spectral sequence associated to the motivic filtration of $\pi_*(v_1^{-1}\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(\mathbb{Z}_2)/2)$ is as follows:



As suggested from the picture, every element is equipped with a "dual". $\pi_*(v_1^{-1}\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(\mathbb{Z}_2)/2)$ is then a Poincaré duality group with top class $v_1^{-1}\partial\lambda_1$. By the étale comparison theorem, this is related to étale cohomology of $\mathbb{Q}_2$, and the example reveals an instance of the Tate duality.

Actually, everything in the image of $\pi_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(\mathbb{Z}_2)/2$ pairs with something not in the image. This is an instance of the Lagrangian refinement.

Observe that $\pi_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(\ell)/(p,v_1) \to \pi_*(v_2^{-1}\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(\ell)/(p,v_1))$ is Lagrangian, i.e. this map fits into a fiber sequence where mapping to the fiber captures the information of Tate duality (see [Bha22]). There are two questions associated with this observation:

Question 18.18. How general is this observation?

Question 18.19. What interpretation can be given to these groups, if any, that is analogous to Galois/étale cohomology?

In fact, there are some special cases already achieved in works by Hahn, Devalapurkar, Raksit, and Yuan.

Another important related area is the prismatization/syntomification/F-gauges (see [Bha22]). Here's the breif idea. The prismatic cohomology can be obtained from a complex of sheaves on the Cartier-Witt stack. For a p-adic formal quasi-syntomic scheme X, one can associate it with a stack X^{Δ} called its prismatization. The Cartier-Witt stack is an example of the prismatization of $\mathrm{Spf}(\mathbb{Z}_p)$. The prismatization fits into a bigger stack called the Nygaard filtered prismatization $X^{\mathcal{N}}$. Gluing together two copies of X^{Δ} inside $X^{\mathcal{N}}$, one obtains another stack called the syntomification of X, denoted X^{Syn}. The derived category of quasi-coherent sheaves on X^{Syn} is the category of prismatic F-gauges. The prismatic F-gauges have strong connection with Galois representations, and allow us to refine Tate duality for Galois representations. For example, the étale realization of a coherent sheaf on

$\mathbb{Z}_p^{\mathrm{Syn}}$, upon inverting p, is a crystalline Galois representation. More details can be check in Bhatt's lecture notes [Bha22].

Returning to our case, let's consider $\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(\mathbb{Z}_p)/p$. After applying π_*, as discussed in Avi's talk (Lecture 12), we see that it relates to mod p syntomic cohomology at $\mathbb{Z}_p$. It corresponds to the F-gauges and furthermore the crystalline Galois representations. If we invert v_1 and look at $v_1^{-1}\,\underline{\mathrm{gr}}^*_{\mathrm{mot}}\,\mathrm{TC}(\mathbb{Z}_p)/p$, then it relates to the Galois cohomology of $\mathbb{Q}_p$, which corresponds to $\mathrm{Gal}(\overline{\mathbb{Q}_p} \mid \mathbb{Q}_p)$-representations. Morally, we want to define the category of quasi-coherent sheaves on stacks presented by the cosimplicial descent $\mathrm{THH}(\mathbb{Z}_p) \to A$, where A is even and the map is eff, but this works except for Nygaard completion. To fix this, we need to develop the even filtration in S^1-equivariant homotopy theory.

Question 18.20. What is the category of F-gauges on ℓ, in purely algebraic terms?

We also give a few rapid fire miscellaneous questions:

Question 18.21. Does the even filtration on G-spectra, where G a non-abelian compact Lie group, recover the Balmer spectrum?

I believe some people are working on the following:

Question 18.22. Can you define motivic filtrations that help when computing Grothendieck-Witt theory or algebraic L-theory?

Analogously we can ask the following two questions:

Question 18.23. Motivic filtrations computing equivariant algebraic K-theory K_G?

Question 18.24. Motivic filtrations computing Efimov's refined TC^-? See the work of Wagner and Meyer–Wagner.

As a final remark, the abelian case of Question 18.21 has already been solved. Hausmann [Hau19] showed that $\pi_*^A \mathrm{MU}_A$ is isomorphic to an A-equivariant Lazard ring L_A for every abelian compact Lie group A. Hausmann-Meier [HM23] showed that the Balmer spectrum of finite A-spectra is homeomorphic to the spectrum of points of the moduli stack $\mathcal{M}_{FG}^A$ of A-equivariant formal groups, which is equivariant to the spectrum of invariant prime ideals of the A-equivariant Lazard ring L_A.

References

[AHL10] Vigleik Angeltveit, Michael Hill, and Tyler Lawson. "Topological Hochschild homology of l and ko". In: *American Journal of Mathematics* 132.2 (2010), pp. 297–330. URL: https://www.jstor.org/stable/40730782.

[AKN24] Benjamin Antieau, Achim Krause, and Thomas Nikolaus. "On the K-theory of $\mathbb{Z}/p^n$". In: *arXiv preprint arXiv:2405.04329* (2024).

[Ang+23] Gabriel Angelini-Knoll, Mark Behrens, Eva Belmont, and Hana Jia Kong. *A deformation of Borel equivariant homotopy*. 2023. arXiv: 2308.01873 [math.AT]. URL: https://arxiv.org/abs/2308.01873.

[Ant+22] Benjamin Antieau, Akhil Mathew, Matthew Morrow, and Thomas Nikolaus. "On the Beilinson fiber square". In: *Duke Mathematical Journal* 171.18 (2022), pp. 3707–3806. DOI: 10.1215/00127094-2022-0037. URL: https://doi.org/10.1215/00127094-2022-0037.

[BCS10] Andrew J. Blumberg, Ralph L. Cohen, and Christian Schlichtkrull. "Topological Hochschild homology of Thom spectra and the free loop space". In: *Geometry & Topology* 14.2 (2010), pp. 1165–1242.

[Ben+23] Shay Ben-Moshe, Shachar Carmeli, Tomer M Schlank, and Lior Yanovski. "Descent and cyclotomic redshift for chromatically localized algebraic K-theory". In: *arXiv preprint arXiv:2309.07123* (2023).

[Bha22] Bhargav Bhatt. *Prismatic F-gauges*. 2022. URL: https://www.math.ias.edu/~bhatt/teaching/mat549f22/lectures.pdf.

[BHS20] Robert Burklund, Jeremy Hahn, and Andrew Senger. "Galois reconstruction of Artin-Tate $\mathbb{R}$-motivic spectra". In: *arXiv e-prints*, arXiv:2010.10325 (Oct. 2020), arXiv:2010.10325. DOI: 10.48550/arXiv.2010.10325. arXiv: 2010.10325 [math.AT].

[Blu10] Andrew J. Blumberg. "Topological Hochschild homology of Thom spectra which are $\mathbb{E}_\infty$-ring spectra". In: *Journal of Topology* 3 (2010), pp. 535–560.

[BM10] Maria Basterra and Michael A. Mandell. "The Multiplication on BP". In: *arXiv e-prints*, arXiv:1101.0023 (Dec. 2010), arXiv:1101.0023. DOI: 10.48550/arXiv.1101.0023. arXiv: 1101.0023 [math.AT].

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. "Topological Hochschild homology and integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 129 (2019), pp. 199–310. ISSN: 0073-8301,1618-1913. DOI: 10.1007/s10240-019-00106-9. URL: https://doi.org/10.1007/s10240-019-00106-9.

[Bou79] A.K. Bousfield. "The localization of spectra with respect to homology". In: *Topology* 18.4 (1979), pp. 257–281. ISSN: 0040-9383. DOI: https://doi.org/10.1016/0040-9383(79)90018-1. URL: https://www.sciencedirect.com/science/article/pii/0040938379900181.

[BS22] Bhargav Bhatt and Peter Scholze. "Prisms and Prismatic Cohomology". In: *arXiv preprint arXiv:1905.08229* (2022).

[Bur+23] Robert Burklund, Jeremy Hahn, Ishan Levy, and Tomer M Schlank. "K-theoretic counterexamples to Ravenel's telescope conjecture". In: *arXiv preprint arXiv:2310.17459* (2023).

[CHY24] Hood Chatham, Jeremy Hahn, and Allen Yuan. "Wilson Spaces, Snaith Constructions, and Elliptic Orientations". In: *arXiv preprint arXiv:1910.04616* (2024).

[Cla+20] Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel. "Descent and vanishing in chromatic algebraic K-theory via group actions". In: *arXiv preprint arXiv:2011.08233* (2020).

[CM21] Dustin Clausen and Akhil Mathew. "Hyperdescent and étale K-theory". In: *Inventiones mathematicae* 225 (2021), pp. 981–1076.

[CSY21] Shachar Carmeli, Tomer M Schlank, and Lior Yanovski. "Chromatic cyclotomic extensions". In: *arXiv preprint arXiv:2103.02471* (2021).

[GAR23] Angelini-Knoll Gabriel, Christian Ausoni, and John Rognes. "Algebraic K-theory of real topological K-theory". In: *arXiv preprint arXiv:2309.11463* (2023).

[GHR23] Jesper Grodal, Mike Hill, and Birgit Richter. *Report No. 34/2023: Homotopy Theory*. 2023. URL: `https://publications.mfo.de/bitstream/handle/mfo/4079/OWR_2023_34.pdf?sequence=4`.

[Goo92] Thomas Goodwillie. "Calculus II: Analytic Functors". In: *K-Theory* 5 (1992), pp. 295–332. URL: `https://people.math.rochester.edu/faculty/doug/otherpapers/goodwillie-calc2.pdf`.

[Hau19] Markus Hausmann. "Global group laws and equivariant bordism rings". In: *arXiv preprint arXiv:1912.07583* (2019).

[Hed21] Alice Petronella Hedenlund. "Multiplicative Tate spectral sequences". In: (2021).

[Hes94] Lars Hesselholt. "Stable topological cyclic homology is topological Hochschild homology". In: *K-theory - Strasbourg* 226 (1994), pp. 175–192. URL: `http://www.numdam.org/item/AST_1994__226__175_0/`.

[HM23] Markus Hausmann and Lennart Meier. "Invariant prime ideals in equivariant Lazard rings". In: *arXiv preprint arXiv:2309.00850* (2023).

[HRW22] Jeremy Hahn, Arpon Raksit, and Dylan Wilson. "A motivic filtration on the topological cyclic homology of commutative ring spectra". In: *arXiv preprint arXiv:2206.11208* (2022).

[HS18] Lars Hesselholt and Peter Scholze. "Arbeitsgemeinschaft: Topological Cyclic Homology". In: *Oberwolfach Rep.* 15.2 (2018), pp. 805–940.

[HW22] Jeremy Hahn and Dylan Wilson. "Redshift and multiplication for truncated Brown-Peterson spectra". In: *Ann. of Math. (2)* 196.3 (2022), pp. 1277–1351. ISSN: 0003-486X,1939-8980. DOI: `10.4007/annals.2022.196.3.6`. URL: `https://doi.org/10.4007/annals.2022.196.3.6`.

[KN18] Achim Krause and Thomas Nikolaus. *Lectures on Topological Hochschild Homology and cyclotomic spectra*. 2018. URL: `https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/Papers/Lectures.pdf`.

[Lan+24] Markus Land, Akhil Mathew, Lennart Meier, and Georg Tamme. "Purity in chromatically localized algebraic K-theory". In: *Journal of the American Mathematical Society* (2024).

[Lev22] Ishan Levy. "The algebraic K-theory of the K (1)-local sphere via TC". In: *arXiv preprint arXiv:2209.05314* (2022).

[LL23] David Jongwon Lee and Ishan Levy. "Topological Hochschild homology of the image of j". In: *arXiv preprint arXiv:2307.04248* (2023).

[LT19] Markus Land and Georg Tamme. "On the K-theory of pullbacks". In: *Annals of Mathematics* 190.3 (2019), pp. 877–930.

[Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: 10.1515/9781400830558. URL: https://doi.org/10.1515/9781400830558.

[Lur10] Jacob Lurie. *Chromatic homotopy theory*. Lecture notes for Harvard Math 252X. 2010.

[Lur17] Jacob Lurie. *Higher algebra*. Unpublished. Available online at https://www.math.ias.edu/~lurie/. Sept. 2017.

[Mad95] Ib Madsen. *Algebraic K-theory and traces*. 1995. URL: https://intlpress.com/site/pub/files/_fulltext/journals/cdm/1995/1995/0001/CDM-1995-1995-0001-a003.pdf.

[Mat21] Akhil Mathew. "On $K(1)$-local TR". In: *Compos. Math.* 157.5 (2021), pp. 1079–1119. ISSN: 0010-437X,1570-5846. DOI: 10.1112/S0010437X21007144. URL: https://doi.org/10.1112/S0010437X21007144.

[Mil58] John Milnor. "The Steenrod algebra and its dual". In: *Ann. of Math. (2)* 67 (1958), pp. 150–171. ISSN: 0003-486X. DOI: 10.2307/1969932. URL: https://doi.org/10.2307/1969932.

[MR99] Mark Mahowald and Charles Rezk. "Brown-Comenetz duality and the Adams spectral sequence". In: *Amer. J. Math.* 121.6 (1999), pp. 1153–1177. ISSN: 0002-9327,1080-6377. URL: http://muse.jhu.edu/journals/american_journal_of_mathematics/v121/121.6mahowald.pdf.

[MSV97] J. McClure, R. Schwänzl, and R. Vogt. "THH$(R) \cong R \otimes S^1$ for $\mathbb{E}_\infty$ ring spectra". In: *J. Pure Appl. Algebra* 121.2 (1997), pp. 137–159.

[NS18] Thomas Nikolaus and Peter Scholze. "On topological cyclic homology". In: *Acta Math.* 221.2 (2018), pp. 203–409.

[Rav86] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*. Vol. 121. Pure and Applied Mathematics. Academic Press, Inc., Orlando, FL, 1986, pp. xx+413. ISBN: 0-12-583430-6; 0-12-583431-4.

[Rav93] Douglas C Ravenel. "Life after the telescope conjecture". In: *Algebraic K-Theory and Algebraic Topology*. Springer, 1993, pp. 205–222.

[RSV22] Nima Rasekh, Bruno Stonek, and Gabriel Valenzuela. "Thom spectra, higher THH and tensors in ∞-categories". In: *Algebraic & Geometric Topology* 22 (4 2022), pp. 1841–1903.

[Sch11] Christian Schlichtkrull. "Higher topological Hochschild homology of Thom spectra". In: *Journal of Topology* 4.1 (2011), pp. 161–189.

[Wil75] W. Stephen Wilson. "The Ω-spectrum for Brown-Peterson cohomology. II". In: *Amer. J. Math.* 97 (1975), pp. 101–123. ISSN: 0002-9327,1080-6377. DOI: 10.2307/2373662. URL: https://doi.org/10.2307/2373662.

[Wil82] W. Wilson. *Brown-Peterson Homology: An Introduction and Sampler*. Dec. 1982. DOI: 10.1090/cbms/048. URL: https://www.ams.org/cbms/048.