

NOTE ON COFIBRATIONS II

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Introduction.

The present paper is a continuation of [7] and contains some results of a general topological nature concerning fibrations and cofibrations. Section 1 is devoted to the proof of a dual of theorem 1 of [7], while the second section contains a characterization of cofibrations and some immediate consequences of this result. Theorem 3 of [7] is strengthened and dualized in section 3, and in the last section we prove that the pull-back of a closed cofibration over a fibration is a cofibration and we prove a conjecture of Per Holm (see [2]), who has also made a number of valuable suggestions. After the work described here was completed Puppe has published his article [5], which slightly overlaps this one.

A few words about notation. The set Y^X of all continuous functions from X to Y is given the compact-open topology. Continuous maps $i: T \rightarrow X$ and $p: Y \rightarrow Z$ induce continuous maps $i^\#: Y^X \rightarrow Y^T$ and $p_\#: Y^X \rightarrow Z^X$ such that $i^\#(f) = fi$ and $p_\#(f) = pf$. We denote by I the closed unit interval $[0, 1]$ with the usual topology and boundary $\dot{I} = \{0, 1\}$. For any space X continuous maps $i_0: X \rightarrow X \times I$, $\pi_0, \pi_1: X^I \rightarrow X$ are defined by $i_0(x) = (x, 0)$, $\pi_0(f) = f(0)$, $\pi_1(f) = f(1)$. By $pr_1: X \times Y \rightarrow X$ and $pr_2: X \times Y \rightarrow Y$ we denote the projections. Further $a \wedge b$ denotes the smaller of two real numbers a and b . All maps considered will be continuous.

We shall have occasion to use the following theorem of “exponential correspondence”.

EXP. *For arbitrary spaces X , Y , and Z there is an injection (not necessarily continuous)*

$$\vartheta: Y^{X \times Z} \rightarrow (Y^X)^Z$$

such that $[\vartheta(f)(z)](x) = f(x, z)$. If X is locally compact and regular, ϑ is a bijection.

See [3, V. 3] for a proof. The maps f and $f' = \vartheta(f)$ are called *associate maps*. An immediate consequence of EXP is that $i: A \rightarrow X$ is a cofibration if and only if every commutative diagram

$$\begin{array}{ccc}
 A & \xrightarrow{F'} & Y^I \\
 i \downarrow & & \downarrow \pi_0 \\
 X & \xrightarrow{f} & Y
 \end{array}$$

can be filled in with a commutativity preserving map $\bar{F}': X \rightarrow Y^I$.

1.

In [7] it was proved that all cofibrations are imbeddings. In the case of a fibration $p: E \rightarrow B$ it will not always be true that $p(E)$ is a quotient space of E (see 2.4.8 of [6] for a counterexample), but we do have the following result.

THEOREM 1. *If $p: E \rightarrow B$ is a surjective fibration with a locally path connected base space B , then p is a quotient map.*

PROOF. The proof is modelled on the proof of theorem 1 of [7].

Consider the subspace

$$\bar{B} = \{(e, \omega) \in E \times B^I \mid \omega(0) = p(e)\}$$

of $E \times B^I$ and define $\bar{p}: E^I \rightarrow \bar{B}$ by $\bar{p}(\omega) = (\omega(0), p\omega)$. It is well known that there exists a section λ of $\bar{p}$ (cf. [4]). The map $\pi_1: E^I \rightarrow E$ also has a section $s: E \rightarrow E^I$ sending each point of E to the constant path at that point. Consequently $\bar{p}$ and π_1 are quotient maps. We define a map $\pi: \bar{B} \rightarrow B$ by $\pi(e, \omega) = \omega(1)$ and so obtain a commutative diagram

$$\begin{array}{ccc}
 E^I & \xrightarrow{\pi_1} & E \\
 \bar{p} \downarrow & & \downarrow p \\
 \bar{B} & \xrightarrow{\pi} & B
 \end{array}$$

Because π_1 and $\bar{p}$ are quotient maps, p is a quotient map if and only if π is a quotient map. We shall prove that π is a quotient map.

Let A be a subset of B such that $\pi^{-1}(A)$ is open in $\bar{B}$ and suppose that $b \in A$. If ω_b is the constant path at b and e is a point of $p^{-1}(b)$, then $(e, \omega_b) \in \pi^{-1}(A)$ and there exists an open set $W \subset B^I$ such that

$$(e, \omega_b) \in (e \times W) \cap \bar{B} \subset \pi^{-1}(A).$$

Because ω_b is a constant path it is easily seen that there exists an open set $U \subset B$ such that

$$\omega_b \in U^I \subset W,$$

U^I being regarded as a subspace of B^I . Now, b belongs to U , and because B is locally path connected the path component V of U containing b is open. If b' is an arbitrary point of V there exists a path ω in U such that $\omega(0)=b$ and $\omega(1)=b'$. Then

$$(e, \omega) \in (e \times W) \cap \bar{B} \subset \pi^{-1}(A)$$

and $b' = \pi(e, \omega) \in A$. Therefore $V \subset A$, and so b is an interior point of A . But b was an arbitrary point of A and consequently A is open.

2.

THEOREM 2. *The pair (X, A) is cofibered if and only if $X \times 0 \cup A \times I$ is a retract of $X \times I$.*

PROOF. If (X, A) is a cofibered pair the identity map

$$X \times 0 \cup A \times I \rightarrow X \times 0 \cup A \times I$$

extends to a retraction

$$r: X \times I \rightarrow X \times 0 \cup A \times I.$$

Conversely, if such a retraction exists, then every continuous map

$$f: X \times 0 \cup A \times I \rightarrow Y$$

has a continuous extension

$$fr: X \times I \rightarrow Y.$$

It remains to show that every function $f: X \times 0 \cup A \times I \rightarrow Y$ whose restrictions $f|X \times 0$ and $f|A \times I$ are continuous is itself continuous. This is an immediate consequence of the following lemma (which is trivial if A is closed).

LEMMA 3. *If (X, A) is a pair such that $X \times 0 \cup A \times I$ is a retract of $X \times I$, then a subset C of $X \times 0 \cup A \times I$ is open in $X \times 0 \cup A \times I$ if and only if $C \cap X \times 0$ and $C \cap A \times I$ are open in $X \times 0$ and $A \times I$ respectively.*

PROOF. The “only if”-part is obvious. To prove the “if”-part let $C \subset X \times 0 \cup A \times I$ be such that $C \cap X \times 0$ and $C \cap A \times I$ are open in $X \times 0$

and $A \times I$ respectively. It is then easily seen that C is the union of $C \cap (A \times \langle 0, 1 \rangle)$ (which is open in $X \times 0 \cup A \times I$) and the set

$$B = U \times 0 \cup \bigcup_{n=1}^{\infty} ((A \cap U_n) \times [0, 1/n]),$$

where $U, U_1, U_2, \dots$ are open subsets of X given by

$$U = \{x \in X \mid (x, 0) \in C\},$$

$$U_n = \bigcup \{V \mid V \text{ open in } X \text{ and } (V \cap A) \times [0, 1/n] \subset C\}.$$

Then $A \cap U = A \cap \bigcup_{n=1}^{\infty} U_n$ and if V is an open subset of X such that $V \cap A \subset U_n$, then $V \subset U_n$.

We prove $U \subset \bigcup_{n=1}^{\infty} U_n$. Suppose $x \in X - \bigcup_{n=1}^{\infty} U_n$. Then $x \in \bar{A}$. Let $t \in \langle 0, 1 \rangle$. We then have

$$r(x, t) \in r(\bar{A} \times t) = A \times t.$$

If $r(x, t)$ belongs to some $U_n \times I$ there must exist open neighborhoods V and W of x and t respectively such that

$$r(V \times W) \subset U_n \times I.$$

We should then have

$$(V \cap A) \times t = r((V \cap A) \times t) \subset U_n \times I,$$

that is, $V \cap A \subset U_n$. But this, in turn, would imply $V \subset U_n$, and so

$$x \in U_n \subset \bigcup_{n=1}^{\infty} U_n,$$

contrary to hypothesis.

Consequently

$$r(x, t) \in \left(A - \bigcup_{n=1}^{\infty} U_n \right) \times I = (A - U) \times I \subset (X - U) \times I$$

for each $t \in \langle 0, 1 \rangle$, and, since r is continuous and $X - U$ is closed,

$$(x, 0) = r(x, 0) \in (X - U) \times I, \quad x \in X - U,$$

which shows that $X - \bigcup_{n=1}^{\infty} U_n \subset X - U$, that is, $U \subset \bigcup_{n=1}^{\infty} U_n$.

Let $V_n = U \cap U_n$, $n = 1, 2, \dots$. Then each V_n is open in X , $U = \bigcup_{n=1}^{\infty} V_n$, $A \cap U_n = A \cap V_n$, and

$$B = (X \times 0 \cup A \times I) \cap \bigcup_{n=1}^{\infty} (V_n \times [0, 1/n])$$

is open in $X \times 0 \cup A \times I$. But then

$$C = B \cup (C \cap (A \times \langle 0, 1 \rangle))$$

is also open in $X \times 0 \cup A \times I$, and the lemma is proved.

If A is a subspace of a space X the mapping cylinder of the inclusion map $A \subset X$ may be identified with the subset $X \times 0 \cup A \times I$ of $X \times I$. Lemma 3 shows that if $X \times 0 \cup A \times I$ is a retract of $X \times I$, then the subspace topology inherited from $X \times I$ is identical with the mapping cylinder topology. These topologies are also identical if A is closed, even if no retraction of $X \times I$ to $X \times 0 \cup A \times I$ exists, but examples are easily constructed to show that they need not be identical for arbitrary pairs (X, A) .

We can now prove

LEMMA 4. *The pair (X, A) is cofibered if and only if there exist a continuous function $\varphi: X \rightarrow I$ such that $A \subset \varphi^{-1}(0)$ and a homotopy $H: X \times I \rightarrow X$ such that*

$$\begin{aligned} H(x, 0) &= x, & x \in X, \\ H(a, t) &= a, & a \in A, t \in I, \end{aligned}$$

and such that $H(x, t) \in A$ whenever $t > \varphi(x)$.

If, in addition, A is a strong deformation retract of X we may assume that φ is everywhere less than 1.

PROOF. If there exists a retraction $r: X \times I \rightarrow X \times 0 \cup A \times I$ we may define φ and H as follows:

$$\begin{aligned} \varphi(x) &= \sup_{t \in I} |t - pr_2 r(x, t)|, & x \in X, \\ H(x, t) &= pr_1 r(x, t), & x \in X, t \in I. \end{aligned}$$

Conversely, given φ and H a retraction $r: X \times I \rightarrow X \times 0 \cup A \times I$ is defined by

$$r(x, t) = \begin{cases} (H(x, t), 0), & t \leq \varphi(x), \\ (H(x, t), t - \varphi(x)), & t \geq \varphi(x). \end{cases}$$

Finally, given φ and H and a strong deformation retraction $D: X \times I \rightarrow X$ of X to A we may replace $\varphi(x)$ and $H(x, t)$ by $\varphi'(x) = \frac{1}{2} \wedge \varphi(x)$ and $H'(x, t) = D(H(x, t), 2t \wedge 1)$.

Note that if $\varphi(x) < 1$, then $H(x, \varphi(x)) \in \overline{H(x \times \langle \varphi(x), 1 \rangle)} \subset \bar{A}$. Thus, replacing $H(x, t)$ by $\bar{H}(x, t) = H(x, t \wedge \varphi(x))$ we obtain

COROLLARY 5. *If (X, A) is a cofibered pair, so is $(X, \bar{A})$.*

We use lemma 4 to prove

THEOREM 6. *If (X, A) and (Y, B) are cofibered pairs with A closed in X , then the product pair*

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$$

is also cofibered. If, in addition, A (or B) is a strong deformation retract of X (or Y), then $X \times B \cup A \times Y$ is a strong deformation retract of $X \times Y$.

PROOF. Let $\varphi: X \rightarrow I$ and $H: X \times I \rightarrow X$ be as described in lemma 4 and let ψ and G be the corresponding maps for (Y, B) . Define $\eta: X \times Y \rightarrow I$ and $F: X \times Y \times I \rightarrow X \times Y$ by

$$\begin{aligned}\eta(x, y) &= \varphi(x) \wedge \psi(y), \\ F(x, y, t) &= (H(x, t \wedge \psi(y)), G(y, t \wedge \varphi(x))) .\end{aligned}$$

Then $X \times B \cup A \times Y \subset \eta^{-1}(0)$ and $F(x, y, t) = (x, y)$ if $t = 0$ or $(x, y) \in X \times B \cup A \times Y$.

Because A is closed $H(x, \varphi(x)) \in A$ whenever $\varphi(x) < 1$. Now suppose that $t \in I$ and $t > \eta(x, y)$. Then either $\varphi(x) \leq \psi(y)$ and $\varphi(x) < t$, in which case $t \wedge \psi(y) \geq \varphi(x)$ and $F(x, y, t) \in A \times Y$, or $\psi(y) < \varphi(x)$ and $\psi(y) < t$, so that $t \wedge \varphi(x) > \psi(y)$ and $F(x, y, t) \in X \times B$. This shows that $F(x, y, t) \in X \times B \cup A \times Y$ whenever $t > \eta(x, y)$, and it follows from lemma 4 that $(X \times Y, X \times B \cup A \times Y)$ is cofibered.

If A (or B) is a strong deformation retract of X (or Y), then we may assume that φ (or ψ) is everywhere less than 1. But then $\eta(x, y) < 1$ for all $(x, y) \in X \times Y$, and so $F(x, y, 1) \in X \times B \cup A \times Y$, which shows that F is a strong deformation retraction of $X \times Y$ to $X \times B \cup A \times Y$.

See [5] for an example showing that $(X \times Y, X \times B \cup A \times Y)$ need not be cofibered if neither A nor B is closed.

In the way of a converse of theorem 6 we have

THEOREM 7. *Suppose that $A \subset X$, that there exists a continuous function $\varphi: X \rightarrow I$ with $A \subset \varphi^{-1}(0)$, and that there exists a point $x_0 \in X - A$ such that $\varphi(x_0) \neq 0$. Then if (Y, B) is a pair such that $(X \times Y, X \times B \cup A \times Y)$ is cofibered, (Y, B) itself is cofibered.*

PROOF. Let $\eta: X \times Y \rightarrow I$ and $F: X \times Y \times I \rightarrow X \times Y$ be functions for $(X \times Y, X \times B \cup A \times Y)$ as described in lemma 4. We may obviously assume that $\varphi(x_0) = 1$, and the functions $G: Y \times I \rightarrow Y$ and $\psi: Y \rightarrow I$ defined by

$$\begin{aligned}G(y, t) &= pr_2 F(x_0, y, t), \\ \psi(y) &= \max(\eta(x_0, y), 1 - \inf_{t \in I} \varphi pr_1 F(x_0, y, t)),\end{aligned}$$

will then satisfy the conditions of lemma 4.

Note, in particular, that (X, A) is cofibered if and only if $(X \times I, X \times 0 \cup A \times I)$ is cofibered, and then $X \times 0 \cup A \times I$ is a strong deformation retract of $X \times I$.

3.

According to 1.4.10 and 1.4.11 of [6] a cofibration $i: A \subset X$ is a homotopy equivalence if and only if A is a strong deformation retract of X (the closedness restriction on A in [6] is unnecessary in our case in view of lemma 3). Correspondingly, a fibration $p: E \rightarrow B$ is a homotopy equivalence if and only if there exists a section $s: B \rightarrow E$ of p such that $sp \simeq \underset{p}{1_E}$ (see 6.2 of [1]).

We shall strengthen theorem 3 of [7] and also prove its dual. But first a definition.

DEFINITION. If $i: A \rightarrow X$ and $p: E \rightarrow B$ are maps, a *map pair* $f = (f'', f'): i \rightarrow p$ is a pair of maps $f'': A \rightarrow E$ and $f': X \rightarrow B$ such that $pf'' = f'i$, that is, the diagram

$$\begin{array}{ccc} A & \xrightarrow{f''} & E \\ i \downarrow & & \downarrow p \\ X & \xrightarrow{f'} & B \end{array}$$

commutes. A map $\bar{f}: X \rightarrow E$ defines a map pair

$$\varrho(\bar{f}) = (\bar{f}i, p\bar{f}): i \rightarrow p.$$

$\bar{f}$ is called a *lifting* of the pair $\varrho(\bar{f})$.

THEOREM 8. *Let $i: A \rightarrow X$ be a map such that $i(A)$ is closed in X . The following are then equivalent.*

- (i) *Every map pair $f: i \rightarrow p$ with $p: E \rightarrow B$ a fibration has a lifting.*
- (ii) *i is a cofibration and a homotopy equivalence.*

When (i) and (ii) hold the lifting $\bar{f}$ of f is unique up to homotopy relative to $i(A)$.

PROOF. (ii) $\Rightarrow$ (i) and the uniqueness property are just theorem 3 of [7]. To prove that (i) $\Rightarrow$ (ii) note that $\pi_0: Y^I \rightarrow Y$ is a fibration for any space Y (2.8.2 of [6]), and so $i: A \rightarrow X$ must be a cofibration, and we may assume that i is an inclusion map. Because $A \rightarrow *$ is a fibration ($*$ de-

notes a one-point space), a retraction $r: X \rightarrow A$ is obtained as a lifting of the map pair

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ i \cap & & \downarrow \\ X & \longrightarrow & * \end{array}$$

The map $p: X^I \rightarrow X \times X$ defined by $p(\omega) = (\omega(0), \omega(1))$ is also a fibration ([6], 2.8.3), and the map pair

$$\begin{array}{ccc} A & \xrightarrow{f''} & X^I \\ i \cap & & \downarrow p \\ X & \xrightarrow{f'} & X \times X \end{array}$$

with $f''(a)(t) = a$, $f'(x) = (x, r(x))$ has a lifting $\bar{f}: X \rightarrow X^I$ associate to a strong deformation retraction of X to A .

In a similar fashion we prove

THEOREM 9. *For a map $p: E \rightarrow B$ the following are equivalent.*

(i) *Every map pair $f: i \rightarrow p$ with $i: A \subset X$ a closed cofibration has a lifting.*

(ii) *p is a fibration and a homotopy equivalence.*

When (i) and (ii) hold the lifting $\bar{f}$ of f is unique up to fiber homotopy over p .

PROOF. (i) $\Rightarrow$ (ii): Applying (i) to map pairs

$$\begin{array}{ccc} X \times 0 & \longrightarrow & E \\ \cap & & \downarrow p \\ X \times I & \longrightarrow & B \end{array}$$

we see that p must be fibration. The pair $(B, \emptyset)$ is a cofibered pair, and a section $s: B \rightarrow E$ of p is obtained as a lifting of

$$\begin{array}{ccc} \emptyset & \subset & E \\ \cap & & \downarrow p \\ B & \longrightarrow & B \\ & 1_B & \end{array}$$

Finally, let $F: E \times I \rightarrow E$ be a lifting of the map pair

$$\begin{array}{ccc} E \times I & \xrightarrow{f''} & E \\ \cap & & \downarrow p \\ E \times I & \xrightarrow{f'} & B \end{array}$$

with $f''(e, 0) = sp(e)$, $f''(e, 1) = e$, $f'(e, t) = p(e)$. Then $F: sp \underset{p}{\simeq} 1_E$.

(ii) $\Rightarrow$ (i): We know that there exists a section s of p and a fiber homotopy $F: sp \underset{p}{\simeq} 1_E$. Let $A \subset X$ be a closed cofibration and consider the map pair

$$\begin{array}{ccc} A & \xrightarrow{f''} & E \\ \cap & & \downarrow p \\ X & \xrightarrow{f'} & B \end{array}$$

Define $F'': X \times 0 \cup A \times I \rightarrow E$ and $F': X \times I \rightarrow B$ by $F''(x, 0) = sf'(x)$, $F''(a, t) = F(f''(a), t)$, and $F'(x, t) = f'(x)$. The diagram

$$\begin{array}{ccc} X \times 0 \cup A \times I & \xrightarrow{F''} & E \\ \cap & & \downarrow p \\ X \times I & \xrightarrow{F'} & B \end{array}$$

is then commutative and has a lifting $\bar{F}: X \times I \rightarrow E$ (theorem 4 of [7]). A lifting $\bar{f}$ of (f'', f') is given by $\bar{f}(x) = \bar{F}(x, 1)$. Finally, any lifting $\bar{f}$ of (f'', f') is fiber homotopic to $sp\bar{f} = sf'$.

For maps $i: A \rightarrow X$ and $p: E \rightarrow B$ the set of map pairs $i \rightarrow p$ may be identified with the fibered product $E^A \times' B^X$ of the maps $i^*: B^X \rightarrow B^A$ and $p_*: E^A \rightarrow B^A$, and the function ϱ mentioned above is then a continuous map from E^X to $E^A \times' B^X$. We have the following analogue of 7.8.10 of [6].

THEOREM 10. *If $i: A \subset X$ is a closed cofibration with X locally compact and regular and $p: E \rightarrow B$ is a fibration, then $\varrho: E^X \rightarrow E^A \times' B^X$ is a fibration, and if i or p is a homotopy equivalence, so is ϱ .*

PROOF. Given a map pair

$$\begin{array}{ccc} Y & \xrightarrow{f'} & E^X \\ i_0 \downarrow & & \downarrow \varrho \\ Y \times I & \xrightarrow{F'} & E^A \times' B^X \end{array}$$

we shall prove the existence of a lifting $\bar{F}': Y \times I \rightarrow E^X$. By EXP there exist maps $f: Y \times X \times 0 \cup Y \times A \times I \rightarrow E$ and $F: Y \times X \times I \rightarrow B$ such that

$$\begin{aligned} f(y, x, 0) &= f'(y)(x), \\ f(y, a, t) &= [pr_1 F'(y, t)](a), \\ F(y, x, t) &= [pr_2 F'(y, t)](x). \end{aligned}$$

The diagram

$$\begin{array}{ccc} Y \times X \times 0 \cup Y \times A \times I & \xrightarrow{f} & E \\ \cap & & \downarrow p \\ Y \times X \times I & \xrightarrow{F} & B \end{array}$$

is then commutative. By theorem 6 $(Y \times X, Y \times A) = (Y, \emptyset) \times (X, A)$ is a cofibered pair, and since $Y \times A$ is closed in $Y \times X$ theorem 4 of [7] gives us a lifting $\bar{F}: Y \times X \times I \rightarrow E$ of (f, F) . The associate map $\bar{F}': Y \times I \rightarrow E^X$ is then a lifting of (f', F') .

Now, suppose that i or p is a homotopy equivalence and let $C \subset Z$ be a closed cofibration. Every map pair

$$(1) \quad \begin{array}{ccc} C & \longrightarrow & E^X \\ \cap & & \downarrow \varrho \\ Z & \longrightarrow & E^A \times' B^X \end{array}$$

corresponds to a map pair

$$(2) \quad \begin{array}{ccc} Z \times A \cup C \times X & \longrightarrow & E \\ \cap & & \downarrow p \\ Z \times X & \longrightarrow & B \end{array}$$

(EXP again), and theorem 6 together with theorem 8 or 9 gives a lifting $Z \times X \rightarrow E$ of (2). The associate map $Z \rightarrow E^X$ is then a lifting of (1). Consequently ϱ is a homotopy equivalence.

The following theorem is related to theorem 10 in very much the same way as theorem 7 is to theorem 6.

THEOREM 11. *Let (X, A) be a topological pair and $p: E \rightarrow B$ a map. Suppose that $\varrho: E^X \rightarrow E^A \times' B^X$ is a fibration and that there exist a continuous function $\varphi: X \rightarrow I$ and a point $x_0 \in X$ such that $A \subset \varphi^{-1}(0)$ and $\varphi(x_0) \neq 0$. Then $p: E \rightarrow B$ is a fibration.*

PROOF. We may assume $\varphi(x_0) = 1$. In order to establish the existence of a lifting of the map pair

$$\begin{array}{ccc} Y & \xrightarrow{f} & E \\ i_0 \downarrow & & \downarrow p \\ Y \times I & \xrightarrow{F} & B \end{array}$$

we define $g: Y \rightarrow E^X$ and $G: Y \times I \rightarrow E^A \times' B^X$ by

$$\begin{aligned} g(y)(x) &= f(y), \\ [pr_1 G(y, t)](a) &= f(y), \\ [pr_2 G(y, t)](x) &= F(y, t \wedge \varphi(x)). \end{aligned}$$

We thus obtain a map pair $(g, G): i_0 \rightarrow \varrho$, and since ϱ is a fibration (g, G) has a lifting $\bar{G}: Y \times I \rightarrow E^X$. The map $\bar{F}: Y \times I \rightarrow E$ defined by $\bar{F}(y, t) = \bar{G}(y, t)(x_0)$ is then a lifting of (f, F) .

If we put $(X, A) = (I, 0)$ it follows that, in the notation used in the proof of theorem 1, $p: E \rightarrow B$ is a fibration if and only if $\bar{p}: E^I \rightarrow \bar{B} \approx E^0 \times' B^I$ is a fibration, and then $\bar{p}$ is a homotopy equivalence, which implies that the lifting function $\lambda: \bar{B} \rightarrow E^I$ for p is unique up to fiber homotopy over $\bar{p}$ (cf. [4]), corresponding to the fact that the retraction $X \times I \rightarrow X \times 0 \cup A \times I$ for a cofibration $A \subset X$ is unique up to homotopy relative to $X \times 0 \cup A \times I$.

It is well known (and an easy consequence of theorem 10) that if X is locally compact and regular and $p: E \rightarrow B$ is a fibration, then $p_\#: E^X \rightarrow B^X$ is also a fibration. Conversely, it follows from theorem 11 (with $A = \emptyset$) that, if X is non-empty and $p_\#: E^X \rightarrow B^X$ is a fibration, then $p: E \rightarrow B$ is also a fibration.

4.

Consider the following situation. The pair (B, A) is cofibered, and $p: E \rightarrow B$ is a map. We denote $p^{-1}(A)$ by $E|A$. In general it need not be true that $(E, E|A)$ is cofibered, but we do have

THEOREM 12. *If (B, A) is a cofibered pair with A closed and $p: E \rightarrow B$ is a fibration, then $(E, E|A)$ is a cofibered pair.*

PROOF. Let $\varphi: B \rightarrow I$ and $H: B \times I \rightarrow B$ be as given by lemma 4. Since p is a fibration there exists a homotopy $\bar{H}: E \times I \rightarrow E$ making commutative the diagram

$$\begin{array}{ccc} E & \xrightarrow{1_E} & E \\ i_0 \downarrow & \nearrow \bar{H} & \downarrow p \\ E \times I & \xrightarrow{H(p \times 1_I)} & B \end{array}$$

Define $\tilde{H}: E \times I \rightarrow E$ by $\tilde{H}(e, t) = \bar{H}(e, t \wedge \varphi p(e))$. $\tilde{H}$ and φp then satisfy the requirements of lemma 4, which completes the proof.

Finally we prove

THEOREM 13. *Suppose that (B, A) is a cofibered pair with A closed, that $p: E \rightarrow B$ is a fibration, and that there exists a section s of p . Suppose further that there exist a continuous function $\psi: E \rightarrow I$ such that $E' = s(B) = \varphi^{-1}(0)$ and a fiber deformation $D: E \times I \rightarrow E$ relative to E' such that $D(\varphi^{-1}([0, 1]) \times 1) \subset E'$.*

Then $(E, E' \cup E|A)$ is a cofibered pair.

PROOF. As before, let $\varphi: B \rightarrow I$ and $H: B \times I \rightarrow B$ be as described in lemma 4. Replacing $D(e, t)$, if necessary, by

$$D'(e, t) = \begin{cases} D(e, t/\psi(e)), & t < \psi(e) \\ D(e, 1), & t \geq \psi(e) \end{cases},$$

it follows that we may assume $D(e, t) \in E'$ whenever $t > \psi(e)$. (E, E') is obviously a cofibered pair, and by theorem 4 of [7] there exists a homotopy $\bar{H}: E \times I \rightarrow E$ such that $\bar{H}(e, 0) = e$, $p\bar{H}(e, t) = H(p(e), t)$, and $\bar{H}(s(b), t) = sH(b, t)$ for $e \in E$, $b \in B$, and $t \in I$. Define $\eta: E \rightarrow I$ and $G: E \times I \rightarrow E$ by

$$\begin{aligned} \eta(e) &= \psi(e) \wedge \varphi p(e), \\ G(e, t) &= \bar{H}(D(e, t \wedge \varphi p(e)), t \wedge \eta(e)). \end{aligned}$$

Then $\eta^{-1}(0) = E' \cup E|A$ and $G(e, t) = e$ if $t = 0$ or $e \in E' \cup E|A$. If $t > \eta(e)$, then either $\psi(e) \geq \varphi p(e)$ and

$$\begin{aligned}
 pG(e, t) &= p\bar{H}(D(e, \varphi p(e)), \varphi p(e)) \\
 &= H(pD(e, \varphi p(e)), \varphi p(e)) \\
 &= H(p(e), \varphi p(e)) \in A,
 \end{aligned}$$

so that $G(e, t) \in E|A$, or $\varphi p(e) > \psi(e)$, in which case

$$G(e, t) = \bar{H}(D(e, t \wedge \varphi p(e)), \psi(e)) \in \bar{H}(E' \times I) = E'.$$

Thus, $G(e, t) \in E' \cup E|A$ whenever $t > \eta(e)$, and by lemma 4 $(E, E' \cup E|A)$ is cofibered.

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