Localization Theorems in Equivariant Cohomology

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December 9, 2025

Abstract

This note reviews Atiyah–Bott localization theorems in equivariant cohomology. The goal is to highlight the following three things: First, that proofs of Atiyah–Bott style localization theorems can be made completely formal. Second, give easy to check conditions for when an Atiyah–Bott style localization theorem holds in the integer graded setting, and show they always hold in the $\mathrm{RO}(G)$-graded setting. Finally, clarify the situation for non-abelian groups.

Contents

1 Introduction

The results in this note are surely known to experts. Moreover, the proof strategy is the same as that of Atiyah–Bott [AB84]. However, it can be difficult to extract the general statement from the literature; in particular it is not always clear which assumptions are strictly needed, and which assumptions are purely in place for simplification purposes.

The purpose of this note threefold:

1. Show that proofs of Atiyah–Bott style localization theorems can be made completely formal.

2. Give easy to check conditions in the integer graded setting for an Atiyah–Bott style localization theorem to hold, and show that Atiyah–Bott style localization theorems always hold in the $\mathrm{RO}(G)$-graded setting.

3. Clarify the situation for non-abelain compact Lie groups.

2 Equivariant Localization Theorems

Equivariant localization theorems are among the most classical of theorems in equivariant homotopy theory. We highly recommend reading Atiyah–Bott [AB84], and watching [Hop14].

Classical equivariant localization theorems for equivariant Borel cohomology, equivariant K-theory, and equivariant bordism, have been proven in [AS65, AB84, BV82, Qui71], [Seg68], and [Die70] respectively. Localization theorems for equivariantly complex oriented cohomology theories have been proved by Greenlees [Gre01]. More recently localization theorems have been proved for $\mathrm{RO}(C_2)$-graded Bredon cohomology by May [May20], and $\mathrm{RO}(C_p)$-graded Bredon cohomology by Basu–Ghosh [BG21].

Before equivariant localization theorems, there were equivariant integration formulae: In favourable situations, equivariant integration formulae describe an integral of some equivariant quantity over a manifold in terms of a finite sum over the fixed points

$$\int_\omega M = \sum_{x \in M^G} f_\omega(x).$$

These ideas are ubiquitous in physics; in work that lead to what is now known as the Witten genus, Witten used equivariant integration formulae to study the index of a would-be Dirac operator on the loop space of a manifold [Wit06]. Arguably, the first instance of an equivariant integration formula goes back to work of Duistermaat–Heckmann [DH82].

Work of Atiyah–Bott [AB84], and simultaneously Berline–Vergne [BV82], explained the Duistermaat–Heckmann formula in terms of a localization theorem in equivariant cohomology.

Roughly speaking, a localization theorem says that, upto inverting some elements, the equivariant cohomology of a suitably nice G-space is determined by the equivariant cohomology of its fixed points. The idea that the equivariant cohomology of a space is closely related to the equivariant cohomology of its fixed points fixed points goes back to at least as far as Borel, but it was Atiyah–Bott that first gave the precise formulation in terms of an isomorphism modulo torision.

Here suitably nice usually means that G is a torus or a finite cyclic group, and the space is usually is a compact manifold with a smooth action of G or a finite G-CW-complex. In all cases, these assumptions are in place exclusively for the purpose of providing a way to decompose the space into orbits, in a cohomologically well behaved manner. After separating the orbits from the fixed points, one kills the contribution from the orbits by inverting classes which annihilate the cohomology of the orbits.

3 Abstract Localization Theorems

We abstract Atiyah–Bott style Localization Theorems to the following setup. Note, this abstract setup is not the most general setup in which a localization theorem is true; it is soley designed to isolate the key aspects of the Atiyah–Bott's proof.

Definition 3.1. We define a *localization context* $(\mathcal{C}, \mathcal{F}, \mathcal{A}, h)$ as the following setup:

- Let $\mathcal{C}$ be a stable ∞-category.
- Let $\mathcal{F}$ be a collection of objects in $\mathcal{C}$.

- Let $\mathcal{A}$ be an ordinary 1-category.
- Let $h\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ which satisfies the following: For any pushout square in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

with A and C in $\mathcal{F}$, it follows that $h(D) \to h(B)$ is an isomorphism.

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Remark 3.2. The example to keep in mind is the following:

- Let $\mathcal{C}$ be Sp^G for $G = T^n$.
- Let $\mathcal{F}$ be the collection of $S^n \times \Sigma^\infty_+ G/H$ for $H \neq G$.
- Let $\mathcal{A}$ be the category of (possibly graded) commutative rings.
- Let $h\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ be Bredon cohomology with euler classes inverted.

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In this setup localization theorems are automatic:

Proposition 3.3 (Abstract Localization)**.** *Fix a localization context $(\mathcal{C}, \mathcal{F}, \mathcal{A}, h)$. Let X_0 in $\mathcal{C}$. Suppose that there is a sequence of pushout squares*

$$\begin{tikzcd} S_n \arrow[r] \arrow[d] & X_n \arrow[d] \\ D_n \arrow[r] & X_{n+1} \end{tikzcd}$$

with S_n and D_n in $\mathcal{F}$. Then $X_0 \to X_n$ induces an isomorphism $h(X_n) \to h(X_0)$ for every n.

Remark 3.4. The notation S_n and D_n above is purely suggestive; S_n and D_n are arbitary objects of $\mathcal{F}$. //

Remark 3.5. In the context of G-spectra, we think of X_0 as the fixed points of some G-spectrum or G-space X which admits a good cell descomposition $X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$. In this context S_n and D_n will be $\mathbb{S}^n \otimes \Sigma^\infty_+ G/H$ and $\Sigma^\infty_+ G/H$ respectively. //

Proof. This follows by induction from the assumption on the pushout squares and the the assumption on h. ☐

No where in Proposition 3.3 did we use that $\mathcal{C}$ is stable. The assumption that $\mathcal{C}$ is stable is included just to make the condition on h easy to check.

Lemma 3.6. *Let $\mathcal{C}$ be a stable ∞-category. Let $\mathcal{A}$ be an abelian category. Let $h\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ be a functor which: sends binary coproducts to binary products in $\mathcal{A}$; and which is exact,* *i.e. if $X \to Y \to Z$ is a cofiber sequence in $\mathcal{C}$ then $h(Z) \to h(Y) \to h(Z)$ is exact at $h(Y)$. Then, for any pushout square in $\mathcal{C}$*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

with A and C such that $h(A) = h(C) = 0$, it follows that $h(D) \to h(B)$ is an isomorphism. In particular, $(\mathcal{C}, \mathcal{F}, \mathcal{A}, h)$ is a localization context when $\mathcal{F}$ is taken to be all $X \in \mathcal{C}$ such that $h(X) = 0$.

Proof. Note, since h is exact, it follows that $\mathcal{F}$ is closed under suspensions. Since $\mathcal{C}$ is stable

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a pushout square if and only if $A \to B \oplus C \to D$ is a cofiber sequence; by rotating, this is also equivalent to $B \oplus C \to D \to \Sigma A$ being a cofiber sequence. The assumption that h is exact and sends binary coproducts to binary products gives a Mayer–Vietoris type sequence. By the assumption that $h(A)$ and $h(C)$ are both zero, and that $\mathcal{A}$ is an abelian category, it follows that $h(D) \to h(B)$ is an isomorphism. Hence $(\mathcal{C}, \mathcal{F}, \mathcal{A}, h)$ is a localization context. $\square$

4 Equivariant Localization Theorems

In this section we apply Proposition 3.3 to prove localization theorems in contexts of interest. We treat integer and representation graded cohomology theories seperately. In the integer graded setting, there is a condition to check in order for a cohomology theory to admit a localization theorem. In the representation graded context localization theorems always hold.

4.1 Integer Gradings

Let E be a homotopy commutative G-ring spectrum. The question of whether the resutling G-equivariant cohomology theory $E_G^*(-) \coloneqq \pi_*(F(-, E)^G)$ admits a Localization Theorem has a pleasing algebro-geometric interpretation.

For any closed subgroup H of G, write $E_H^* \coloneqq E_G^*(G/H)$; write $f_H \colon E_G^* \to E_H^*$ for the map induced by $G/H \to G/G$. Let $f_H^* \colon \operatorname{Spec}(E_H^*) \to \operatorname{Spec}(E_G^*)$ be the map of Zariksi spectra induced by f_H.

Theorem 4.1. *Let G be an abelian compact Lie group. Let X be a finite G-CW-complex. Suppose that*

$$\bigcup_{H \lneq G} f_H^* \operatorname{Spec}(E_H^*)$$

does not cover all of $\operatorname{Spec}(E_G^*)$. Then there is a prime ideal $\mathfrak{p}$ in $\operatorname{Spec}(E_G^*)$ such that $E_G^*(X)_{\mathfrak{p}} \to E_G^*(X^G)_{\mathfrak{p}}$ is an isomorphism of $E_{G\mathfrak{p}}^*$-algebras.

Proof. Given such a $\mathfrak{p}$ it is clear that $(\mathrm{Sp}^G, \mathcal{F}, \mathrm{CRing}_{E^*_{G\mathfrak{p}}}, E^*_G(-)_{\mathfrak{p}})$ forms a localization context, where $\mathcal{F}$ is taken to be all $\mathbb{S}^n \otimes \Sigma^\infty_+ G/H$ with $H \neq G$: Indeed, by Lemma 3.6, the only thing to check is that $E^*_G(G/H) = 0$ for $H \neq G$, but this follows by assumption on $\mathfrak{p}$. Hence the result follows from Proposition 3.3. □

The above statement is not sharp. Let us highlight the ways this is not sharp:

- Instead of looking at just the map $X^G \hookrightarrow X$ we can consider $X^H \hookrightarrow X$ for any closed subgroup G.
- Abstractly, the condition that G is abelian can be relaxed, in this case $X^H \hookrightarrow X$ should be replaced by $G \cdot X^H \hookrightarrow X$. However, in practice, for integer graded theories, there will almost never be a suitable prime ideal to localize at in the case G is not abelian.
- The condition that X is a finite G-CW-complex can be significantly relaxed; instead we can consider finite-dimensional G-CW-complexes with finitely many orbit types. This is essentially sharp in the sense that, if either one of finite-dimensional or finitely many orbit types are dropped then there exist G-spaces which are counter examples to any reasonable version of a localization theorem.
- Localizition with respect to a prime ideal of $\mathrm{Spec}(E^*_G)$ can be replaced with inverting a multiplicatively closed subset of E^*_G.

The following is a sharp version.

Theorem 4.2. *Let G be a compact Lie group. Fix H be a closed subgroup of G. Let X be a finite-dimensional G-CW-complex with finitely many orbit types. Let $\Theta_H(X)$ be the collection of isotropy groups of X which H is not subconjugate to. If $S \subset E^*_G$ is a multiplicatively closed subset such that $S \cap \mathrm{Ann}\, E^*_K \neq \emptyset$ for all $K \in \Theta_H(X)$, then*

$$S^{-1}E^*_G(X) \to S^{-1}E^*_G(G \cdot X^H)$$

is an isomorphism of $S^{-1}E^*_G$-algebras.

Proof. We take $\mathcal{F}$ to be all $\coprod_{i\in I} \mathbb{S}^n \otimes \Sigma^\infty_+ G/K$ with $K \in \Theta_H(X)$, and where I is allowed to be an infinite set. Then $(\mathrm{Sp}^G, \mathcal{F}, \mathrm{CRing}_{S^{-1}E^*_G}, S^{-1}E^*_G)$ forms a localization context. Again, by Lemma 3.6, the only thing to check is that $S^{-1}E^*_G(\coprod_{i\in I} \Sigma^\infty_+ G/K) = 0$ for $K \in \Theta_H(X)$, but this follows by assumption on S. Note, since $\coprod_{i\in I} \Sigma^\infty_+ G/K \to G/K$ induces a ring map $E^*_K \to E^*_G(\coprod_{i\in I} \Sigma^\infty_+ G/K)$ it suffices to observe that the assumption on S implies that $S^{-1}E^*_K = 0$. Now the result follows from Proposition 3.3. □

4.2 Representation Gradings

This result is essentially in tom Dieck [tD87], the main difference is we treat $\mathrm{RO}(G)$-gradings more carefully using the approach in [Qui], and we make the statement slightly sharper. We can replace the integer graded theory E^*_G with a representation graded theory $E^\star_G$ everywhere in the statement, and proof, of Theorem 4.2.

Theorem 4.3. *Let G be a compact Lie group. Fix H be a closed subgroup of G. Let X be a finite-dimensional G-CW-complex with finitely many orbit types. Let $\Theta_H(X)$ be the collection* *of isotropy groups of X which H is not subconjugate to. If $S \subset E_G^\star$ is a multiplicatively closed subset such that $S \cap \operatorname{Ann} E_K^\star \neq \emptyset$ for all $K \in \Theta_H(X)$, then*

$$S^{-1}E_G^\star(X) \to S^{-1}E_G^\star(G \cdot X^H)$$

is an isomorphism of $S^{-1}E_G^\star$-algebras.

Proof. After replacing every instance of E_G^* with a representation graded theory $E_G^\star$ in the proof of Theorem 4.2 the proof is verbatim. ☐

The main difference is that in the representation graded setting we can always find a suitable multiplicative closed subetset to invert.

Proposition 4.4. *Let $E_G^\star$ be a representation graded cohomology theory. For any $K \leq G$ there exists an element $e(V_K) \in E_G^\star$ such that $e(V_K)$ annhilates $E_G^\star(G/K)$. In particular, for any family of subgroups Θ there the collection $(e(V_K))_{K\in\Theta}$ we can form a multiplicatively closed subset S such that $S \cap \operatorname{Ann} E_K^\star \neq \emptyset$ for all $K \in \Theta$.*

Proof. By the Peter–Weyl Theorem, for any closed subgroup K there is a representation V_K and an equivariant embedding $f\colon G/K \hookrightarrow V_K \setminus \{0\}$. Applying [tD87, Proposition 3.10, pg. 194] to this embedding gives a class $e(V_K)$ such that $e(V_K)$ annhilates $E_G^\star(G/K) = E_K^\star$. The multiplicatively closed subset S for any family of subgroups Θ is then taken to be the multiplicatively closed subset generated by $e(V_K)$ as K varies through Θ. ☐

Theorem 4.5. *Let G be a compact Lie group. Fix H be a closed subgroup of G. Let X be a finite-dimensional G-CW-complex with finitely many orbit types. Let Θ be the collection of isotropy groups of X which H is not subconjugate to. Then*

$$E_G^\star(X)[e(V_K)^{-1}|K \in \Theta] \to E_G^\star(G \cdot X^H)[e(V_K)^{-1}|K \in \Theta]$$

is an isomorphism of $E_G^\star[e(V_K)^{-1}|K \in \Theta]$-algebras.

Proof. This follows from Theorem 4.3 with the multiplicatively closed subset S taken to be the generated by the euler classes from Proposition 4.4. ☐

4.3 Non-Abelian Compact Lie Groups

Let G be a non-abelian compact Lie group. Fix a maximial torus T inside G. It is often said that Atiyah–Bott style localization theorems do not hold for non-abelain G. The main reason for this is the following example.

Example 4.6. The Atiyah–Bott localization theorem cannot hold for integer graded G-equivariant Bredon cohomology $H_G^*(-)$ with coefficents $\mathbb{C}$. Consider the G-space $X = G/T$. This is a perfectly reasonable finite G-CW-complex. Note that $H_G^*(X) = H_T^*$. Note that $X^G = \emptyset$, so $H_G^*(X^G) = 0$. We have that H_G^* is the invariants of H_T^* under the action of the Weyl group, that is, $H_G^* \cong (H_T^*)^W$. Recall that $H_T^* \cong \mathbb{C}[x_1, \ldots, x_n]$ where $n = \dim(T)$. So it follows that $H_G^*(X)$ is a torsion free H_G^*-module. In particular, there is no multiplicatively closed subset $S \subset H_G^*$ such that $S^{-1}H_G^*(X) \to S^{-1}H_G^*(X^G) \cong 0$ is an isomorphism. //

However, as shown in Proposition 4.4, in the representation graded setting $E_G^\star(G/H)$ is always a torsion $E_G^\star$-module.

References

[AB84] Michael F Atiyah and Raoul Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984. 1, 2

[AS65] MF Atiyah and GB Segal. Equivariant cohomology and localization. *Lecture notes. Coventry, UK: University of Warwick*, 1965. 2

[BG21] Samik Basu and Surojit Ghosh. Bredon cohomology of finite dimensional C_p-spaces. *Homology, Homotopy and Applications*, 23(2):33–57, 2021. 2

[BV82] Nicole Berline and Michèle Vergne. Classes caractéristiques équivariantes. formule de localisation en cohomologie équivariante. *CR Acad. Sci. Paris*, 295(2):539–541, 1982. 2

[DH82] Johannes J Duistermaat and Gerrit J Heckman. On the variation in the cohomology of the symplectic form of the reduced phase space. *Inventiones mathematicae*, 69(2):259–268, 1982. 2

[Die70] Tammo Tom Dieck. Bordism of G-manifolds and integrality theorems. *Topology*, 9(4):345–358, 1970. 2

[Gre01] J. P. C. Greenlees. Equivariant formal group laws and complex oriented cohomology theories. *Homology, Homotopy and Applications*, 3(2):225–263, 2001. 2

[Hop14] Michael Hopkins. Equivariant homotopy and localization. Talk at the Introductory Workshop: Algebraic Topology, Simons Laufer Mathematical Sciences Institute (SLMath), 2014. Available at: `https://www.slmath.org/workshops/685/schedules/17899` (accessed 2025-12-04). 2

[May20] Clover May. A structure theorem for $\mathrm{RO}(C_2)$–graded Bredon cohomology. *Algebraic amp; Geometric Topology*, 20(4):1691–1728, July 2020. 2

[Qui] Ryan Quinn. A convenient approach to gradings in homotopy theories. Unpublished note. 5

[Qui71] Daniel Quillen. The spectrum of an equivariant cohomology ring: I. *Annals of Mathematics*, 94(3):549–572, 1971. 2

[Seg68] Graeme Segal. Equivariant k-theory. *Publications Mathématiques de l'IHÉS*, 34:129–151, 1968. 2

[tD87] Tammo tom Dieck. *Transformation groups*, volume 8. Walter de Gruyter, 1987. 5, 6

[Wit06] Edward Witten. The index of the Dirac operator in loop space. In *Elliptic Curves and Modular Forms in Algebraic Topology: Proceedings of a Conference held at the Institute for Advanced Study Princeton, Sept. 15–17, 1986*, pages 161–181. Springer, 2006. 2