# Truncated Brown–Peterson spectra as generalized Thom spectra

Ryan Quinn

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**Abstract**

We construct $\mathrm{H}\mathbb{Z}$ as an $\mathbb{E}_1$-MU-Thom spectrum, and ku as an $\mathbb{E}_1$-$\mathrm{MU}[y]$-Thom spectrum where $\mathrm{MU}[y]$ is the free $\mathbb{E}_1$-MU-algebra on a a class in degree 2. More generally, fix a prime $p$ and let $\mathrm{MU}_{(p)}[y_1, y_2, \ldots, y_{n-1}]$ denote $\mathrm{MU}_{(p)}$ adjoin polynomial generators $y_i$ in degree $|v_i|$. For all $n \geq 0$ we construct the truncated Brown–Peterson spectrum $\mathrm{BP}\langle n\rangle$ as an $\mathbb{E}_1$-$\mathrm{MU}_{(p)}[y_1, y_2, \ldots, y_{n-1}]$-Thom spectrum.

This note is expository, the content is well known to experts. This note grew out of an attempt to better understand a result of Basu–Sagave–Schlichtkrull.

## Contents



## 1 Introduction

Constructing spectra via Thom spectra methods is a convenient way to build multiplicative structures. Thom spectra straddle the line between geometrically defined cohomology theories, and 'designer' spectra. A good instantiation of Thom spectra as designer spectra is the Basu–Sagave–Schlichtkrull approach to constructing quotients of ring spectra.

Let's first recall how to construct quotients of ring spectra. For simplicity we assume that $R$ is an $\mathbb{E}_2$-ring[1] spectrum. Fix an element $x \in \pi_n(R)$ of which we want to form the quotient $R/x$. Such an element is equivalent to a map of spectra $x\colon \mathbb{S}^n \to R$. We form the quotient from an $R$-module viewpoint as follows: By the free forgetful $R$-module adjunction, this is adjoint to map of $R$-modules $\Sigma^n R \to R$. The quotient $R/x$ is defined by the cofiber sequence

$$\Sigma^n R \to R \to R/x.$$

This approach highlights the universal property of $R/x$; an $R$-module map $R/x \to A$ is equivalent to an $R$-module map $R \to A$ along with a null-homotopy in $R$-modules[2] of $\Sigma^n R \to R \to A$.

[1] Homotopy associative is sufficient for this first construction.

[2] Warning, it is not sufficient to produce a null-homotopy in spectra.

Alternatively,[3] we may form the quotient from an algebra perspective. Again we start from the map spectra $x\colon \mathbb{S}^n \to R$. Recall that the free $\mathbb{E}_1$-$R$-algebra functor is given by $R \otimes \mathrm{free}^{\mathbb{E}_1}(-)$. Using the free forgetful $\mathbb{E}_1$-$R$-algebra adjunction we get a map of $\mathbb{E}_1$-$R$-algebras $\tilde{x}\colon R \otimes \mathrm{free}^{\mathbb{E}_1}(\mathbb{S}^n) \to R$. We suggestively write $R[x]$ for $R \otimes \mathrm{free}^{\mathbb{E}_1}(\mathbb{S}^n)$ equipped with the augmentation $\tilde{x}\colon R \otimes \mathrm{free}^{\mathbb{E}_1}(\mathbb{S}^n) \to R$. The quotient $R/x$ can then be constructed as the relative tensor product $R \otimes_{R[x]} R$ where the left copy of $R$ is given an $R[x]$-module structure via $\tilde{x}$, and the right copy of $R$ is given an $R[x]$-module structure by base changing the augmentation $\mathrm{free}^{\mathbb{E}_1}(\mathbb{S}^n) \to \mathbb{S}$ to $R$.

The fact that both approaches produce equivalent $R$-modules follows from the cofiber sequence

$$\Sigma^n \mathbb{S}[x] \to \mathbb{S}[x] \to \mathbb{S}.$$

of $\mathbb{S}[x]$-modules. Indeed, applying $R \otimes -$ gives a cofiber sequence of $R[x]$-modules. Base changing this along the augmentation $R[x] \to R$ gives a cofiber sequence

$$\Sigma^n R \to R \to R \otimes_{R[x]} R.$$

So both $R \otimes_{R[x]} R$ and $R/x$ are the cofiber of the multiplication by $x$. In particular, they are equivalent.

To start moving towards the Basu–Sagave–Schlichtkrull approach, we need two observations:

- The Snaith splitting gives an equivalence of $\mathbb{E}_1$-algebras between $\mathrm{free}^{\mathbb{E}_1}(\mathbb{S}^n)$ and $\Sigma^\infty_+ \Omega\Sigma S^n$, whence there is an equivalence of $\mathbb{E}_1$-$R$-algebras between $R \otimes \mathrm{free}^{\mathbb{E}_1}(\mathbb{S}^n)$ and $R[\Omega\Sigma S^n](:= R \otimes \Sigma^\infty_+ \Omega\Sigma S^n)$. In particular, there is an equivalence of $R$-modules

$$R/x \cong R \otimes_{R[\Omega\Sigma S^n]} R.$$

- Let $X$ be a connected pointed space. Fix a map of pointed spaces $f\colon X \to BGL_1R$. There is an induced map of loop spaces $\Omega X \to GL_1(R)$. The Thom spectrum of $f$ is equivalent to the relative tensor product $R \otimes_{R[\Omega X]} R$ where the left copy of $R$ has an $R[\Omega X]$-module structure induced from $\Omega X \to GL_1 R$, and the right copy of $R$ has an $R[\Omega X]$-module structure induced from $\Omega X \to *$. See [ARB+13]

Combinging these observations gives the following step towards the Basu–Sagave–Schlichtkrull approach.

**Lemma 1.1.** *Let $x \in \pi_n(R)$ with $n > 0$. Then $R/x$ is equivalent to the Thom spectrum of a map $S^{n+1} \to BGL_1(R)$.*

*Proof.* For $n > 0$, the following things are equivalent:

- an element $x \in \pi_n(R)$;
- a map of spectra $\mathbb{S}^n \to R$;
- a map of pointed spaces $S^n \to \Omega^\infty R$;
- a map of pointed spaces $S^n \to GL_1 R$;
- a map of pointed spaces $\Sigma S^n \to BGL_1 R$;

[3]This second construction requires an $\mathbb{E}_1$-ring structure.

- a map of grouplike $\mathbb{E}_1$-spaces $\Omega\Sigma S^n \to GL_1 R$;
- a map of $\mathbb{E}_1$-spaces $\Omega\Sigma S^n \to \Omega^\infty R$;
- a map of $\mathbb{E}_1$-ring spectra $\Sigma^\infty_+ \Omega\Sigma S^n \to R$;
- a map of $\mathbb{E}_1$-$R$-algebras $R[\Omega\Sigma S^n] \to R$.

In particular an element $x \in \pi_n(R)$ with $n > 0$ determines an $R[\Omega\Sigma S^n]$-module structure on $R$. Now we apply the observation that the Thom spectrum of map $X \to BGL_1 R$ is equivalent to the relative tensor product $R \otimes_{R[\Omega X]} R$ for $X = S^{n+1}$ to get the result. $\square$

Basu–Sagave–Schlichtkrull upgrade this in two ways. First, they allow one to quotient multiple elements. Second, and more importantly, they produce a multiplicative structure on the resulting Thom spectra. Doing this requires further assumptions on $R$. In [BSS17], they assume that $R$ is an $\mathbb{E}_\infty$-ring with homotopy groups concentrated in even degrees. As pointed out in [HW18], using the machinery of [AB18], the $\mathbb{E}_\infty$-assumption can easily be dropped.

**Theorem 1.2** (Basu–Sagave–Schlichtkrull, [BSS17])**.** *Let $R$ be an $\mathbb{E}_2$-ring spectrum with homotopy concentrated in even degrees. Let $x_1, x_2, \ldots, x_n$ be a sequence of elements with $x_i \in \pi_{2i}(R)$. The quotient $R/(x_1, x_2, \ldots, x_n)$ can be constructed as an $\mathbb{E}_1$-$R$-Thom spectrum over $SU(n+1)$, i.e., there is grouplike $\mathbb{E}_1$-algebra map $SU(n) \to BGL_1(R)$ such that $\mathrm{Th}(SU(n+1) \to BGL_1(R))$ is an $\mathbb{E}_1$-$R$-algebra whose underlying $R$-module is equivalent to $R/(x_1, x_2, \ldots, x_n)$.*

I found this result quite surprising at first. For the convenience of the reader, we sketch some intuition. The Thom spectrum of a map $SU(n+1) \to BGL_1(R)$ is equivalent to the relative tensor product $R \otimes_{R[\Omega SU(n)]} R$. The following splitting makes the appearance of $SU(n+1)$ less surprising

$$R \otimes \Sigma^\infty_+ \Omega SU(n+1) \cong R \otimes \Sigma^\infty_+ \Omega S^3 \otimes \Sigma^\infty_+ \Omega S^5 \otimes \ldots \otimes \Sigma^\infty_+ \Omega S^{2n+1}.$$

This splitting is standard: In the case that $R = H\mathbb{Z}$ the equivalence follows from a direct homology computation. Since both homologies are primitively generated polynomial rings, an abstract equivalence of homology rings is enough to construct the desired equivalence of spectra. We specify an equivalence by sending the generator in degree $2k+1$ to the generator in $2k+1$. The remaining cases follow by the Atiyah–Hirzebruch spectral sequence.

To tie this together with quotients, recall that the free $\mathbb{E}_1$-algebra on $S^k$ is given by $\Sigma^\infty_+ \Omega S^{k+1}$. Write

$$R[x_1, x_2, \ldots, x_n] \coloneqq R \otimes \Sigma^\infty_+ \Omega S^3 \otimes \Sigma^\infty_+ \Omega S^5 \otimes \cdots \otimes \Sigma^\infty_+ \Omega S^{2n+1}.$$

Then $R/(x_1, x_2, \ldots, x_n)$ is defined as the relative tensor product

$$R \otimes_{R[x_1, x_2, \ldots, x_n]} R$$

where the $R[x_1, x_2, \ldots, x_n]$-module structure on the left copy of $R$ is given informally by sending the formal generator $x_i$ maps to element $x_i$, and the $R[x_1, x_2, \ldots, x_n]$-module structure on the right copy of $R$ is given by the augmentation sending the formal generator $x_i$ to the element 0.

Actually turning this intuition into a rigorous proof requires work. In particular one needs to make sure that the $R[\Omega SU(n+1)]$-module structure on $R$ coming from a map

$SU(n+1) \to BGL_1(R)$ is compatible with the $R[x_1, x_2, \ldots, x_n]$-module structure on $R$ used in defining the quotient.

Theorem 1.2 immediately generalizes to case of taking quotients by an infinite sequence of elements.

**Theorem 1.3** (Basu–Sagave–Schlichtkrull)**.** *Let $R$ be an $\mathbb{E}_2$-ring spectrum with homotopy concentrated in even degrees. Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of elements with $x_i \in \pi_{2i}(R)$. The quotient $R/(x_1, x_2, x_3, \ldots)$ can be constructed as an $\mathbb{E}_1$-$R$-Thom spectrum over $SU$.*

*Proof.* This is not explicitly stated in [BSS17], but it is certainly known to the authors. It is clear that their proof works mutatis mutandis for infinite sequences of elements. Alternatively, the infinite case follows from the finite case by taking the colimit along the filtration of $SU$ by $SU(n)$. □

# 2 Constructing truncated Brown–Peterson as Thom spectra

Now we consider some applications of Basu–Sagave–Schlichtkrull's construction. We emphasize again that these examples are certainly known to the experts.

Since $\mathrm{H}\mathbb{Z}$ is equivalent to $\mathrm{MU}/(x_1, x_2, x_3, \ldots)$ we can immediately apply Theorem 1.3 to construct $\mathrm{H}\mathbb{Z}$ as $\mathbb{E}_1$-MU-Thom spectrum. However, we *cannot* apply Theorem 1.3 to construct ku as an $\mathbb{E}_1$-MU-Thom spectrum; the Basu–Sagave–Schlichtkrull construction demands that we kill elements in every even degree; in particular we cannot skip over $x_1$.

To circumvent this, we will adjoin an extra generator in degree 2 to MU. Let $\mathrm{MU}[y]$ denote the free $\mathbb{E}_1$-MU-algebra on a class in degree 2. Then we have an equivalence between ku and $\mathrm{MU}[y]/(y, x_2, x_3, \ldots)$. To apply Theorem 1.3 to construct ku as an $\mathbb{E}_1$-$\mathrm{MU}[y]$-Thom spectrum, it remains to see that $\mathrm{MU}[y]$ admits an $\mathbb{E}_2$-structure and has homotopy concentrated in even degrees; this follows from, for example, [HW18] or [HW22].

This idea of adjoining 'dummy' generators generalizes and allows us to 'skip' certain generators, although at the cost of working relative to a bigger base.

Fix a prime $p$. Moving to the $p$-local case we can use this to construct $\mathrm{BP}\langle n\rangle$ for $n \geq 0$.

**Theorem 2.1.** *Let $n \geq 0$. Let $\mathrm{MU}_{(p)}[y_1, y_2, \ldots, y_{n-1}]$ denote $\mathrm{MU}_{(p)}$ adjoin polynomial generators $y_i$ in degree $|v_i|$ for $0 < i < n$. Then there is a loop space map*

$$SU \to BGL_1(\mathrm{MU}_{(p)}[y_1, y_2, \ldots, y_{n-1}])$$

*whose Thom spectrum is $\mathrm{BP}\langle n\rangle$.*

*Proof.* The fact that $\mathrm{MU}_{(p)}[y_1, y_2, \ldots, y_{n-1}]$ is an $\mathbb{E}_2$-ring with even homotopy follows from [HW18]. The result follows Theorem 1.3. □

## References


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