

LOCAL FIELDS AND EXTRAORDINARY K -THEORY

JACK MORAVA

ABSTRACT. We describe integral lifts $K(L)$, indexed by local fields L of degree $n = [L : \mathbb{Q}_p]$, of the extraordinary cohomology theories $K(n)$, and apply the generalized character theory of Hopkins, Kuhn and Ravenel to identify $K(L)(BG) \otimes \mathbb{Q}$, for a finite group G , as a ring of functions on a certain scheme $\mathfrak{C}_L G$ étale over L , whose points are conjugacy classes of homomorphisms from the valuation ring of L to G . When L is $\mathbb{Q}_p$ this specializes to a classical theorem of Artin and Atiyah.

Introduction The $2(p^n - 1)$ -periodic mod p cohomology functors $K(n)$ play a useful role in our understanding of stable homotopy theory, indexing its thick subcategories of finite objects [12]. This note considers certain integral lifts $K(L)$ of these functors, indexed now by local number fields L with $n = [L : \mathbb{Q}_p]$ (more precisely: by Lubin-Tate formal group laws associated to these fields), taking values in compact topological modules over the valuation ring $\mathfrak{o}_L$ of L .

Following a suggestion in [11 §1.3], the principal result below applies the generalized character theory of Hopkins, Kuhn, and Ravenel to identify the rationalization of $K(L)(BG)$ (for G a finite group), as a ring of functions (with values in the maximal abelian extension of L) on the set

$$C_L G := \mathrm{Hom}(\mathfrak{o}_L, G)/G^{\mathrm{conj}}$$

of conjugacy classes of homomorphisms $\mathfrak{o}_L \rightarrow G$. When $L = \mathbb{Q}_p$ this recovers a classical result [27 Theorem 25] in the representation theory of finite groups.

This (very compressed) account is organized as follows: §1 gets the necessary local number theory out of the way, though it is not really used until §3. The second section summarizes some properties of the classical $K(n)$'s, while the third section uses the Baas-Sullivan construction, together with old work of Hazewinkel, to construct the proposed lifts relatively explicitly, for unramified fields. I hope this will correct some of the confusion in [20].

Date: 23 July 2012.

1991 *Mathematics Subject Classification.* 11S31, 55N22.

This work has been supported for many years by the NSF.

Section four reformulates the basics of generalized character theory in terms of the familiar fact that

$$H^1(\mathbb{Z}^n, \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^n,$$

and §5 recalls enough of the theory of level structures on formal groups to state the main technical result [§5.4]. §6 is devoted to unbridled speculation.

Acknowledgements The work behind this summary has taken too long for it to be practical for me to thank my friends and colleagues adequately for their support. Instead, I will just remark that it was motivated by recent developments in classfield theory [3] and in the study of power operations [24] in algebraic topology.

1. THE LOCAL BACKGROUND

Fix a prime p and an integer $n \geq 1$, and let $q = p^n$.

In what follows, $\mathbb{F}_q$ will denote the field with q elements, $W(\mathbb{F}_q)$ its ring of Witt vectors, and $\mathbb{Q}_q = W(\mathbb{F}_q) \otimes \mathbb{Q}$ the quotient field of the latter: which is the unique unramified extension of degree n of the field $\mathbb{Q}_p$ of p -adic rationals. It can be constructed by adjoining the $(q-1)$ th roots of unity to $\mathbb{Q}_p$, and the homomorphism

$$\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p) \rightarrow \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p) \cong \mathbb{Z}/n\mathbb{Z}$$

(defined by the action of the Galois group on the residue field) is an isomorphism.

I'll make constant use of Lubin and Tate's constructive approach to classfield theory. In that framework [14, 26],

1) Artin's local reciprocity law asserts that the maximal abelian extension L^{ab} of a local number field L has Weil group

$$L^\times \cong \mathbb{W}(L^{\mathrm{ab}}/L) \subset \mathrm{Gal}(L^{\mathrm{ab}}/L)$$

(a canonical dense subgroup of the Galois group [29]). Moreover,

2) the maximal totally ramified extension L^{trab} of L in L^{ab} can be constructed by adjoining the torsion elements of the group of points of a Lubin-Tate formal group LT for L (ie, with values in an algebraic closure of L); and

3) this group $\mathrm{LT}_{\mathrm{tors}}$ of torsion points is canonically isomorphic (as Galois module) to the quotient $L/\mathfrak{o}_L$ (where $\mathfrak{o}_L$ is the valuation ring of L , with the Weil group $L^\times$ acting by multiplication, via the projection

$$L^\times = \mathbb{Z} \times \mathfrak{o}_L^\times \rightarrow \mathfrak{o}_L^\times$$

defined by the valuation on L).

For example, if $n = 1$ we recover the local Kronecker-Weber theorem, which asserts that the maximal abelian extension of $\mathbb{Q}_p$ has Galois group isomorphic to the profinite completion of $\mathbb{Q}_p^\times$, and is obtained by adjoining all roots of unity (ie, the torsion points of the multiplicative group), to $\mathbb{Q}_p$. [The associated projection

$$\chi : \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \rightarrow \text{Gal}(\mathbb{Q}_p^{\text{trab}}/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$$

is usually called the cyclotomic character.]

2. OUR STORY SO FAR

2.1 Recall [13, 22, 31] that for $p > 2, 3$ there are multiplicative (graded-commutative) 2-periodic cohomology functors

$$K(n)^*(-, \mathbb{F}_p) : (\text{Spaces}) \rightarrow (\mathbb{F}_p - \text{Mod})$$

such that $K(n)^*(\text{pt}, \mathbb{F}_p) = \mathbb{F}_p$ when $*$ is even, and $= 0$ for $*$ odd; thus finite CW-spaces are mapped to finite-dimensional vector spaces. These theories have Chern classes c for complex line bundles, which defines a formal group law

$$\mathbf{F} : K(n)^*(B\mathbb{T}, \mathbb{F}_p) = \mathbb{F}_p[[c]] ,$$

the mod p reduction of the formal group law F associated to Honda's logarithm.

The spectra representing these theories are in some sense the 'residue fields' associated to certain (multiplicative, periodic) ring-spectra E_n (with $E_n^*(\text{pt}) = \mathbb{Z}_p[[v_1, \dots, v_{n-1}]][[u^{\pm 1}]]$) constructed from BP by Landweber's exact functor theorem. Those cohomology theories can be understood as taking values in quasicohherent sheaves of modules over the Lubin-Tate moduli stack of deformations of $\mathbf{F}$ – which is, roughly, the transformation groupoid [19]

$$[\text{Spf } E_n^*(\text{pt})/\text{Aut}(\mathbf{F})]$$

defined by the natural action of the group(scheme) of automorphisms of $\mathbf{F}$ on its space $\text{Spf } E_n^*(\text{pt})$ of deformations [7, 23]. Similarly, $\text{Aut}(\mathbf{F})$ acts as multiplicative automorphisms (ie, as cohomology operations) on the 'fiber' $K(n)$ at the point of the moduli stack defined by $v_i \mapsto 0$, $1 \leq i \leq n-1$.

2.2 This action is most concisely described over the corresponding geometric point

$$\overline{\mathbf{F}} : \text{Spec } \overline{\mathbb{F}_p} \rightarrow \text{Spec } E_n^*(\text{pt}) ,$$

by interpreting $\text{Aut}(\mathbf{F})$ to be the (pro)étale groupscheme over $\mathbb{F}_p$ defined by the action of $\mathbb{Z} \subset \hat{\mathbb{Z}} = \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ on the group $\mathfrak{o}_D^\times$ of strict units

$$1 \longrightarrow \mathfrak{o}_D^\times \longrightarrow \mathfrak{o}_D^\times \rtimes \mathbb{Z} = D^{\text{grd}_p} \longrightarrow \mathbb{Z} \longrightarrow 0$$

of the division algebra

$$D = \mathbb{Q}_q \langle F \rangle / (F^n - p)$$

(where $Fa = a^\sigma F$ if $a \in \mathbb{Q}_q$, with $\sigma \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ the Frobenius generator); thus the group $\text{Aut}(\mathbf{F})(\overline{\mathbb{F}}_p)$ of points acts continuously on

$$K(n)^*(-, \mathbb{F}_p) \otimes \overline{\mathbb{F}}_p := K(n)^*(-, \overline{\mathbb{F}}_p)$$

by multiplicative operations.

Note that in general, a copy of $\mathbb{Z}_p^\times$ (ie of the stable p -adic Adams operations, cf §3.4) sits naturally in $\text{Aut}(\mathbf{F})(\mathbb{F}_p)$, as the center of $\text{Aut}(\mathbf{F})$.

The pro-Sylow p -subgroup

$$1 \rightarrow \mathbb{S}(D) \rightarrow \text{Aut}(\mathbf{F}) = \mathbb{S}(D) \rtimes \mu_{q-1} \rightarrow \mu_{q-1} \rightarrow 1$$

of the strict units of D splits, and the action of the subgroup μ_{q-1} of prime-to- p roots of unity defines a $2(q-1)$ -periodic refinement of the grading on $K(n)^*(-, \mathbb{F}_p)$, recovering the usual convention that $v_n = u^{q-1}$ and $|v_k| = 2(p^k - 1)$.

2.3 This action of $D^\times$ does not, however, exhaust the cohomology operations on $K(n)^*(-, \mathbb{F}_p)$, which is constructed using Baas-Sullivan theory. This provides $K(n)$ with a (co)action of an exterior algebra $E(Q_i \mid 0 \leq i \leq n-1)$ of Bockstein operations (corresponding to the departed v_i 's)¹, as well as a universal-coefficient spectral sequence

$$\text{Tor}_*^{E_n}(E_n^*(-), K(n)^*(\text{pt}, \mathbb{F}_p)) \Rightarrow K(n)^*(-, \mathbb{F}_p) .$$

As $\mathbb{S}(D)$ -module, $E(Q_*)$ is the exterior algebra on the ‘normal bundle’ m_E/m_E^2 of $\mathbf{F}$ in $\text{Spf } E_n^*$ (isomorphic, aside from a copy of the arithmetic Bockstein Q_0 , to Lubin and Tate’s second cohomology group $H_s^2(\mathbf{F})$ of $\mathbf{F}$ (which controls its infinitesimal deformations [15])).

3. AN APPLICATION OF HAZEWINKEL’S FUNCTIONAL EQUATION

3.1 Proposition: The series

$$\log_F(X) = X + \sum_{1 \leq k} \prod_{1 \leq i \leq k} (1 - p^{q^i - 1})^{-1} \frac{X^{q^k}}{p^k} \in \mathbb{Q}[[X]] .$$

satisfies the equation

$$p \log_F(X) = \log_F(pX) + \log_F(X^q) .$$

¹The Bocksteins corresponding to $i > n$ are killed by inverting v_n .

Proof: The assertion is clear modulo terms of degree greater than one. On the other hand if we compare coefficients of X^{q^k} for $k > 0$, the statement becomes

$$p^{1-k} \prod_{1 \leq i \leq k} (1 - p^{q^i-1})^{-1} = p^{q^k-k} \prod_{1 \leq i \leq k} (1 - p^{q^i-1})^{-1} + p^{1-k} \prod_{1 \leq i \leq k-1} (1 - p^{q^i-1}) .$$

Clearing denominators and multiplying by p^k simplifies this to

$$p = p^{q^k} + p(1 - p^{q^k-1}) ,$$

which is obvious.

Corollary:

$$g(X) := p^{-1} \log_F(pX) = X + \sum_{1 \leq k} p^{q^k-k-1} \prod_{1 \leq i \leq k} (1 - p^{q^i-1})^{-1} X^{q^k}$$

has p -adically integral coefficients (cf eg $p = 2$, $n = 1$, $k = 1$).

3.2 This proposition can be restated as the assertion

$$\log_F(X) = g(X) + p^{-1} \log_F(X^q) .$$

Since g is p -adically integral, this is an instance of **Hazewinkel's functional equation** [8 §5.2], from which it follows that

$$F(X, Y) = \log_F^{-1}(\log_F(X) + \log_F(Y))$$

is a p -typical formal group law over $\mathbb{Z}_{(p)}$ with $\log_F$ as its logarithm².

If we regard F as a group law over $W(\mathbb{F}_q)$, it further follows from the functional equation lemma that

$$a \mapsto [a]_{\mathbb{F}}(X) = \log_F^{-1}(a \log_F(X)) : W(\mathbb{F}_q) \rightarrow \text{End}_{W(\mathbb{F}_p)}(\mathbf{F})$$

is an isomorphism.

Corollary: $[p]_F(X) = pX +_F X^q$.

This is just a restatement of the proposition, but it implies that the reduction $\mathbf{F}$ of F modulo p is Honda's formal group law of height n .

The group of continuous automorphisms of the formal Hopf algebra structure on $W(\overline{\mathbb{F}}_p)[[X]]$ defined by F contains

$$W(\mathbb{F}_q)^{\times} \rtimes \mathbb{Z} \subset D^{\times} = (\text{End}_{\overline{\mathbb{F}}_p}(\mathbf{F})) \otimes \mathbb{Q}^{\times}$$

as a dense subgroup, with $\mathbb{Z}$ acting on the ring of Witt vectors through

$$\mathbb{Z} \rightarrow \hat{\mathbb{Z}} = \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \rightarrow \text{Gal}(\mathbb{F}_q/\mathbb{F}_p) = \mathbb{Z}/n\mathbb{Z}$$

²The case $g = 0$ of Hazewinkel's lemma yields Honda's logarithm $\sum p^{-k} X^{q^k}$.

as powers of Frobenius. This identifies the subgroup of $D^\times$ above as the Weil group

$$1 \rightarrow \mathbb{Q}_q^\times = W(\mathbb{F}_q)^\times \times \mathbb{Z} \rightarrow \mathbb{W}(\mathbb{Q}_q^{\text{ab}}/\mathbb{Q}_p) \rightarrow \text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p) = \mathbb{Z}/n\mathbb{Z} \rightarrow 1$$

of the maximal abelian extension of $\mathbb{Q}_q := W(\mathbb{F}_q) \otimes \mathbb{Q}$.

3.3 We can extend F in another way, to a **graded** formal group law $\mathbf{F}$ over $\mathbb{Z}_p[u, u^{-1}]$, by defining

$$\mathbf{F}(X, Y) = u^{-1}F(uX, uY),$$

with $[p]_{\mathbf{F}}(X) = pX +_{\mathbf{F}} u^{q-1}X^q$. Being p -typical, $\mathbf{F}$ is classified by the homomorphism

$$\mathbf{F} : \text{BP}^* = \mathbb{Z}_p[v_i \mid 1 \leq i < \infty] \rightarrow \mathbb{Z}_p[u, u^{-1}]$$

which sends the polynomial generator v_n to u^{q-1} and all the other v_i 's to 0; where the v_i are Araki's generators, satisfying

$$[p]_{\text{BP}}(X) = \sum_{\text{BP}} v_k X^{p^k}$$

(so $v_0 = p$). The associated genus of complex-oriented manifolds sends

$$\mathbb{C}P^{q^k-1} \mapsto \prod_{1 \leq i \leq k} (1 - p^{q^i-1})^{-1} \cdot (q/p)^k \in \mathbb{Z}_{(p)}$$

and is zero on the other projective spaces.

3.4 The Baas-Sullivan construction, applied to the specialization $v_i \rightarrow 0$, $i \neq 0, n$ of BP associated to the group law F of §1, defines a natural ‘integral lift’ $K(n)^*(-, \mathbb{Z}_p)$ of $K(n)^*(-, \mathbb{F}_p)$. The resulting theories are multiplicative (in the weak sense considered here) even when $p = 2$ or 3 .

The normalizer

$$W(\mathbb{F}_q)^\times \rtimes \mathbb{Z} = \mathbb{W}(\mathbb{Q}_q^{\text{ab}}/\mathbb{Q}_p) \subset D^\times$$

(of the units of $\mathbb{Q}_q$ inside the units of D) acts on

$$K(n)^*(-, W(\overline{\mathbb{F}}_p)) := K(n)^*(-, \mathbb{Z}_p) \otimes W(\overline{\mathbb{F}}_p)$$

with $W(\mathbb{F}_q)$ as endomorphisms of F , and $\mathbb{Z}$ acting via its embedding in $\hat{\mathbb{Z}} = \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ (lifting the action of $D^\times$ on $K(n)^*(-, \overline{\mathbb{F}}_p)$ described in §2.2). In particular, it follows from the cell decomposition of $B\mathbb{T} = \mathbb{C}P^\infty$ that

$$K(n)^*(S^{2k}, W(\overline{\mathbb{F}}_p)) \cong W(\overline{\mathbb{F}}_p)^{\otimes k}$$

as $W(\mathbb{F}_q)^\times \rtimes \mathbb{Z}$ -modules. This looks a lot like a Tate twist ...

$K(1)^*(-, W(\mathbb{F}_p))$ is thus the p -adic completion of classical complex K -theory [2], with $\mathbb{Z}_p^\times$ acting as (p -adically completed) stable Adams operations; $K(1)^*(-, \mathbb{F}_p)$ is then its usual mod p reduction.

3.5 This story generalizes to local fields L which are not necessarily unramified. A Lubin-Tate group [5, 26] for such a field has a p -typification, classified by a ring homomorphism

$$F_L : BP^* \rightarrow \mathfrak{o}_L[u]$$

as above, but now sending v_i to some $w_i(L)u^{p^i-1}$ with $w_i(L) \in \mathfrak{o}_L$. The corresponding sequence

$$\dots, \tilde{v}_i = v_i - w_i(L)u^{p^i-1}, \dots \in \mathfrak{o}_L \otimes BP^*[u]$$

is regular ($\{\tilde{v}_i\}$ is just as good a set of polynomial generators for $\mathfrak{o}_L \otimes BP^*[u]$ over $\mathfrak{o}_L$ as $\{v_i\}$ is), so the Baas-Sullivan-Koszul construction [17, appendix] defines, as above, an $\mathfrak{o}_L \otimes BP[u, u^{-1}]$ -module-valued cohomology theory $K(L)^*(-)$, with $K(L)^*(B\mathbb{T})$ canonically isomorphic to the Lubin-Tate group chosen for L .

These are thus formal $\mathfrak{o}_L$ -module spectra [25]; but it seems likely that the normalizer [18] of $L^\times$ in $D^\times$ acts as stable multiplicative endomorphisms of $K(L) \otimes_{\mathfrak{o}_L} \mathfrak{o}_{L^{\text{nr}}}$ (with L^{nr} the maximal unramified extension of L).

Under this convention, $K(n)^*(-, W(\mathbb{F}_q))$ becomes $K(\mathbb{Q}_q)^*(-)$; in particular, $K(\mathbb{Q}_p) \cong K(\mathbb{C}) \otimes \mathbb{Z}_p \dots$

4. GENERALIZED CHERN CLASSES FOR FINITE GROUPS

4.1 The exponential sequence

$$s \mapsto \mathbf{e}(s) = \exp(2\pi i s) : 0 \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{C}^\times \rightarrow \mathbb{R}/\mathbb{Q} \times \mathbb{R} \rightarrow 0$$

identifies the Picard group of complex topological line bundles on the classifying space BG of a finite group G as its first cohomology group

$$L \mapsto [L] \in \text{Pic}_{\mathbb{C}}(BG) = H^1(BG, \mathbb{C}^\times) \cong H^1(G, \mathbb{Q}/\mathbb{Z}) .$$

with coefficients in $\mathbb{Q}/\mathbb{Z}$.

The set

$$C_n G = \text{Hom}(\mathbb{Z}^n, G) / G^{\text{conj}}$$

of conjugacy classes of commuting n -tuples of elements of G is the quotient of the set of homomorphisms $\gamma : \mathbb{Z}^n \rightarrow G$ under the equivalence relation $g, \gamma \mapsto g \circ \gamma \circ g^{-1}$ defined by conjugation.

4.2 The group $\text{Gl}_n(\mathbb{Z}) = \text{Aut}(\mathbb{Z}^n)$ acts naturally on $C_n(G)$, defining a transformation groupoid

$$[C_n G / \text{Gl}_n(\mathbb{Z})] .$$

Assigning to $[\gamma : \mathbb{Z}^n \rightarrow G]$ the group

$$D(\mathbb{Z}^n) = \text{Hom}(\mathbb{Z}^n, \mathbb{Q}/\mathbb{Z})$$

dual to $\mathbb{Z}^n$ defines a functor

$$[C_n G / \mathrm{Gl}_n(\mathbb{Z})] \rightarrow (\mathrm{Ab}) ;$$

Grothendieck's fibered category of elements associated to this functor is the pullback category

$$\begin{array}{ccc} \{C_n G / \mathrm{Gl}_n(\mathbb{Z})\} & \cdots \cdots \cdots \rightarrow & (\mathrm{Ab})_* \\ \downarrow & & \downarrow \\ [C_n G / \mathrm{Gl}_n(\mathbb{Z})] & \longrightarrow & (\mathrm{Ab}) \end{array}$$

defined by the forgetful functor from the category of pointed abelian groups.

Let $\Gamma_{\mathrm{Gl}_n(\mathbb{Z})} C_n G$ be the group of sections of this fibered category. In fact we will be most interested in the p -analog

$$\Gamma_{\mathrm{Gl}_n(\mathbb{Z}_p)} C_{n,p} G$$

of this construction, defined by homomorphisms $\gamma : \mathbb{Z}_p^n \rightarrow G$ from free modules over the ring of p -adic integers.

4.3 Proposition The correspondence

$$(L, \gamma) \mapsto \gamma^*[L] \in H^1(\mathbb{Z}^n, \mathbb{Q}/\mathbb{Z}) \cong D(\mathbb{Z}^n)$$

defines a homomorphism

$$\mathfrak{c} : \mathrm{Pic}_{\mathbb{C}}(BG) \rightarrow \Gamma_{\mathrm{Gl}_n(\mathbb{Z})} C_n G .$$

Example If $n = 1$ and γ is a conjugacy class in G , then $\gamma^*[L] \in \mathbb{Q}/\mathbb{Z} \subset \mathbb{C}^\times$ defines the classical Chern class

$$\mathrm{Pic}(BG) \rightarrow (1 + \tilde{R}_{\mathbb{C}}(G))^\times$$

in complex K -theory.

5. LEVEL STRUCTURES

5.1 Following Hopkins, Kuhn, and Ravenel, let E^* be a complex-oriented multiplicative cohomology theory such that $E^*(\mathrm{pt})$ is an evenly graded complete local domain, with residue field of positive characteristic p and quotient field of characteristic zero, and with formal group law

$$F : E^*[[x]] \cong E^*(B\mathbb{T})$$

of finite height $n \geq 1$.

If R^* is an evenly-graded local $E^*(\mathrm{pt})$ -algebra, let

$$F(R) = \mathrm{Hom}_{E^*-\mathrm{loc}}(E^*(B\mathbb{T}), R^*)$$

(abusing gradings as usual) be the group of points of F , with values in the the maximal ideal of R^* . If A is a finite abelian group, a homomorphism

$$\phi : A \rightarrow {}_{p^r}F(R)$$

corresponds to a homomorphism

$$\Phi : R[[x]]/([p^r](x)) \rightarrow \text{Fns}(A, R)$$

of Hopf algebras. There is a universal example

$$\phi_{\text{univ}} : (\mathbb{Z}/p^r\mathbb{Z})^n \rightarrow F(E^*BD(\mathbb{Z}/p^r\mathbb{Z})) .$$

of such a thing.

5.2 Proposition [1 §2.4.3, 8, 11 §6] The functor

$$R \rightarrow \mathbf{L}_r(R) = \{ \phi \in \text{Hom}((\mathbb{Z}/p^r\mathbb{Z})^n, {}_{p^r}F(R)) \mid \Phi \text{ is an iso} \}$$

is represented, in the category of local $E^*(\text{pt}) \otimes \mathbb{Q} := E\mathbb{Q}$ -algebras, by the localization

$$\mathcal{L}_r(E^*) = S_r^{-1} E^*BD((\mathbb{Z}/p^r\mathbb{Z})^n) ,$$

where S_r is the multiplicatively closed subset of the ring on the right, generated by

$$\{ \phi_{\text{univ}}(\alpha)^*(x) \mid 0 \neq \alpha \in D((\mathbb{Z}/p^r\mathbb{Z})^n) \} .$$

Moreover, $\mathcal{L}_r(E^*)$ is finite and faithfully flat over $E^*(\text{pt}) \otimes \mathbb{Q}$, with an action of $\text{Gl}_n(\mathbb{Z}/p^r\mathbb{Z})$ such that

$$\mathcal{L}(E^*) = \varinjlim \mathcal{L}_r(E^*) ,$$

as $\text{Gl}_n(\mathbb{Z}_p)$ -module, has $E\mathbb{Q}$ as ring of invariants.

With these definitions, we can state the main result of HKR theory:

Theorem C There is a natural isomorphism

$$E^*(BG) \otimes \mathbb{Q} \xrightarrow{\sim} \text{Fns}_{\text{Gl}_n(\mathbb{Z}_p)}(C_{n,p}(G), \mathcal{L}(E^*))$$

(with the subscript denoting the set of $\text{Gl}_n(\mathbb{Z}_p)$ -equivariant maps).

5.3 For a connected space X , the complex orientation on E^* defines a group homomorphism

$$\text{Pic}_{\mathbb{C}}(X) = \pi_0 \text{Maps}(X, B\mathbb{T}) \rightarrow \text{Hom}_{E^*-\text{loc}}(E^*B\mathbb{T}, E^*X) = F(E^*X) ;$$

so composing with the character map above defines

$$F(E^*BG) \rightarrow F(\text{Fns}_{\text{Gl}_n(\mathbb{Z}_p)}(C_{n,p}G, \mathcal{L}(E))) .$$

In the dual language of schemes this map has target

$$\text{Mor}_{\text{Sch}/\text{Spf } E\mathbb{Q}}(C_{n,p}G \times_{\text{Gl}_n(\mathbb{Z}_p)} \text{Spf } \mathcal{L}(E^*), \text{Spf } E\mathbb{Q}^*B\mathbb{T})$$

so by adjointness we get a group homomorphism

$$\text{Pic}_{\mathbb{C}}(BG) \rightarrow \text{Fns}_{\text{Gl}_n(\mathbb{Z}_p)}(C_{n,p}G, F(\mathcal{L}(E^*))) .$$

On the other hand, Yoneda says

$$\begin{aligned} F(\mathcal{L}(E^*)) &= \text{Mor}(\text{Spf } \mathcal{L}(E^*), \text{Spf } E\mathbb{Q}^*B\mathbb{T}) \\ &= \text{NatTrans}_{E\mathbb{Q}-\text{loc}}(\text{Iso}(D(\mathbb{Z}_p^n), F(-)_{\text{tors}}), F(-)_{\text{tors}}) , \end{aligned}$$

so the evaluation map

$$\Gamma_{\mathrm{Gl}_n(\mathbb{Z}_p)} C_{n,p} G \times \mathrm{Iso}(D(\mathbb{Z}_p^n), F(-)_{\mathrm{tors}}) \rightarrow \mathrm{Fns}(C_{n,p} G, F(-)_{\mathrm{tors}})$$

(which sends a function from tuples to $D(\mathbb{Z}_p^n)$, together with an isomorphism of $D(\mathbb{Z}_p^n)$ with F_{tors} , to a function from tuples to torsion points) renders the diagram below commutative:

5.4 Proposition

$$\begin{array}{ccc} \mathrm{Pic}_{\mathbb{C}}(BG) & \xrightarrow{\epsilon} & \Gamma_{\mathrm{Gl}_n(\mathbb{Z}_p)} C_{n,p} G \\ \downarrow c & & \downarrow \\ F(E^* BG) & \longrightarrow & \mathrm{Fns}_{\mathrm{Gl}_n(\mathbb{Z}_p)}(C_{n,p} G, F(\mathcal{L}(E^*))) \end{array}$$

6. A GENERALIZATION OF THE ARTIN - ATIYAH THEOREM

6.1 Taking $E = K(L)$ in §5.3 identifies $\mathrm{Spf} \mathcal{L}(E)$ with $\mathrm{Iso}(D(\mathbb{Z}_p^n), \mathrm{LT}_{\mathrm{tors}})$; but by §1.3 this is isomorphic, as a functor on local $\mathfrak{o}_L$ -algebras, to $\mathrm{Iso}(\mathbb{Z}_p^n, \mathfrak{o}_L)$. The natural Galois action on these groups of points identifies the obvious $\mathfrak{o}_L^\times$ -action on the right, with that of the Galois group $\mathrm{Gal}(L^{\mathrm{trab}}/L)$ of the maximal totally ramified abelian extension of L on the left.

Proposition

$$\begin{aligned} \mathrm{Spec} K(L)(BG) \otimes \mathbb{Q} &\cong C_{n,p} G \times_{\mathrm{Gl}_n(\mathbb{Z}_p)} \mathrm{Iso}(\mathbb{Z}_p^n, \mathfrak{o}_L) \\ &\cong \mathrm{Hom}(\mathfrak{o}_L, G)/G^{\mathrm{conj}} := \mathfrak{C}_L G \end{aligned}$$

regarded as an étale scheme over $\mathrm{Spec} L$.

Here the ring of functions on this scheme is the twisted group algebra of functions $f : C_L G \rightarrow L^{\mathrm{trab}}$ such that

$$f(\alpha \cdot \phi) = [\alpha](f(\phi)) ,$$

with $\alpha \in \mathfrak{o}_L^\times$ acting on ϕ by premultiplication, and on L^{trab} through Artin reciprocity.

Example If $n = 1$ this is the set of conjugacy classes $\hat{G}$ of G , understood as an étale scheme over $\mathbb{Q}_p$ with Galois action

$$\alpha, \gamma \mapsto \chi(\alpha) \cdot \gamma : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \times \hat{G} \rightarrow \hat{G}$$

defined by the cyclotomic character.

6.2 If L is Galois over $\mathbb{Q}_p$, the Weil group

$$1 \rightarrow L^\times \rightarrow \mathbb{W}(L^{\mathrm{ab}}/\mathbb{Q}_p) \rightarrow \mathrm{Gal}(L/\mathbb{Q}_p) \rightarrow 1$$

acts (naturally in G) on $\mathfrak{C}_L G$, sending $w \in \mathbb{W}$, f to the function

$$f^w(\phi) := [w](f(w^{-1}(\phi))) ;$$

where w acts on ϕ by projection to $\text{Gal}(L/\mathbb{Q}_p)$, and on L^{trab} as a subfield of L^{ab} . Indeed, if $\alpha \in \mathfrak{o}_L^\times$ as above, then

$$\begin{aligned} f^w(\alpha \cdot \phi) &= [w](f(w^{-1}(\alpha \cdot \phi))) = [w](f(w^{-1}(\alpha) \cdot w^{-1}(\phi))) \\ &= [w]([w^{-1}(\alpha)](f(w^{-1}(\phi)))) = [\alpha](f^w(\phi)) . \end{aligned}$$

For such L , $\mathfrak{C}_L G$ is thus in some sense defined over $\mathbb{Q}_p$.

This suggests regarding $\mathfrak{C}_L G$ as naturally indexed by the commutative subfields of a division algebra with center $\mathbb{Q}_p$ [21, 30 appendix 3] – which fits well with the noncommutative approach to class field theory suggested in [6].

6.3 Examples

$$\mathfrak{C}_L(G_0 \times G_1) \cong \mathfrak{C}_L(G_0) \times_{\text{Spec}_{\text{et}} L} \mathfrak{C}_L(G_1) .$$

We also have

$$\text{Hom}(\mathfrak{o}_L, \mathbb{Z}/p^\nu \mathbb{Z}) \cong {}_{p^\nu}(L/\mathfrak{o}_L)$$

if $G = \mathbb{Z}/p^\nu \mathbb{Z}$, via the pairing

$$x, y \mapsto \text{Tr}_{L/\mathbb{Q}_p}(xy) \bmod p : \mathfrak{o}_L \times L/\mathfrak{o}_L \rightarrow \mathbb{Q}_p/\mathbb{Z}_p .$$

If we identify L^{trab} with functions of finite support from $\mathfrak{o}_L^\times$ to $\mathbb{Q}_p$ by the normal basis theorem, then we have

$$\begin{aligned} \mathfrak{C}_L(\mathbb{Z}/p^\nu \mathbb{Z})(\overline{L}) &= \text{Spec Fns}_{\mathfrak{o}_L^\times}({}_{p^\nu}(L/\mathfrak{o}_L), \mathfrak{o}_L\{\mathfrak{o}_L^\times\}) \\ &\cong \text{Spec Fns}({}_{p^\nu}(L/\mathfrak{o}_L), \mathbb{Q}_p) \cong {}_{p^\nu}(L/\mathfrak{o}_L) . \end{aligned}$$

Finally, it seems likely that the Tate-Borel cohomology of a finite group G fits in an extension

$$0 \rightarrow K(L)^*(BG) \rightarrow t_G^* K(L) \cong K(L)^*(BG) \otimes \mathbb{Q} \rightarrow K(L)_{-* -1}(BG) \rightarrow 0$$

generalizing [28, 31]. Applied to the p -divisible system $\{\mathbb{Z}/p^\nu \mathbb{Z}\}$, this suggests that $t_G^* K(L)$ is essentially the universal additive extension [4, 10 §11, 16] of the Lubin-Tate group of L .

6.4 Quillen's work on the algebraic K -theory of a classical ring R can be interpreted as a construction of the best representable approximation to the functor which assigns to a space X , the Grothendieck group of flat bundles of R -modules over X . Conceivably the natural transformation

$$\mathfrak{C}_L \pi_1 \rightarrow \text{Spec}_{\text{et}} K(L) \otimes \mathbb{Q}$$

has a similar characterization.

REFERENCES

1. M Ando, Isogenies of formal group laws and power operations in the cohomology theories E_n , *Duke Math. J.* **79** (1995) 423 - 485
2. MF Atiyah, DO Tall, Group representations, Λ -rings and the J -homomorphism, *Topology* **8** (1969) 253 - 297
3. J Borger, B de Smit, Galois theory and integral models of Λ -rings, [arXiv:0801.2352](#)
4. P Cartier, Relèvements des groupes formels commutatifs, *Seminaire Bourbaki* 359, Springer LNM **179** (1971) 217 - 230
5. —, Groupes de Lubin-Tate généralisés, *Invent. Math.* **35** (1976) 273 - 284.
6. A Connes, C Consani, On the arithmetic of the BC-system, [arXiv:1103.4672](#)
7. PG Goerss, MJ Hopkins, Moduli spaces of commutative ring spectra, in **Structured ring spectra** 151 - 200, LMS Lecture Notes 315, CUP 2004
8. JPC Greenlees, NP Strickland, Varieties and local cohomology for chromatic group cohomology rings, *Topology* **38** (1999) 1093 - 1139
9. M Hazewinkel, On formal groups: the functional equation lemma and some of its applications, in *Journées de Géométrie Algébrique de Rennes I*, 73 - 82, *Astérisque* 63, Soc. Math. France, Paris 1979
10. MJ Hopkins, B Gross, Equivariant vector bundles on the Lubin-Tate moduli space, in **Topology and representation theory** 23 - 88, *Contemp. Math.* **158**, AMS (1994)
11. —, NJ Kuhn, DC Ravenel, Generalized group characters and complex oriented cohomology theories. *JAMS* **13** (2000) 553 - 594
12. —, JH Smith, Nilpotence and stable homotopy theory II, *Ann. of Math.* **148** (1998) 1 - 49
13. M Hovey, NP Strickland, Morava K -theories and localisation. *Mem. AMS* **139** (1999), no. 666
14. S Lang, **Cyclotomic fields I and II** Springer Graduate Texts in Mathematics 121 (1990)
15. J Lubin, J Tate, Formal moduli for one-parameter formal Lie groups, *Bull. Soc. Math. France* **94** (1966) 49 - 59
16. B Mazur, W Messing, Universal extensions and one dimensional crystalline cohomology. Springer LNM **370** (1974)
17. J Morava, A product for the odd-primary bordism of manifolds with singularities, *Topology* **18** (1979) 177 - 186
18. —, The Weil group as automorphisms of the Lubin-Tate group, *Journées de Géométrie Algébrique de Rennes I*, 169 - 177, *Astérisque* 63, Soc. Math. France, Paris (1979)
19. —, Noetherian localisations of categories of cobordism comodules. *Ann. of Math.* **121** (1985) 1 - 39
20. —, Some Weil group representations motivated by algebraic topology, in **Elliptic curves and modular forms in algebraic topology** 94 - 106, Springer LNM **1326** (1988)
21. —, Stable homotopy and local number theory, in **Algebraic analysis, geometry, and number theory** 291 - 305, JHU Press, Baltimore (1989)
22. DC Ravenel, **Complex cobordism and stable homotopy groups of spheres**, Pure and Applied Mathematics **121**, Academic Press 1986
23. C Rezk, Notes on the Hopkins-Miller theorem, in **Homotopy theory via algebraic geometry and group representations** 313 - 366, *Contemp. Math.* **220**, AMS 1998
24. —, The congruence criterion for power operations in Morava E -theory, in *Homology, Homotopy Appl.* **11** (2009) 327 - 379
25. A Salch, Structure of the moduli stack of one-dimensional formal Λ -modules, [arXiv:1005.0119](#)

- 26. JP Serre, Local class field theory, in **Algebraic Number Theory** (Brighton) 128 - 161, Thompson 1987
- 27. —, **Linear representations of finite groups**. Springer Graduate Texts in Mathematics 42 (1977)
 - 1. 28]28 NP Strickland, $K(n)$ -local duality for finite groups and groupoids, *Topology* 39 (2000) 733 - 772.
- 29. J Tate, Number theoretic background, in **Automorphic forms, representations and L -functions** 3 - 26, Proc. Sympos. Pure Math. XXXIII, AMS 1979
- 30. A Weil, **Basic number theory**, third edition, Springer Grundlehren der Mathematischen Wissenschaften 144 (1974)
- 31. G Wilson, K -theory invariants for unitary G -bordism, *Quart. J. Math.* 24 (1973) 499 - 526
- 32. U Würgler, Morava K -theories: a survey, in **Algebraic topology**(Poznan') 111 - 138, Springer LNM 1474 (1991)

THE JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218

E-mail address: jack@math.jhu.edu