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Article

The Relative Brauer Group of $K(1)$-Local Spectra

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Using profinite Galois descent, we compute the Brauer group of the $K(1)$-local category relative to Morava E-theory. At odd primes this group is generated by a cyclic algebra formed using any primitive $(p-1)$st root of unity, but at the prime two is a group of order 32 with nontrivial extensions; we give explicit descriptions of the generators, and consider their images in the Brauer group of KO. Along the way, we compute the relative Brauer group of completed KO, using the étale locally trivial Brauer group of Antieau, Meier and Stojanoska.

1 Introduction

The classification of central simple algebras over a field is a classical question in number theory, and by the Wedderburn theorem any such algebra is a matrix algebra over some division algebra, determined up to isomorphism. If one identifies those algebras that arise as matrix algebras over the same division ring, then tensor product defines a group structure on the resulting set of equivalence classes. This relation is *Morita equivalence*, and the resulting group is the *Brauer group* $\mathrm{Br}(K)$. One formulation of class field theory is the determination of $\mathrm{Br}(K)$ in the case that K is a number field.

One consequence of Wedderburn's theorem is that every central simple algebra is split by some extension L/K; in fact, one can take L to be Galois. This opens the door to cohomological descriptions of the Brauer group: by Galois descent, one obtains the first isomorphism in

$$\mathrm{Br}(K) \cong H^1(K, \mathrm{PGL}_\infty) \cong H^2(K, \mathbb{G}_m), \tag{1}$$

where PGL_∞ denotes the Galois module $\varinjlim \mathrm{PGL}_k(K^s)$. The second isomorphism is [40, Proposition X.9], and follows from Hilbert 90. This presents $\mathrm{Br}(K)$ as an étale cohomology group, and allows the use of cohomological techniques in its determination. Conversely, it gives a concrete interpretation of 2-cocycles, analogous to the relation between 1-cocycles and Picard elements.

Globalising this picture was a deep problem in algebraic geometry, initiated by the work of Azumaya, Auslander and Goldman, and the Grothendieck school. There are two ways to proceed, resulting in two variants: the group $\mathrm{Br}(X)$ of Azumaya algebras, and the more computable group $\mathrm{Br}'(X)$. These are in general related by an injective map $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X)$, but surjectivity is known to fail for arbitrary schemes (e.g., 12, Corollary 3.11). A key insight of Toën was to pass to *derived* Azumaya algebras, in which case this map becomes an isomorphism in full generality [42]. Toën's work shows that even the

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Brauer groups of classical rings are most naturally studied in the context of derived or homotopical algebraic geometry. This initiated a study of Brauer groups through the techniques of higher algebra, and in recent years they have become objects of intense study in homotopy theory. In this context, the basic definitions appear in [4]: as in the classical case, the Brauer group of a ring spectrum R classifies Morita equivalence classes of Azumaya algebras. This gives a useful class of noncommutative algebras over an $\mathbb{E}_\infty$-ring: for example, Hopkins and Lurie [21] computed Brauer groups as part of a program to classify algebra structures on Morava K-theory over Lubin-Tate theory. A more categorical perspective is to view the Brauer space as the Picard space of modules over $\mathrm{Mod}_R \in \mathrm{CAlg}(\mathbf{Pr}^L)$; in this guise it is the target for (invertible) factorisation homology, as an extended quantum field theory [3, 13, 19, 26, 39]. Computations of Brauer groups of ring spectra of particular interest appear in [1, 2, 15].

In this document we study Brauer groups in the monochromatic setting. Recall that Hopkins and Lurie focus on the $K(h)$-local Brauer group $\mathrm{Br}(E_h)$ of E_h, constructing a certain filtration on it whose associated graded they compute; in particular, their work shows that this group is highly nontrivial, in contrast to the Picard case. Our starting motivation here was to extend this to a computation of the Brauer group $\mathrm{Br}_h := \mathrm{Br}(\mathcal{S}p_{K(h)})$ of the entire $K(h)$-local category, which classifies $K(h)$-local Azumaya algebras up to Morita equivalence. As a first step, in this document we complement [21] by studying the *relative* Brauer group $\mathrm{Br}_h^0 := \mathrm{Br}(\mathcal{S}p_{K(h)} \mid E_h)$, which classifies those $K(h)$-local Azumaya algebras that become trivial after basechange to E_h. Combining the two computations gives access to the group Br_h up to extension problems, which we do not attempt to address here. The group Br_h^0 also has a concrete interpretation in terms of chromatic homotopy theory: it classifies twists of the $\mathbb{G}_h$-action on the ∞-category $\mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)})$. For the standard action (by basechange along the Goerss-Hopkins-Miller action on E_h), one has $\mathcal{S}p_{K(h)} \simeq \mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)})^{h\mathbb{G}_h}$ (see [31, Proposition 10.10] and [35, Theorem A.II] for two formulations). Taking fixed points for a twisted action therefore gives a twisted version of the $K(h)$-local category.

Our main theorems give the computation of the relative Brauer group at height one:

Theorem A (Lemma 5.16). At the prime two, $\mathrm{Br}_1^0 \cong \mathbb{Z}/8\{Q_1\} \oplus \mathbb{Z}/4\{Q_2\}$.

(i) The $\mathbb{Z}/4$-factor is mapped injectively to $\mathrm{Br}(KO_2)$, and $Q_2^{\otimes 2} \otimes KO_2$ is the image of the generator under $\mathbb{Z}/2 \cong \mathrm{Br}(KO \mid KU) \to \mathrm{Br}(KO_2 \mid KU_2)$. The $\mathbb{Z}/8$ factor is the relative Brauer group $\mathrm{Br}(\mathcal{S}p_{K(1)} \mid KO_2)$.

(ii) $Q_4 := Q_1^{\otimes 4}$ is the cyclic algebra formed using the C_2-Galois extension $\mathbb{S}_{K(1)} \to KU_2^{h(1+4\mathbb{Z}_2)}$ and the *strict* unit $1+\varepsilon \in \pi_0\mathrm{GL}_1(\mathbb{S}_{K(1)}) = (\mathbb{Z}_2[\varepsilon]/(2\varepsilon, \varepsilon^2))^\times$.

We have indexed generators on the filtration in which they are detected in the descent spectral sequence, which we recall later in the introduction. For the final part, note [4, §4] that cyclic algebras are defined using *strict* units: that is, maps of spectra $u : \mathbb{Z} \to \mathfrak{gl}_1(E)$. We give a construction of cyclic algebras from strict units in Section 3, and using this we show that they are detected in the HFPSS by a *symbol* in the sense of [40, Chapter XIV]. This allows us to deduce when they give rise to nontrivial Brauer classes.

Any strict unit has an underlying unit, and we abusively denote these by the same symbol. The unit $1+\varepsilon$ was shown to be strict in [10], which is what gives rise to the claimed representative for the class Q_4 at the prime two. Likewise, at odd primes the roots of unity are strict, which leads to our second main computation:

Theorem B (Lemma 5.1). At odd primes, $\mathrm{Br}_1^0 \cong \mathbb{Z}/(p-1)$. A generator is given by the cyclic algebra $(KU_p^{h(1+p\mathbb{Z}_p)}, \chi, \omega)$, where $\chi : \mu_{p-1} \cong \mathbb{Z}/(p-1)$ is a character and $\omega \in (\pi_0\mathbb{S}_{K(1)})^\times \cong \mathbb{Z}_p^\times$ is a primitive $(p-1)$st root of unity.

We now give an outline of the computation. Since Grothendieck, the main approach to computing Brauer groups has been *étale* or *Galois descent*, and this is the case for us too. Namely, recall that at any height h, Morava E-theory defines a $K(h)$-local Galois extension $\mathbb{S}_{K(h)} \to E_h$, with profinite Galois group $\mathbb{G}_h$. In [35], we used condensed mathematics to prove a Galois descent statement of the form

$$\mathcal{S}p_{K(h)} \simeq \mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)})^{h\mathbb{G}_h},$$

and deduced from this a homotopy fixed point spectral sequence for Picard and Brauer groups, extending the Galois descent results of Mathew and Stojanoska [32] and Gepner and Lawson [15]. For our purposes, the main computational upshot of that paper is:

Theorem 1.1. There is a descent spectral sequence

$$E_2^{s,t} = H^s(\mathbb{G}_h, \pi_t \mathfrak{pic}(E_h)) \implies \pi_{t-s}\mathfrak{pic}(\mathcal{S}p_{K(h)}),$$

whose (-1)-stem gives an upper bound on Br_h^0. In a large range, there is an explicit comparison of differentials with the $K(h)$-local E_h-Adams spectral sequence.

A more precise form of the theorem is recalled in Section 4. In the present paper, we determine this spectral sequence completely at height one, at least for $t-s \geq -1$. Unsurprisingly, the computation looks very different in the cases $p = 2$ and $p > 2$, and the former represents the majority of our work. To prove a lower bound on Br_1^0, we also prove a realisation result in the spirit of Toën's theorem. Namely, we show in a very general context that all classes on the E_∞-page may be represented by Azumaya algebras:

Theorem C (Lemma 2.10). Let $\mathcal{C}$ be a symmetric monoidal ∞-category, and suppose that $\mathcal{C}$ is generated under colimits by dualisable objects. If $A \in \mathrm{CAlg}(\mathcal{C})$ is a faithful dualisable Galois extension of the unit, then the map

$$\mathrm{Br}(\mathcal{C} \mid A) \to \mathrm{Br}'(\mathcal{C} \mid A) := \{\mathcal{D} : \mathrm{Mod}_A(\mathcal{D}) \simeq \mathrm{Mod}_A(\mathcal{C})\} \subset \mathrm{Pic}(\mathrm{Mod}_{\mathcal{C}}(\mathbf{Pr}^L))$$

sending an Azumaya algebra to its module ∞-category is an isomorphism.

Most pertinently, the chromatic localisations of spectra give examples of such ∞-categories; see also [1, §6.3] and [15, §6.4] for similar results in the case of compact unit. This accounts immediately for most of the classes in Br_1^0, by sparsity in the descent spectral sequence. At the prime two there is one final computation necessary, which is the relative Brauer group of KO_2. Recall that the group $\mathrm{Br}(KO \mid KU)$ was computed by Gepner and Lawson [15]. In Section 5.2, we prove:

Theorem D (Lemma 5.3). The relative Brauer group of KO_2 is $\mathrm{Br}(KO_2 \mid KU_2) \cong \mathbb{Z}/4$. The basechange map from $\mathrm{Br}(KO \mid KU) \cong \mathbb{Z}/2$ is injective.

The generator, which we denote P_2, may be thought of as a cyclic algebra for the unit $1+4\zeta$, where ζ is a topological generator of $\mathbb{Z}_2$ (see Section 3). We remark that this unit is *not* strict, and so the cyclic algebra cannot be constructed by hand; instead, we invoke Theorem C to prove that the possible obstructions vanish. We then show that P_2 survives the descent spectral sequence for the extension $\mathbb{S}_{K(1)} \to KO_2$, and therefore gives rise to the class $Q_2 \in \mathrm{Br}_1^0$.

As an aside, we observe that Theorem D implies the following result, which may be of independent interest:

Theorem E. There is no C_2-equivariant splitting $\mathfrak{gl}_1 KU_2 \simeq \tau_{\leq 3}\mathfrak{gl}_1 KU_2 \oplus \tau_{\geq 4}\mathfrak{gl}_1 KU_2$.

1.1 Outline of the paper

The first two sections give some general results on Brauer groups, especially on representing elements of the cohomological Brauer group. To begin, in Section 2 we prove that the property of "admitting a compact generator" satisfies $\mathcal{S}p_{K(h)}$-linear Galois descent, which allows us to lift certain E_∞-classes to Azumaya algebras. In Section 3 we give the construction of a certain kind of cyclic algebra, which makes clear where they are detected in the descent spectral sequence; this later allows us to assert that the cyclic algebras we form are distinct. We also give a construction of Brauer classes from 1-cocycles, which we use at the prime two. In Section 4 we specialise to the case of $\mathcal{S}p_{K(1)}$. We begin by computing the E_2-page of the descent spectral sequences and most differentials, giving an upper bound on the relative Brauer groups. Using Section 2, the computation at odd primes follows easily. In Section 5 we compute the relative Brauer group $\mathrm{Br}(KO_2 \mid KU_2)$, and use this to complete the computation of Br_1^0 at the prime two.

1.2 Notation and conventions

- We will freely use the language of ∞-categories (modeled as quasi-categories) as pioneered by Joyal and Lurie [25, 27, 29]. In particular, all (co)limits are ∞-categorical. Most commonly, we will be in the context of a presentably symmetric monoidal stable ∞-category with unit $\mathbf{1}$, and we use the term *stable homotopy theory* to mean such an object. All our computations take place internally to the $K(h)$-local category, and so the symbol $\otimes$ will generally denote the $K(h)$-local smash product.

- We will only consider spectra with group actions, meaning functors $BG \to \mathcal{S}p$ when G is finite or sheaves on $BG_{\text{proét}}$ when G is profinite. When G is a profinite group, we will write $H^*(G, M)$ for *continuous* group cohomology.
- We work at a fixed prime p and height h (mostly one). As such, p and h are often implicit in the notation: thus we write E, K and $\mathbb{G}$ for E_h, $K(h)$, and $\mathbb{G}_h$, respectively. We denote the $K(h)$-local sphere by $\mathbf{1}_K$. We use the symbol $\mathbb{S}$ for the both the sphere and its p-completion, according to context. To avoid ambiguity, we are explicit in some cases: for example, KU will always mean *integral* K-theory.
- We follow the conventions of [35]. In particular, we direct the reader to [35, §2.2] for details about the proétale site and the sheaves $\underline{E}$ and $\mathfrak{pic}(\underline{E})$. If M is a topological G-module, we will write $\underline{M}$ for the associated proétale abelian group. Given a descendable G-Galois extension $\mathbf{1} \to A$ in a stable homotopy theory (possibly profinite), we will implicitly use the associated (hypercomplete) sheaf $\mathcal{A} \in \mathrm{Sh}(BG_{\text{proét}}, \mathcal{C})$, writing $A^{hG} := \Gamma\mathcal{A}$. Using [35, §3.1], we will also form the sheaf $\mathrm{Mod}_{\mathcal{A}}(\mathcal{C}) \in \mathrm{Sh}(BG_{\text{proét}}, \mathrm{Pr}^{L,\mathrm{smon}})$, and hence the Picard sheaf $\mathfrak{pic}(\mathcal{A}) \in \mathrm{Sh}(BG_{\text{proét}}, \mathcal{S}p_{\geq 0})$. In this case the descent spectral sequence reads

$$H^s(BG_{\text{proét}}, \pi_t\mathfrak{pic}(\mathcal{A})) \implies \pi_{t-s}\mathfrak{pic}(A)^{hG}, \tag{2}$$

- and the E_2-term can be identified with continuous cohomology of the G-module $\pi_t\mathfrak{pic}(A)$ as long as $\pi_t\mathfrak{pic}(\mathcal{A}) = \underline{\pi_t\mathfrak{pic}(A)}$ and $\pi_t\mathfrak{pic}(A)$ satisfies for example the conditions of [7, Lemma 4.3.9].
- When indexing spectral sequences, we will always use s for filtration, t for internal degree, and $t-s$ for stem. We abbreviate “homotopy fixed points spectral sequence” to HFPSS. We write “Picard spectral sequence” for the descent spectral sequence (2).
- We fix once and for all a regular cardinal κ such that (i) $\mathbb{S}_{K(1)} \in \mathcal{S}p_{K(1)}$ is κ-compact; (ii) $|\mathbb{G}_1| < \kappa$. The cohomological Brauer space $\mathfrak{Br}'(\mathcal{S}p_{K(1)})$ will by definition be the Picard space of $\mathcal{S}p_{K(1)} \in \mathrm{CAlg}(\mathrm{Pr}^L_\kappa)$, which is a *small* space since Pr^L_κ is presentable. As noted in Lemma 2.10, for *relative* Brauer classes this is no restriction.

2 Descent for Compact Generators

If $\mathcal{C}$ is a stable homotopy theory, we will be interested in the group $\mathrm{Br}(\mathbf{1})$ of Azumaya algebras in $\mathcal{C}$. It will often be technically convenient to first compute the related group

$$\mathrm{Br}'(\mathbf{1}) := \mathrm{Pic}(\mathrm{Mod}_{\mathcal{C}}),$$

where $\mathrm{Mod}_{\mathcal{C}} := \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^L_\kappa)$ and κ is chosen to be large enough that $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_\kappa)$. In this first section we study the relation between the two groups.

Given a Galois extension $\mathbf{1} \to A$ in $\mathcal{C}$, we showed in [35, §5] that the Picard spectral sequence computes the subgroup

$$\mathrm{Br}'(\mathbf{1} \mid A) = \pi_0(B\mathfrak{Pic}(A))^{hG} = \mathbf{ker}(\mathrm{Br}'(\mathbf{1}) \to \mathrm{Br}'(A)) \subset \mathrm{Br}'(\mathbf{1})$$

of the cohomological Brauer group. To relate this to the Brauer-Azumaya group classifying Azumaya algebras in $\mathcal{C}$, we prove a descent result for compact generators valid in the K-local setting. This is entirely analogous to the theory of [1, §6.3] and [15, §6.4].

Definition 2.1. Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category, and $\mathcal{D} \in \mathrm{Mod}_{\mathcal{C}}$. An object $D \in \mathcal{D}$ is $\mathcal{C}$-*compact* if the functor

$$\underline{\mathrm{Map}}_{\mathcal{D}}(D, -) : \mathcal{D} \to \mathcal{C} \tag{3}$$

preserves filtered colimits. We say that D is a $\mathcal{C}$-*generator* if the functor (3) is conservative; when $\mathcal{C}$ is stable, it is equivalent that (3) detects zero objects. A $\mathcal{C}$-*compact generator* of $\mathcal{D}$ is an object

$D \in \mathcal{D}$ that is both $\mathcal{C}$-compact and a $\mathcal{C}$-generator, and we shall write $\mathcal{D}^{\mathrm{ecg}} \subset \mathcal{D}$ for the full subcategory of such D. (These might also be called *e*nriched *c*ompact *g*enerators.)

Example 2.2. The K-local sphere is an $\mathcal{S}p_K$-compact generator of $\mathcal{S}p_K$. More generally, we always have $\mathbf{1} \in \mathcal{C}^{\mathrm{ecg}}$ since in this case (3) is the identity functor.

Our first objective is to show that Schwede-Shipley theory goes through in the presence of a $\mathcal{C}$-compact generator.

Definition 2.3. Let $\mathcal{C}$ be a stable homotopy theory. We say $\mathcal{C}$ is *rigidly generated* if it is generated under colimits by dualisable objects. That is, the localising category generated by $\mathcal{C}^{\mathrm{dbl}}$ is $\mathcal{C}$ itself.

Example 2.4.

(i) $\mathcal{S}p$ is generated under colimits by shifts of $\mathbf{1}$, and so rigidly generated.
(ii) If $\mathcal{C}$ is a rigidly generated stable homotopy theory and $L\colon \mathcal{C} \to \mathcal{C}'$ a monoidal localisation, then $\mathcal{C}'$ is rigidly generated. Thus, $\mathcal{S}p_K$ is rigidly generated.
(iii) For a compact Lie group G, the ∞-category $\mathcal{S}^G$ of G-spaces is generated under colimits by orbits G/H (e.g., 30, Theorem 1.8). Its stabilisation $\mathcal{S}p^G_{\mathscr{U}}$ at any G-universe $\mathscr{U}$ (as defined in [16, Corollary C.7]) is generated under colimits by shifts $\Sigma^{-V}\Sigma^\infty_{\mathscr{U}} G/H_+$ as V ranges over representations in $\mathscr{U}$, and if $\mathscr{U}$ is complete then these are dualisable by virtue of the Wirthmüller isomorphism [18, (4.16)]. Thus, the ∞-category $\mathcal{S}p^G$ of genuine G-spectra is rigidly generated.

Proposition 2.5 (Enriched Schwede-Shipley). Let $\mathcal{C}$ be a rigidly generated stable homotopy theory and $\mathcal{D} \in \mathrm{Mod}_{\mathcal{C}}$. Suppose that $D \in \mathcal{D}^{\mathrm{ecg}}$, and write $A := \underline{\mathrm{End}}_{\mathcal{D}}(D) \in \mathrm{Alg}(\mathcal{C})$. Then there is a $\mathcal{C}$-linear equivalence

$$\mathcal{D} \simeq \mathrm{LMod}_A(\mathcal{C}).$$

Proof. The object $D \in \mathcal{D}$ determines canonically a $\mathcal{C}$-linear left adjoint $F\colon \mathcal{C} \to \mathcal{D}$, with right adjoint $G := \underline{\mathrm{Map}}_{\mathcal{D}}(D, -)$. According to [27, Proposition 4.8.5.8], it is enough to check the following:

(i) G preserves colimits of simplicial objects: in fact G preserves all colimits. Indeed, G preserves filtered colimits since D is $\mathcal{C}$-compact, and finite colimits as it is a right adjoint between stable ∞-categories.
(ii) G is conservative: this is by definition of $\mathcal{C}$-compact generators.
(iii) for every $D' \in \mathcal{D}$ and $C \in \mathcal{C}$, the map

$$C \otimes FG(D') = C \otimes G(D') \otimes D \to C \otimes D'$$

(iii) is adjoint to an equivalence $C \otimes G(D') \xrightarrow{\sim} G(C \otimes D')$. But by (i), the functor G preserves all colimits, so by rigid generation we reduce to C dualisable. In this case, the desired equivalence is the composite

$$C \otimes \underline{\mathrm{Map}}_{\mathcal{D}}(D, D') \simeq \underline{\mathrm{Map}}_{\mathcal{C}}(C^\vee, \underline{\mathrm{Map}}_{\mathcal{D}}(D, D')) \simeq \underline{\mathrm{Map}}_{\mathcal{D}}(C^\vee \otimes D, D') \simeq \underline{\mathrm{Map}}_{\mathcal{D}}(D, C \otimes D').$$

■

To use this to produce Azumaya algebras, we first need to be able to produce $\mathcal{C}$-compact generators.

Remark 2.6. Recall [31, Definition 3.18] that in a stable homotopy theory $\mathcal{C}$, we say $A \in \mathrm{CAlg}(\mathcal{C})$ is *descendable* if the thick $\otimes$-ideal generated by A contains $\mathbf{1}$. For example, any faithful finite Galois extension is descendable [31, Corollary 6.15], and E is descendable in $L_n\mathcal{S}p$ (and hence in $\mathcal{S}p_K$) by [36, §8].

In this context, we will have a good supply of generators thanks to the following result:

Proposition 2.7 (Descent for $\mathcal{C}$-compact generators). Let $\mathbf{1} \to A$ be a descendable extension in a rigidly generated stable homotopy theory $\mathcal{C}$, and suppose that A is dualisable. Then $\mathcal{D} \in \mathrm{Mod}_{\mathcal{C}}$ admits a $\mathcal{C}$-compact generator if and only if $\mathrm{Mod}_A(\mathcal{D}) := \mathcal{D} \otimes_{\mathcal{C}} \mathrm{Mod}_A(\mathcal{C})$ admits one.

The proof relies on the following basic lemma:

Lemma 2.8. Suppose $\mathcal{C} \in \mathrm{CAlg}(\mathbf{Pr}^L)$ and $A \in \mathcal{C}$ is dualisable. Then A is faithful if and only if $A^\vee$ is.

Proof. Assume that A is faithful, and that $A^\vee \otimes X = 0$; the converse is given by taking duals. Then the identity on $A \otimes X$ factors as

$$A \otimes X \to A^\vee \otimes A \otimes A^\vee \otimes X \to A \otimes X,$$

and in particular $A \otimes X = 0$. By faithfulness of A, this implies $X = 0$. ■

Proof. (Lemma 2.7) As in [15], we will make use of the adjunctions

$$i_! : \mathcal{C} \rightleftarrows \mathrm{Mod}_A(\mathcal{C}) : i^* \quad \text{and} \quad i^* : \mathrm{Mod}_A(\mathcal{C}) \rightleftarrows \mathcal{C} : i_*$$

and the adjunctions (denoted by the same symbols) between $\mathcal{D}$ and $\mathrm{Mod}_A(\mathcal{D})$. We begin by proving that the adjunctions $i_! \dashv i^* \dashv i_*$ are $\mathcal{C}$-linear. For $i_!$ we observed $\mathcal{C}$-linearity in the proof of Lemma 2.5. To see $\mathcal{C}$-linearity for i^* it is enough to prove that the canonical map

$$\theta : C \otimes i^*M \to i^*(C \otimes M)$$

is an equivalence for every C and M, and by rigid generation we reduce to C dualisable. As in [29, Remark D.7.4.4] one checks that $\mathrm{Map}_{\mathcal{C}}(C', \theta)$ is the composite equivalence

$$\begin{aligned} \mathrm{Map}(C', C \otimes i^*M) &\simeq \mathrm{Map}(C' \otimes C^\vee, i^*M) \\ &\simeq \mathrm{Map}(i_!(C' \otimes C^\vee), M) \\ &\simeq \mathrm{Map}(i_!C' \otimes C^\vee, M) \\ &\simeq \mathrm{Map}(i_!(C'), C \otimes M) \\ &\simeq \mathrm{Map}(C', i^*(C \otimes M)) \end{aligned}$$

for any $C' \in \mathcal{C}$, which gives the claim.

Given linearity, the proof of the proposition is straightforward. If $\mathcal{D}$ admits a $\mathcal{C}$-compact generator D, one checks that $i_!D \in \mathrm{Mod}_A(\mathcal{D})^{\mathrm{ecg}}$: indeed, $i_!D$ is $\mathcal{C}$-compact because D is so and i^* preserves colimits, while $i_!D$ generates because D does and i^* is conservative. Conversely, suppose we have $D \in \mathrm{Mod}_A(\mathcal{D})^{\mathrm{ecg}}$, and consider $i^*D \in \mathcal{D}$. By dualisability of A, the right adjoint

$$i_* = \underline{\mathrm{Map}}_{\mathcal{C}}(A, -) \simeq A^\vee \otimes - : \mathcal{C} \to \mathrm{Mod}_A(\mathcal{C})$$

preserves colimits, and hence the right adjoint $i_* : \mathrm{Mod}_A(\mathcal{D}) \to \mathcal{D}$ does too. As a result, i^*D is $\mathcal{C}$-compact. On the other hand, if $X \in \mathcal{D}$ and $\underline{\mathrm{Map}}_{\mathcal{D}}(i^*D, X) = 0$, then $\underline{\mathrm{Map}}_{\mathrm{Mod}_A(\mathcal{D})}(D, i_*X) = 0$ and so

$$i_*X = A^\vee \otimes X = 0.$$

Now faithfulness of $A^\vee$ implies that $X = 0$. ■

Example 2.9. If $A \to B$ is an E-local Galois extension of ring spectra with stably dualisable Galois group G, then Rognes [37, Proposition 6.2.1] shows that B is dualisable over A. For example, this covers the following cases:

(i) $E = \mathbb{S}$ and G is finite or compact Lie.
(ii) $E = \mathbb{F}_p$ and G is p-compact.
(iii) $E = \mathrm{K}$ and $G = K(\pi, m)$ for π a finite p-group and $m \leq h$.

Corollary 2.10 (Br = Br′). Let $\mathbf{1} \to A$ be a faithful dualisable Galois extension in a rigidly generated stable homotopy theory $\mathcal{C}$, and $\mathcal{Q} \in \pi_0 B\mathfrak{Pic}(A)^{hG}$ a relative Brauer-Grothendieck class. Then $\mathcal{Q}$ is represented by some Azumaya algebra Q whose basechange to A is (Morita) trivial: that is,

$$\mathcal{Q} \simeq \mathrm{Mod}_Q(\mathcal{C}) \in \mathrm{Mod}_{\mathcal{C}},$$

and $\mathrm{Mod}_{A\otimes Q}(\mathcal{C}) \simeq \mathrm{Mod}_A(\mathcal{C})$. Thus, the map $\mathrm{Br}(\mathbf{1} \mid A) \to \mathrm{Br}'(\mathbf{1} \mid A)$ is an isomorphism.

Proof. We claim that $\mathcal{Q}$ is κ-compactly generated, so that $\mathcal{Q} \in \mathrm{Pic}(\mathrm{Mod}_{\mathcal{C}})$. Given this, the result follows from Lemma 2.7: by assumption, $\mathrm{Mod}_A(\mathcal{Q}) \simeq \mathrm{Mod}_A(\mathcal{C})$, and so

$$A \in \mathrm{Mod}_A(\mathcal{C})^{\mathrm{ecg}} \simeq \mathrm{Mod}_A(\mathcal{Q})^{\mathrm{ecg}}.$$

By descent for compact generators we obtain $D \in \mathcal{Q}^{\mathrm{ecg}}$, and so Schwede-Shipley theory yields a $\mathcal{C}$-linear equivalence $\mathcal{Q} \simeq \mathrm{Mod}_Q(\mathcal{C})$, where $Q = \underline{\mathrm{End}}_{\mathcal{Q}}(D)$.

For the claim, we note that as in [31, Corollary 3.42], $\mathcal{Q}$ is the limit of the cosimplicial diagram $\mathcal{Q} \otimes_{\mathcal{C}} \mathrm{Mod}_{A^{\otimes \bullet+1}}(\mathcal{C})$, and for $j \geq 1$ we have $\mathcal{Q} \otimes \mathrm{Mod}_{A^{\otimes j}}(\mathcal{C}) \simeq \mathrm{Mod}_{A^{\otimes j}}(\mathcal{C}) \in \mathrm{Pr}^L_{\kappa}$. Thus $\mathcal{Q} \in \mathrm{Pr}^L_{\kappa}$. ■

Remark 2.11. In a previous version, we claimed that Morava E-theory is dualisable in $\mathcal{S}p_{\mathrm{K}}$. As pointed out to us by Maxime Ramzi, while E is Spanier-Whitehead self-dual [6, 41] and hence *reflexive*, it is not dualisable: for example, $\mathrm{K}_*\mathrm{E}$ would otherwise be finite by [23, Theorem 8.6]. We will bypass this issue at height one by showing that all generators of $\mathrm{Br}'(\mathcal{S}p_{\mathrm{K}} \mid \mathrm{E})$ are in fact trivialised in a *finite* Galois extension of the sphere, and hence lift to Azumaya algebras by Lemma 2.10; we do not know if $\mathrm{Br}^0_h \cong \mathrm{Br}'(\mathcal{S}p_{\mathrm{K}} \mid \mathrm{E})$ at arbitrary height.

3 Explicit Generators

In this section, we give some explicit constructions of Azumaya algebras from Galois extensions. Most of this section works in an arbitrary stable homotopy theory $\mathcal{C}$. We will use these constructions in Section 5 to describe generators of the group Br^0_1, and hence to solve extension problems.

3.1 $\mathbb{Z}_p$-extensions

We will begin with a straightforward construction for extensions with Galois group $\mathbb{Z}_p$, using the fact that $\mathbf{cd}(\mathbb{Z}_p) = 1$.

Remark 3.1. Let G be a profinite group, and $\mathcal{B} \in \mathrm{Sh}(BG_{\text{proét}}, \mathcal{S}_*)$. Under suitable assumptions on $\mathcal{B}$, décalage gives an isomorphism between the descent spectral sequence and the spectral sequence for the Čech nerve of $G \to *$ [35, Appendix A]. Moreover, the homotopy groups of the latter can often be identified with the complex of continuous cochains with coefficients in $\pi_t B$, yielding an isomorphism on the E_2-pages

$$H^*(G, \pi_t B) \to H^*(BG_{\text{proét}}, \pi_t \mathcal{B}). \tag{4}$$

For example, this is the case for the sheaf $\mathfrak{pic}(\underline{E})$ for any Morava E-theory $E(k, \Gamma)$.

Lemma 3.2. Let $G = \mathbb{Z}_p$ or $\widehat{\mathbb{Z}}$, and write $\zeta \in G$ for a topological generator. Suppose $\mathcal{C}$ is a stable homotopy theory and $\mathcal{B} \in \mathrm{Sh}(BG_{\text{proét}}, \mathcal{S}_*)$, with $B := \mathcal{B}(G/*)$ and $B^{hG} := \Gamma\mathcal{B}$. If the canonical map (4) is an isomorphism, then

$$B^{hG} \simeq \mathbf{Eq}\left(id, \zeta : B \rightrightarrows B\right). \tag{5}$$

Proof. Write B' for the equaliser in (5). The G-map $(id, \zeta)\colon G \to G \times G$ gives rise to maps

$$B \rightrightarrows \mathcal{B}(G \times G) \to B \tag{6}$$

factoring $id, \zeta\colon B \rightrightarrows B$, and the identification $B^{hG} \simeq \lim \mathcal{B}(G^{\bullet+1})$ gives a distinguished nullhomotopy in (6) after precomposing with the coaugmentation $\eta\colon B^{hG} \to B$. Thus η factors through $\theta\colon B^{hG} \to B'$. Taking fibres, the descent spectral sequence for $\mathcal{B}$ implies that $\pi_*\theta$ fits in a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & (\pi_{t+1}B)_G \arrow[r] \arrow[d, equal] & \pi_t B^{hG} \arrow[r] \arrow[d, "\pi_t\theta"] & (\pi_t B)^G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & (\pi_{t+1}B)_G \arrow[r] & \pi_t B' \arrow[r] & (\pi_t B)^G \arrow[r] & 0 \end{tikzcd}$$

and is therefore an equivalence. ■

Construction 3.3. Suppose $\mathbf{1} \to A$ is a descendable Galois extension in a stable homotopy theory $\mathcal{C}$, with group $G = \mathbb{Z}_p$ or $\widehat{\mathbb{Z}}$. Then $\mathcal{B} := B\mathfrak{Pic}(A) \in \mathrm{Sh}(BG_{\text{proét}}, \mathcal{S}_*)$, and in good cases $\mathcal{B}$ satisfies the assumption that (4) be an isomorphism: for example, this is the case whenever each $\pi_t\mathfrak{pic}(A)$ is the limit of a tower of finite sets by [7, Lemma 4.3.9]. Thus,

$$\mathfrak{Br}'(\mathbf{1} \mid A) \simeq \mathrm{Eq}\left(id, \zeta^* : B\mathfrak{Pic}(A) \rightrightarrows B\mathfrak{Pic}(A)\right),$$

and so a relative cohomological Brauer class is given by the data of an A-linear equivalence

$$\xi : \mathrm{Mod}_A(\mathcal{C}) \xrightarrow{\sim} \zeta^*\mathrm{Mod}_A(\mathcal{C}).$$

In fact, if $X \in \mathrm{Pic}(A)$, then we may form the A-linear composite $\zeta_!^X : \mathrm{Mod}_A \xrightarrow{X\otimes_A -} \mathrm{Mod}_A \xrightarrow{\zeta_!} \zeta^*\mathrm{Mod}_A$. This gives an isomorphism

$$(A, -) := \zeta_!^{(-)} : \mathrm{Pic}(A)_G \cong H^1(G, \mathrm{Pic}(A)) \to \mathrm{Br}'(\mathbf{1} \mid A).$$

Example 3.4. At the prime 2, the extension $\mathbf{1}_K \to KO_2$ is a descendable $\mathbb{Z}_2$-Galois extension. As a result, Lemma 3.3 applies, and we can form the Brauer class (KO_2, X) associated to any $X \in \mathrm{Pic}(KO_2)$ as above. For example, since KO_2 is 8-periodic one can form the cohomological Brauer class

$$(KO_2, \Sigma KO_2) \in \mathrm{Br}'(\mathbf{1}_K \mid KO_2).$$

Example 3.5. Let $KO_2^{\mathrm{nr}} := \varinjlim_n (KO_2)_{W(\mathbb{F}_{2^n})}$ be the ind-étale KO_2-algebra given by the maximal unramified extension of $\pi_0 KO_2 = \mathbb{Z}_2$; since étale extensions are uniquely determined by their π_0, one can also describe this as $KO_2^{\mathrm{nr}} = KO_2 \otimes_{\mathbb{S}} \mathbb{SW}$, where $\mathbb{SW} = W^+(\overline{\mathbb{F}}_2)$ denotes the spherical Witt vectors [28, §5.2]. The extension $KO_2 \to KO_2^{\mathrm{nr}}$ is a descendable Galois extension: indeed, $KO_2 \otimes_{\mathbb{S}} (-)$ preserves finite limits, and $\mathbb{S} \to \mathbb{SW}$ is descendably $\widehat{\mathbb{Z}}$-Galois. As a result, Lemma 3.3 applies, and we can form the Brauer class (KO_2^{nr}, X) associated to any $X \in \mathrm{Pic}(KO_2^{\mathrm{nr}})$ as above. For example, since KO_2^{nr} is 8-periodic one can form the cohomological Brauer class

$$(KO_2^{\mathrm{nr}}, \Sigma KO_2^{\mathrm{nr}}) \in \mathrm{Br}'(KO_2 \mid KO_2^{\mathrm{nr}}).$$

In fact, $(KO_2^{\mathrm{nr}}, \Sigma KO_2^{\mathrm{nr}})$ is an element of the *étale locally trivial* Brauer group $\mathrm{LBr}(KO_2)$ of [2]; we discuss this in Section 5.

3.2 Cyclic algebras

Suppose that $\mathcal{C}$ is a stable homotopy theory and $\mathbf{1} \to A$ a finite Galois extension in $\mathcal{C}$ with group G. Suppose also given the following data:

(i) an isomorphism $\chi : G \cong \mathbb{Z}/k$.
(ii) a *strict* unit $u \in \pi_0\mathbb{G}_m(\mathbf{1})$.

In this section, we will use this to define a relative Azumaya algebra $(A, \chi, u) \in \mathrm{Br}(\mathcal{C} \mid A)$.

Let us first recall the construction when $\mathcal{C}$ is the category of modules over a classical ring. Then we begin with a G-Galois extension $R \to A$ of rings, and define a G-action on the matrix algebra $M_k(A)$ by letting $\sigma := \chi^{-1}(1)$ act as follows: first act termwise by σ using the G-action on A, and then conjugate by the matrix

$$\widetilde{u} := \begin{bmatrix} 0 & & & u \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \tag{7}$$

Since $\widetilde{u}^k = uI_k \in \mathrm{GL}_k A$ is central, this gives a well-defined action on $M_k A$. Passing to fixed points, we obtain the cyclic algebra

$$(A, \chi, u) \in \mathrm{Br}(R \mid A). \tag{8}$$

When R is a field it is well-known (see, e.g., [11]) that under the isomorphism

$$\mathrm{Br}(R \mid A) \cong H^2(G, A^\times),$$

the cyclic algebra (A, χ, u) maps to the cup-product $\beta(\chi) \cdot u$, where β denotes the Bockstein homomorphism

$$\chi \in \mathrm{Hom}(G, \mathbb{Z}/k) = H^1(G, \mathbb{Z}/k) \xrightarrow{\beta} H^2(G, \mathbb{Z}),$$

and we use the $\mathbb{Z}$-module structure on $A^\times$. Indeed, this follows from the exact sequence

$$1 \to A^\times \to \mathrm{GL}_k(A) \to \mathrm{PGL}_k(A) \to 1$$

and the resulting commutative square of cohomology groups

$$\begin{tikzcd} H^1(G, \mathbb{Z}/k) \arrow[r, "\beta"] \arrow[d, "\widetilde{u}"'] & H^2(G, \mathbb{Z}) \arrow[d, "u"] \\ H^1(G, \mathrm{PGL}_k(A)) \arrow[r, "\delta"'] & H^2(G, A^\times) \end{tikzcd} \tag{9}$$

As a result, we obtain an isomorphism

$$\widehat{H}^0(G, A^\times) = A^\times / N_e^G A^\times \to \mathrm{Br}(R \mid A)$$

sending $u \mapsto (A, \chi, u)$. We will prove an analogous result for cyclic algebras in arbitrary stable homotopy theories, which will allow us to detect permanent cycles in the descent spectral sequence; conversely, this will allow us to assert that the cyclic algebras we construct are nontrivial.

The square (9) motivates the following definition:

Definition 3.6. Given a Galois extension $\mathbf{1} \to A$ with group G in a rigidly generated stable homotopy theory $\mathcal{C}$, and given $\chi : G \cong \mathbb{Z}/k$ and $u \in \pi_0 \mathrm{Map}_{\mathbb{E}_2}(\mathbb{Z}, \mathrm{GL}_1(\mathbf{1}))$, define the composite G-map

$$(A, u) : B\mathbb{Z}/k \xrightarrow{\beta} B^2\mathbb{Z} \xrightarrow{B^2 u} B^2\mathrm{GL}_1(A) \xrightarrow{\tau_{\geq 2}} B\mathfrak{Pic}(A).$$

The relative Brauer class $(A, \chi, u) \in \mathrm{Br}(\mathbf{1} \mid A)$ is defined to be the image of χ under the map on fixed points

$$(A, -, u) := \pi_0(A, u)^{hG} : \mathrm{Hom}(G, \mathbb{Z}/k) = \pi_0 B\mathbb{Z}/k^{BG} \to \pi_0(B\mathfrak{Pic}(A))^{hG} = \mathrm{Br}'(\mathbf{1} \mid A) \cong \mathrm{Br}(\mathbf{1} \mid A).$$

The final isomorphism is inverse to the map $\mathrm{Br}(\mathbf{1} \mid A) \xrightarrow{\sim} \mathrm{Br}'(\mathbf{1} \mid A)$ considered in Section 2.

Remark 3.7. Baker, Richter and Szymik use similar data to define an Azumaya algebra $A(A, \chi, u)$ as the fixed points of an appropriate twist of the action on the matrix algebra $M_k A$. We believe that when u is a *strict* unit the two constructions agree; this question will be investigated in future work.

Theorem 3.8. Suppose given an $\mathbb{E}_2$-unit $u \neq 1 \in \pi_0 \mathrm{Map}_{\mathbb{E}_2}(\mathbb{Z}, \mathfrak{gl}_1(\mathbf{1}))$. Its image in $\pi_0 \mathrm{GL}_1(\mathbf{1})$ is detected in the HFPSS for $\mathfrak{pic}(\mathcal{C}) \simeq \mathfrak{pic}(A)^{hG}$ by a class $v \in E_2^{s,s+1}$, and we assume that one of the following holds:

(i) v is in positive filtration;
(ii) v is in filtration zero, and has nonzero image in $\widehat{H}^0(G, (\pi_0 A)^\times)$.

Then the algebra (A, χ, u) is detected by

$$\beta(\chi) \cdot v \in H^{s+2}(G, \pi_{s+1}\mathfrak{pic}(A)) = E_2^{s+2,s+1}.$$

In particular, $\beta(\chi) \cdot v$ is a permanent cycle. If it survives to E_∞ then $(A, \chi, u) \neq \mathbf{1} \in \mathrm{Br}(\mathbf{1} \mid A)$.

Proof. We consider the map of spectral sequences induced by the G-map (A, u). It is standard that the map induced by β is indeed the Bockstein, and we claim that the map

$$u_* : H^2(G, \mathbb{Z}) \to H^s(G, \pi_{s-1}\mathfrak{Pic}(A))$$

induced on E_2-pages by $B^2 u \colon B^2\mathbb{Z} \to B\mathfrak{Pic}(A)$ agrees with $v \cdot -$. For this, we consider the (equivariant) action map

$$m : \mathbb{Z} \times \mathrm{Map}_{\mathbb{E}_2}(\mathbb{Z}, \mathrm{GL}_1(A)) \to B^2\mathrm{GL}_1(A).$$

Then $u_*\beta(\chi) = m(\beta(\chi), \overline{v})$, where $\overline{v}$ in the E_2 page of the HFPSS for $\mathrm{Map}_{\mathbb{E}_2}(\mathbb{Z}, \mathrm{GL}_1(A))$ detects u. This fits in a diagram

$$\begin{tikzcd}
\mathbb{Z} \times \mathrm{Map}_{\mathbb{E}_2}(\mathbb{Z}, \mathrm{GL}_1(A)) \arrow[r] \arrow[d] & \mathrm{GL}_1(A) \arrow[d, equal] \\
\mathbb{Z} \times \mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}, \mathrm{GL}_1(A)) \arrow[r] & \mathrm{GL}_1(A)
\end{tikzcd}$$

in which the bottom map induces the $H^*(G, \mathbb{Z})$-module structure on the E_2-page for $\mathrm{GL}_1(A) \simeq \mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}, \mathrm{GL}_1(A))$. Since $\overline{v} \mapsto v$, we see by going down and left in the square above that $m(\beta(\chi), \overline{v}) = \beta(\chi) \cdot v$.

The class $\beta(\chi) \cdot v$ is nonzero since Tate cohomology is $\beta(\chi)$-periodic, which gives the result. ■

4 The Descent Spectral Sequence

We now specialise to height one. In this short section, we record the descent spectral sequence that will be the starting point for our computations. At any characteristic (p, h), this arises as the descent spectral sequence for the sheaf

$$\mathfrak{pic}(\underline{E}) \in \mathrm{Sh}(B\mathbb{G}_{\text{proét}}, \mathcal{S}p_{\geq 0})$$

of [35, §3.2]. The main input from *op. cit.* is the following theorem:

Theorem 4.1 ([35], Theorem A and Proposition 5.11).

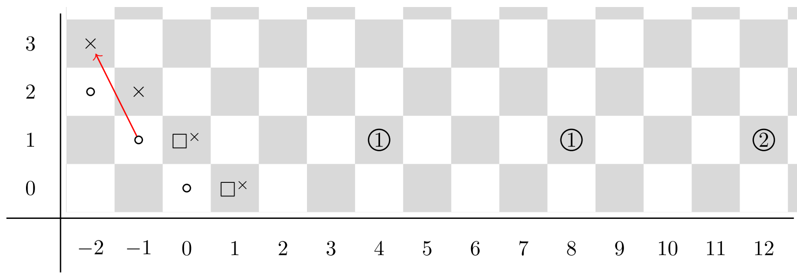


Fig. 1. The height one Picard spectral sequence for odd primes (implicitly at $p = 3$). Classes are labelled as follows: $\circ = \mathbb{Z}/2$, $\square^\times = \mathbb{Z}[p]^\times$, $\times = \mu_{p-1}$, and circles denote p-power torsion (labelled by the torsion degree). Since $\mathrm{Pic}_1 \cong \mathrm{Pic}_1^{\mathrm{alg}} \cong \mathbb{Z}_p^\times \times \mathbb{Z}/2$, no differentials can hit the (-1)-stem. Differentials with source in stem $t - s \le -2$ have been omitted.

1) There is a strongly convergent spectral sequence

$$E_2^{s,t} = H^s(\mathbb{G}, \pi_t \mathfrak{pic}(E)) \implies \pi_{t-s}\mathfrak{pic}(\mathcal{S}p_K). \tag{10}$$

2) Its (-1)-stem converges to $\mathrm{Br}'(\mathbf{1} \mid E)$.
3) Differentials on the E_r page agree with those in the K-local E-Adams spectral sequence in the region $t \ge r + 1$, and for classes $x \in E_r^{r,r}$ we have $d_r(x) = d_r^{\mathrm{ASS}}(x) + x^2$.

In [35,§4], we used this spectral sequence to recover the computation of $\mathrm{Pic}_1 := \mathrm{Pic}(\mathcal{S}p_{K(1)})$ (due to [22]). In this case, Morava E-theory is the p-completed complex K-theory spectrum KU_p, acted upon by $\mathbb{G} \cong \mathbb{Z}_p^\times$ via Adams operations ψ^a.

4.1 Odd primes

We first consider the case $p > 2$. The starting page of the Picard spectral sequence is recorded below:

Lemma 4.2 ([35], Lemma 4.15). At odd primes, the starting page of the descent spectral sequence is given by

$$E_2^{s,t} = H^s(\mathbb{Z}_p^\times, \pi_t\mathfrak{pic}(KU_p)) = \begin{cases} \mathbb{Z}/2 & t = 0 \text{ and } s \ge 0 \\ \mathbb{Z}_p^\times & t = 1 \text{ and } s = 0, 1 \\ \mu_{p-1} & t = 1 \text{ and } s \ge 2 \\ \mathbb{Z}/p^{\nu_p(t)+1} & t = 2(p-1)t' + 1 \neq 1 \text{ and } s = 1 \end{cases} \tag{11}$$

This is displayed in Figure 1. In particular, the spectral sequence collapses for degree reasons at the E_3-page.

Proposition 4.3. At odd primes, Br_1^0 is isomorphic to a subgroup of μ_{p-1}.

Proof. The only possible differentials are d_2-differentials on classes in the (-1)-stem; note that there are no differentials *into* the (-1)-stem, since every E_2-class in the 0-stem is a permanent cycle. The generator in $E_2^{1,0}$ supports a d_2, since this is the case for the class in $E_2^{1,0}$ of the descent spectral sequence for the C_2-action on KU [15, Prop. 7.15], which is displayed in Figure 3a). The cospan of Galois extensions

$$KU_p^{h\mathbb{Z}_p^\times} \to KU_p^{hC_2} \leftarrow KU^{hC_2}$$

allows us to transport this differential (see also Figure 4). Thus,

$$\mathrm{Br}'(\mathcal{S}p_K \mid E) \cong \mu_{p-1}.$$

■

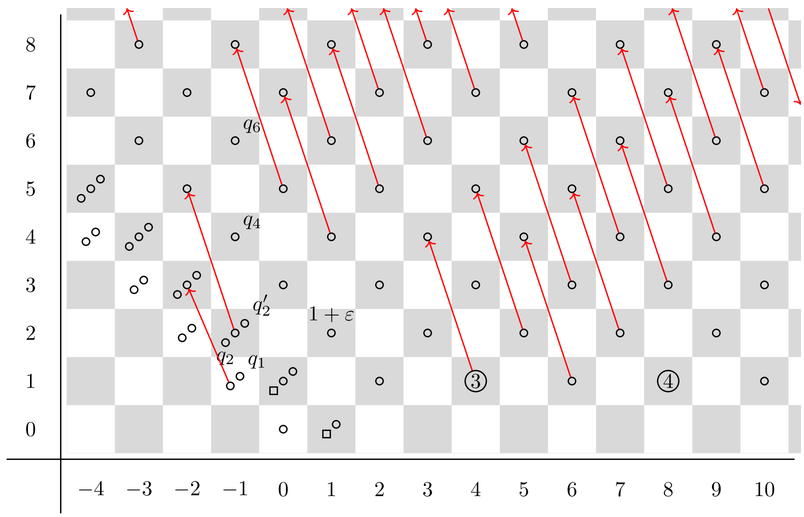
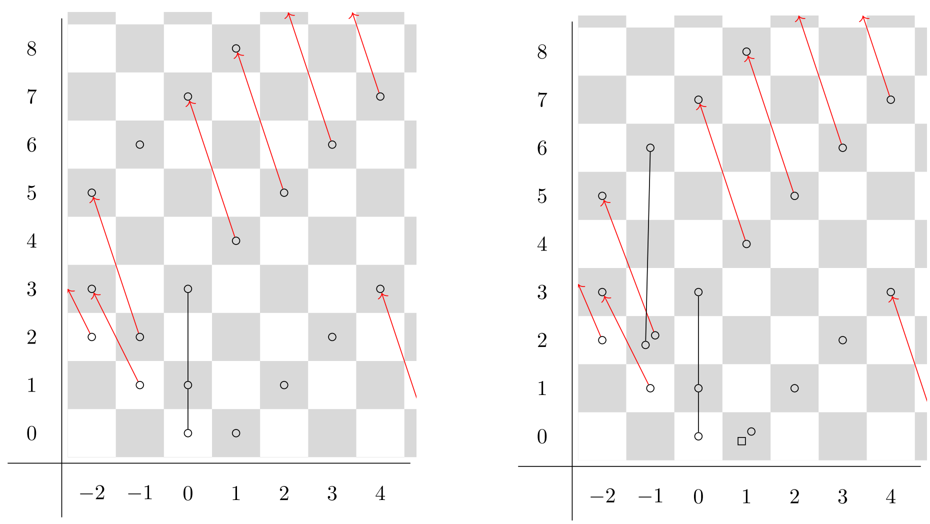


Fig. 2. The Picard spectral sequence for the Galois extension $\mathbf{1}_K \to E = KU_2$ at $p = 2$. We know that all remaining classes in the 0-stem survive, by comparing to the algebraic Picard group. Thus, the only differentials that remain to compute are those *out* of the (-1)-stem; those displayed can be transported from the descent spectral sequence for $\mathfrak{Pic}(KO)^{hC_2}$—see Figure 3a and 3b. We have not displayed possible differentials out of stem ≤ -2.



(a) The Picard spectral sequence for KO, as in [15, Figure 7.2].

(b) The Picard spectral sequence for KO_2.

Fig. 3. The Picard spectral sequences for KO and KO_2 respectively. Differentials in Figure 3b come from the comparison with Figure 3a; see Section 5 for the extension in the (-1)-stem.

In Section 5 we will show that this bound is achieved using the results of the previous sections.

4.2 The case $p = 2$

We now proceed with the computation of the (-1)-stem for the even prime.

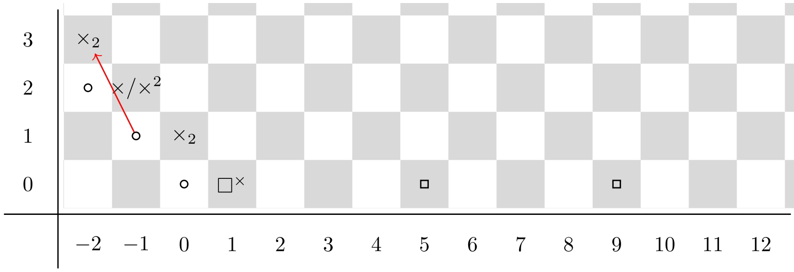


Fig. 4. The HFPSS for $\mathfrak{pic}(KO_p) \simeq \mathfrak{pic}(KU_p)^{C_2}$ at odd primes. Here $\circ = \mathbb{Z}/2$, $\times = \mu_{p-1}$, $\times_2 = \mu_{p-1}[2] = C_2$ and $\square = \mathbb{Z}_p$.

Lemma 4.4 ([35], Lemma 4.17). We have

$$H^s(\mathbb{Z}_2^\times, \mathrm{Pic}(KU_2)) = \begin{cases} \mathbb{Z}/2 & s = 0 \\ (\mathbb{Z}/2)^2 & s \geq 1 \end{cases}$$

$$H^s(\mathbb{Z}_2^\times, (\pi_0 KU_2)^\times) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}/2 & s = 0 \\ \mathbb{Z}_2 \oplus (\mathbb{Z}/2)^2 & s = 1 \\ (\mathbb{Z}/2)^3 & s \geq 2 \end{cases}$$

The resulting spectral sequence is displayed in Figure 2.

Proposition 4.5. At the prime two, $|\mathrm{Br}_1^0| \leq 32$.

Proof. In [35, §4], we determined the following differentials:

- in degrees $t \geq 3$, differentials agree with the well-known pattern of Adams differentials (e.g., 5, Figure 3).
- the class in bidegree $(s, t) = (3, 3)$, which supports a d_3 in the Adams spectral sequence, is a permanent cycle.

By comparing with the Adams spectral sequence, any classes in the (-1)-stem that survive to E_∞ are in filtration at most six; on the E_2-page, there are seven such generators. By comparing to the HFPSS for $\mathfrak{Br}'(KO_2 \mid KU_2) = (B\mathfrak{Pic}(KU_2))^{hC_2}$ as in Section 4.1, we obtain the following differentials:

- a d_2 on the class in $H^1(C_2, \mathrm{Pic}(KU_2)) \subset H^1(\mathbb{Z}_2^\times, \mathrm{Pic}(KU_2))$,
- a d_3 on the class in $H^2(C_2, (\pi_0 KU_2)^\times) \subset H^2(\mathbb{Z}_2^\times, (\pi_0 KU_2)^\times)$.

This gives the claimed upper bound. ■

In Section 5 we will show that this bound is *also* achieved.

Remark 4.6. For later reference, we name the following generators:

(i) $q_1 \in E_2^{1,0}$ is the generator of $H^1(1 + 4\mathbb{Z}_2, \mathbb{Z}/2) \subset H^1(\mathbb{Z}_2^\times, \mathbb{Z}/2)$,
(ii) $q_2 \in E_2^{2,1}$ is the generator of $H^2(C_2, 1 + 4\mathbb{Z}_2) \subset H^2(\mathbb{Z}_2^\times, \mathbb{Z}_2^\times)$,
(iii) $q_2' \in E_2^{2,1}$ is the generator of $H^1(C_2, \mathbb{Z}_2^\times) \otimes H^1(\mathbb{Z}_2, \mathbb{Z}_2^\times) \subset H^2(\mathbb{Z}_2^\times, \mathbb{Z}_2^\times)$,
(iv) q_4 is the unique class in $E_2^{4,3}$,
(v) q_6 is the unique class in $E_2^{6,5}$.

While q_6 survives to E_∞ by sparsity in Figure 2, the other classes are sources of possible differentials. We will show that in fact all are permanent cycles.

5 Computing Br_1^0

We are now ready to complete the proofs of Theorem A and B. At odd primes, we will see that the cyclic algebra construction of Section 3 gives all possible Brauer classes. On the other hand, when $p = 2$ not all classes on the E_∞-page of the descent spectral sequence will be detected in this way. In this case, we will first compute the relative Brauer group $\mathrm{Br}(KO_2 \mid KU_2)$.

5.1 Odd primes

When $p \geq 3$, Figure 1 shows that $\mathrm{Br}_1^0 \subset \mu_{p-1}$. In fact, we can deduce from Lemma 3.8 that this inclusion is an equality:

Theorem 5.1. Let p be an odd prime, and choose $\chi : \mu_{p-1} \cong \mathbb{Z}/p-1$. There is an isomorphism

$$\mu_{p-1} \xrightarrow{\sim} \mathrm{Br}_1^0,$$

given by the cyclic algebra construction $\omega \mapsto (KU_p^{h(1+p\mathbb{Z}_p)}, \chi, \omega)$.

Proof. By Lemma 4.3 there is an inclusion $\mathrm{Br}_1^0 \subset \mu_{p-1}$. Since $H^2(\mu_{p-1}, (\pi_0 B)^\times) \cong \mu_{p-1}\{\beta(\chi) \cdot \omega\}$, Lemma 3.8 implies it is enough to show that the roots of unity in $(\pi_0 \mathbf{1}_K)^\times = \mathbb{Z}_p^\times$ have $\mathbb{E}_2$ lifts. But we have a commuting square

$$\begin{tikzcd}
\pi_0\mathbb{G}_m(\mathbb{S}_p) \arrow[r] \arrow[d] & \pi_0\mathbb{G}_m(\mathbf{1_K}) \arrow[d] \\
\pi_0 GL_1(\mathbb{S}_p) \arrow[r, equal] & \pi_0 GL_1(\mathbf{1_K})
\end{tikzcd}$$

and at odd primes the roots of unity are strict in $\mathbb{S}_p$ by [9, Theorem A]. ■

Remark 5.2. In fact, Lemma 5.1 also follows from Lemma 2.10. Indeed, Figure 1 shows that

$$\mathrm{Br}'(\mathcal{S}p_K \mid KU_p) \cong H^2(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) \cong H^2(\mu_{p-1}, \mathbb{Z}_p^\times).$$

In particular, this group is killed in the μ_{p-1}-Galois extension $\mathbf{1}_K \to KU_p^{h(1+p\mathbb{Z}_p)}$, since the group

$$\mathrm{Br}'(KU_p^{h(1+p\mathbb{Z}_p)} \mid KU_p) = \pi_0 B\mathfrak{Pic}(KU_2)^{h(1+p\mathbb{Z}_p)}$$

is concentrated in filtration $s \leq 1$ of the Picard spectral sequence for the $(1+p\mathbb{Z}_p)$-action. Since the extension $\mathbf{1}_K \to KU_p^{h(1+p\mathbb{Z}_p)}$ is finite, Lemma 2.10 yields the second isomorphism below:

$$\mathrm{Br}'(\mathcal{S}p_K \mid KU_p) \cong \mathrm{Br}'(\mathcal{S}p_K \mid KU_p^{h(1+p\mathbb{Z}_p)}) \cong \mathrm{Br}(\mathcal{S}p_K \mid KU_p^{h(1+p\mathbb{Z}_p)}) \subset \mathrm{Br}_1^0.$$

5.2 Completed K-theory

In this section we use Galois descent to compute the Brauer group $\mathrm{Br}(KO_p \mid KU_p)$. This builds on the integral case computed by Gepner and Lawson [15], and we will therefore also determine the completion maps

$$\mathrm{Br}(KO \mid KU) \to \mathrm{Br}(KO_p \mid KU_p).$$

The computation for $p = 2$ will be important for our main computation: we will show that the relative Brauer classes of KO_2 descend to $\mathbf{1}_K$, which will help us determine the group Br_1^0. Therefore, we start with the computation in this case:

Theorem 5.3. At the prime two we have

$$\mathrm{Br}(KO_2 \mid KU_2) \simeq \mathbb{Z}/4,$$

and the completion map from $\mathrm{Br}(KO \mid KU)$ is injective.

Since the extension $KO_2 \to KU_2$ is finite, it follows by combining Figure 3b with Lemma 2.10 that $|\mathrm{Br}(KO_2 \mid KU_2)| = 4$. To prove the theorem, we need to prove there is a nontrivial extension between $E_\infty^{2,1} = \mathbb{Z}/2$ and $E_\infty^{5,6} = \mathbb{Z}/2$ in Figure 3b, and we do this by reducing to computations of étale cohomology.

Definition 5.4. Recall the *étale locally trivial Brauer group* $\mathrm{LBr}(KO_2) \subset \mathrm{Br}(KO_2)$ of [2]; more generally, Antieau, Meier and Stojanoska define

$$\mathrm{LBr}(R) := \pi_0 \Gamma B\mathfrak{Pic}_{\mathcal{O}_R}$$

for any commutative ring spectrum R, where $B\mathfrak{Pic}_{\mathcal{O}_R}$ is the sheafification of $B\mathfrak{Pic}(\mathcal{O}_R)$ on the étale site of $\mathbf{Spec}\, R$. (Note that this is equivalent to the étale site of $\mathbf{Spec}\, \pi_0 R$.) Explicitly [2, Lemma 2.17] this is the group of Brauer classes that are trivialised in some faithful étale extension $R \to R'$ in the sense of [29, Definition 7.5.0.4]. Likewise, for an extension $R \to S$ of commutative ring spectra we write

$$\mathrm{LBr}(R \mid S) := \mathbf{ker}(\mathrm{LBr}(R) \to \mathrm{LBr}(S)) \subset \mathrm{Br}(R \mid S).$$

The group $\mathrm{LBr}(R)$ is sometimes more computationally tractable than $\mathrm{Br}(R)$: for example, one can often reduce to étale cohomology of $\mathbf{Spec}\, \pi_0 R$, which gives access to the standard cohomological toolkit. When $R = KO_p$, this allows us to use Gabber-Huber rigidity in the proof of Lemma 5.3.

Remark 5.5. In the setting of unlocalised $\mathbb{E}_\infty$-rings, one always has compact generators and hence $\mathrm{LBr}(R) \cong \mathrm{LBr}'(R)$. One may (rightly) worry about the difference between the groups of unlocalised and of $K(1)$-local Brauer classes, since the results of [2] pertain to the former. While we do not know if the two groups agree in general (even for nice even-periodic rings), in our applications this is taken care of by restricting to *relative* Brauer classes. Indeed, in that case we are computing the space

$$\mathfrak{Br}'(\mathbf{1} \mid A) = B\mathfrak{Pic}(\mathrm{Mod}_A(\mathcal{C}))^{hG},$$

and by [20,Remark 3.7] the canonical map

$$\iota_A : \mathfrak{Pic}(\mathrm{Mod}_A) \to \mathfrak{Pic}(\mathrm{Mod}_A(\mathcal{S}p_K))$$

is an equivalence of infinite loop-spaces in the following cases:

- $A = E(k, \Gamma)$ is any Morava E-theory,
- A admits a descendable extension $A \to B$ for which ι_B is an equivalence.

For example, this means that the two possible meanings of the expression $\mathrm{LBr}'(KO_2)$ agree. In fact, by Lemma 2.10 we know that any element of $\mathrm{LBr}'(KO_2 \mid KU_2)$ lifts to an Azumaya algebra (in both the localised and unlocalised setting). This means that there is no ambiguity in writing $\mathrm{LBr}(KO_2 \mid KU_2)$ below.

We begin with a preliminary computation; the following is essentially [2, Proposition 3.8]:

Proposition 5.6. The étale sheaf $\pi_0 \mathfrak{pic}(\mathcal{O}_{KO_2})$ fits in a nonsplit extension

$$0 \to i_* \mathbb{Z}/2 \to \pi_0 \mathfrak{pic}(\mathcal{O}_{KO_2}) \to \mathbb{Z}/4 \to 0,$$

where $i\colon \mathbf{Spec}\, \mathbb{F}_2 \to \mathbf{Spec}\, \mathbb{Z}_2$ is the inclusion of the closed point. Moreover, $i^* \pi_0 \mathfrak{pic}(\mathcal{O}_{KO_2}) \cong \mathbb{Z}/8$.

Proof. We specify the necessary adjustments from the case of integral K-theory KO. Recall that Antieau, Meier and Stojanoska compute the sheaf $\pi_0\mathfrak{pic}(\mathcal{O}_{KO})$ on $\mathrm{Spec}\,KO = \mathrm{Spec}\,\mathbb{Z}$, using the sheaf-valued HFPSS for

$$\mathfrak{pic}(\mathcal{O}_{KO}) \simeq \mathfrak{pic}(\mathcal{O}_{KU})^{hC_2},$$

which is [2, Figure 1]. The same figure gives the HFPSS for

$$\mathfrak{pic}(\mathcal{O}_{KO_2}) \simeq \mathfrak{pic}(\mathcal{O}_{KU_2})^{hC_2},$$

as long as one correctly interprets the symbols as in [2, Table 1], replacing $\mathcal{O} = \mathcal{O}_{\mathbb{Z}}$ with $\mathcal{O}_{\mathbb{Z}_2}$. The proofs of [2, Lemmas 3.5 and 3.6] go through verbatim to give the 0-stem in the E_∞-page, so that $\pi_0\mathfrak{pic}(\mathcal{O}_{KO_2})$ admits a filtration

$$\begin{tikzcd} F^2 \arrow[r, hook] \arrow[d, equal] & F^1 \arrow[r, hook] \arrow[d, two heads] & \pi_0\mathfrak{pic}(\mathcal{O}_{KO_2}) \arrow[d, two heads] \\ i_*\mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 \end{tikzcd}$$

Now the determination of the extensions follows as in [2, Proposition 3.8], by using the exact sequence

$$H^1(\mathrm{Spec}\,\mathbb{Z}_2, \mathbb{G}_m) = \mathrm{Pic}(\mathbb{Z}_2) \to \mathrm{Pic}(KO_2) \to H^0(\mathrm{Spec}\,\mathbb{Z}_2, \pi_0\mathfrak{pic}(\mathcal{O}_{KO_2}))$$

of [2, Proposition 2.25], and the fact that $\mathbb{Z}_2$ is local so has trivial Picard group. In particular, since i^* is exact we have that $i^*\pi_0\mathfrak{pic}(\mathcal{O}_{KO_2})$ admits a filtration by three copies of $\mathbb{Z}/2$, and a surjection from the constant sheaf $\mathbb{Z}/8$. ■

Proof. (Lemma 5.3) By inspection of the HFPSS for the C_2 action on $\mathfrak{Pic}(KU_2)$ (Figure 3b), the Brauer group is of order four. It remains to prove there is a nontrivial extension between the two $\mathbb{Z}/2$-generators. In fact, we will prove that

$$\mathrm{Br}(KO_2 \mid KU_2) \supset \mathrm{LBr}(KO_2 \mid KU_2) \cong \mathbb{Z}/4.$$

When $\pi_0 R$ is a regular complete local ring with finite residue field, the exact sequence [2, Proposition 2.25] simplifies to an isomorphism

$$\mathrm{LBr}(R) \cong H^1(\mathrm{Spec}\,\pi_0 R, \pi_0\mathfrak{pic}(\mathcal{O}_R)),$$

since the cohomology of $\mathbb{G}_m$ vanishes [34, 1.7(a)]. One has that $\pi_0\mathfrak{pic}(\mathcal{O}_{KU_2}) \simeq \mathbb{Z}/2$ is constant since KU_2 is even periodic with regular Noetherian π_0, while $\pi_0\mathfrak{pic}(\mathcal{O}_{KO_2})$ is torsion by Lemma 5.6. We can therefore use Gabber-Huber rigidity [14, 24] to compute

$$\mathrm{LBr}(KO_2) \cong H^1(\mathrm{Spec}\,\mathbb{Z}_2, \pi_0\mathfrak{pic}(\mathcal{O}_{KO_2})) \cong H^1(\mathrm{Spec}\,\mathbb{F}_2, i^*\pi_0\mathfrak{pic}(\mathcal{O}_{KO_2})),$$
$$\mathrm{LBr}(KU_2) \cong H^1(\mathrm{Spec}\,\mathbb{Z}_2, \pi_0\mathfrak{pic}(\mathcal{O}_{KU_2})) \cong H^1(\mathrm{Spec}$$
$$mathbbF_2, i^*\pi_0\mathfrak{pic}(\mathcal{O}_{KU_2})).$$

Since $i^*\pi_0\mathfrak{pic}(\mathcal{O}_{KO_2}) \simeq \mathbb{Z}/8$, we obtain

$$\mathrm{LBr}(KO_2) \cong \mathbb{Z}/8 \qquad \text{and} \qquad \mathrm{LBr}(KU_2) \cong \mathbb{Z}/2,$$

which implies that $\mathrm{LBr}(KO_2 \mid KU_2) \cong \mathbb{Z}/4$ or $\mathbb{Z}/8$; but $|\mathrm{LBr}(KO_2 \mid KU_2)| \le 4$, so we are done. ■

Remark 5.7. The generator of $\mathrm{LBr}(KO_2)$ is the Azumaya algebra $(KO_2^{\mathrm{nr}}, \Sigma KO_2^{\mathrm{nr}})$ constructed in Lemma 3.5, and the generator of $\mathrm{LBr}(KO_2 \mid KU_2)$ is $(KO_2^{\mathrm{nr}}, \Sigma^2 KO_2^{\mathrm{nr}})$.

Remark 5.8. In [33, Lemma V.3.1], May proves a splitting of infinite loop-spaces

$$BU^{\otimes} \simeq \tau_{\leq 3} BU^{\otimes} \times \tau_{\geq 4} BU^{\otimes},$$

where the monoidal structure is given by tensor product of vector spaces. The map $BU(1) = \tau_{\leq 3} BU \to BU$ is induced from a map at the level of bipermutative categories, and classifies the canonical line bundle. One can ask if this extends to a splitting of $\mathfrak{gl}_1 KU$, and if this splitting happens C_2-equivariantly.

As an aside to Lemma 5.3, we deduce that the completed, equivariant analogue of this splitting fails:

Corollary 5.9. There is no C_2-equivariant splitting

$$\mathfrak{gl}_1 KU_2 \simeq \tau_{\leq 3} \mathfrak{gl}_1 KU_2 \oplus \tau_{\geq 4} \mathfrak{gl}_1 KU_2.$$

Proof. By comparing the HFPSS for the C_2-action on $B^2\mathrm{GL}_1(KU_2)$ with Figure 3b, one computes that $\pi_0(B^2\mathrm{GL}_1(KU_2))^{hC_2} = \mathbb{Z}/4$. Suppose now that a splitting as above did exist. Then

$$\pi_0(B^2\mathrm{GL}_1(KU_2))^{hC_2} = \pi_0(\Sigma^2 \mathfrak{gl}_1(KU_2))^{hC_2} \simeq \pi_0(\tau_{\leq 5} \Sigma^2 \mathfrak{gl}_1(KU_2))^{hC_2} \oplus \pi_0(\tau_{\geq 6} \Sigma^2 \mathfrak{gl}_1(KU_2))^{hC_2}.$$

However, the generators in the (-1)-stem of the E_∞-page of Figure 3b are in filtrations two and six, and in particular both factors in the right-hand splitting are nontrivial: indeed, the generator in filtration two maps to a nonzero element in $\pi_0(\tau_{\leq 5} \Sigma^2 \mathfrak{gl}_1(KU_2))^{hC_2}$, and the generator in filtration six is in the image of $\pi_0(\tau_{\geq 6} \Sigma^2 \mathfrak{gl}_1(KU_2))^{hC_2}$. This contradicts the fact that $\pi_0(B^2\mathrm{GL}_1(KU_2))^{hC_2}$ is cyclic. ■

We do not know if $\mathfrak{gl}_1 KU$ (or its 2-completion) splits equivariantly: note that the obstruction in the case of KU_2 comes in the form of an extension on a class originating in $H^2(\mathbb{Z}_2^\times, 1 + 4\mathbb{Z}_2) \subset H^2(\mathbb{Z}_2^\times, \mathbb{Z}_2^\times)$.

Let us briefly also mention the case when p is odd; this will not be necessary for the computation of Br_1^0 at odd primes.

Proposition 5.10. When p is odd, we have

$$\mathrm{Br}(KO_p \mid KU_p) \cong \mathbb{Z}/2,$$

and the map $\mathrm{Br}(KO \mid KU) \to \mathrm{Br}(KO_p \mid KU_p)$ is zero.

Proof. Since $\mathbb{Z}_p$ is local away from 2, the E_2-page takes the form in Figure 4, from which $\mathrm{Br}(KO_p \mid KU_p) \cong \mu_{p-1}/\mu_{p-1}^2 \cong \mathbb{Z}/2$ follows by Lemma 2.10. To describe the generators, note that the roots of unity $\mu_{p-1} \subset \pi_0 KO_p^\times$ are strict, since they are so in the p-complete sphere [9]. Choosing $\chi : C_2 \cong \mathbb{Z}/2$, Lemma 3.8 implies that the cyclic algebra construction

$$\omega \mapsto (KU_p, \chi, \omega)$$

yields $\beta(\chi) \cdot \mu_{p-1} = H^2(C_2, \mu_{p-1}) \cong \mathrm{Br}(KO_2 \mid KU_2)$. The map from $\mathrm{Br}(KO \mid KU)$ is zero, since $\mathrm{Br}(KO_2 \mid KU_2)$ is detected in filtration 2 and $\mathrm{Br}(KO \mid KU)$ in filtration 6. ■

5.3 The case $p = 2$

Putting together the work of the previous sections, we complete the computation of Br_1^0 at the prime two.

5.3.1 *Descent from* KO_2

Lemma 5.3 yields

$$\pi_t \mathfrak{Br}(KO_2 \mid KU_2) = \pi_t \mathfrak{Br}'(KO_2 \mid KU_2) = \begin{cases} \mathbb{Z}/4\{(KO_2^{\mathrm{nr}}, \Sigma^2 KO_2^{\mathrm{nr}})\} & t = 0 \\ \mathbb{Z}/8\{\Sigma KO_2\} & t = 1 \\ \mathbb{Z}_2^\times & t = 2 \\ \pi_{t-2} KO_2 & t \geq 3 \end{cases}$$

We will use this to compute the group $\mathrm{Br}(\mathbf{1}_K \mid KU_2)$ by Galois descent along the $\mathbb{Z}_2$-Galois extension $\mathbf{1}_K \to KO_2$. Namely, we use the iterated fixed points formula

$$\mathfrak{Br}'(\mathbf{1}_K \mid KU_2) \simeq (B\mathfrak{Pic}(KU_2)^{hC_2})^{h(1+4\mathbb{Z}_2)} = \mathfrak{Br}(KO_2 \mid KU_2)^{h(1+4\mathbb{Z}_2)}.$$

to form the descent spectral sequence

$$E_2^{s,t} = H^s(\mathbb{Z}_2, \pi_t \mathfrak{Br}(KO_2 \mid KU_2)) \implies \pi_{t-s} \mathfrak{Br}'(\mathbf{1}_K \mid KU_2).$$

More precisely, this is the descent spectral sequence for $p_* B\mathfrak{Pic}(\underline{E}) \in \mathrm{Sh}(B(\mathbb{Z}_2^\times/C_2)_{\text{proét}})$, where p is the projection. To see that it takes the form stated, we compute the proétale homotopy groups as below.

Lemma 5.11. Let $p\colon \mathbb{Z}_2^\times \twoheadrightarrow G = \mathbb{Z}_2^\times/C_2$. Then,

$$\pi_t p_* \mathfrak{pic}(\underline{E}) = \underline{\pi_t \mathfrak{pic}(KO_2)}.$$

Proof. Write $i\colon G \cong (1+4\mathbb{Z}_2) \hookrightarrow \mathbb{Z}_2^\times$. The adjunction $i^* \dashv i_*$ is monadic (c.f. [35, §2.2.4]), and provides an equivalence

$$\mathrm{Sh}(B\mathbb{G}_{\text{proét}}) \simeq \mathrm{Sh}(BG_{\text{proét}})^{BC_2}.$$

Under this identification we have $p_* \simeq (i^*(-))^{hC_2}$ and so obtain a HFPSS of proétale abelian groups

$$H^s(C_2, \pi_t i^* B\mathfrak{Pic}(\underline{E})) \implies \pi_{t-s} p_* B\mathfrak{Pic}(\underline{E}).$$

Note that $\pi_t i^* B\mathfrak{Pic}(\underline{E}) = \underline{\pi_t B\mathfrak{Pic}(E)}$, and so the E_2-page looks identical to the HFPSS for $\mathfrak{pic}(KO_2^{\mathrm{nr}})$, replacing every abelian group M there by $\underline{M}$. By passing to G-modules in [38, Corollary 4.9] we see that the functor $M \mapsto \underline{M}$ on derived categories is fully faithful, so in particular one has

$$\mathrm{Ext}^*_{BG_{\text{proét}}}(\underline{\mathbb{Z}/2}, \underline{M}) = \mathrm{Ext}^*_G(\mathbb{Z}/2, M),$$

where $M = \mathbb{Z}/2$ or $\mathbb{Z}_2(j)$. Thus, the spectral sequence is determined the spectral sequence of underlying G-spaces (i.e., when we evaluate on a G-torsor). This is what we computed in Lemma 5.3. ■

Remark 5.12. The same proof works for KO_2^{nr} in place of KO_2, giving an isomorphism

$$\pi_t p_* \mathfrak{pic}(\underline{E}) \cong \underline{\pi_t \mathfrak{pic}(KO_2^{\mathrm{nr}})} \in \mathrm{Ab}(B(G \times \widehat{\mathbb{Z}})_{\text{proét}}),$$

where now $\underline{E}$ is based on algebraically closed Lubin-Tate theory $E = E(\overline{\mathbb{F}}_p, \Gamma_h)$.

Since $\mathbb{Z}_2$ has cohomological dimension one, there is no room for differentials and the spectral sequence collapses immediately. To determine $\mathrm{Br}'(\mathcal{S}p_K \mid KU_2)$, what remains to compute is the following:

- The group $E_2^{0,1} = \mathrm{Br}(KO_2 \mid KU_2)^{1+4\mathbb{Z}_2}$,
- The extension between the groups $E_\infty^{0,-1} = E_2^{0,-1} \cong \mathrm{Br}(KO_2 \mid KU_2)^{1+4\mathbb{Z}_2}$ and $E_\infty^{1,0} = E_2^{1,0} \cong \mathbb{Z}/8\{(KO_2, \Sigma KO_2)\}$.

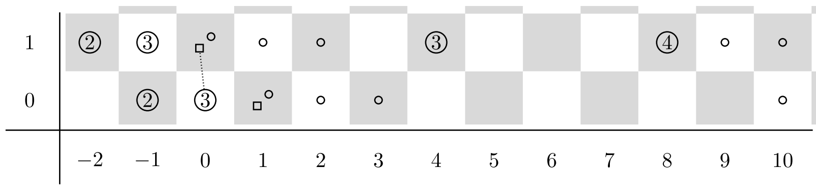


Fig. 5. The descent spectral sequence for $\mathfrak{Br}'(\mathbf{1}_K \mid KU_2) \simeq \mathfrak{Br}'(KO_2 \mid KU_2)^{h(1+4\mathbb{Z}_2)}$. To match other figures, we have shifted everything in degree by one (so one may think of this as the spectral sequence for $\Sigma^{-1}\mathrm{br}'$). The extension in the 0-stem is $4 \in \mathrm{Ext}(\mathbb{Z}/8, \mathbb{Z}_2) \simeq \mathbb{Z}/8$, which gives $\pi_0\Sigma^{-1}\mathrm{br}'(\mathbf{1}_K \mid KU_2) = \mathrm{Pic}_1 = \mathbb{Z}_2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2$.

This is achieved in the next couple of results, and the result is displayed in Figure 5. Note that we have shifted degrees by one to match other figures, so that the relative Brauer group is still computed by the (-1)-stem.

Proposition 5.13. We have $\mathrm{Br}(KO_2 \mid KU_2)^{1+4\mathbb{Z}_2} = \mathbb{Z}/4$, so the map

$$\mathrm{Br}'(\mathbf{1}_K \mid KU_2) \to \mathrm{Br}(KO_2 \mid KU_2)$$

is surjective.

Proof. It suffices to prove that $\psi^*(KO_2^{\mathrm{nr}}, \Sigma^2 KO_2^{\mathrm{nr}}) \simeq (KO_2^{\mathrm{nr}}, \Sigma^2 KO_2^{\mathrm{nr}})$, where $\psi = \psi^\ell$ is the Adams operation for a topological generator $\ell \in 1 + 4\mathbb{Z}_2$. By Lemma 3.3, this class is given by

$$\left(\varphi_!^{\Sigma KO_2^{\mathrm{nr}}} : \mathrm{Mod}_{KO_2^{\mathrm{nr}}} \to \varphi^*\mathrm{Mod}_{KO_2^{\mathrm{nr}}}\right) \in \mathbf{Eq}\left(id, \varphi^* : B\mathfrak{Pic}(KO_2^{\mathrm{nr}}) \rightrightarrows B\mathfrak{Pic}(KO_2^{\mathrm{nr}})\right),$$

where $\varphi = KO_2 \otimes \varphi_2$ is the Frobenius on $KO_2^{\mathrm{nr}} = KO_2 \otimes_{\mathbb{S}} \mathbb{SW}$. In particular, note that $\psi \otimes \mathbb{SW}$ commutes with the φ. Thus, the proposition follows from the square

$$\begin{tikzcd}
\mathrm{Mod}_{KO_2^{\mathrm{nr}}} \arrow[r, "\Sigma^2"] \arrow[dd, "\psi_!"'] & \mathrm{Mod}_{KO_2^{\mathrm{nr}}} \arrow[r, "\varphi_!"] & \varphi^*\mathrm{Mod}_{KO_2^{\mathrm{nr}}} \arrow[d, "\psi_!"] \\
& & \psi^*\varphi^*\mathrm{Mod}_{KO_2^{\mathrm{nr}}} \arrow[d, "\sim"] \\
\psi^*\mathrm{Mod}_{KO_2^{\mathrm{nr}}} \arrow[r, "\Sigma^2"'] & \psi^*\mathrm{Mod}_{KO_2^{\mathrm{nr}}} \arrow[r, "\varphi_!"'] & \varphi^*\psi^*\mathrm{Mod}_{KO_2^{\mathrm{nr}}}
\end{tikzcd}$$

whose commutativity is witnessed by the natural equivalence

$$\psi_!\varphi_!\Sigma^2 \simeq \varphi_!\psi_!\Sigma^2 \simeq \varphi_!\Sigma^2\psi_!.$$

■

The next result will be used in solving the extension problem in $\mathrm{Br}'(\mathbf{1} \mid KU_2)$.

Proposition 5.14. The cohomological Brauer group relative to KO_2^{nr} is

$$\mathrm{Br}'(\mathbf{1}_K \mid KO_2^{\mathrm{nr}}) = \mathbb{Z}/8 \oplus \mathbb{Z}/8.$$

Proof. Note that $\mathbf{1}_K \to KO_2^{\mathrm{nr}}$ is the descendable $\mathbb{Z}_2^\times \widehat{\mathbb{Z}}$ Galois extension corresponding to the proétale spectrum $p_*\underline{E}$ of Lemma 5.12. We will compute the relative Brauer group $\mathrm{Br}'(\mathbf{1}_K \mid KO_2^{\mathrm{nr}})$ by means of the descent spectral sequence

$$H^s(\mathbb{Z}_2 \times \widehat{\mathbb{Z}}, \pi_t B\mathfrak{Pic}(KO_2^{\mathrm{nr}})) \implies \pi_{t-s}\mathfrak{Br}'(\mathbf{1}_K \mid KO_2^{\mathrm{nr}}),$$

which collapses at the E_3 page since $\mathbb{Z}_2 \times \widehat{\mathbb{Z}}$ has cohomological dimension two for profinite modules. To compute the E_2-page, note that

$$\mathrm{Pic}(KO_2^{\mathrm{nr}}) = \mathbb{Z}/8\{\Sigma KO_2^{\mathrm{nr}}\} \quad \text{and} \quad \pi_0 \mathrm{GL}_1(KO_2^{\mathrm{nr}}) = \mathbb{W}^\times,$$

by C_2-Galois descent from $KU_2^{\mathrm{nr}} := E(\overline{\mathbb{F}}_2, \widehat{\mathbb{G}}_m) \simeq KU_2 \otimes_{\mathbb{S}} \mathbb{S}\mathbb{W}$. The action on $\mathrm{Pic}(KO_2^{\mathrm{nr}})$ is trivial, while the action on $\pi_0 \mathrm{GL}_1(KO_2^{\mathrm{nr}})$ is trivial for the $\mathbb{Z}_2$-factor, and Frobenius for the $\widehat{\mathbb{Z}}$-factor. In particular, note that $H^0(\widehat{\mathbb{Z}}, \mathbb{W}^\times) = \mathbb{Z}_2^\times$; using the $\widehat{\mathbb{Z}}$-equivariant splitting $\mathbb{W}^\times \simeq \overline{\mathbb{F}}_2^\times \times U_2$ (where $U_2 = \{x : \nu(x-1) \geq 2\} \cong \mathbb{W}$), we see that

$$\begin{aligned} H^1(\widehat{\mathbb{Z}}, \mathbb{W}^\times) &\cong H^1(\widehat{\mathbb{Z}}, \overline{\mathbb{F}}_2^\times) \oplus H^1(\widehat{\mathbb{Z}}, \mathbb{W}) \\ &\cong 0 \oplus \varinjlim H^1(\mathbb{Z}/n, \mathbb{W}(\mathbb{F}_{2^n})) \\ &= 0 \end{aligned}$$

by Hilbert 90. This implies that $H^2(\mathbb{Z}_2 \times \widehat{\mathbb{Z}}, \pi_0 \mathrm{GL}_1(KO_2^{\mathrm{nr}})) = 0$, since $\mathbf{cd}(\mathbb{Z}_2) = \mathbf{cd}(\widehat{\mathbb{Z}}) = 1$ for profinite coefficients. The (-1)-stem of the E_2-page is hence concentrated in filtration one, and agrees with the E_∞-page in this range. Thus,

$$\mathrm{Br}'(\mathbf{1}_K \mid KO_2^{\mathrm{nr}}) = H^1(\mathbb{Z}_2 \times \widehat{\mathbb{Z}}, \mathbb{Z}/8) = \mathbb{Z}/8 \oplus \mathbb{Z}/8.$$

■

Putting the pieces together, we can compute the group $\mathrm{Br}(\mathbf{1} \mid KU_2)$:

Theorem 5.15. The relative cohomological Brauer group at the prime two is

$$\mathrm{Br}'(\mathcal{S}p_K \mid KU_2) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/4.$$

Proof. Based on Figure 5, what remains is to compute the extension from $\mathrm{Br}'(\mathbf{1}_K \mid KO_2) \cong \mathbb{Z}/8\{(KO_2, \Sigma KO_2)\}$ to $\mathrm{Br}(KO_2 \mid KU_2)^{1+4\mathbb{Z}_2} \cong \mathbb{Z}/4\{(KO_2^{\mathrm{nr}}, \Sigma^2 KO_2^{\mathrm{nr}})\}$. But both $(KO_2, \Sigma KO_2)$ and $(KO_2^{\mathrm{nr}}, \Sigma^2 KO_2^{\mathrm{nr}})$ split over KO_2^{nr}, so that the inclusion of $\mathrm{Br}'(\mathbf{1}_K \mid KU_2)$ in $\mathrm{Br}'(\mathbf{1}_K)$ factors as

$$\begin{tikzcd} & \mathrm{Br}'(\mathbf{1_K} \mid KO_2^{\mathrm{nr}}) \arrow[d, hook] \\ \mathrm{Br}'(\mathbf{1_K} \mid KU_2) \arrow[ur, dashed] \arrow[r, hook] & \mathrm{Br}'(\mathbf{1_K}) \end{tikzcd}$$

Thus $\mathrm{Br}'(\mathbf{1}_K \mid KU_2) \hookrightarrow \mathbb{Z}/8 \oplus \mathbb{Z}/8$, which implies the claim. ■

5.3.2 *Generators at the prime two.*

To deduce Theorem A from Lemma 5.15 we will appeal to the results of Section 2.

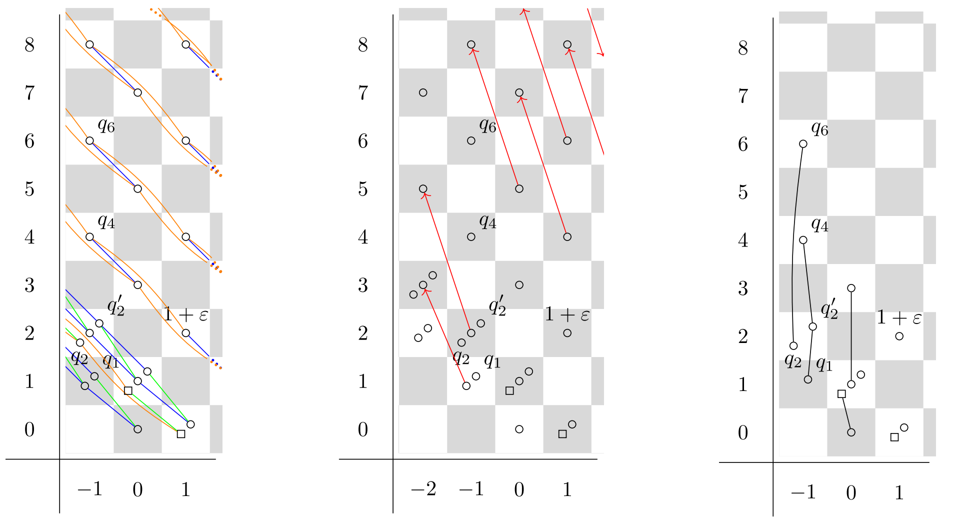
Theorem 5.16. The relative Brauer group at the prime two is

$$\mathrm{Br}_0^1 = \mathbb{Z}/8 \oplus \mathbb{Z}/4.$$

Proof. To lift the cohomological Brauer classes generating $\mathrm{Br}'(\mathcal{S}p_K \mid KU_2) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/4$ to Azumaya algebras, it is enough by Lemma 2.10 to prove that they are trivialised in some finite extension of $\mathbf{1}_K$. Recall from the previous subsection that:

1) The generator of the $\mathbb{Z}/4$-factor is $(KO_2, \Sigma KO_2) \in \mathrm{Br}'(\mathcal{S}p_K \mid KO_2)$ (Lemma 3.4), and detected by $2 \in \mathbb{Z}/8 \cong H^1(\mathbb{Z}_2, \mathrm{Pic}(KO_2))$. Since 2 is in the kernel of

$$H^1(\mathbb{Z}_2, \mathrm{Pic}(KO_2)) \to H^1(4\mathbb{Z}_2, \mathrm{Pic}(KO_2)),$$



(a) Multiplications by χ (in blue), $\beta(\chi)$ (orange), and π (green).

(b) Differentials out of the (-1)-stem.

(c) Extensions in the E_∞-page.

Fig. 6. Detailed view of the Picard spectral sequence (Figure 2) in low degrees.

2) this cohomological Brauer class is trivialised in the $\mathbb{Z}/4$-Galois extension $KO_2^{h(1+16\mathbb{Z}_2)}$, so

$$(KO_2, \Sigma^2 KO_2) \in \mathrm{Br}'(\mathbf{1}_K \mid KO_2^{h(1+16\mathbb{Z}_2)}) \cong \mathrm{Br}(\mathbf{1}_K \mid KO_2^{h(1+16\mathbb{Z}_2)}) \subset \mathrm{Br}_1^0.$$

3) Similarly, the generator of the $\mathbb{Z}/8$-factor is detected in the descent spectral sequence for $\mathbf{1}_K \to KO_2^{\mathrm{nr}}$ by $(1,0) \in \mathbb{Z}/8 \oplus \mathbb{Z}/8 \cong H^1(\widehat{\mathbb{Z}} \times \mathbb{Z}_2, \mathbb{Z}/8)$. We claim that this generator is trivialised in the extension $(KO_2^{\mathrm{nr}})^{h(8\widehat{\mathbb{Z}}\times\mathbb{Z}_2)} \simeq \mathbf{1}_K \otimes_{\mathbb{S}} \mathbb{SW}_8$, where $\mathbb{SW}_8 := W^+(\mathbb{F}_{2^8})$. Indeed, since

$$(1,0) \in \ker\left(H^1(\widehat{\mathbb{Z}} \times \mathbb{Z}_2, \mathbb{Z}/8) \to H^1(8\widehat{\mathbb{Z}} \times \mathbb{Z}_2, \mathbb{Z}/8)\right)$$

4) and the relative cohomological Brauer group $\mathrm{Br}'(\mathbf{1}_K \otimes_{\mathbb{S}} \mathbb{SW}_8 \mid KO_2^{\mathrm{nr}})$ is concentrated in filtration $s \leq 1$ of the Picard spectral sequence for $\mathbf{1}_K \otimes_{\mathbb{S}} \mathbb{SW}_8 \to KO_2^{\mathrm{nr}}$, we see that

$$[(1,0)] \in \mathrm{Br}'(\mathbf{1}_K \mid \mathbf{1}_K \otimes_{\mathbb{S}} \mathbb{SW}_8) \cong \mathrm{Br}(\mathbf{1}_K \mid \mathbf{1}_K \otimes_{\mathbb{S}} \mathbb{SW}_8) \subset \mathrm{Br}_1^0.$$

■

Remark 5.17. Using Lemma 5.15, we can also completely determine the behaviour of the Picard spectral sequence Figure 2. Recall the E_2-generators specified in Remark 4.6.

- Under the base-change to KO_2, the generators q_2 and q_6 map to the E_2-classes representing $P_2, P_6 = P_2^2 \in \mathrm{Br}(KO_2 \mid KU_2)$. The splitting in Lemma 5.16 of the surjection $\mathrm{Br}_1^0 \twoheadrightarrow \mathrm{Br}(KO_2 \mid KU_2)$ of Lemma 5.13 gives a canonical choice of classes $Q_2, Q_6 = Q_2^2 \in \mathrm{Br}_1^0$ lifting these. In particular, q_2 must also be a permanent cycle.
- Since $\mathrm{Br}_1^0 \supset \mathrm{Br}(\mathbf{1}_K \mid KO_2) \cong \mathbb{Z}/8$, the classes q_1, q_2' and q_4 must also survive, and detect Brauer classes Q_1, Q_2' and Q_4 trivialised over KO_2. We have $Q_2' = Q_1^2$ and $Q_4 = Q_1^4$ for this choice.

In Figure 6 we have displayed the E_∞-page of the descent spectral sequence, including extensions. For the purposes of constructing explicit generators, we have also included the module structure over $H^*(\mathbb{Z}_2^\times, \mathbb{Z}/2)$ and $H^*(C_2, \mathbb{Z})$, as appropriate; in particular, in Figure 6a we display multiplications by the generators

- $\chi \in H^1(C_2, \mathbb{Z}/2) \cong \mathrm{Hom}(C_2, \mathbb{Z}/2)$,
- $\pi \in H^1(1 + 4\mathbb{Z}_2, \mathbb{Z}/2) \cong \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}/2)$,
- $\beta(\chi) \in H^2(C_2, \mathbb{Z})$.

Remark 5.18. From the form of the spectral sequence, it follows that the class in $E_2^{7,5}$ survives to E_∞—this should have implications for the nonconnective Brauer spectrum of $\mathcal{S}p_{\mathrm{K}}$, as defined in [19].

Finally, we consider the consequences of Lemma 3.8 at the prime two. The units $\mathbb{Z}/4^\times \subset \mathbb{Z}_2^\times \subset \pi_0 \mathbf{1}_{\mathrm{K}}^\times$ are not strict: for example, they are not strict in Morava E-theory by [8, Theorem 8.17]. In fact, we expect that the descent spectral sequence for $\mathbb{G}_m(\mathbf{1}_{\mathrm{K}}) \simeq \mathbb{G}_m(E(\bar{\mathbb{F}}_2, \widehat{\mathbb{G}}_m))^{h(\widehat{\mathbb{Z}} \times \mathbb{Z}_2^\times)}$ will yield

$$\pi_0 \mathbb{G}_m(\mathbf{1}_{\mathrm{K}}) \cong \mathbb{Z}/2\,\{1+\varepsilon\} \subset (\pi_0 \mathbf{1}_{\mathrm{K}})^\times = (\mathbb{Z}_2[\varepsilon]/(2\varepsilon, \varepsilon^2))^\times.$$

This will be discussed in future work. Nevertheless, we have the following corollary of Lemma 3.8:

Corollary 5.19. For any $\chi : C_2 \cong \mathbb{Z}/2$, we have

$$Q_1^4 = Q_4 := (KU^{h(1+4\mathbb{Z}_2)}, \chi, 1+\varepsilon) \in \mathrm{Br}_1^0.$$

Proof. The unit $1+\varepsilon$ is strict by [10, Corollary 5.5.5], so the result follows from Lemma 3.8 since $q_4 = \beta(\chi) \cup (1+\varepsilon)$. ∎

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