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Author(s): J. Milnor
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ON THE COBORDISM RING Ω^* AND A COMPLEX ANALOGUE, PART I.*

By J. MILNOR.

This paper will prove that the cobordism groups Ω^i, defined by Thom [15], have no odd torsion.[1] Furthermore, it is shown that certain related groups $\pi_{i+2n}M(U_n)$ have no torsion at all; providing that n is large. The proofs are based on a spectral sequence due to J. F. Adams [1, 2].

The following is a brief summary of Thom's constructions. Let G be a subgroup of the orthogonal group O_n. (More generally one could start with any Lie group G, together with a specified representation into O_n.) Beginning with a universal bundle for G we can form:

1) The weakly associated bundle having the disk D^n as fibre. Let $\pi\colon E \to B(G)$ denote the projection map of this bundle.

2) The weakly associated bundle having the sphere S^{n-1} as fibre. Let $\partial E \subset E$ denote the total space.

The *Thom space* $M(G)$ is now defined as the identification space obtained from E by collapsing ∂E to a point.

Taking G to be the rotation group $SO_n \subset O_n$, Thom showed that the homotopy group $\pi_{i+n}M(SO_n)$ is independent of n, providing that n is large. He showed that this group is isomorphic to the "cobordism group" Ω^i; and determined its structure up to torsion. The 2-torsion subgroup of Ω^i has recently been determined by C. T. C. Wall. Hence the assertion that Ω^i has no odd torsion completes the description of this group.

Let $M(U_n)$ denote the Thom space for the unitary group $U_n \subset O_{2n}$. In Part II of this paper it will be shown that the stable homotopy group $\pi_{i+2n}M(U_n)$ can be interpreted as a "complex cobordism group." Part I will determine the structure of this group without attempting to interpret it.

* Received July 27, 1959.

[1] Added in proof. This result has been obtained independently by B. G. Averbuch, *Doklady Akademii Nauk SSSR*, vol. 125 (1959), pp. 11-14. The results on complex cobordism have been obtained independently by Novikov.

The first section proves several lemmas concerning the Steenrod algebra, which are needed later. The second section describes the Adams spectral sequence, which relates the cohomology module of any space to the stable homotopy groups of the space. Sections 3 and 4 complete the argument by computing the cohomology modules of $M(U_n)$ and $M(SO_n)$ respectively.

1. Lemmas concerning the Steenrod algebra. Let A denote the Steenrod algebra corresponding to a fixed prime p. (See Cartan [6], Adem [3].) The Bochstein coboundary operation will be denoted by $Q_0 \in A^1$. The two-sided ideal generated by Q_0 in A will be denoted by (Q_0).

LEMMA 1. *The Steenrod algebra contains a subalgebra A_0 with the following properties.*

(i) *A_0 is a Grassmann algebra over Z_p with generators $Q_0, Q_1, \cdots$ of odd dimension.*

(ii) *A is free as a right A_0-module.*

(iii) *The identity map of A induces an isomorphism between the left A-modules $A \otimes_{A_0} Z_p$ and $A/(Q_0)$.*

[*Explanation of* (iii). The field Z_p is considered as a left A_0-module with $Q_i Z_p = 0$. Hence $A \otimes_{A_0} Z_p$ is the quotient of A by the left ideal $AQ_0 + AQ_1 + AQ_2 + \cdots$.]

Proof for the case p odd. We will first prove the corresponding statements with left and right interchanged. According to Milnor [10, Theorem 4a]:

(1) There is a basis for A over Z_p consisting of elements $Q_0{}^{e_0} Q_1{}^{e_1} \cdots \mathfrak{P}^R$. Here the integers $e_0, e_1, \cdots$ should be 0 or 1, and almost all zero. The letter R stands for a sequence $(r_1, r_2, \cdots)$ of non-negative integers, almost all zero.

[*Explanation.* The element $\mathfrak{P}^R$ is a complicated polynomial in the Steenrod operations, with dimension $\sum r_j(2p^j - 2)$. For the special case $R = (r, 0, 0, 0, \cdots)$ the element $\mathfrak{P}^R$ is equal to the Steenrod operation $\mathfrak{P}^r$. The element Q_i of dimension $2p^i - 1$ can be defined inductively by the rule $Q_{i+1} = \mathfrak{P}^{p^i} Q_i - Q_i \mathfrak{P}^{p^i}$.]

Furthermore:

(2) The elements Q_i are odd dimensional, and satisfy $Q_i Q_j + Q_j Q_i = 0$, $Q_i Q_i = 0$.

Thus the Q_i generate a Grassmann algebra which may be denoted by $A_0 \subset A$. Clearly A is free as a left A_0-module, with basis $\{\mathcal{P}^R\}$.

Consider the right ideal $Q_0A + Q_1A + Q_2A + \cdots$. The following identity (see [10, Theorem 4a]) proves that this is also a left ideal. Define $p^i\Delta_j$ as the sequence $(0, \cdots, 0, p^i, 0, \cdots)$ with p^i in the j-th place.

(3) $\mathcal{P}^R Q_i$ is equal to $Q_i \mathcal{P}^R + \sum Q_{i+j} \mathcal{P}^{R-p^i\Delta_j}$, to be summed over all $j > 0$ for which $R - p^i\Delta_j$ is a sequence of non-negative integers. (That is, all j for which $r_j \geqq p^i$.)

Thus $Q_0A + Q_1A + \cdots$ is a two-sided ideal which contains Q_0, and therefore contains (Q_0).

As a special case of (3), the identity $\mathcal{P}^{\Delta_j} Q_0 = Q_0 \mathcal{P}^{\Delta_j} + Q_j$ is valid. Thus the elements Q_j belong to the ideal (Q_0). This proves that the ideal $Q_0A + Q_1A + \cdots$ is equal to (Q_0). Dividing A by these ideals, it follows that $Z_p \otimes_{A_0} A$ is isomorphic to $A/(Q_0)$.

This proves Lemma 1 for p odd, except that right and left have been interchanged. To complete the proof it is only necessary to recall:

(4) There exists an anti-automorphism of A; that is, a Z_p-isomorphism $c\colon A \to A$ satisfying

$$c(xy) = (-1)^{\dim x \dim y} c(y) c(x).$$

Furthermore, c carries Q_i into $-Q_i$.

This is proved in [10, § 7]. Clearly Lemma 1 follows (for p odd).

LEMMA 2. *The elements $\mathcal{P}^R \in A$ yield a basis over Z_p for the quotient algebra $A/(Q_0)$.*

Proof for p odd. Recall that $\{\mathcal{P}^R\}$ forms a basis for A, considered as a left A_0-module. Hence it forms a basis for $Z_p \otimes_{A_0} A = A/(Q_0)$ over Z_p, which completes the proof.

Conventions. The sum $R + R'$ of two sequences is defined as the term by term sum, and nR denotes the sequence $(nr_1, nr_2, \cdots)$. The binomial coefficient (R, R') is defined as the product over i of $(r_i + r'_i)!/r_i!r'_i!$. The symbol Δ_j stands for a sequence with 1 in the j-th place and zero elsewhere.

Proof of Lemmas 1 *and* 2 *for the case* $p = 2$. The Steenrod algebra over Z_2 has a basis consisting of elements Sq^R of dimension $r_1 + 3r_2 + 7r_3 + \cdots$. (See [10, Appendix 1].) Define $\mathcal{P}^R$ to be Sq^{2R} and define Q_{i-1} to be Sq^{Δ_i}. (For example $Q_0 = \mathrm{Sq}^{\Delta_1} = \mathrm{Sq}^1$ which checks with the definition of Q_0 as the

Bochstein coboundary operator.) Then we will prove Assertions (1), (2), (3) and (4) above. Using these, the proof of Lemmas 1 and 2 can be carried out just as for p odd.

The formula for products $\mathrm{Sq}^{R'}\mathrm{Sq}^{R}$ is rather complicated; however the following special case will suffice.

(5) If E is a sequence satisfying $e_i \leqq 1$, then $\mathrm{Sq}^E\mathrm{Sq}^R$ is equal to $(E,R)\,\mathrm{Sq}^{E+R}$.

For a proof see [10, Corollary 4 and Appendix 1]. As examples, taking $E = \Delta_{i+1}$, $R = \Delta_{j+1}$ we find that $Q_iQ_j = Q_jQ_i$, and that $Q_iQ_i = 0$. This proves Assertion (2) for the case $p = 2$.

By induction the product $Q_0{}^{e_1}Q_1{}^{e_2}\cdots$ is equal to Sq^E. Furthermore, a binomial coefficient of the form $(E, 2R)$ is always odd, hence $\mathrm{Sq}^E\mathfrak{P}^R = \mathrm{Sq}^E\mathrm{Sq}^{2R}$ is equal to Sq^{E+2R}. Since every sequence can be written uniquely in the form $E + 2R$, it follows that these elements form a basis for A over Z_2. This proves Assertion (1).

Proof of Assertion (3) *for* $p = 2$. Direct application of the general product rule [10, Theorem 4b] shows that

$$\mathrm{Sq}^{2R}\mathrm{Sq}^{\Delta_{i+1}} = \mathrm{Sq}^{\Delta_{i+1}}\mathrm{Sq}^{2R} + \sum \mathrm{Sq}^{2R-2^{i+1}\Delta_j+\Delta_{i+1+j}},$$

to be summed over all $j \geqq 1$ for which $r_j \geqq 2^i$. On the other hand, using Assertion (5), the j-th term on the right can be written as

$$\mathrm{Sq}^{\Delta_{i+1+j}}\mathrm{Sq}^{2R-2^{i+1}\Delta_j} = Q_{i+j}\mathfrak{P}^{R-2^i\Delta_j}.$$

Thus $\mathfrak{P}^R Q_i = Q_i\mathfrak{P}^R + \sum Q_{i+j}\mathfrak{P}^{R-2^i\Delta_j}$, as required.

Since Assertion (4) is also true for $p = 2$, this completes the proof of Lemmas 1 and 2.

[*Remark.* There is one essential difference between the case p odd and the case $p = 2$. For p odd the elements $\mathfrak{P}^R$ span a subalgebra of A isomorphic to $A/(Q_0)$; but for $p = 2$ there is no such subalgebra. This can be seen using the identity $\mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1 \neq 0$.]

The symbol Δ_0 will denote the sequence $(0, 0, \cdots)$.

LEMMA 3. *If p is odd, then the cohomology operations $\mathfrak{P}^R$ have the following properties.*

(1) *For $x, y \in H^*(X; Z_p)$ the element $\mathfrak{P}^R(xy)$ is equal to*

$$\sum_{R_1+R_2=R} (\mathfrak{P}^{R_1}x)(\mathfrak{P}^{R_2}y).$$

(2) *For a 2-dimensional cohomology class* $t \in H^2(X; Z_p)$, *the element* $\mathcal{P}^R t$ *is equal to* t^{p^i} *if* $R = \Delta_i$; *and is zero if* R *is not equal to one of the sequences* $\Delta_0, \Delta_1, \Delta_2, \cdots$.

Proof. The first assertion follows from [10, Lemma 9]. For the special case $R = r\Delta_1$, the second assertion is well known. That is:

$$\mathcal{P}^0 t = t, \quad \mathcal{P}^1 t = t^p, \quad \mathcal{P}^r t = 0 \text{ for } r > 1.$$

But every $\mathcal{P}^R$ is a "polynomial" in the Steenrod operations $\mathcal{P}^r$. Proceeding by induction on the complexity of this polynomial, we see that $\mathcal{P}^R t$ must have the form kt^i, where $k \in Z_p$ is some constant, and $2i$ is the dimension.

To evaluate k it is sufficient to consider one example. As example, let X be the $2i$-skeleton of the Eilenberg-MacLane complex $K(Z_p, 1)$. According to [10, Lemmas 4, 6] we have:

$$\lambda(t) = t \otimes \xi_0 + t^p \otimes \xi_1 + \cdots;$$

hence

$$\mathcal{P}^R t = \sum_i \langle \mathcal{P}^R, \xi_i \rangle t^{p^i}.$$

Using the definition of $\mathcal{P}^R$, this is equal to t^{p^i} if $R = \Delta_i$ and is zero otherwise. This completes the proof.

For the prime $p = 2$, both assertions of Lemma 3 would be false. However the following modified assertions are proved by the same method:

(1′) $\operatorname{Sq}^R(xy) = \sum_{R_1 + R_2 = R} (\operatorname{Sq}^{R_1} x)(\operatorname{Sq}^{R_2} y)$.

(2′) If $a \in H^1(X; Z_2)$, then $\operatorname{Sq}^{\Delta_i} a = a^{2^i}$; and $\operatorname{Sq}^R a = 0$ for R not of the form Δ_i.

Using these statement the following result will be proved.

LEMMA 3′. *Let* $p = 2$ *and let* $H^*(X; Z_2)$ *be a cohomology ring which is annihilated by the operation* $Q_0 = \operatorname{Sq}^1$. *The assertions* (1) *and* (2) *of Lemma 3 are valid as originally stated.*

Proof of (1). If R_1 is a sequence containing some odd integer, then Sq^{R_1} belongs to the ideal (Q_0) (compare the proof of Lemma 1), and therefore annihilates the cohomology of X. Thus in formula (1′) above, it is sufficient to consider sequences R_1 and R_2 which are "even." This proves assertion (1).

Proof of (2). It will be convenient to weaken the hypothesis on X, and assume only that $\operatorname{Sq}^1 t = 0$. Then just as in the proof of Lemma 3, it follows that $\mathfrak{P}^R t$ has the form kt^i. In order to determine the constant $k \in Z_2$, it is sufficient to consider the example of a real projective space X, with $t = a^2$, Using (1′) and (2′) it is seen that $\mathfrak{P}^R t$ equals t^{2^i} for $R = \Delta_i$ and equals zero otherwise. This completes the proof of Lemma 3′.

2. The spectral sequence of Adams. Let X, Y be finite CW-complexes with base point denoted by o; and let A be the Steenrod algebra for some fixed prime p. Thus the cohomology group $H^*(X \bmod o; Z_p)$ is a graded left A-module.

The m-fold *suspension* $S^m X$ is obtained from the product $X \times I^m$ by collapsing $(X \times \partial I^m) \cup (o \times I^m)$ to a point. Here I^m denotes the unit m-cube. The *stable track group* $\{X, Y\}_n$ is the direct limit under suspension of the group of homotopy classes of maps $S^{m+n} X \to S^m Y$. (The integer n may be positive or negative.)

Theorem of Adams. *There exists a spectral sequence $\{E_r^{st}, d_r\}$ determined by X, Y and p such that*

$$E_2^{st} = \operatorname{Ext}_A^{st}(H^*(Y \bmod o; Z_p), H^*(X \bmod o; Z_p))$$

and such that

$$E_\infty^{st} = B^{st}/B^{s+1\, t+1},$$

where $\{X, Y\}_n = B^{0\, n} \supset B^{1\, n+1} \supset B^{2\, n+2} \supset \cdots$ is a certain filtration. The intersection $\bigcap_s B^{s\, n+s}$ of these groups is equal to the subgroup of $\{X, Y\}_n$ consisting of elements whose order is finite and prime to p. Each succeeding term E_{r+1} of the spectral sequence is equal to the homology of E_r with respect to the differential operator

$$d_r \colon E_r^{st} \to E_r^{s+r\, t+r-1};$$

and E_∞ is the limit as $r \to \infty$ of E_r.

The functor $\operatorname{Ext}_A^{st}$ is defined as follows. If M and N are graded left A-modules let $\operatorname{Hom}_A^t(M, N) = \operatorname{Ext}_A^{0t}(M, N)$ denote the group of A-homomorphisms $M \to N$ of degree $-t$. Choose a projective resolution

$$\cdots \to P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \to M \to 0,$$

where the A-homomorphisms d have degree zero. Then $\operatorname{Ext}_A^{st}(M, N)$ is defined as the homology group (kernel modulo image) of the sequence

$$\operatorname{Hom}_A^t(P_{s-1}, N) \xrightarrow{d^*} \operatorname{Hom}_A^t(P_s, N) \xrightarrow{d^*} \operatorname{Hom}_A^t(P_{s+1}, N).$$

It will be convenient to add an E_1 term to the spectral sequence by defining $E_1{}^{st} = \operatorname{Hom}_A{}^t(P_s, N)$, $d_1 = d^*$.

For the special case $X = S^0$ this theorem is proved in Adams [1]. The more general case is proved by the same argument. It is only necessary to replace the homotopy group $\{S^0, \ \}_n$ by the track group $\{X, \ \}_n$ throughout. See Adams [2].

More generally the finite complex Y may be replaced by a "spectrum" in the sense of Lima [9] and Spanier [13]; or by an "object in the stable category" in the sense of Adams [2]. For our purpose the following definition will be convenient. A *stable object* $\mathbf{Y}$ is a sequence of CW-complexes $(Y_0, Y_1, \cdots)$ such that each suspension SY_i is a subcomplex of Y_{i+1}. The imbedding $SY_i \subset Y_{i+1}$ must be explicitly given.

Given such an object, define the chain group $C_n(\mathbf{Y})$ as the direct limit under suspension of the chain groups $C_{n+i}(Y_i \bmod o)$. Homology and cohomology groups are then defined as usual. Similarly, for any finite complex X define $\{X, \mathbf{Y}\}_n = \text{dir.lim.}\{S^iX, Y_i\}_n$. The abbreviation $\pi_n\mathbf{Y}$ will sometimes be used for $\{S^0, Y\}_n$.

Remark. The suspension homomorphism of chain groups should be defined by the correspondence

$$\alpha \to \alpha \times \iota, \text{ for } \alpha \in C_*(Y_i \bmod o),\ \iota \in C_1(I \bmod \partial I),$$

so as to commute with boundary homomorphisms.

Examples. Any finite complex Y may be defined with the stable object

$$\mathbf{Y} = (Y, SY, S^2Y, \cdots).$$

We will see later that the suspension of the Thom space $M(SO_n)$ is imbedded naturally as a subcomplex of $M(SO_{n+1})$. Hence the *stable Thom object*

$$\boldsymbol{M}(SO) = (o, M(SO_1), M(SO_2), \cdots)$$

is defined. Note that the track group

$$\{S^0, \boldsymbol{M}(SO)\}_n = \text{dir. lim. } \pi_{n+i}(M(SO_i))$$

is isomorphic to the cobordism group Ω^n.

Assertion. The theorem of Adams remains valid if the finite complex Y is replaced by any stable object $\mathbf{Y}$; providing that the following *finiteness condition* is satisfied. The groups $C_n(\mathbf{Y}; Z)$ should be finitely generated, and should vanish for n less than some constant.

This can be proved in two ways. One can simply take the direct limit of the spectral sequences for the "finite sub-objects" of $\mathbf{Y}$; or the theorem can be proved from the beginning in the stable category. See Adams [2]. The second approach is preferable, since the proof is much easier in the stable category. Details will not be given.

Using the Adams spectral sequence we will prove the following key result. Let $\mathbf{Y}$ be an object such that $H^n(\mathbf{Y};Z_p)$ is zero for n odd. Then $H^*(\mathbf{Y};Z_p)$ is annihilated by the element Q_0, and hence can be considered as a graded module over the quotient algebra $A/(Q_0)$.

THEOREM 1. *If $H^*(\mathbf{Y};Z_p)$ is a free $A/(Q_0)$-module with even dimensional generators, and if $C_*(\mathbf{Y};Z)$ satisfies the finiteness condition, then the stable homotopy group $\{S^0,\mathbf{Y}\}_n$ contains no p-torsion.*

The idea of the proof is to compute the spectral sequence for the track group $\{X,\mathbf{Y}\}_n$, where X is a "co-Moore space" having cohomology groups $H^i(X \bmod o;Z)$ equal to Z_p for $i=k$ and equal to zero for $i\neq k$.

The following universal coefficient theorem has been proved by Peterson [11]. *There exists an exact sequence*

$$0\to\{S^k,\mathbf{Y}\}_n\otimes Z_p\to\{X,\mathbf{Y}\}_n\to\operatorname{Tor}(\{S^k,\mathbf{Y}\}_{n-1},Z_p)\to 0.$$

An immediate consequence is the following.

LEMMA 4. *If $\{S^0,\mathbf{Y}\}_n$ contains p-torsion, then $\{X,\mathbf{Y}\}_m$ must be nontrivial for two consecutive values of m.*

On the other hand, assuming that $H^*(\mathbf{Y};Z_p)$ is a free $A/(Q_0)$-module on even dimensional generators, we will see that $\{X,\mathbf{Y}\}_m$ is zero for m odd. This will prove Theorem 1.

Construction of an A-free resolution for $H^*(\mathbf{Y};Z_p)$.

First consider the Grassmann algebra A_0 and the A_0-module Z_p. According to Cartan's theory of constructions, to each Grassmann algebra A_0 there corresponds a twisted polynomial algebra P and a differential operator d on $A_0\otimes P$ so that this tensor product becomes acyclic. If A_0 has generators $Q_0,Q_1,\cdots$, then P has a basis over Z_p consisting of elements $b(r_0,r_1,\cdots)$ of dimension $\sum r_i(\dim Q_i+1)$. The integers $r_0,r_1,\cdots$ should be non-negative and almost all zero. The differential operator d is defined as follows. (In order to make the signs come out correctly, we let d act on the right.) For any $a\in A_0$:

$$a \otimes b(r_0, r_1, \cdots) d = \sum a Q_i \otimes b(r_0, \cdots, r_i - 1, r_{i+1}, \cdots),$$

summed over all i for which $r_i > 0$.

Proof that $A_0 \otimes P$ is acyclic. For a Grassmann algebra on one generator, see Cartan [5, p. 704, I]. But a Grassmann algebra with finitely many generators in each dimension can be considered as a tensor product of Grassmann algebras with one generator. Hence the conclusion follows by applying the Künneth theorem.

This conclusion can be formulated as follows. Let F_s be the free A_0-module generated by those symbols $b(r_0, r_1, \cdots)$ for which $r_0 + r_1 + \cdots = s$. Then $A_0 \otimes P$ can be considered as the direct sum $F_0 + F_1 + \cdots$. The augmentation $\epsilon\colon F_0 \to Z_p$ is the A_0-homomorphism defined by $b(0, 0, \cdots) \to 1$. It follows that *the sequence*

$$\cdots \xrightarrow{d} F_1 \xrightarrow{d} F_0 \xrightarrow{\epsilon} Z_p \to 0$$

is an A_0-free resolution of Z_p.

Now apply the functor $A \otimes_{A_0}$ to this exact sequence. Since A is free as a right A_0-module, we obtain an exact sequence

$$\cdots \to A \otimes_{A_0} F_1 \to A \otimes_{A_0} F_0 \to A \otimes_{A_0} Z_p \to 0$$

of left A-modules. Furthermore, each $A \otimes_{A_0} F_s$ is a free A-module. Thus we have constructed an A-free resolution of $A \otimes_{A_0} Z_p$.

According to Lemma 1, the A-module $A/(Q_0)$ is isomorphic to $A \otimes_{A_0} Z_p$. Hence in order to form an A-free resolution of any $A/(Q_0)$-free module, it is sufficient to take the direct sum of a number of copies of the above resolution. This proves the following.

LEMMA 5. *Let $H^*(\mathbf{Y}; Z_p)$ be a free module over $A/(Q_0)$ with basis $\{y_\alpha\}$. Then there exists an A-free resolution*

$$\cdots \to F_1' \to F_0' \to H^*(\mathbf{Y}; Z_p) \to 0,$$

where each F_s' has a basis consisting of elements $b_\alpha(r_0, r_1, \cdots)$, with $r_0 + r_1 + \cdots = s$. The dimension of such a basis element is equal to $\dim y_\alpha + \sum 2r_i(p^i - 1) + s$.

[*Explanation.* The integer s has been added to the dimension of $b_\alpha(r_0, r_1, \cdots)$ so that the homomorphisms $d'\colon F_s' \to F_{s-1}'$ will have degree zero.]

Now consider the complex X consisting of a circle with a 2-cell attached by a map of degree p. Let

$$x \in H^1(X \operatorname{mod} o; Z_p), \qquad Q_0 x \in H^2(X \operatorname{mod} o; Z_p)$$

be generators. Then the term

$$E_1^{st} = \operatorname{Hom}_A{}^t(F_s', H^*(X \operatorname{mod} o; Z_p))$$

of the spectral sequence for $\{X, \boldsymbol{Y}\}$ has a basis consisting of the following elements.

(1) For each $b_\alpha(r_0, r_1, \cdots)$ of dimension $t+1$, the homomorphism $h_\alpha(r_0, r_1, \cdots)$ which carries this basis element into x and carries the other basis elements into zero.

(2) For each $b_\alpha(r_0, r_1, \cdots)$ of dimension $t+2$, the homomorphism $h_\alpha'(r_0, r_1, \cdots)$ which carries this basis element into $Q_0 x$ and carries the other basis elements into zero.

The boundary operator $d_1\colon E_1^{st} \to E_1^{s+1\,t}$ is given by

$$d_1 h_\alpha(r_0, r_1, \cdots) = h_\alpha'(r_0 + 1, r_1, \cdots),$$

and

$$d_1 h_\alpha'(r_0, r_1, \cdots) = 0.$$

Thus E_2^{st} has as basis the set of elements $h_\alpha'(0, r_1, r_2, \cdots)$, with total dimensions $t - s$ equal to $\dim y_\alpha + \sum 2 r_i(p^i - 1) - 2$.

If the integers $\dim y_\alpha$ are all even, then everything in the spectral sequence is even dimensional. It follows that $\{X, \boldsymbol{Y}\}_m$ is zero for m odd. Together with Lemma 4, this completes the proof of Theorem 1.

3. Computation of $H^*(B(U_n); Z_p)$ and $H^*(\boldsymbol{M}(U); Z_p)$. This section will complete the study of $M(U_n)$ by constructing a stable object

$$\boldsymbol{M}(U) = (o, o, M(U_1), SM(U_1), M(U_2), SM(U_2), \cdots);$$

and showing that $H^*(\boldsymbol{M}(U); Z_p)$ is a free module over $A/(Q_0)$, with even dimensional generators, for any prime p.

The proof of this assertion is an immediate generalization of the argument which Thom used to compute the non-orientable cobordism group. In our terminology, Thom showed that $H^*(\boldsymbol{M}(O); Z_2)$ is a free A-module. (See [15, pp. 39-42].)

First a description of $H^*B(U_n)$. *The coefficient group Z_p is to be* *understood, where p is some fixed prime.* (However, integer coefficients could equally well be used.) Let $T_n \subset U_n$ be the n-torus consisting of diagonal unitary matrices. There is a natural map $B(T_n) \to B(U_n)$ of classifying spaces. The cohomology algebra $H^*B(T_n)$ is a polynomial algebra on generators $t_1, \cdots, t_n$ of dimension 2. According to Borel and Serre [4] we may identify $H^*B(U_n)$ with the subalgebra consisting of all symmetric polynomials.

A basis for $H^{2r}B(U_n)$ over Z_p is given as follows. Let $\omega = i_1 \cdots i_k$ range over all partitions of r such that the "length" k is less than or equal to n. (A partition of r is an unordered sequence of positive integers with sum r.) *Define* $s(\omega)$ as the "smallest" symmetric polynomial which contains the term $t_1^{i_1} \cdots t_k^{i_k}$.

[The notation $\sum t_1^{i_1} \cdots t_k^{i_k}$ is commonly used. A more precise definition would be the following. Consider all distinct monomials which can be obtained from $t_1^{i_1} \cdots t_k^{i_k}$ by permuting the n variables; and let $s(\omega)$ denote their sum. It is clear that these elements $s(\omega)$ from a basis for the vector space of symmetric polynomials.]

Next we must study the Thom complex $M(U_n)$. For a group $G \subset SO_m$ recall that $M(G)$ is the quotient space $E/\partial E$, where E is an oriented m-disk bundle over $B(G)$. Any CW-cell subdivision of $B(G)$ induces a cell subdivision of $M(G)$ as follows. For each open i-cell e of $B(G)$, the inverse image e' in $E - \partial E$ is an $(i+m)$-cell. Clearly, $M(G)$ is the disjoint union of these cells e', together with the base point. It is not difficult to verify that $M(G)$ thus becomes a CW-complex.

Let $G \times 1$ denote the group G, considered as a subgroup of SO_{m+1}. The CW-complex $M(G \times 1)$ can be identified with the suspension $SM(G)$ as follows. Let D^m denote the m-disk and I the unit interval. Map $D^m \times I$ onto D^{m+1} by the correspondence

$$(x_1, \cdots, x_m), y \to x_1, \cdots, x_m, (2y - 1)(1 - x_1^2 - \cdots - x_m^2)^{\frac{1}{2}}.$$

This correspondence gives rise to a map f of $E \times I$ onto the total space E_1 of the associated $(m+1)$-disk bundle. Since f carries $(\partial E \times I) \cup (E \times \partial I)$ onto the boundary ∂E_1, it follows that f gives rise to a map $f'\colon SM(G) \to M(G \times 1)$. But f is a relative homeomorphism, hence f' is a homeomorphism.

The Thom isomorphism

$$\phi\colon H^iB(G) \to H^{i+m}(M(G) \bmod o)$$

is defined as follows. (see [14, Théorème I. 4]). The cohomology of $M(G) \bmod o$ will be identified with the cohomology of $E \bmod \partial E$. It can be verified that $H^m(E \bmod \partial E; Z)$ is an infinite cyclic group, with standard generator u. The isomorphism ϕ is now defined by the formula $\phi(a) = \pi^*(a)u$, where $\pi\colon E \to B(G)$ denotes the projection map. It follows from this definition that the following diagram is commutative:

$$\begin{array}{ccc} H^{i+m}(M(G) \bmod o) & \xrightarrow{S} & H^{i+m+1}(M(G \times 1) \bmod o) \\ \uparrow \phi & & \uparrow \phi \\ H^i B(G) & = & H^i B(G \times 1). \end{array}$$

Here S denotes the cohomology suspension, defined using the cohomology cross product.

Now let us specialize to the case $G = U_n \subset SO_{2n}$. The classifying space $B(U_n)$ has a standard cell subdivision due to Ehresmann [7] and $B(U_n)$ is a subcomplex of $B(U_{n+1})$. Hence $M(U_n)$ is a CW-complex and the two-fold suspension

$$S^2 M(U_n) = M(U_n \times 1 \times 1)$$

is a subcomplex of $M(U_{n+1})$. Thus

$$\boldsymbol{M}(U) = (0, 0, M(U_1), SM(U_1), M(U_2), \cdots)$$

is a stable object. The track group $\{S^0, \boldsymbol{M}(U)\}_k$ is clearly isomorphic to the stable homotopy group $\pi_{k+2n}(M(U_n))$, with n large.

On the other hand the complexes $B(U_1) \subset B(U_2) \subset \cdots$ have a union $B(U)$ which is again a CW-complex. The isomorphisms

$$\phi\colon H^i B(U_n) \to H^{i+2n}(M(U_n) \bmod o)$$

give rise, in the limit, to an isomorphism

$$\phi\colon H^i B(U) \to H^i \boldsymbol{M}(U).$$

It follows that $H^*\boldsymbol{M}(U)$ has a basis over Z_p consisting of the elements $\phi s(\omega)$, where ω ranges over all partitions.

THEOREM 2. *The cohomology $H^*\boldsymbol{M}(U)$ with coefficient group Z_p is a free module over $A/(Q_0)$, having as basis the elements $\phi s(\lambda)$, where λ ranges over all partitions which contain no integer of the form $p^j - 1$.*

Together with Theorem 1, and the fact that $\boldsymbol{M}(U)$ has no odd dimensional cohomology, this clearly implies the following.

THEOREM 3. *The groups $\{S^0, \boldsymbol{M}(U)\}_m$ have no torsion.*

The full structure of these stable homotopy groups can now be determined, using the fact that the stable Hurewicz homomorphism

$$\{S^0, \boldsymbol{Y}\}_m \rightarrow H_m(\boldsymbol{Y}; Z)$$

is a $\mathscr{C}$-isomorphism, where $\mathscr{C}$ denotes the class of finite groups. (See Serre [12] for definitions. This particular assertion is not in Serre's paper, but is well known.)

COROLLARY. *The group $\{S^0, \boldsymbol{M}(U)\}_m = \pi_m \boldsymbol{M}(U)$ is zero for m odd, and is free abelian for $m = 2n$, the number of generators being equal to the number of partitions of n.*

The proof of Theorem 2 will be based on a peculiar partial ordering of partitions, due to Thom. Given a sequence $R = (r_1, r_2, \cdots)$, define ω_R as the partition of $\sum r_j(p^j - 1)$ consisting of r_j copies of $p^j - 1$ for each $j \geqq 1$. Thus every partition ω can be written uniquely in the form $\lambda\omega_R$, where $\lambda = h_1 \cdots h_l$ contains no integer of the form $p^j - 1$. Let l denote the length of λ and let $\Sigma = h_1 + \cdots + h_l$ denote the sum of the integers in λ. Similarly, given a second partition ω', define l' and Σ'.

Definition. ω' is *less than* ω if $l' < l$, or if $l' = l$ and $\Sigma' > \Sigma$. (Note that integers of the form $p^j - 1$ are completely ignored in this definition.)

LEMMA 6. *The cohomology operation $\mathscr{P}^R$ carries $\phi s(\lambda) \in H^*\boldsymbol{M}(U)$ into $\phi s(\lambda\omega_R)$ plus a linear combination of elements $\phi s(\omega')$ with ω' less than $\lambda\omega_R$.*

Proof. It is clearly sufficient to prove the corresponding assertion for $H^*M(U_n)$, where n is large (say $n \geqq l + r_1 + r_2 + \cdots$), but finite. Consider the cross-section

$$f: B(U_n) \rightarrow E, \partial E$$

of the $2n$-disk bundle, determined by the center points of the disks. The induced cohomology homomorphism f^* carries the fundamental cohomology class $u \in H^{2n}(E \bmod \partial E)$ into the characteristic class

$$c_n = t_1 \cdots t_n = s(1 \cdots 1) \in H^{2n}B(U_n).$$

(See Thom [14], Borel and Serre [4].) Hence f^* carries the general element $\phi(a) = \pi^*(a)u \in H^{i+2n}(E \bmod \partial E)$ into the cup product $ac_n \in H^{i+2n}B(U_n)$. But the correspondence $a \rightarrow ac_n$ is a monomorphism; hence f^* is a monomorphism. Thus in order to prove Lemma 6 it is sufficient to prove the following.

Assertion. *$\mathcal{P}^R(s(\lambda)c_n)$ is equal to $s(\lambda\omega_R)c_n$ plus a linear combination of elements $s(\omega')c_n$ with ω' less than $\lambda\omega_R$.*

Consider a typical monomial $t_1^{a_1}\cdots t_n^{a_n}$ of the sum $s(\lambda)c_n$. Here l of the integers $a_1,\cdots,a_n$ are equal to the integers $1+h_1,\cdots,1+h_l$ in some order; while the remaining $n-l$ integers a_i are equal to 1. According to Lemma 3 we have

$$\mathcal{P}^R(t_1^{a_1}\cdots t_n^{a_n})=\sum_{R_1+\cdots+R_n=R}(\mathcal{P}^{R_1}t_1^{a_1})\cdots(\mathcal{P}^{R_n}t_n^{a_n}).$$

This formula is valid even for the case $p=2$, since $B(U_n)$ has no odd dimensional cohomology. (See Lemma 3′.) The expression $\mathcal{P}^{R_i}t_i^{a_i}$ is equal to some constant k_i times $t_i^{b_i}$, where $b_i\geqq a_i$. The case $b_i=a_i$ can occur only if $R_i=0$.

Each such monomial $(k_1\cdots k_n)t_1^{b_1}\cdots t_n^{b_n}$ contributes to a symmetric polynomials $s(\omega')c_n$, where ω' denotes the partition obtained from the sequence $b_1-1,\cdots,b_n-1$ by deleting zero. We wish to choose $R_1,\cdots,R_n$ so that this partition ω' is as "large" as possible, in the sense of the partial ordering. The first requirement is that as few as possible of the integers b_i-1 should be of the form p^j-1. But if $a_i=1$, and if the constant k_i is non-zero, then $\mathcal{P}^{R_i}t_i^{a_i}$ is necessarily of the form $t_i^{p^j}$. (See Lemma 3.) Thus the best we can do is to *choose $R_1,\cdots,R_n$ so that b_i is a power of p only if $a_i=1$.*

The second requirement in order to make ω' "large" is that the sum of all b_i-1 for which b_i is not a power of p should be as small as possible. Evidently, the best we can do in this direction is to *choose $R_i=0$ whenever $a_i>1$*; so that b_i will be equal to a_i whenver $a_i>1$.

Now consider the sum of all terms $(\mathcal{P}^{R_1}t_1^{a_1})\cdots(\mathcal{P}^{R_n}t_n^{a_n})$ for which this last condition (that R_i must be equal to zero whenever $a_i>1$) is satisfied. Each such term has the form $t_1^{b_1}\cdots t_n^{b_n}$, where l of the integers $b_1,\cdots,b_n$ are equal to $1+h_1,\cdots,1+h_l$ in some permutation; and the remaining $n-l$ integers b_i are powers of p. Recall that $\mathcal{P}^{R_i}t_i$ is equal to $t_i^{p^j}$ if $R_i=\Delta_j$ and is zero otherwise. Hence the relation $R_1+\cdots+R_n=R=(r_1,r_2,\cdots)$ implies that a given power p^j, $j\geqq 1$, must occur exactly r_j times in the sequence $b_1,\cdots,b_n$. The integer 1 must therefore occur $n-l-r_1-r_2-\cdots$ times in the sequence $b_1,\cdots,b_n$. Taking the sum of all monomials $t_1^{b_1}\cdots t_n^{b_n}$ which satisfy these conditions, we obtain exactly the polynomial $s(\lambda\omega_R)c_n$. This completes the proof of Lemma 6.

Proof of Theorem 2. The equations

$$\mathcal{P}^R\phi s(\lambda)=\phi s(\lambda\omega_R)+\sum(\text{constant})\phi s(\lambda'\omega_{R'}),$$

with all λ' less than λ, can be solved inductively, giving rise to equations:

$$\phi s(\lambda\omega_R) = \mathcal{P}^R\phi s(\lambda) + \sum(\text{constant})\mathcal{P}^{R'}\phi s(\lambda'),$$

with all λ' less than λ. (Only a finite number of terms are involved, since $H^*\boldsymbol{M}(U)$ is finitely generated in each dimension.) But the elements $\phi s(\lambda\omega_R)$ are known to form a Z_p-basis for $H^*\boldsymbol{M}(U)$. Therefore the elements $\mathcal{P}^R\phi s(\lambda)$ also form a Z_p-basis for $H^*\boldsymbol{M}(U)$. Since $\{\mathcal{P}^R\}$ is a basis for the vector space $A/(Q_0)$ over Z_p, this implies that the elements $\phi s(\lambda)$ form an $A/(Q_0)$-basis for $H^*\boldsymbol{M}(U)$. This completes the proof of Theorem 2, and hence Theorem 3.

4. Cohomology computations for $B(SO_{2n})$ and $M(SO)$. Consider the torus $T_n \subset U_n \subset SO_{2n}$, and the corresponding homomorphism

$$H^*(B(SO_{2n});Z_p) \to H^*(B(T_n);Z_p).$$

According to Borel and Serre [4], if p is odd, then the first algebra may be identified with the subalgebra of the second consisting of all polynomials $a + t_1 \cdots t_n b$, where a and b are symmetric polynomials in the elements $t_1^2, \cdots, t_n^2$. Thus a basis for $H^*(B(SO_{2n});Z_p)$ over Z_p is given by the elements $s(\omega)$ and $s(\omega)t_1 \cdots t_n$, where $\omega = i_1 \cdots i_k$, $k \leqq n$, is a partition into even integers. Letting n tend to infinity, *a Z_p-basis for $H^*(B(SO);Z_p)$ is given by the elements $s(\omega)$, where ω ranges over all partitions into even integers.*

Carrying out an argument completely analogous to that in Section 3, we construct a stable object

$$\boldsymbol{M}(SO) = (o, M(SO_1), M(SO_2), \cdots),$$

and prove the following.

THEOREM 4. *Let p be an odd prime, and let $\lambda = h_1 \cdots h_l$ range over all partitions into integers h_i which are even and not of the form $p^j - 1$. Then $H^*(\boldsymbol{M}(SO);Z_p)$ is the free $A/(Q_0)$-module having as basis the elements $\phi s(\lambda)$.*

Together with Theorem 1 this proves the following

THEOREM 5. *The cobordism groups $\Omega^i = \pi_i(\boldsymbol{M}(SO))$ contain no odd torsion.*

C. T. C. Wall has recently proved that an element in the 2-torsion subgroup of Ω^i is completely determined by its Stiefel-Whitney numbers. Together with Theorem 5, this proves the following conjecture of Thom.

COROLLARY 1. *If the Stiefel-Whitney numbers and the Pontrjagin numbers of a compact, oriented, differentiable manifold V^i are all zero, then V^i is a boundary.*

As special cases:

COROLLARY 2. *Suppose that V^i can be imbedded in euclidean space so as to have trivial normal bundle. Then V^i is a boundary.*

The proof is clear.

COROLLARY 3. *Suppose that $H_*(V^i; Z_2)$ is isomorphic to $H_*(S^i; Z_2)$. Then V^i is a boundary.*

Proof. The Stiefel-Whitney number $w_i[V^i]$ is equal to the Euler characteristic reduced modulo 2; hence is zero. If $i = 4n$, then the Pontrjagin number $p_n[V^i]$ is zero by the index theorem (Hirzebruch [8]). Since the other characteristic numbers of V^i are trivially zero, it follows that V^i is a boundary.

Concluding Remarks. There are other homotopy groups which may be accessible, using the Adams spectral sequence. For example, the symplectic groups $Sp(n) \subset SO_{4n}$ give rise to a stable object

$$\boldsymbol{M}(Sp) = (o, o, o, o, M(Sp(1)), SM(Sp(1)), S^2M(Sp(1)), S^3M(Sp(1)), M(Sp(2)), \cdots).$$

Assertion. The groups $\pi_i\boldsymbol{M}(Sp)$ have no odd torsion.

This can be proved directly from the spectral sequence; or can be derived from Theorem 5, using the natural map $\boldsymbol{M}(Sp) \to \boldsymbol{M}(SO)$.

Problem. Can one compute the spectral sequence for $\pi_*\boldsymbol{M}(Sp)$ corresponding to the prime $p = 2$?

Similarly, the representations $Spin(n) \to SO_n$ give rise to a stable object.

$$\boldsymbol{M}(Spin) = (o, M(Spin(1)), M(Spin(2)), \cdots).$$

Again there is no odd torsion; but the case $p = 2$ seems difficult. As a final question, consider the stable object $\boldsymbol{M}(SU)$ corresponding to the special unitary group.

Problem. What can be said about $\pi_*\boldsymbol{M}(SU)$?

REFERENCES.

[1] J. F. Adams, "On the structure and applications of the Steenrod algebra," *Commentarii Mathematici Helvetici*, vol. 32 (1958), pp. 180-214.

[2] ———, "Sur la théorie de l'homotopie stable," *Bulletin de la Société Mathématique de France*, vol. 87 (1959), pp. 277-280.

[3] J. Adem, "The relations on Steenrod powers of cohomology classes," *Algebraic geometry and topology*, Princeton, 1957, pp. 191-238.

[4] A. Borel and J.-P. Serre, "Groupes de Lie et puissances réduites de Steenrod," *American Journal of Mathematics*, vol. 75 (1953), pp. 409-448.

[5] H. Cartan, "Sur les groupes d'Eilenberg-MacLane II," *Proceedings of the National Academy of Sciences U. S. A.*, vol. 40 (1954), pp. 704-707.

[6] ———, "Sur l'iteration des opérations de Steenrod," *Commentarii Mathematici Helvetici*, vol. 29 (1955), pp. 40-58.

[7] C. Ehresmann, "Sur la topologie de certains espaces homogenes," *Annals of Mathematics*, vol. 35 (1934), pp. 396-443.

[8] F. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie*, Springer, 1956.

[9] E. L. Lima, "Stable Postnikov invariants," *American Mathematical Society Notices*, vol. 5 (1958), abstract number 544-27, p. 215.

[10] J. Milnor, "The Steenrod algebra and its dual," *Annals of Mathematics*, vol. 67 (1958), pp. 150-171.

[11] F. Peterson, "Generalized cohomotopy groups," *American Journal of Mathematics*, vol. 78 (1956), pp. 259-281.

[12] J.-P. Serre, "Groupes d'homotopie et classes de groupes abéliens," *Annals of Mathematics*, vol. 58 (1953), pp. 258-294.

[13] E. Spanier, "Function spaces and duality," *Annals of Mathematics*, vol. 70 (1959), pp. 338-378.

[14] R. Thom, "Espaces fibrés en sphères et carrés de Steenrod," *Annales Scientifiques de l'Ecole Normale superieure*, vol. 69 (1952), pp. 109-181.

[15] ———, "Quelques propriétés globales de variétés différentiables," *Commentarii Mathematici Helvetici*, vol. 28 (1954), pp. 17-86.