ROOT INVARIANTS AND PERIODICITY IN STABLE HOMOTOPY THEORY

MARK MAHOWALD AND PAUL SHICK

1. *Introduction and statement of results*

The goal of this paper is to relate two concepts of recent interest in stable homotopy theory. The first is the 'root invariant' (called the 'Mahowald invariant' in [9]). This is connected with Lin's proof of the Segal conjecture for $\mathbb{Z}/2$ [4] which shows that

$$\lim_{\leftarrow k} \pi_*(\mathbb{R}P^{\infty}_{-k}) \cong \pi_*(S^{-1})^{\wedge}.$$

We will give a precise definition of this invariant below. The second concept is the existence of infinite systematic (or 'periodic') families of elements in stable homotopy. These are associated with nonnilpotent self maps of finite complexes. (All complexes and spectra are objects in the stable category, localized at a prime p. For simplicity, we will state our results and proofs in terms of the prime 2.) The recent proof of the nilpotence conjecture [1] shows that the only possible nonnilpotent self maps of a 2-local finite complex X are v_n self maps, for some n; that is, maps detected in $K(n)_*(X)$ by some power of v_n, where $K(n)$ is the nth Morava K-theory. Calculations of various sorts lead to the following slogan:

The root invariant of v_n-periodic homotopy is v_n-torsion.

We will prove a version of this slogan. The first task is to give some precision to the idea of 'v_n-periodic' homotopy. Following Hopkins and Smith [2], we define a v_n self map as follows.

DEFINITION 1.1. A self map $v: \Sigma^t X \to X$ is a v_n self map if $K(m)_*(v)$ is zero for $m < n$, an isomorphism for $m = n$, and nilpotent for $m > n$.

If a finite complex X has a v_n self map, v, then the homotopy theory $[X, \_]_*$ is a module over $\mathbb{Z}[v]$. Thus we can speak of the v-periodic or v-torsion homotopy in $[X, Y]_*$. Further, Hopkins and Smith have shown that v_n self maps are essentially unique, in the following result.

THEOREM 1.2 [2]. *If X and Y are finite complexes with v_n self maps $v: \Sigma^t X \to X$ and $u: \Sigma^j Y \to Y$, and $f: X \to Y$ is any map, then there exist positive integers k and l such that the following diagram commutes:*

$$\begin{array}{ccc} \Sigma^{kt} X & \xrightarrow{f} & \Sigma^{jl} Y \\ \downarrow{\scriptstyle v^k} & & \downarrow{\scriptstyle u^l} \\ X & \xrightarrow{f} & Y. \end{array}$$

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Taking the identity map $i: X \to X$ shows that any two v_n self maps of X have iterates which are homotopic. Thus, for any spectrum Y, we can form the localization

$$[X, Y]_*(v_n^{-1}) = \varinjlim([X, Y]_* \xrightarrow{\cdot v} [X, Y]_* \xrightarrow{\cdot v} \dots)$$

in a well defined manner, independent of the choice of v_n self map, v. The kernel of the natural map $[X, Y]_* \to [X, Y]_*(v_n^{-1})$ is the v_n-torsion homotopy of Y (with coefficients in X) and the image is the v_n-periodic homotopy of Y. Both these terms refer to the homotopy of Y with coefficients in the finite complex X and do not refer to the ordinary homotopy groups $\pi_*(Y)$.

We now recall the root invariant and related notions. For any integer j, define $\mathbb{R}P_j$ to be the Thom spectrum of j times the canonical line bundle over $\mathbb{R}P^\infty$, the real projective space. (For odd primes, replace $\mathbb{R}P^\infty$ with $B\Sigma_p$, the classifying space of the symmetric group.) There is a naturally defined inverse system of spectra

$$\dots \longrightarrow \mathbb{R}P_{-k-1} \longrightarrow \mathbb{R}P_{-k} \longrightarrow \dots.$$

Lin showed that $\operatorname{holim}_{k\to\infty}(\mathbb{R}P_{-k}) \simeq (S^{-1})^\wedge$, the sphere completed at the prime 2. Jones and Wegmann showed in [3] that for any finite complex X, $\operatorname{holim}_{k\to\infty}(\mathbb{R}P_{-k} \wedge X) \simeq \Sigma^{-1}X$. Using this, we can define the root invariant as follows.

DEFINITION 1.3. Let $\alpha \in [X, S^0]_j$ be a homotopy class. Then there is a smallest integer k such that the composite

$$\Sigma^{j-1}X \xrightarrow{\alpha} S^{-1} \longrightarrow \mathbb{R}P_{-k}$$

is essential. This determines a map $R(\alpha)$ so that we have the following commutative diagram (from [7]).

$$\begin{array}{ccc} \Sigma^{j-1}X & \xrightarrow{\alpha} & S^{-1} \\ \big\downarrow R(\alpha) & & \big\downarrow \\ S^{-k} & \xrightarrow{i} & \mathbb{R}P_{-k}. \end{array}$$

The map $R(\alpha)$ is called the *root invariant* of α and, as in the case of Hopf invariants, there is some indeterminacy involved in its definition. This idea is discussed at some length in [6]. Our main result is the following.

THEOREM 1.4. *Suppose X is a finite complex with a v_n self map, v. Then* $\operatorname{holim}_{k\to\infty}[(\mathbb{R}P_{-k} \wedge X)(v^{-1})]$ *is contractible.*

Note that if one takes the homotopy limit first and then inverts the self map, the resulting complex is just $\Sigma^{-1}X(v_n^{-1})$, the desuspension of the mapping telescope, which is not contractible. By considering the Spanier-Whitehead dual of X, we easily derive the following.

COROLLARY 1.5. $\varprojlim_k [X, \mathbb{R}P_{-k}](v_n^{-1}) = 0.$

This is an instance of the slogan discussed above. Thus, if $\alpha \in [X, S^0]_*$ is v_n-periodic, then each root invariant, $R(\alpha)$, is v_n-torsion, at least when considered in $[X, \mathbb{R}P_{-k}]$. In particular, an entire v_n-periodic family is not all borne on the same sphere with $v^k\alpha$ having root invariant $v^kR(\alpha)$. If the telescope conjecture is true, then except for finitely many α, $R(\alpha)$ is v_n-torsion in $[X, S^0]_*$. This affirmatively answers part of a conjecture of [6]. Hopkins and Wegmann also conjectured the above results.

Another interpretation of these results can be stated by using Lin's theorem to construct an Atiyah–Hirzebruch spectral sequence for the homotopy of X (or for its Spanier–Whitehead dual). If we use an Atiyah–Hirzebruch spectral sequence to calculate $\pi_*(\mathbb{R}P_{-k} \wedge X)$, we have an E_1 term described in terms of the homotopy of X. Precisely, $E_1^{s,t} = \pi_{s+t}(X)$, $s \leqslant k$. So, to calculate the homotopy of the $\operatorname{holim}_{k\to\infty}(\mathbb{R}P_{-k} \wedge X)$, we have a spectral sequence with E_1 depending on all the homotopy of X. The spectral sequence converges to $\pi_*(X)$, but the 'v_n-free' classes do not survive the spectral sequence. Thus each v_n-periodic homotopy class has a representative in this spectral sequence which is v_n-torsion.

This result is easy to see in the case where X is the $\mathbb{Z}/2$ Moore space, M. The proof in this case contains the crucial idea for the general case, so we sketch it here to help the reader understand the general case. Let $v: \Sigma^8 M \to M$ be the Adams map, which is an example of a v_1 self map. Note that the 'times 2' map on $\mathbb{R}P_{4k-1}$ yields the following cofibration sequence:

$$\mathbb{R}P_{4k-1} \xrightarrow{2} \mathbb{R}P_{4k-1} \longrightarrow \mathbb{R}P_{4k-1} \wedge M.$$

This degree 2 map factors through the pinch map $p: \mathbb{R}P_{4k-1} \to \mathbb{R}P_{4k+1}$. Let $f: \mathbb{R}P_{4k+1} \to \mathbb{R}P_{4k-1}$ be the other factor. Let C_{4k} denote the cofiber of f. Then the pinch maps in the inverse system factor

$$\mathbb{R}P_{4k-1} \wedge M \longrightarrow C_{4k} \longrightarrow \mathbb{R}P_{4k+3} \wedge M$$

where C_{4k} has mod 2 cohomology $H^*(C_{4k})$ free over A_1, the subalgebra of the Steenrod algebra generated by Sq^1 and Sq^2. (All cohomology groups are assumed to have mod p coefficients, where the prime is usually taken to be 2.) So for each k, C_{4k} has a v_1 self map which is trivial. Therefore the mapping telescope $C_{4k}(v_1^{-1})$ is contractible for each k and the homotopy limit $\operatorname{holim}_{k\to\infty}[(\mathbb{R}P_{4k-1} \wedge M)(v^{-1})]$ is contractible. This establishes the theorem for this particular example.

2. *Reduction to the main lemma*

Let A_n denote the subalgebra of the Steenrod algebra generated by $\{\mathrm{Sq}^1, \mathrm{Sq}^2, \ldots, \mathrm{Sq}^{2^n}\}$. *Let* $B_n = A_n/Q_n A_n$, where Q_n is the nth Milnor generator of the Steenrod algebra. Thus B_n is known as 'half A_n'. We say that a complex X is B_n-free if its mod 2 cohomology is free as a module over B_n.

MAIN LEMMA 2.1. *Let X be a B_n-free finite complex with a v_n self map $v: \Sigma^t X \to X$. Then* $\lim_{k\to\infty}[X, \mathbb{R}P_{-k}](v^{-1}) = 0$.

To prove that the lemma implies Theorem 1.4, we recall the following result of Hopkins and Smith. Let $\mathscr{C}_0$ denote the full subcategory of the stable category with objects consisting of finite CW spectra. Let $\mathscr{C}_n$ be the full subcategory of $\mathscr{C}_0$ consisting of $K(n-1)_*$-acyclics. The collection $\{\mathscr{C}_i\}$ forms a decreasing filtration of finite spectra. That the filtration is given by proper inclusions is shown by the construction of complexes with cohomology free over B_n given independently by Mitchell [7] and Smith [9]. By results of Hopkins *et al.* in [2], each complex $X \in \mathscr{C}_n$ has a v_n self map (although that map is trivial if X is $K(n)_*$-acyclic). The following remarkable theorem appears in [2].

THEOREM 2.2 [2]. *Let $\mathscr{C}$ be a full subcategory of $\mathscr{C}_0$ such that*

(*a*) *if X is any object in $\mathscr{C}$, then any retract of X is also in $\mathscr{C}$;*

(*b*) *if $X \to Y \to Z$ is a cofibration sequence with X and Y in $\mathscr{C}$, then Z is in $\mathscr{C}$. Then the category $\mathscr{C} = \mathscr{C}_n$ for some n.*

To apply this theorem to our situation, let $\mathscr{C}$ be the category of finite spectra X with v_n self maps such that $\operatorname{holim}_{k\to\infty}[(\mathbb{R}P_{-k} \wedge X)(v_n^{-1})]$ is contractible. Clearly $\mathscr{C} \subset \mathscr{C}_n$. It remains only to show that $\mathscr{C}$ is closed under conditions (*a*) and (*b*) of the theorem, and to show that $\mathscr{C}$ is nonempty.

To show that $\mathscr{C}$ is closed under retracts, it suffices to show that if $X \vee \Sigma X$ in the category implies that X is, also. Let $X \vee \Sigma X$ have a v_n self map v such that

$$\operatorname{holim}_{k\to\infty}[(X \vee \Sigma X) \wedge \mathbb{R}P_{-k}(v^{-1})]$$

is contractible. Then X has a v_n self map which factors through the v_n self map for $X \vee \Sigma X$. Hence $X \in \mathscr{C}$.

Let $X \to Y \to Z$ be a cofibration sequence with X and Y in $\mathscr{C}$. Then the functorial nature of $[\_, \mathbb{R}P_{-k}](v_n^{-1})$ shows that Z is in $\mathscr{C}$. This completes the proof of Theorem 1.4, then, once we have established the main lemma, which shows that $\mathscr{C}$ is nonempty.

3. *Proof of the Main Lemma*

Our proof uses the classical Adams spectral sequence (clASS) for $[X, \mathbb{R}P_{-k}]_*$ and its localizations. To begin, note that if $H^*(X)$ is free over B_n, then $H^*(DX)$ is also free as a B_n-module, where DX denotes the Spanier–Whitehead dual of X. Further, $\operatorname{Ext}_A(\mathbb{Z}/2, H^*X)$, the E_2 term of the clASS converging to $[X, S^0]$, has a v_n operator, given by composition with the class $v_n \in \operatorname{Ext}^1_A(H^*X, H^*X)$. Also, since H^*X is free over B_n, $\operatorname{Ext}_A(\mathbb{Z}/2, H^*X)$ has a vanishing line of slope $1/|v_n| = 1/(2^{n+1}-1)$. This makes the following lemma clear.

LEMMA 3.1. *There is a localized clASS*

$$\operatorname{Ext}_A(\mathbb{Z}/2, H^*X)(v_n^{-1}) \Rightarrow [X, S^0](v_n^{-1}).$$

Here the Ext-group is localized with respect to the class $v_n \in \operatorname{Ext}^1_A(H^*X, H^*X)$, while the group $[X, S^0]$ is localized with respect to any v_n self map of X.

Let P_k denote $H^*\mathbb{R}P_k$ for any integer k. Then $\operatorname{Ext}_A(P_k, H^*X)(v_n^{-1})$ converges to $[X, \mathbb{R}P_k](v_n^{-1})$ since $\operatorname{Ext}_A(P_k, H^*X)$ also has a $1/|v_n|$ vanishing line.

We calculate Ext_A by means of a spectral sequence based on $A \otimes_{A_n} \mathbb{Z}/2$ resolutions. Let J denote $A//A_n = A \otimes_{A_n} \mathbb{Z}/2$, and I its augmentation ideal. Then for any A-module M, we have an exact sequence

$$0 \to M \to J \otimes M \to J \otimes I \otimes M \to \dots \to J \otimes I^\sigma \otimes M \to \dots.$$

Applying the functor $\operatorname{Ext}_A(\_\_, N)$ to this, we obtain a spectral sequence converging to $\operatorname{Ext}_A(M, N)$ with $E_1^{\sigma,s,t} = \operatorname{Ext}_A^{s,t}(J \otimes I^\sigma \otimes M, N)$. The $E_\infty^{\sigma,s,t}$ term contributes to $\operatorname{Ext}_A^{\sigma+s,t}(M, N)$. This spectral sequence is strongly convergent in the sense that for each σ, s, t, there is a k such that $E_r^{\sigma,s,t}$ is constant for $r > k$. Note that the bottom class in I is in degree $2^{n+1}-1$, whereas the degree of v_n is 1 less. This allows us to construct a spectral sequence with v_n inverted, giving

$$E_1^{\sigma,s,t}(v_n^{-1}) = \operatorname{Ext}_A^{s,t}(J \otimes I^\sigma \otimes M, N)(v_n^{-1}) \Rightarrow \operatorname{Ext}_A^{\sigma+s,t}(M, N)(v_n^{-1}).$$

We now describe how this spectral sequence behaves with respect to the pinch maps which form the inverse system $\{\mathbb{R}P_k\}$. Let $\mathbb{P} = \mathbb{Z}/2[x, x^{-1}]$, with $|x| = 1$, which is made into an A-module by $Sq^i x^j = \binom{j}{i} x^{i+j}$, so that $\mathbb{P} = \lim_{k\to\infty} P_{-k}$. Let $F_{k,n}$ denote the A_n-submodule generated by $\{x^j : j > k\}$ [5]. The $\mathbb{P}/F_{k,n}$ is a cofinal object for the direct system of groups $\{P_j\}$ in the sense that for $k \equiv -1 \bmod (2^{n+1})$, there are A_n-module maps a and b

$$P_{j+r2^{n+1}} \xrightarrow{b} \mathbb{P}/F_{j,n} \xrightarrow{a} P_j$$

so that the composition is the map induced in cohomology by the pinch map. Here $r2^{n+1}$ must be at least as large as the degree of the top cell in $A_n//A_{n-1}$. These A_n-maps induce

$$\mathrm{Ext}_A^{s,t}(J \otimes I^\sigma \otimes P_j, H^*X)(v_n^{-1}) \to \mathrm{Ext}_A^{s,t}(J \otimes I^\sigma \otimes \mathbb{P}/F_{j,n}, H^*X)(v_n^{-1})$$
$$\longrightarrow \mathrm{Ext}_A^{s,t}(J \otimes I^\sigma \otimes P_{j+r2^{n+1}}, H^*X)(v_n^{-1}).$$

By the main result of [5], $J \otimes \mathbb{P}/F_{j,n}$ is isomorphic to a direct sum of $A//A_{n-1}$. Since H^*X is free over B_n (and hence is free over A_{n-1}), the middle Ext-group is isomorphic to $\mathrm{Ext}_A(M, \mathbb{Z}/2)$, where M is a free A-module. Localizing with respect to v_n must annihilate this term, so that the inverse limit of the E_1 terms of the spectral sequences is zero. Therefore the homotopy limit has the homotopy groups of a point. This proves the theorem.

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Mathematics Department
Northwestern University
Evanston, Illinois 60201
USA

Mathematics Department
John Carroll University
University Heights, Ohio 44118
USA