

EQUIVARIANT GENERATING HYPOTHESES FOR FINITE GROUPS

SIHAO MA, XIAOLIN DANNY SHI, ZHOULI XU, AND SHANGJIE ZHANG

ABSTRACT. We disprove Bohmann’s equivariant generating hypothesis for every nontrivial finite group G , even when all $\mathrm{RO}(H)$ -graded homotopy groups at every subgroup H are tested. For each fixed G and prime p dividing $|G|$, we construct ghosts on finite G -spectra with arbitrarily long nonzero composition powers. We also prove that the homotopy-module functors are nonfull and construct non-equivalent finite G -spectra with isomorphic full homotopy modules.

Our constructions use circle power maps and cyclic permutations of products of projective spaces, and are motivated by Ma–Xu’s categorical method in the motivic setting and the projective-space power maps.

CONTENTS

1. Introduction	1
2. A four-cell ghost for C_p	4
3. Representation-twisted fixed points and nonvanishing	11
4. A four-cell ghost for all finite groups	13
5. Arbitrarily long ghosts	15
6. Nonfullness and failure of classification	19
7. Homological consequences	22
References	23

1. INTRODUCTION

Freyd’s generating hypothesis asks whether stable homotopy groups detect maps between finite spectra [Fre66].

Conjecture 1.1 (Freyd’s classical generating hypothesis). *Let $f : X \rightarrow Y$ be a map between finite spectra. If*

$$\pi_n(f) : \pi_n X \longrightarrow \pi_n Y$$

is zero for every $n \in \mathbb{Z}$, then $f = 0$ in the stable homotopy category.

The conjecture remains open. Freyd proved that the conjectured faithfulness would imply fullness [Fre66, Proposition 9.7]: for finite X and Y , the natural map

$$[X, Y] \longrightarrow \mathrm{Hom}_{\pi_* S}(\pi_* X, \pi_* Y)$$

would be an isomorphism, where module homomorphisms preserve degrees. Graded homotopy modules would then recover both maps and stable homotopy types of finite spectra.

Bohmann formulated the equivariant generating hypothesis in 2010, using integer-graded homotopy groups at every subgroup [Boh10, Conjecture 1.2]. We disprove it for every nontrivial finite group. The failure persists after testing all representation-graded homotopy groups at every subgroup, and the resulting ghost ideal is nonnilpotent for each fixed nontrivial finite group. We also prove that the homotopy-module functors are nonfull and construct non-equivalent finite G -spectra with isomorphic full homotopy modules.

2020 *Mathematics Subject Classification.* Primary 55P91, Secondary 55P42, 55Q10.

Key words and phrases. Equivariant generating hypothesis, finite equivariant spectra, ghosts.

Fix a finite group G . We work with G -spectra indexed on a complete universe [LMS86, Chapters I–II]. Finite G -spectra are the compact objects in this category. They form the thick subcategory generated by the orbit spectra $\Sigma_G^\infty G/H_+$, $H \leq G$ [BS17, Section 2, (A)]. Write $[X, Y]^G$ for stable maps and put

$$\pi_W^H(X) = [S^W, \text{Res}_H^G X]^H \quad (H \leq G, W \in \text{RO}(H)).$$

The H -Mackey functor $\pi_W^H(X)$ is defined on orbits by

$$\pi_W^H(X)(H/L) = [S^{\text{Res}_L^H W}, \text{Res}_L^G X]^L \quad (L \leq H).$$

We use $*$ for integer degrees and $\star$ for representation degrees: $\pi_* X$ denotes the integer-graded G -Mackey functor, and $\pi_*^H(X)$ the $\text{RO}(H)$ -graded family of H -Mackey functors. We consider the following three versions of the generating hypothesis on the compact category.

Conjecture 1.2 (Equivariant generating hypotheses for finite groups). *Let G be finite and let $f : X \rightarrow Y$ be a map between finite G -spectra. Each of the following formulations asserts that its stated vanishing condition implies $f = 0$.*

(1) **Bohmann’s integer-graded Mackey GH.** *The condition is*

$$\pi_*(f) = 0.$$

(2) **$\text{RO}(G)$ -graded Mackey GH.** *The condition is*

$$\pi_*^G(f) = 0.$$

(3) **All-subgroup representation-orbit GH.** *The condition is that, for every $H \leq G$,*

$$\pi_*^H(f) = 0.$$

By induction–restriction adjunction, these conditions test maps from $G/H_+ \wedge S^n$, $G/H_+ \wedge S^V$, and $G_+ \wedge_H S^W$, respectively, where $H \leq G$, $n \in \mathbb{Z}$, $V \in \text{RO}(G)$, and $W \in \text{RO}(H)$. All three families consist of compact objects and generate the finite G -spectra as a thick subcategory. They increase from parts (1) to (3), since ordinary suspensions are representation suspensions and

$$G/H_+ \wedge S^V \simeq G_+ \wedge_H S^{\text{Res}_H^G V}.$$

A counterexample to part (3) therefore disproves all three. A map satisfying the vanishing condition for one of these families is called a *ghost* for that family [Chr98, Definition 2.5 and Section 7.1]. For part (3), we use the term *representation-orbit ghost*.

For finite G , Greenlees–May [GM95, Appendix A, Theorem A.4] proved faithfulness and fullness of the integer-graded homotopy Mackey functor on all rational G -spectra. All three hypotheses therefore hold rationally. Bohmann [Boh10, Theorem 1.7] gave a counterexample on compact rational S^1 -equivariant spectra in the sense of Conjecture 1.2(1).

For each of the three detector families, the homotopy-module functor records homotopy groups together with the operations induced by stable maps between the detecting spectra.

Convention. Unless otherwise specified, for the rest of this paper G is a finite group with $|G| \neq 1$ and p is a prime dividing $|G|$.

Bohmann [Boh10, Theorem 1.5] proved that integer-graded Mackey GH for finite G implies fullness over the graded sphere Green functor. In this paper, we prove nonfaithfulness and nonfullness for all three homotopy-module functors.

Theorem A (Nonfaithfulness and nonfullness).

- (1) *All three assertions in Conjecture 1.2 are false, both integrally and in the category of finite p -local G -spectra.*
- (2) *All three homotopy-module functors on finite G -spectra are nonfull, integrally and p -locally.*

If ℓ is a prime not dividing $|G|$, each of the three ℓ -local versions of Conjecture 1.2 is equivalent to the classical nonequivariant ℓ -local generating hypothesis. This follows from the geometric fixed-point splitting [Wim19] and averaging over the Weyl groups.

The equivariant homotopy of a finite G -spectrum X contains more information than the module $\pi_*^G X$ over the equivariant stable homotopy ring $\pi_*^G S$. The homotopy groups at all subgroups also carry restrictions, transfers, and all other operations induced by stable maps between the detecting spectra. Morphisms of homotopy modules commute with all these operations (see Section 6). Even with all this structure, our proof of nonfullness gives the following result.

Theorem B (Homotopy modules). *There exist non-equivalent finite G -spectra A and B that are p -local and whose homotopy Mackey functors admit, for all $H \leq G$ and $W \in \text{RO}(H)$, isomorphisms*

$$\pi_W^H(A) \cong \pi_W^H(B)$$

compatible with all stable operations.

The nonfaithfulness in Theorem A(1) already occurs on a spectrum with four free orbit cells. We construct these examples in Sections 2–4 (Theorem 2.1).

Our second construction gives nonzero powers of arbitrary length for each fixed group.

Theorem C (Arbitrarily long ghost powers). *For every $m \geq 0$ and $n \geq 1$, there are a compact object Y in the p -local G -equivariant stable homotopy category and an endomorphism $f : Y \rightarrow Y$ such that*

$$\pi_W^H(f) = 0 \quad (H \leq G, W \in \text{RO}(H)), \quad p^m \cdot f^n \neq 0.$$

In fact, we can further show that $(S^{-W} \wedge \text{Res}_H^G f)^H$ is null as a map of spectra for every $H \leq G$ and $W \in \text{RO}(H)$.

These ghost powers give simultaneous lower bounds for the flat, projective, and injective dimensions of all three homotopy modules on the same finite spectrum.

Theorem D (Homological consequences). *For every integer $n \geq 1$, there is a finite p -local G -spectrum Y_n such that the homotopy modules*

$$\pi_* Y_n, \quad \pi_*^G(Y_n), \quad \{\pi_*^H(Y_n)\}_{H \leq G}$$

each have flat, projective, and injective dimension at least n in their respective p -local module categories. Consequently, each of these categories has infinite weak global dimension and infinite global dimension.

Constructions and proofs. The starting point for this paper is the work of the first and third authors [MX26] on the algebraic and motivic generating hypotheses. Their constructions produce finite ghosts with arbitrarily long nonzero composition powers. Two ideas from that work guide our equivariant constructions: natural operations that vanish on the detecting objects, and projective-space models supporting long ghost powers.

The first idea is the following naturality principle.

Lemma 1.7. *Let $\mathcal{T}$ be an additive category with a collection $\mathcal{P}$ of objects. Suppose $\Theta : \text{id}_{\mathcal{T}} \Rightarrow \text{id}_{\mathcal{T}}$ is natural and $\Theta_P = \text{id}_P$ for every $P \in \mathcal{P}$. Then $1 - \Theta_X$ induces zero on $\text{Hom}_{\mathcal{T}}(P, X)$ for every $P \in \mathcal{P}$ and $X \in \mathcal{T}$.*

Proof. For $a : P \rightarrow X$, naturality gives $(1 - \Theta_X)a = a - a\Theta_P = 0$. □

In a triangulated category, include the relevant shifts among $\mathcal{P}$. In that work, formal multiplication gives natural coefficient automorphisms Θ fixing the shifted units, so $1 - \Theta$ is a ghost. A rank-two comodule supports a nonzero square-zero ghost of order p on an object built from four shifts of BP_* . Projective-space coefficients give arbitrarily long nonzero powers. Here the group action supplies the natural automorphism.

Proposition 1.8 (A natural cyclic ghost operation). *Let g generate C_p , and let g_X be its action on a C_p -spectrum X . The natural operation*

$$\Psi_X = (p - [C_p/e])(1 - g_X)$$

is a representation-orbit ghost with null underlying map. It is nilpotent on every finite C_p -spectrum, but its nilpotence exponents on finite spectra are unbounded. These statements also hold p -locally.

The factors serve complementary purposes: $p - [C_p/e]$, where $[C_p/e] \in A(C_p)$ is the class of the free orbit, vanishes on underlying spectra, while $1 - g_X$ vanishes on C_p -geometric fixed points. Their product therefore vanishes on representation spheres, and naturality makes Ψ_X a ghost. It remains to find a finite X with $\Psi_X \neq 0$.

Second, the projective-space coefficient examples in that work realize as finite cellular motivic spectra with long ghost powers through the Gheorghe–Wang–Xu comparison [GWX21]. These examples suggest $\mathbb{CP}^\infty$ as a natural object on which to seek nontrivial ghosts. To introduce a natural C_p -action, we pass to its p -fold product $(\mathbb{CP}^\infty)^p$, equipped with cyclic permutation of the factors. We prove that every power of Ψ is nonzero on $\Sigma_+^\infty(\mathbb{CP}^\infty)^p$ and remains nonzero on a sufficiently large finite stage $\Sigma_+^\infty(\mathbb{CP}^d)^p$. Induction then gives Theorem C for all nontrivial finite groups.

To construct the four-cell ghosts, we start with power maps on a circle with a free C_p -action and then smash with a Moore spectrum. The case $p = 2$ requires a modification. Longer ghosts also yield nonfullness, inequivalent spectra with isomorphic homotopy modules, and unbounded homological dimensions.

Organization. Sections 2–4 prove the four-cell Theorem 2.1, which implies Theorem A(1). Section 2 constructs the C_p -examples from circle power maps and explains the intuition behind the construction. Section 3 proves that the maps vanish after every representation twist and passage to fixed points, and detects their nonvanishing using chains with C_p -action. Section 4 passes to arbitrary finite groups by using the induction functor.

Section 5 proves Proposition 1.8 and Theorem C. It constructs the natural cyclic ghost, detects its powers on $(\mathbb{CP}^\infty)^p$, and passes to finite complexes and arbitrary finite groups. Section 6 uses double ghosts to prove Theorem A(2) and distinguishes spectra with isomorphic homotopy modules to prove Theorem B. Section 7 derives the homological dimension bounds of Theorem D from arbitrarily long ghost powers.

Acknowledgments. The second author is partially supported by NSF Grant No. DMS-2404828 and NSF CAREER Grant No. DMS-2541934. The third author is partially supported by the AMS Centennial Research Fellowship and NSF Grant DMS-2506247. The fourth author gratefully acknowledges that this research is supported in part by the Pacific Institute for the Mathematical Sciences.

2. A FOUR-CELL GHOST FOR C_p

We establish the following result in Sections 2–4. In this section, we construct the example for C_p . Section 4 extends the construction to all nontrivial finite groups by induction.

Theorem 2.1 (Four-cell ghosts). *There are a finite G -spectrum X with four free G -orbit cells and an endomorphism $f : X \rightarrow X$ of exact additive order p such that, for every $H \leq G$ and every $W \in \text{RO}(H)$, the non-equivariant spectrum map*

$$(S^{-W} \wedge \text{Res}_H^G f)^H : (S^{-W} \wedge \text{Res}_H^G X)^H \longrightarrow (S^{-W} \wedge \text{Res}_H^G X)^H \quad (2.1)$$

is null, while its image $f : X \rightarrow X$ in $\text{Fun}(BG, \text{Sp})$ remains nonzero.

Here $\text{Fun}(BG, \text{Sp})$ is the category of Borel G -spectra, consisting of ordinary spectra with homotopy coherent G -action. Equation (2.1) asserts nullity of the entire map, a stronger condition than vanishing on homotopy groups. The examples can be chosen with X already p -local and $f^2 = 0$.

2.1. A four-cell counterexample. We give the construction of the four-cell counterexample in this section.

For $r > 1$, let

$$S/r = \text{cofib}(r : S \longrightarrow S)$$

denote the mod- r Moore spectrum. Let $C_p = \langle g \rangle$ denote the cyclic group of order p for some prime p , and let λ be the real representation underlying the complex character $g \mapsto e^{2\pi i/p}$. Let

$$a_\lambda : S^0 \longrightarrow S^\lambda$$

denote the Euler class of λ . There is a cofiber sequence

$$S^0 \xrightarrow{a_\lambda} S^\lambda \xrightarrow{i} \text{Cof}(a_\lambda) \xrightarrow{\delta} S^1, \quad (2.2)$$

where $\text{Cof}(a_\lambda)$ can be identified with $\Sigma S(\lambda)_+$.

Recall the Burnside ring of C_p

$$A(C_p) = \mathbb{Z}\{1, [C_p/e]\}, \quad (2.3)$$

where

$$[C_p/e]^2 = p[C_p/e], \quad a_\lambda \cdot [C_p/e] = 0.$$

The marks homomorphism gives an injection

$$A(C_p) \hookrightarrow \mathbb{Z}^2, \quad x + y[C_p/e] \mapsto (x + py, x). \quad (2.4)$$

Moreover let

$$N := 1 + g + \cdots + g^{p-1} \in \mathbb{Z}[C_p]$$

be the norm in the group ring. It satisfies

$$(1 - g)N = 0, \quad N^2 = p \cdot N.$$

Consider the map

$$\begin{aligned} q_{p+1} : S(\lambda) &\longrightarrow S(\lambda) \\ z &\mapsto z^{p+1}. \end{aligned}$$

In particular, it is C_p -equivariant at the space level. Abusing the notation, we let q_{p+1} also denote its stabilization on $\text{Cof}(a_\lambda)$.

Lemma 2.2. *There is a commutative diagram of cofiber sequences:*

$$\begin{array}{ccccccc} S^0 & \xrightarrow{a_\lambda} & S^\lambda & \xrightarrow{i} & \text{Cof}(a_\lambda) & \xrightarrow{\delta} & S^1 \\ \downarrow 0 & & \downarrow -[C_p/e] & & \downarrow 1 - q_{p+1} & & \downarrow 0 \\ S^0 & \xrightarrow{a_\lambda} & S^\lambda & \xrightarrow{i} & \text{Cof}(a_\lambda) & \xrightarrow{\delta} & S^1. \end{array}$$

In particular,

$$\begin{aligned} (1 - q_{p+1})i &= -i[C_p/e], \\ \delta(1 - q_{p+1}) &= 0. \end{aligned} \quad (2.5)$$

Proof. Consider the standard cofiber sequence

$$S(\lambda)_+ \longrightarrow S^0 \xrightarrow{a_\lambda} S^\lambda.$$

The map q_{p+1} on $S(\lambda)$ and the identity on S^0 induce a map on its mapping cone S^λ . This map is the one-point compactification of the radial extension of q_{p+1} , which we denote by

$$\begin{aligned} F_{p+1} : \lambda &\longrightarrow \lambda, \\ F_{p+1}(re^{i\theta}) &= re^{i(p+1)\theta}, \\ F_{p+1}(0) &= 0. \end{aligned}$$

It is proper and C_p -equivariant. Taking one-point compactification, $\widehat{F}_{p+1}$ fixes 0 and ∞ , so

$$\widehat{F}_{p+1}a_\lambda = a_\lambda.$$

Using the identification $[S^\lambda, S^\lambda]^{C_p} \cong A(C_p)$, we compute its stable class from its degrees on underlying and C_p -fixed points. The underlying map has degree $p+1$, since q_{p+1} winds the circle $p+1$ times. On C_p -fixed points, $S^{\lambda^{C_p}} = S^0$, and $\widehat{F}_{p+1}$ is the identity. These degrees are $p+1$ and 1, respectively. Since the map $A(C_p) \hookrightarrow \mathbb{Z}^2$ above is injective, we obtain

$$[\widehat{F}_{p+1}] = 1 + [C_p/e].$$

Rotating the cofiber sequence gives a map of (2.2) with components 1, $\widehat{F}_{p+1}$, q_{p+1} , and 1, where the third component is $\Sigma(q_{p+1})_+$ under $\text{Cof}(a_\lambda) \simeq \Sigma S(\lambda)_+$. Consequently, the middle and right squares give, respectively,

$$\begin{aligned} q_{p+1}i &= i(1 + [C_p/e]), \\ \delta q_{p+1} &= \delta. \end{aligned}$$

Subtracting this map of cofiber sequences from the identity proves the two asserted relations:

$$\begin{aligned} (1 - q_{p+1})i &= i - i(1 + [C_p/e]) = -i[C_p/e], \\ \delta(1 - q_{p+1}) &= \delta - \delta = 0. \end{aligned}$$

□

We are now ready to introduce our four-cell model for the counterexample.

Definition 2.3. For p odd, define

$$X_p := \text{Cof}(a_\lambda) \wedge S/p,$$

$$f_p = 1 - q_{p+1} : X_p \rightarrow X_p.$$

For $p = 2$, let σ denote the C_2 -sign representation, and define

$$X_2 := \text{Cof}(a_{2\sigma}) \wedge S/4,$$

$$f_2 = 2 \cdot (1 - q_3) : X_2 \rightarrow X_2.$$

Figure 1 shows cell diagrams of X_p and X_2 . Circles denote cells, with their dimensions labeled inside. Each oval containing two cells denotes a Moore spectrum. Lines without arrows denote attaching maps, labeled alongside, and horizontal arrows denote morphisms.

We also note the following.

Lemma 2.4. *There is a cofiber sequence of free cells:*

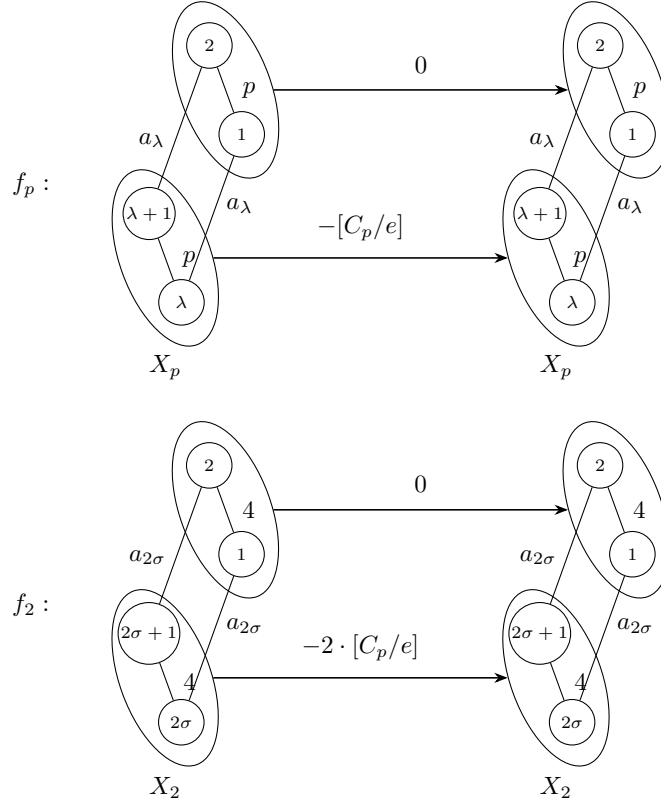
$$\Sigma(C_p)_+ \xrightarrow{1-g} \Sigma(C_p)_+ \xrightarrow{j} \text{Cof}(a_\lambda) \xrightarrow{\partial} \Sigma^2(C_p)_+. \quad (2.6)$$

Here g acts by right translation. Moreover, there is a commutative diagram of cofiber sequences:

$$\begin{array}{ccccccc} \Sigma(C_p)_+ & \xrightarrow{1-g} & \Sigma(C_p)_+ & \xrightarrow{j} & \text{Cof}(a_\lambda) & \xrightarrow{\partial} & \Sigma^2(C_p)_+ \\ -\Sigma N \downarrow & & 0 \downarrow & & \downarrow 1-q_{p+1} & & \downarrow -\Sigma^2 N \\ \Sigma(C_p)_+ & \xrightarrow{1-g} & \Sigma(C_p)_+ & \xrightarrow{j} & \text{Cof}(a_\lambda) & \xrightarrow{\partial} & \Sigma^2(C_p)_+. \end{array}$$

In particular,

$$(1 - q_{p+1})j = 0, \quad \partial(1 - q_{p+1}) = -(\Sigma^2 N)\partial. \quad (2.7)$$

FIGURE 1. A cell diagram for X_p and X_2 .

Proof. For the free-cell structure and attaching map $1 - g$, see [BHZ26, Section 2.4]. The map q_{p+1} fixes the vertices, so $q_{p+1}j = j$. It sends the chosen oriented edge to itself followed by one full turn, giving $1 + N$ on the relative one-cells. The induced map of cofiber sequences therefore has components $1 + \Sigma N$, 1 , q_{p+1} , and $1 + \Sigma^2 N$. In particular, $\partial q_{p+1} = (1 + \Sigma^2 N)\partial$. Subtracting from the identity gives the diagram and both relations. $\square$

Since $\Phi^e(\text{Cof}(a_\lambda))$ splits as $\Sigma S \vee \Sigma^2 S$, we have

$$\Phi^e(q_{p+1}) = \begin{pmatrix} 1 & 0 \\ 0 & p+1 \end{pmatrix}. \quad (2.8)$$

2.2. Proof of Theorem 2.1 for C_p . The proof relies on two propositions, whose proofs will be given in Subsection 3.1 and Subsection 3.2, respectively.

The first proposition claims that f_p induces trivial maps on the H -fixed points of the mapping spectrum $\text{Map}(S^V, \text{Res}_H^{C_p} X_p)$, and in particular induces trivial maps on π_V^H , for every $H \leq C_p$ and $V \in \text{RO}(H)$.

Proposition 2.5. *The power maps on the spectra in Definition 2.3 satisfy the following statements.*

(1) *For an odd prime p ,*

$$(S^{-V} \wedge \text{Res}_H^{C_p} f_p)^H = 0 \quad (H \leq C_p, V \in \text{RO}(H)).$$

(2) *At $p = 2$,*

$$(S^{-V} \wedge \text{Res}_H^{C_2} f_2)^H = 0 \quad (H \leq C_2, V \in \text{RO}(H)).$$

The second proposition detects f_p on ordinary homology spectra while retaining the homotopy coherent C_p -action. The resulting map of chains is nonzero in the derived category of modules over the group ring, so f_p itself is nonzero.

Proposition 2.6.

(1) Let p be odd and let

$$C = (0 \longrightarrow \mathbb{F}_p[C_p] \xrightarrow{1-g} \mathbb{F}_p[C_p] \longrightarrow 0)$$

be concentrated in degrees one and zero. Define $\tau : C \longrightarrow C$ by

$$\tau_1 = -(1 + g + \cdots + g^{p-1}), \quad \tau_0 = 0.$$

Then, $H\mathbb{F}_p \wedge \Phi^e(X_p)$ with a residual C_p -action is represented by $\Sigma C \oplus \Sigma^2 C$, and $H\mathbb{F}_p \wedge \Phi^e(f_p)$ is represented by $\Sigma \tau \oplus \Sigma^2 \tau$. This map is nonzero in $D(\mathbb{F}_p[C_p])$.

(2) Let $p = 2$ and let

$$C = (0 \longrightarrow (\mathbb{Z}/4)[C_2] \xrightarrow{1-g} (\mathbb{Z}/4)[C_2] \longrightarrow 0)$$

be concentrated in degrees one and zero. Define $\tau : C \longrightarrow C$ by

$$\tau_1 = -2 \cdot (1 + g), \quad \tau_0 = 0.$$

Then, $H\mathbb{Z}/4 \wedge \Phi^e(X_2)$ with a residual C_2 -action is represented by $\Sigma C \oplus \Sigma^2 C$, and $H\mathbb{Z}/4 \wedge \Phi^e(f_2)$ is represented by $\Sigma \tau \oplus \Sigma^2 \tau$. This map is nonzero in $D((\mathbb{Z}/4)[C_2])$.

In both cases, f_p remains nonzero in the Borel category, although the induced map on ordinary homology groups is zero.

Proof of Theorem 2.1 for C_p . By Lemma 2.4, $\text{Cof}(a_\lambda)$ has two free C_p -orbit cells. Smashing with the two-cell Moore spectrum S/p for odd p , or $S/4$ for $p = 2$, gives four free orbit cells in X_p . Here $\lambda = 2\sigma$ when $p = 2$. By Proposition 2.5,

$$(S^{-V} \wedge \text{Res}_H^{C_p} f_p)^H = 0 \quad (H \leq C_p, V \in \text{RO}(H)).$$

Taking π_0 gives $\pi_V^H(f_p) = 0$ for every H and V . Proposition 2.6 shows that f_p is nonzero, even in the Borel category. The maps are therefore counterexamples to all three generating hypotheses.

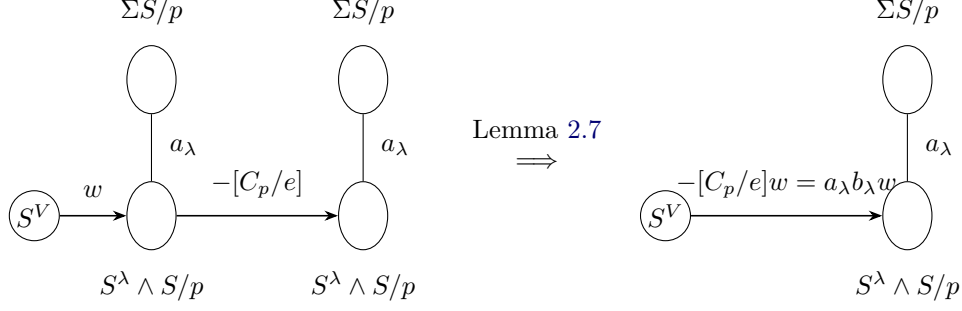
The spectra are finite and already p -local, so they also give the p -local counterexamples. For odd p , Lemma 3.1, smashed with $\text{Cof}(a_\lambda)$, gives $p \cdot \text{id}_{X_p} = 0$, so the additive order of f_p is exactly p . At two, the same lemma gives $4 \cdot \text{id}_{X_2} = 0$. Together with Definition 2.3, this gives $2 \cdot f_2 = 4 \cdot (1 - q_3) = 0$, so the additive order of f_2 is exactly two. □

2.3. A cell diagram interpretation. We provide the intuition for Propositions 2.5 and 2.6 in this section. The displayed cofiber sequence and Figures 2–3 use the odd-prime notation. At $p = 2$, replace S/p by $S/4$, λ by 2σ , b_λ by $\eta_{C_2}^2$, and $-[C_p/e]$ by $-2[C_2/e]$.

Consider the cofiber sequence (2.2) smashed with S/p :

$$S/p \xrightarrow{a_\lambda} S^\lambda \wedge S/p \xrightarrow{i} X_p \xrightarrow{\delta} \Sigma S/p.$$

For a map $\ell \in \pi_V^{C_p} X_p$, either its image $\delta \ell$ in the top Moore spectrum $\Sigma S/p$ is nonzero, or exactness makes ℓ factor through the bottom Moore spectrum $S^\lambda \wedge S/p$. First suppose that ℓ maps into the bottom part, that is, $\delta \ell = 0$.

FIGURE 2. Maps factor through the bottom Moore spectrum (odd p).**Lemma 2.7.**

- (1) Let p be odd. There is a unique $b_\lambda \in \pi_\lambda^{C_p} S_{(p)}$ with

$$a_\lambda b_\lambda = p - [C_p/e].$$

After smashing with S/p , this gives $[C_p/e] = -a_\lambda b_\lambda$ as a map on $S^\lambda \wedge S/p$.

- (2) Let $p = 2$. Then

$$a_{2\sigma} \eta_{C_2}^2 = 4 - 2 \cdot [C_p/e].$$

After smashing with $S/4$, this gives $2 \cdot [C_p/e] = -a_{2\sigma} \eta_{C_2}^2$ as a map on $S^{2\sigma} \wedge S/4$. Here $\eta_{C_2} \in \pi_\sigma^{C_2} S$ has the convention $a_\sigma \eta_{C_2} = 2 - [C_p/e]$.

Proof. For odd p , apply $[-, S_{(p)}]^{C_p}$ to (2.2). By Lemma 2.4 and the fact that $\pi_1 S_{(p)} = \pi_2 S_{(p)} = 0$, we have the exact sequence

$$0 \longrightarrow \pi_\lambda^{C_p} S_{(p)} \xrightarrow{a_\lambda} A(C_p)_{(p)} \xrightarrow{\text{rank}} \mathbb{Z}_{(p)}.$$

Since the kernel of the rank map is $\mathbb{Z}_{(p)}(p - [C_p/e])$, there exists b_λ satisfying $a_\lambda b_\lambda = p - [C_p/e]$. Injectivity of a_λ gives uniqueness. Smashing this identity with S/p and using $p \cdot \text{id}_{S/p} = 0$ from Lemma 3.1 gives $[C_p/e] = -a_\lambda b_\lambda$.

At $p = 2$, the relation $a_\sigma^2 \eta_{C_2} = 2 \cdot a_\sigma$ from [BXZ24, Section 2.2 and Lemma 4.2], together with $a_\sigma \eta_{C_2} = 2 - [C_p/e]$ (see also [BC25, Section 8] for this normalization in the 2-local setting), gives

$$a_{2\sigma} \eta_{C_2}^2 = 2 \cdot a_\sigma \eta_{C_2} = 4 - 2 \cdot [C_p/e].$$

Smashing with $S/4$ and using $4 \cdot \text{id}_{S/4} = 0$ from Lemma 3.1 gives the claimed factorization of $2 \cdot [C_p/e]$. $\square$

As a result, if $\ell: S^V \rightarrow X_p$ has $\delta\ell = 0$, choose w with $\ell = iw$. For odd p , Lemma 2.7(1) gives

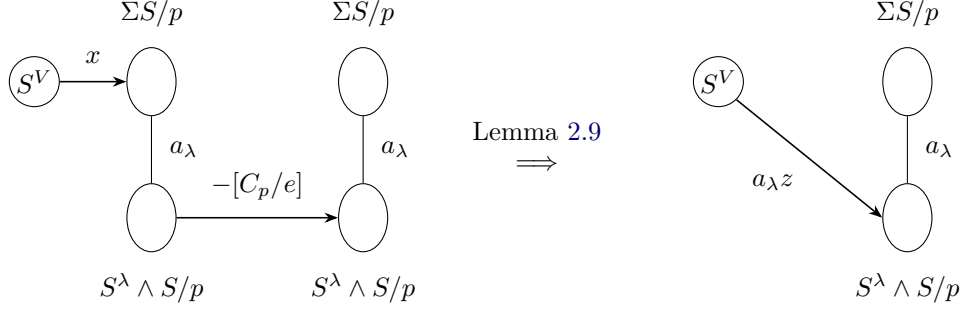
$$f_p \ell = -i[C_p/e]w = ia_\lambda b_\lambda w = 0.$$

For $p = 2$, part (2) gives

$$f_2 \ell = -i(2 \cdot [C_p/e])w = ia_{2\sigma} \eta_{C_2}^2 w = 0.$$

The induced map is zero, as illustrated in Figure 2.

Next, suppose that ℓ has nonzero image in the top Moore spectrum $\Sigma S/p$.

FIGURE 3. Case of a nonzero image in the top Moore spectrum (odd p).

Lemma 2.8. *There is an extension*

$$\bar{t} : \text{Cof}(a_\lambda) \longrightarrow S^\lambda, \quad \bar{t}i = [C_p/e], \quad 1 - q_{p+1} = -i\bar{t}.$$

Proof. By Lemma 2.2, $\delta(1 - q_{p+1}) = 0$, so exactness gives $\bar{t} : \text{Cof}(a_\lambda) \rightarrow S^\lambda$ with $-i\bar{t} = 1 - q_{p+1}$. The middle square gives $i(\bar{t}i - [C_p/e]) = 0$, so exactness shows that $\bar{t}i - [C_p/e]$ factors through a_λ and has underlying degree zero. Also $(\bar{t}i - [C_p/e])a_\lambda = 0$. Since $\Phi^{C_p}(a_\lambda) = 1$, its C_p -fixed-point degree is zero. Injectivity of the mark homomorphism therefore gives $\bar{t}i = [C_p/e]$. $\square$

Smash the extension in Lemma 2.8 with S/p at odd primes and with $S/4$ at two. We fix the resulting maps

$$\bar{t} : X_p \longrightarrow S^\lambda \wedge S/p, \quad \bar{t}_2 : X_2 \longrightarrow S^{2\sigma} \wedge S/4,$$

which satisfy

$$\bar{t}i = [C_p/e], \quad f_p = -i\bar{t}, \quad \bar{t}_2i = [C_p/e], \quad f_2 = -i(2 \cdot \bar{t}_2).$$

For odd p and $x \in \pi_{V-1}^{C_p}(S/p)$ with $a_\lambda x = 0$, define the restricted Toda bracket [CF17, Definition 3.7]:

$$\langle [C_p/e], a_\lambda, x \rangle_{\bar{t}} := \left\{ \bar{t} \circ \tilde{x} \left| \begin{array}{l} \tilde{x} : S^V \longrightarrow X_p, \\ \delta \circ \tilde{x} = \Sigma x \end{array} \right. \right\}.$$

Here the extension $\bar{t}$ is fixed, while the lift $\tilde{x}$ varies. At $p = 2$, for $x \in \pi_{V-1}^{C_2}(S/4)$ with $a_{2\sigma}x = 0$, we similarly define

$$\langle 2 \cdot [C_p/e], a_{2\sigma}, x \rangle_{2 \cdot \bar{t}_2} := \left\{ (2 \cdot \bar{t}_2) \circ \tilde{x} \left| \begin{array}{l} \tilde{x} : S^V \longrightarrow X_2, \\ \delta \circ \tilde{x} = \Sigma x \end{array} \right. \right\}.$$

The following divisibility statement is a consequence of Proposition 2.5, whose proof appears in Section 3.

Lemma 2.9.

(1) *Let p be odd. For every $V \in \text{RO}(C_p)$ and $x \in \pi_{V-1}^{C_p}(S/p)$ with $a_\lambda x = 0$,*

$$\langle [C_p/e], a_\lambda, x \rangle_{\bar{t}} \subseteq a_\lambda \cdot \pi_V^{C_p}(S/p).$$

(2) *Let $p = 2$. For every $V \in \text{RO}(C_2)$ and $x \in \pi_{V-1}^{C_2}(S/4)$ with $a_{2\sigma}x = 0$,*

$$\langle 2 \cdot [C_p/e], a_{2\sigma}, x \rangle_{2 \cdot \bar{t}_2} \subseteq a_{2\sigma} \cdot \pi_V^{C_2}(S/4).$$

Proof. For any lift $\tilde{x} : S^V \rightarrow X_p$, Proposition 2.5 and the chosen extensions give

$$i\tilde{x} = -f_p\tilde{x} = 0 \quad (p \text{ odd}), \quad i(2 \cdot \bar{t}_2)\tilde{x} = -f_2\tilde{x} = 0 \quad (p = 2).$$

For odd p , smash the Euler cofiber sequence (2.2) with S/p and apply $[S^V, -]^{C_p}$. The resulting exact sequence contains

$$\pi_V^{C_p}(S/p) \xrightarrow{a_\lambda} [S^V, S^\lambda \wedge S/p]^{C_p} \xrightarrow{i_*} [S^V, X_p]^{C_p}.$$

Since $i\bar{t}\tilde{x} = 0$, the class $\bar{t}\tilde{x}$ lies in $\ker(i_*) = \text{im}(a_\lambda \cdot)$. We can write $\bar{t}\tilde{x} = a_\lambda z$ for some $z \in \pi_V^{C_p}(S/p)$. At $p = 2$, the same argument using $S/4$ and $a_{2\sigma}$ gives $2 \cdot \bar{t}_2\tilde{x} = a_{2\sigma}z$ for some $z \in \pi_V^{C_2}(S/4)$. $\square$

Now suppose $\ell : S^V \rightarrow X_p$ with $x = \Sigma^{-1}\delta\ell \neq 0$. For odd p , Lemma 2.9(1) gives $\bar{t}\ell = a_\lambda z$ for some $z \in \pi_V^{C_p}(S/p)$, so

$$f_p\ell = -i\bar{t}\ell = -ia_\lambda z = 0.$$

For $p = 2$, part (2) gives $2 \cdot \bar{t}_2\ell = a_{2\sigma}z$ for some $z \in \pi_V^{C_2}(S/4)$, and

$$f_2\ell = -i(2 \cdot \bar{t}_2\ell) = -ia_{2\sigma}z = 0.$$

These two cases express the vanishing of f_p on $\pi_V^{C_p}$ in terms of Euler divisibility. Figure 3 records the second case.

The map f_p itself is nontrivial. For odd p , smashing with $H\mathbb{F}_p$ in the C_p -Borel category splits the Moore factor as $H\mathbb{F}_p \wedge S/p \simeq H\mathbb{F}_p \vee \Sigma H\mathbb{F}_p$. The resulting map on equivariant chains is nonzero in $D(\mathbb{F}_p[C_p])$, even though its map on ordinary homology is zero. For $p = 2$, the analogous calculation uses $H\mathbb{Z}/4$.

3. REPRESENTATION-TWISTED FIXED POINTS AND NONVANISHING

3.1. Representation-twisted fixed points. We prove Proposition 2.5 in this subsection. We start by recalling the following classical results.

Lemma 3.1.

(1) For p an odd prime, we have $\pi_1(S/p) = \pi_2(S/p) = 0$, $[\Sigma S/p, S/p] = 0$, and

$$p \cdot \text{id}_{S/p} = 0.$$

(2) For $p = 2$, we have

$$4 \cdot \text{id}_{S/4} = 0.$$

Proof. In the exact sequence

$$\pi_2 S \xrightarrow{p} \pi_2 S \rightarrow \pi_2(S/p) \rightarrow \pi_1 S \xrightarrow{p} \pi_1 S \rightarrow \pi_1(S/p) \rightarrow \pi_0 S \xrightarrow{p} \pi_0 S,$$

since $\pi_1 S = \pi_2 S = \mathbb{Z}/2$ and p is odd, multiplication by p is an isomorphism on π_1 and π_2 . Since multiplication by p is injective, we have $\pi_2(S/p) = \pi_1(S/p) = 0$. The long exact sequence

$$\pi_2(S/p) \rightarrow \pi_2(S/p) \rightarrow [\Sigma S/p, S/p] \rightarrow \pi_1(S/p) \rightarrow \pi_1(S/p)$$

then implies $[\Sigma S/p, S/p] = 0$. For the order of the identity of Moore spectra, see [Oka84, p. 257] for example. $\square$

For $C_p = \langle g \rangle$ and $V \in \text{RO}(C_p)$, we first compute the induced map of g after twisting by a representation sphere.

Lemma 3.2. For $V \in \text{RO}(C_p)$, there is an equivalence

$$(S^{-V} \wedge (C_p)_+)^{C_p} \simeq S^{-|V|}. \quad (3.1)$$

Under this equivalence, right translation by g on $(C_p)_+$ corresponds to a nonequivariant map of spheres with the following degree:

- (1) For odd p , the degree is 1.
- (2) For $p = 2$, let $V = a + b\sigma$. Then the degree is $(-1)^b$.

Proof. Since $S^{-V} \wedge (C_p)_+$ is a free C_p -spectrum, (3.1) follows from the Adams isomorphism:

$$(S^{-V} \wedge (C_p)_+)^{C_p} \simeq (S^{-V} \wedge (C_p)_+)/C_p \simeq S^{-|V|}.$$

Since $g^p = e$, the degree d of the action of g satisfies $d^p = 1$. In particular, d must be ± 1 . If p is odd, this forces $d = 1$. For $p = 2$ and $V = a + b\sigma$, the action on each trivial summand has degree 1, whereas the action on σ is the antipodal action and has degree -1 . Its degree on S^{-V} is therefore $(-1)^b$. $\square$

For every C_p -spectrum Z , there is a natural equivalence

$$(S/r \wedge Z)^{C_p} \simeq S/r \wedge Z^{C_p}, \quad (3.2)$$

since smash-product and taking C_p -fixed points both preserve cofiber sequences.

Proof of Proposition 2.5. For $H = e$, by (2.8), the underlying map $1 - q_{p+1}$ on $\text{Cof}(a_\lambda) \simeq \Sigma S \vee \Sigma^2 S$ can be identified with the matrix

$$(1 - q_{p+1})^e = \begin{pmatrix} 0 & 0 \\ 0 & -p \end{pmatrix} : \Sigma S \vee \Sigma^2 S \rightarrow \Sigma S \vee \Sigma^2 S.$$

For odd p , smashing with S/p therefore identifies $(f_p)^e$ with

$$\begin{pmatrix} 0 & 0 \\ 0 & -p \cdot \text{id}_{\Sigma^2 S/p} \end{pmatrix}.$$

By Lemma 3.1(1), $p \cdot \text{id}_{S/p} = 0$, so f_p is null nonequivariantly. For $p = 2$, the map is

$$(f_2)^e = (2(1 - q_3))^e = \begin{pmatrix} 0 & 0 \\ 0 & -4 \cdot \text{id}_{\Sigma^2 S/4} \end{pmatrix}$$

which is zero by Lemma 3.1(2). Therefore, smashing with $S^{-|V|}$ is null as well.

For $H = C_p$, by (3.2), we may consider the map on fixed points before smashing with the Moore spectrum.

- (1) Suppose p is odd. Applying Lemma 3.2(1) to the cofiber sequence in Lemma 2.4, we have $(S^{-V} \wedge \text{Cof}(a_\lambda))^{C_p} \simeq S^{1-|V|} \vee S^{2-|V|}$. Since each right translation g^i has degree one, the norm N acts by p . Lemma 2.4 now gives

$$(S^{-V} \wedge (1 - q_{p+1}))^{C_p} = \begin{pmatrix} 0 & \beta \\ 0 & -p \end{pmatrix}, \quad (3.3)$$

for some $\beta \in \pi_1 S = \mathbb{Z}/2$. After smashing with S/p , multiplication by $-p$ is zero by Lemma 3.1. As $[\Sigma S/p, S/p] = 0$ in Lemma 3.1, $\beta \wedge \text{id}_{S/p}$ is null as well.

- (2) Let $p = 2$ and $V = a + b\sigma$. If b is even, Lemma 3.2(2) and Lemma 2.4 give the same matrix form as (3.3). Then $2\beta = 0$ plus $4 \cdot \text{id}_{S/4} = 0$ implies that $(S^{-V} \wedge f_2)^{C_2}$ is null.

If b is odd, Lemma 3.2(2) identifies

$$(S^{-V} \wedge \text{Cof}(a_{2\sigma}))^{C_2} \simeq \Sigma^{1-|V|} S/2.$$

By Lemma 2.4, $1 - q_3$ vanishes on the bottom cell, so it factors as

$$\Sigma^{1-|V|} S/2 \xrightarrow{\partial} S^{2-|V|} \xrightarrow{h} \Sigma^{1-|V|} S/2.$$

Since $h \in \pi_1(S/2) = \mathbb{Z}/2$, $2h = 0$. Therefore $2(1 - q_3)$ is null, even before smashing with $S/4$. $\square$

3.2. Nonvanishing and chains with C_p -action. In this subsection, we prove Proposition 2.6.

Proof of Proposition 2.6. In each case, C is a bounded complex of free modules, so τ is zero in the derived category exactly when it is chain null-homotopic.

- (1) Suppose p is odd, and let C and τ be as in Proposition 2.6(1). We use $\mathbb{F}_p$ -chains with C_p -action to represent $H\mathbb{F}_p \wedge X_p$ in $D(\mathbb{F}_p[C_p])$ [SS03, Theorem 5.1.6]. The free-cell sequence in Lemma 2.4 identifies the chains of $\text{Cof}(a_\lambda)$ with ΣC . The two-cell complex of S/p has zero differential over $\mathbb{F}_p$ and trivial C_p -action. Tensoring therefore represents $H\mathbb{F}_p \wedge X_p$ by $\Sigma C \oplus \Sigma^2 C$. Since q_{p+1} acts only on the circle factor, Lemma 2.4 represents $H\mathbb{F}_p \wedge f_p$ by $\Sigma\tau \oplus \Sigma^2\tau$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{F}_p[C_p] & \xrightarrow{1-g} & \mathbb{F}_p[C_p] & \longrightarrow & 0 \\ & & \tau_1 \downarrow & \swarrow h & \downarrow \tau_0=0 & & \\ 0 & \longrightarrow & \mathbb{F}_p[C_p] & \xrightarrow{1-g} & \mathbb{F}_p[C_p] & \longrightarrow & 0. \end{array}$$

A chain nullhomotopy of τ would have a single component $h : C_0 \rightarrow C_1$, given by multiplication by an element $h \in \mathbb{F}_p[C_p]$ satisfying

$$(1-g)h = 0, \quad h(1-g) = -(1+g+\cdots+g^{p-1}).$$

The left sides agree by commutativity, but $1+g+\cdots+g^{p-1} \neq 0$. Therefore τ is nonzero in $D(\mathbb{F}_p[C_p])$, and f_p remains nonzero in the Borel category.

- (2) Let $p = 2$, and let C and τ be as in Proposition 2.6(2). We use $\mathbb{Z}/4$ -chains with C_2 -action to represent $H\mathbb{Z}/4 \wedge X_2$ in $D(\mathbb{Z}/4[C_2])$. The chains of $\text{Cof}(a_{2\sigma})$ are ΣC . The two-cell complex of $S/4$ has zero differential over $\mathbb{Z}/4$ and trivial C_2 -action, so tensoring represents $H\mathbb{Z}/4 \wedge X_2$ by $\Sigma C \oplus \Sigma^2 C$. The map $2(1-q_3)$ acts only on the circle factor, so Lemma 2.4 represents $H\mathbb{Z}/4 \wedge f_2$ by $\Sigma\tau \oplus \Sigma^2\tau$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/4[C_2] & \xrightarrow{1-g} & \mathbb{Z}/4[C_2] & \longrightarrow & 0 \\ & & \tau_1 \downarrow & \swarrow h & \downarrow \tau_0=0 & & \\ 0 & \longrightarrow & \mathbb{Z}/4[C_2] & \xrightarrow{1-g} & \mathbb{Z}/4[C_2] & \longrightarrow & 0. \end{array}$$

A chain nullhomotopy would require an element $h \in \mathbb{Z}/4[C_2]$ satisfying

$$(1-g)h = 0, \quad h(1-g) = -2(1+g).$$

The left sides agree, but $-2(1+g) \neq 0$ in $\mathbb{Z}/4[C_2]$. Therefore τ is nonzero in $D(\mathbb{Z}/4[C_2])$, and f_2 remains nonzero in the Borel category.

In both cases, the underlying map is zero by Proposition 2.5 with $H = e$, so the induced map on ordinary homology is zero. $\square$

4. A FOUR-CELL GHOST FOR ALL FINITE GROUPS

In this section, we finish the proof of Theorem 2.1 for all nontrivial finite groups.

Lemma 4.1. *Let G be a finite group, let $H \leq G$, and let $f : X \rightarrow Y$ be a map of H -spectra. The induction functor $\text{Ind}_H^G(-)$ has the following properties.*

- (1) If

$$(S^{-V} \wedge \text{Res}_K^H f)^K = 0 \quad (K \leq H, V \in \text{RO}(K)),$$

then $\text{Ind}_H^G f$ has the same vanishing property at every subgroup of G .

- (2) Induction preserves ghosts for each of the three detector families in Conjecture 1.2.
 (3) If $f \neq 0$, then $\text{Ind}_H^G f \neq 0$.

(4) If $X = Y$, then for every $n \geq 1$,

$$(\mathrm{Ind}_H^G f)^n = 0 \iff f^n = 0.$$

(5) The maps f and $\mathrm{Ind}_H^G f$ have the same additive order.

Proof. Let $L \leq G$ and $W \in \mathrm{RO}(L)$. For each representative g of $L \backslash G/H$, let $K_g = H \cap g^{-1}Lg$, and let W_g denote the restriction of W along $K_g \rightarrow L$, $k \mapsto gkg^{-1}$. The double-coset decomposition, the projection formula, and the identification of induction with coinduction give an equivalence

$$(S^{-W} \wedge \mathrm{Res}_L^G \mathrm{Ind}_H^G X)^L \simeq \bigvee_{[g] \in L \backslash G/H} (S^{-W_g} \wedge \mathrm{Res}_{K_g}^H X)^{K_g},$$

natural in X . Each summand of the map induced by f is null by hypothesis. This proves (1). Taking π_0 proves (2) for the all-subgroup representation-orbit family. Restricting W to integer degrees proves the integer-graded case. For the $\mathrm{RO}(G)$ -graded case, take $W = \mathrm{Res}_L^G V$ with $V \in \mathrm{RO}(G)$. Each W_g is then isomorphic to the restriction of $\mathrm{Res}_H^G V$, so the $\mathrm{RO}(H)$ -graded ghost condition on f applies.

Restricting back to H , the identity double coset gives X as a direct summand of $\mathrm{Res}_H^G \mathrm{Ind}_H^G X$ and f as the corresponding block of $\mathrm{Res}_H^G \mathrm{Ind}_H^G f$. Therefore induction preserves nonzero maps. Since induction is additive and preserves composition, applying this argument to integer multiples and composition powers proves (3)–(5). The same splitting holds in the Borel category. $\square$

Proof of Theorem 2.1. Let p be a prime divisor of $|G|$, and choose a subgroup $K \leq G$ of order p . Identify K with C_p , and let

$$X = \mathrm{Ind}_K^G X_p, \quad f = \mathrm{Ind}_K^G f_p.$$

By Propositions 2.5 and 2.6, all representation-twisted fixed-point maps of f_p are null at every subgroup of C_p , and f_p remains nonzero in the Borel category. Lemma 4.1 gives the same properties for f . The spectrum X is finite and already p -local, since X_p is. Induction carries each free K -orbit cell to one free G -orbit cell, so X has four free orbit cells. The prime-order case in Subsection 2.2 and Lemma 4.1(5) give f exact additive order p .

Taking π_0 of the representation-twisted fixed-point maps of f gives

$$[S^W, \mathrm{Res}_H^G f]^H = 0 \quad (H \leq G, W \in \mathrm{RO}(H)).$$

Since $f \neq 0$ and X is already p -local, this also proves Theorem A(1), integrally and p -locally. $\square$

Proposition 4.2. *For every nontrivial finite group G and every prime $p \mid |G|$, the map $f : X \rightarrow X$ constructed in the proof of Theorem 2.1 satisfies*

$$f \circ f = 0.$$

Proof. We first prove the assertion for f_p . Let L denote S/p for odd p and $S/4$ for $p = 2$. Smashing (2.6) with L gives a cofiber sequence

$$(C_p)_+ \wedge \Sigma L \xrightarrow{j_p} X_p \xrightarrow{\partial_p} (C_p)_+ \wedge \Sigma^2 L.$$

We have $f_p j_p = 0$ by (2.7), and $\mathrm{Res}_e^{C_p} f_p = 0$ by Proposition 2.5. Let $F_p = (C_p)_+ \wedge \Sigma^2 L$. Exactness gives a factorization $f_p = b_p \partial_p$ for some $b_p : F_p \rightarrow X_p$. By induction–restriction adjunction, $f_p b_p = 0$, since $\mathrm{Res}_e^{C_p} f_p = 0$. Therefore $f_p^2 = 0$.

The assertion for $f = \mathrm{Ind}_K^G f_p$ now follows from Lemma 4.1(4). $\square$

5. ARBITRARILY LONG GHOSTS

In this section, we construct a natural ghost for C_p , whose powers are nontrivial on $\Sigma_+^\infty(BS^1)^p$. We then pass this construction to finite products of complex projective spaces, producing nontrivial arbitrarily long ghosts on finite C_p -spectra. By Lemma 4.1, this construction can be promoted to every nontrivial finite group which proves Theorem C.

Fix a prime p and a generator g of C_p .

5.1. A natural ghost. Recall that $A(C_p) = \mathbb{Z}\{1, [C_p/e]\}$. The construction uses the canonical action of $A(C_p)$ on C_p -spectra. Recall that restriction along $i : e \hookrightarrow C_p$ and geometric fixed points induce ring homomorphisms $i^*, \Phi : A(C_p) \rightarrow \mathbb{Z}$, respectively. They together give an injection

$$(i^*, \Phi) : A(C_p) \hookrightarrow \mathbb{Z}^2, \quad x + y[C_p/e] \mapsto (x + py, x). \quad (5.1)$$

which coincides with the marks homomorphism. Multiplying the cyclic operation $1 - g$ by $p - [C_p/e]$ gives the following natural ghost.

Proposition 5.1. *For every C_p -spectrum X , the natural endomorphism*

$$\Psi_X = (p - [C_p/e])(1 - g_X) : X \rightarrow X$$

satisfies

$$(S^{-V} \wedge \text{Res}_H^{C_p} \Psi_X)^H = 0 \quad (H \leq C_p, V \in RO(H))$$

as a map of classical spectra. In particular, Ψ_X is a ghost.

Proof. For $H = e$, the assertion follows from $i^*(p - [C_p/e]) = 0$. Let $H = C_p$ and $V \in RO(C_p)$. We first show that

$$\Psi_{S^V} = (p - [C_p/e])(1 - g) : S^V \rightarrow S^V$$

is null. Under the identification $[S^V, S^V]^{C_p} \cong A(C_p)$, we regard Ψ_{S^V} as an element of the Burnside ring. By the case $H = e$, it remains to check its image under Φ . For an actual representation, g acts identically on its fixed subspace, and this remains true for virtual representations since geometric fixed points preserve smash products and duals of representation spheres [MM02, V, Corollary 4.6 and Proposition 4.7]. We obtain $\Phi(g) = 1$ and

$$\Phi(\Psi_{S^V}) = p(1 - \Phi(g)) = 0.$$

Injectivity of (5.1) gives $\Psi_{S^V} = 0$.

Let Y be a classical spectrum with trivial C_p -action. The diagonal action gives

$$\Psi_{S^V \wedge Y} = \Psi_{S^V} \wedge \text{id}_Y = 0.$$

For every C_p -equivariant map $a : S^V \wedge Y \rightarrow X$, we have the following commutative diagram by naturality

$$\begin{array}{ccc} S^V \wedge Y & \xrightarrow{a} & X \\ \Psi_{S^V \wedge Y} = 0 \downarrow & & \downarrow \Psi_X \\ S^V \wedge Y & \xrightarrow{a} & X, \end{array}$$

so $\Psi_X \circ a = 0$. On the other hand, in the adjunction [MM02, V, Proposition 3.4],

$$[S^V \wedge Y, X]^{C_p} \cong [Y, (S^{-V} \wedge X)^{C_p}],$$

$\Psi_X \circ a$ is the adjoint of

$$Y \xrightarrow{\tilde{a}} (S^{-V} \wedge X)^{C_p} \xrightarrow{(S^{-V} \wedge \Psi_X)^{C_p}} (S^{-V} \wedge X)^{C_p}, \quad (5.2)$$

for some $\tilde{a}$ adjoint to a . The composite (5.2) is therefore zero for every Y and $\tilde{a}$. Taking $Y = (S^{-V} \wedge X)^{C_p}$ and $\tilde{a} = \text{id}$ proves the assertion. $\square$

Remark 5.2 (Powers). For the endomorphism Ψ_X of Proposition 5.1 on any C_p -spectrum X , one has

$$\Psi_X^n = p^{n-1}(p - [C_p/e])(1 - g)^n \in p^{n-1+\lfloor (n-1)/(p-1) \rfloor} [X, X]^{C_p} \quad (n \geq 1).$$

For $p = 2$, this simplifies to $\Psi_X^n = 4^{n-1}\Psi_X$. The powers tend to zero in the p -adic topology. Theorem 5.4 will give an example for which no positive power is annihilated by a power of p .

The induction argument of Section 4 transports this construction to larger finite groups.

Corollary 5.3. *Let G be a finite group containing C_p . For every C_p -spectrum X , let Ψ_X be the endomorphism of Proposition 5.1. Then the induced endomorphism $\text{Ind}_{C_p}^G \Psi_X = G_+ \wedge_{C_p} \Psi_X$ satisfies*

$$(S^{-V} \wedge \text{Res}_H^G \text{Ind}_{C_p}^G \Psi_X)^H = 0 \quad (H \leq G, V \in RO(H))$$

as a map of classical spectra. In particular, $\text{Ind}_{C_p}^G \Psi_X$ is a ghost.

Proof. Apply Lemma 4.1 to the vanishing in Proposition 5.1. $\square$

5.2. A nonnilpotent ghost. Having constructed a natural ghost on every C_p -spectrum, we now ask whether it can be nonzero. The following theorem shows that on $\Sigma_+^\infty(BS^1)^p$ it is not nilpotent. In fact, no positive power is annihilated by a power of p .

We equip $(S^1)^p$ with the C_p -action

$$g(x_0, \dots, x_{p-1}) = (x_1, \dots, x_{p-1}, x_0),$$

which induces the same coordinate permutation on $(BS^1)^p$.

Theorem 5.4. *For $X = \Sigma_+^\infty(BS^1)^p$, the ghost*

$$\Psi_X = (p - [C_p/e])(1 - g) : X \longrightarrow X$$

satisfies

$$p^m \Psi_X^n \neq 0 \quad \text{in} \quad [\Sigma_+^\infty(BS^1)^p, \Sigma_+^\infty(BS^1)^p]^{C_p} \quad (m \geq 0, n \geq 1).$$

Proof. The May–Snaith–Zelevski splitting [MSZ89, Theorem 2.3] requires a finite source group. We therefore restrict along a C_p -equivariant embedding $\varphi : C_{p^2} \hookrightarrow (S^1)^p$ and prove that $p^m \Psi_X^n \varphi \neq 0$.

Write $C_{p^2} = \langle h \rangle$. Choose a primitive p^2 -th root of unity $\zeta \in S^1$, and define

$$\varphi : C_{p^2} \longrightarrow (S^1)^p, \quad \varphi(h) = (\zeta, \zeta^{1+p}, \dots, \zeta^{1+p(p-1)}).$$

The congruence $(1 + p)^j \equiv 1 + pj \pmod{p^2}$ shows that $h \mapsto h^{1+p}$ has order p . This defines a C_p -action on C_{p^2} by $gh = h^{1+p}$. Since $(1 + pi)(1 + pj) \equiv 1 + p(i + j) \pmod{p^2}$, we have

$$\varphi(g^j h) = \varphi(h^{1+pj}) = (\zeta^{1+pj}, \zeta^{1+p(j+1)}, \dots, \zeta^{1+p(j+p-1)}) = g^j \varphi(h).$$

This proves that φ is C_p -equivariant.

We use a splitting of the mapping spectrum to separate the maps $g^j \varphi$. For $L \leq C_{p^2}$ and $\rho : L \rightarrow (S^1)^p$, define

$$W_\rho = (C_{p^2} \times (S^1)^p) / \text{graph}(\rho).$$

Since $C_{p^2} \times (S^1)^p$ is abelian, the splitting of May–Snaith–Zelevski [MSZ89, Theorem 2.3] takes the form

$$\xi : \bigvee_{\substack{L \leq C_{p^2} \\ \rho : L \rightarrow (S^1)^p}} (\Sigma_+^\infty BW_\rho)_p^\wedge \xrightarrow{\sim} F\left(\Sigma_+^\infty BC_{p^2}, \Sigma_+^\infty (BS^1)^p\right)_p^\wedge. \quad (5.3)$$

Here F is the classical mapping spectrum. The wedge is finite, so p -completion may be taken summandwise.

The C_p -action on the source preserves each subgroup L . The map from the ρ -summand to the $g\rho g^{-1}$ -summand is induced by

$$g : W_\rho \longrightarrow W_{g\rho g^{-1}}, \quad [(u, x)] \longmapsto [(gu, gx)],$$

where $u \in C_{p^2}$ and $x \in (S^1)^p$. The target carries the conjugation action $\psi \mapsto g\psi g^{-1}$. We check that ξ is C_p -equivariant by describing its summand maps. Before completion, the map on the ρ -summand is adjoint to the composite [LMM82, Theorem 8 and Remarks 9]

$$\Sigma_+^\infty B(C_{p^2} \times W_\rho) \longrightarrow \Sigma_+^\infty B(C_{p^2} \times (S^1)^p) \xrightarrow{B \operatorname{pr}_{(S^1)^p}} \Sigma_+^\infty (BS^1)^p. \quad (5.4)$$

To construct the first map in (5.4), consider the homomorphism

$$\kappa_\rho : C_{p^2} \times (S^1)^p \longrightarrow C_{p^2} \times W_\rho, \quad (u, x) \longmapsto (u, [(u, x)]).$$

Since $[(1, x)] = 1$ forces $x = 1$, the map κ_ρ is injective. The surjective homomorphism

$$C_{p^2} \times W_\rho \longrightarrow C_{p^2}/L, \quad (u, [(u', x)]) \longmapsto u'u^{-1}L$$

has kernel $\operatorname{im}(\kappa_\rho)$, so $B\kappa_\rho$ is modeled by a covering of degree $[C_{p^2} : L]$. Then the first arrow in (5.4) is the transfer along this covering.

The identities

$$\kappa_{g\rho g^{-1}}(gu, gx) = (g \times g)\kappa_\rho(u, x), \quad \operatorname{pr}_{(S^1)^p}(gu, gx) = g \operatorname{pr}_{(S^1)^p}(u, x)$$

ensure the compatibility of κ_ρ and $\operatorname{pr}_{(S^1)^p}$ with the C_p -actions. By the functoriality of the bar construction and the transfer, the family (5.4) is coherently C_p -equivariant. Adjunction and completion therefore make (5.3) an equivalence in $\operatorname{Fun}(BC_p, \operatorname{Sp})$.

We now consider the summands with $L = C_{p^2}$ and $\rho = g^j\varphi$, $0 \leq j < p$. The first coordinates of $g^j\varphi(h)$ are the distinct roots ζ^{1+pj} , so the maps $g^j\varphi$ correspond to distinct summands. The C_p -equivariance of φ gives $g(g^j\varphi)g^{-1} = g^j\varphi$, so each summand is C_p -invariant. To compute the action and the adjoint (5.4) of the $g^j\varphi$ -summand, consider the isomorphism

$$W_{g^j\varphi} \xrightarrow{\cong} (S^1)^p, \quad [(u, x)] \longmapsto (g^j\varphi(u))^{-1}x \quad (5.5)$$

whose inverse is $x \mapsto [(1, x)]$. The identity

$$(g^j\varphi(gu))^{-1}gx = g((g^j\varphi(u))^{-1}x)$$

shows that (5.5) is C_p -equivariant for the quotient action on $W_{g^j\varphi}$ and cyclic permutation on $(S^1)^p$. Under this identification, $\kappa_{g^j\varphi}$ is the isomorphism

$$\kappa_{g^j\varphi} : C_{p^2} \times (S^1)^p \xrightarrow{\cong} C_{p^2} \times (S^1)^p, \quad (u, x) \longmapsto (u, (g^j\varphi(u))^{-1}x).$$

The transfer in (5.4) is therefore induced by its inverse, so the adjoint of this summand is induced by

$$C_{p^2} \times (S^1)^p \longrightarrow (S^1)^p, \quad (u, x) \longmapsto g^j\varphi(u)x. \quad (5.6)$$

The projection pr_j onto each of these invariant summands is C_p -equivariant. To extract its bottom-sphere coordinate, define

$$\begin{aligned} \theta_j : F\left(\Sigma_+^\infty BC_{p^2}, \Sigma_+^\infty (BS^1)^p\right)_p^\wedge &\xrightarrow{\xi^{-1}} \bigvee_{L, \rho} (\Sigma_+^\infty BW_\rho)_p^\wedge \\ &\xrightarrow{\operatorname{pr}_j} (\Sigma_+^\infty BW_{g^j\varphi})_p^\wedge \xrightarrow{\text{collapse}} S_p^\wedge, \end{aligned} \quad (5.7)$$

where ξ^{-1} is the inverse in $\operatorname{Fun}(BC_p, \operatorname{Sp})$ and the last arrow is induced by $BW_{g^j\varphi} \rightarrow *$, with trivial C_p -action on $S_p^\wedge$.

For $\alpha \in [\Sigma_+^\infty BC_{p^2}, \Sigma_+^\infty (BS^1)^p]^{C_p}$, the completed adjoint of α is a morphism in $\operatorname{Fun}(BC_p, \operatorname{Sp})$,

$$S_p^\wedge \longrightarrow F\left(\Sigma_+^\infty BC_{p^2}, \Sigma_+^\infty (BS^1)^p\right)_p^\wedge. \quad (5.8)$$

Write $[\alpha]$ for its class in $\pi_0((F(\Sigma_+^\infty BC_{p^2}, \Sigma_+^\infty (BS^1)^p)_p^\wedge)^{hC_p})$. Restricting (5.6) along $u \mapsto (u, 1)$ gives $g^j\varphi$. For $\alpha = g^j\varphi$, composing (5.8) with ξ^{-1} gives the bottom-sphere inclusion into the

$g^j\varphi$ -summand of (5.3), induced by its fixed bar vertex. Its projection to the i -th summand is zero for $i \neq j$. For $i = j$, the remaining composite in (5.7) is induced by $* \rightarrow BW_{g^j\varphi} \rightarrow *$, so it is the identity. These identities hold in $\text{Fun}(BC_p, \text{Sp})$, so the induced homotopy-fixed-point classes satisfy

$$(\theta_i^{hC_p})_*[g^j\varphi] = \delta_{ij} \cdot 1, \quad 0 \leq i, j < p. \quad (5.9)$$

The Segal conjecture for C_p [Car84]

$$A(C_p)_p^\wedge \xrightarrow{\cong} \pi_0((S_p^\wedge)^{hC_p}). \quad (5.10)$$

enables us to track the Burnside factor in Ψ_X . Naturality of the sphere action and adjunction identify scalar multiplications of elements in $A(C_p)$ with precomposition on $S_p^\wedge$, which commutes with postcomposition by θ_j . Under (5.10), the coordinate maps $\alpha \mapsto (\theta_j^{hC_p})_*[\alpha]$ are $A(C_p)$ -linear.

The homomorphism Φ in (5.1) extends p -adically to

$$\Phi : A(C_p)_p^\wedge = \mathbb{Z}_p\{1, [C_p/e]\} \longrightarrow \mathbb{Z}_p, \quad x + y[C_p/e] \longmapsto x.$$

Applying Φ to these coordinates gives the additive homomorphism

$$\begin{aligned} \Xi : [\Sigma_+^\infty BC_{p^2}, \Sigma_+^\infty (BS^1)^p]^{C_p} &\longrightarrow \mathbb{Z}_p[C_p] = \mathbb{Z}_p[g]/(g^p - 1), \\ \alpha &\longmapsto \sum_{j=0}^{p-1} \Phi((\theta_j^{hC_p})_*[\alpha]) g^j. \end{aligned} \quad (5.11)$$

The map Ξ is $A(C_p)$ -linear for the action on its target induced by Φ , and (5.9) gives $\Xi(g^i\varphi) = g^i$ for $0 \leq i < p$. Using $g^p = 1$, $\Phi(p - [C_p/e]) = p$, and Remark 5.2, we obtain

$$\Xi(p^m \Psi_X^n \varphi) = p^{m+n} \Xi((1 - g)^n \varphi) = p^{m+n} (1 - g)^n. \quad (5.12)$$

Since the right-hand side is nonzero, $p^m \Psi_X^n \varphi \neq 0$, so $p^m \Psi_X^n \neq 0$. $\square$

5.3. Finite stages. We now pass from $\Sigma_+^\infty BS^1$ to finite spectra. Using the standard model $BS^1 = \mathbb{C}P^\infty$, the coordinate inclusions give

$$\Sigma_+^\infty (BS^1)^p \simeq \text{hocolim}_{k \geq 0} \Sigma_+^\infty (\mathbb{C}P^k)^p.$$

These inclusions are C_p -equivariant and commute with Ψ . We show that restriction along them detects every nonzero $p^m \Psi^n$ supplied by Theorem 5.4.

Theorem 5.5. *For every $m \geq 0$ and $n \geq 1$, the ghost Ψ on $\Sigma_+^\infty (\mathbb{C}P^k)^p$ satisfies $p^m \Psi^n \neq 0$ for all sufficiently large k .*

Proof. The Milnor exact sequence for the displayed homotopy colimit is

$$\begin{aligned} 0 &\longrightarrow \varprojlim_k^1 [\Sigma \Sigma_+^\infty (\mathbb{C}P^k)^p, \Sigma_+^\infty (BS^1)^p]^{C_p} \\ &\xrightarrow{j} [\Sigma_+^\infty (BS^1)^p, \Sigma_+^\infty (BS^1)^p]^{C_p} \\ &\xrightarrow{i} \varprojlim_k [\Sigma_+^\infty (\mathbb{C}P^k)^p, \Sigma_+^\infty (BS^1)^p]^{C_p} \longrightarrow 0. \end{aligned}$$

To show that i is injective, it suffices to show that the groups $[\Sigma \Sigma_+^\infty (\mathbb{C}P^k)^p, \Sigma_+^\infty (BS^1)^p]^{C_p}$ are finitely generated and torsion for all k . An inverse system of finite groups satisfies the Mittag-Leffler condition, so its $\varprojlim^1$ vanishes.

Using the tom Dieck splitting [MSZ89, Theorem 4.1], $\pi_n^{C_p}(\Sigma_+^\infty (BS^1)^p)$ and $\pi_n^e(\Sigma_+^\infty (BS^1)^p)$ are finite direct sums of some classical stable homotopy groups of spaces of finite type, so these homotopy groups are finitely generated. Since $\Sigma \Sigma_+^\infty (\mathbb{C}P^k)^p$ is a finite C_p -spectrum, induction over its orbit cells shows that the mapping groups are finitely generated as well.

It remains to prove torsion. Since each $\Sigma \Sigma_+^\infty (\mathbb{C}P^k)^p$ is finite, consider tensoring the mapping group with $\mathbb{Q}$. The underlying spaces $(\mathbb{C}P^k)^p$, $(BS^1)^p$ and their fixed-point diagonals $\mathbb{C}P^k$, BS^1 all have rational homology concentrated in even degrees. By the rational equivariant splitting [GM95, Appendix A, Theorem A.4], the extra suspension on $\Sigma_+^\infty (\mathbb{C}P^k)^p$ makes both classical rational mapping groups zero. It follows that the mapping groups $[\Sigma \Sigma_+^\infty (\mathbb{C}P^k)^p, \Sigma_+^\infty (BS^1)^p]^{C_p}$ are torsion.

This proves that i is injective. By Theorem 5.4, for each fixed $m \geq 0$ and $n \geq 1$, there is an integer $k_0 \geq 0$ such that the composite

$$\Sigma_+^\infty (\mathbb{C}P^{k_0})^p \xrightarrow{\text{inc}} \Sigma_+^\infty (BS^1)^p \xrightarrow{p^m \Psi^n} \Sigma_+^\infty (BS^1)^p$$

is nonzero. For every $k \geq k_0$, this composite factors through the self-map $p^m \Psi^n$ of $\Sigma_+^\infty (\mathbb{C}P^k)^p$, since the inclusions commute with Ψ . This self-map is nonzero for every $k \geq k_0$. $\square$

Applying Lemma 4.1, these finite examples can be extended to arbitrary finite groups while preserving their nonzero powers.

Corollary 5.6. *Let G be a finite group containing C_p . On $\text{Ind}_{C_p}^G \Sigma_+^\infty (\mathbb{C}P^k)^p$, the ghost $\text{Ind}_{C_p}^G \Psi$ has zero underlying map. For every fixed $m \geq 0$ and $n \geq 1$, we have*

$$p^m (\text{Ind}_{C_p}^G \Psi)^n \neq 0$$

for all sufficiently large k . Consequently every nontrivial finite group admits ghosts with arbitrarily long nonzero powers on finite spectra.

Proof. Corollary 5.3 gives the ghost and underlying vanishing assertions. Induction preserves finite spectra and detects nonzero maps by Lemma 4.1. Since

$$\text{Ind}_{C_p}^G (p^m \Psi^n) = p^m (\text{Ind}_{C_p}^G \Psi)^n,$$

Theorem 5.5 gives the finite-stage nonvanishing. Every nontrivial finite group contains a subgroup of prime order, which proves the final assertion. $\square$

Corollary 5.6 proves the integral nonvanishing required in Theorem C. We complete that theorem by proving finite p -power additive order and nonvanishing after p -localization.

Proof of Theorem C. Fix $m \geq 0$ and $n \geq 1$. Choose $C_p \leq G$ and a sufficiently large k as in Corollary 5.6 for these m and n . The ghost lemma [Chr98, Theorem 3.5] implies that Ψ is nilpotent in the C_p -equivariant homotopy category. Using (2.3) and $g^p = 1$, we obtain

$$p^p \Psi = \sum_{j=2}^p (-1)^j \binom{p}{j} p^{p-j} \Psi^j.$$

Iteration gives p -power torsion, since the right side is divisible by Ψ^2 . Moreover, induction preserves additive order by Lemma 4.1.

Mapping groups between finite G -spectra localize by tensoring with $\mathbb{Z}_{(p)}$. The nonzero p -primary class $p^m (\text{Ind}_{C_p}^G \Psi)^n$ therefore remains nonzero after p -localization. The ghost condition is also preserved by localization. $\square$

6. NONFULLNESS AND FAILURE OF CLASSIFICATION

In this section, we use the constructions and results of Section 5 to prove Theorem A(2) and Theorem B.

In the integral or p -local G -stable homotopy category, let

$$\begin{aligned} \mathcal{C}_\mathbb{Z} &= \text{add}\{G/H_+ \wedge S^n : H \leq G, n \in \mathbb{Z}\}, \\ \mathcal{C}_{\text{RO}} &= \text{add}\{G/H_+ \wedge S^V : H \leq G, V \in \text{RO}(G)\}, \\ \mathcal{C}_{\text{all}} &= \text{add}\{G_+ \wedge_H S^W : H \leq G, W \in \text{RO}(H)\}. \end{aligned} \tag{6.1}$$

Here add denotes the full subcategory of finite direct sums of the displayed spectra, with a small skeleton understood. In the p -local case, the detectors and their morphisms are p -localized. For any of these categories, set

$$\text{Mod } \mathcal{C} = \text{Add}(\mathcal{C}^{\text{op}}, \text{Ab}), \quad h_{\mathcal{C}}(X) = [-, X]^G|_{\mathcal{C}}. \quad (6.2)$$

Here $\text{Add}(\mathcal{C}^{\text{op}}, \text{Ab})$ denotes the category of additive functors from $\mathcal{C}^{\text{op}}$ to the category Ab of abelian groups. The restriction to $\mathcal{C}$ means that $h_{\mathcal{C}}(X)(P) = [P, X]^G$ for each detector $P \in \mathcal{C}$, with maps in $\mathcal{C}$ acting by precomposition. Module maps commute with these operations. This is the presheaf formulation of Bohmann–May [BM12, Section 1]. For the first two categories, it agrees with modules over the corresponding graded sphere Green functor [LM06, Sections 3–5]. When $\mathcal{C}$ is fixed, write $h = h_{\mathcal{C}}$.

Exactness in $\text{Mod } \mathcal{C}$ is computed at each detector. A map f satisfies $h(f) = 0$ precisely when it is a $\mathcal{C}$ -ghost. Each $\mathcal{C}$ is closed under integer suspension and desuspension, so every integer suspension of a ghost is again a ghost. Moreover, each $\mathcal{C}$ contains the integer-suspended orbits, so h reflects isomorphisms by the equivariant Whitehead theorem [MM02, Definition III.3.2 and Theorem III.4.2]. Fullness on finite spectra means that the natural map

$$[X, Y]^G \longrightarrow \text{Hom}_{\text{Mod } \mathcal{C}}(h(X), h(Y)) \quad (6.3)$$

is surjective for every pair of finite spectra X, Y .

For any $\mathcal{C}$ -ghost $w : X \rightarrow Z$, a distinguished triangle

$$X \xrightarrow{w} Z \xrightarrow{i} E \xrightarrow{\delta} \Sigma X$$

gives a short exact sequence

$$0 \longrightarrow h(Z) \xrightarrow{h(i)} h(E) \xrightarrow{h(\delta)} h(\Sigma X) \longrightarrow 0 \quad (6.4)$$

because $h(w) = h(\Sigma w) = 0$. The following lemma makes the splitting in Pirashvili–Redondo [PR08, Theorem 1] explicit when w is a composite of two ghosts.

Lemma 6.1. *Let $u : X \rightarrow Y$ and $v : Y \rightarrow Z$ be $\mathcal{C}$ -ghosts, and put $w = vu$. Then the short exact sequence (6.4) associated to a distinguished triangle completing w splits in $\text{Mod } \mathcal{C}$.*

Proof. Complete u to a distinguished triangle. Since $w = vu$, the identity of X and the map v extend to a morphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{j} & D & \xrightarrow{q} & \Sigma X \\ \parallel & & \downarrow v & & \downarrow t & & \parallel \\ X & \xrightarrow{w} & Z & \xrightarrow{i} & E & \xrightarrow{\delta} & \Sigma X. \end{array}$$

Applying h gives the solid arrows in the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & h(Y) & \xrightarrow{h(j)} & h(D) & \xrightarrow{h(q)} & h(\Sigma X) \longrightarrow 0 \\ & & \downarrow h(v)=0 & & \downarrow h(t) & \nearrow s & \parallel \\ 0 & \longrightarrow & h(Z) & \xrightarrow{h(i)} & h(E) & \xrightarrow{h(\delta)} & h(\Sigma X) \longrightarrow 0. \end{array}$$

Exactness and $h(t)h(j) = h(i)h(v) = 0$ give the dashed module map s with $sh(q) = h(t)$. Since $\delta t = q$, we have $h(\delta)s h(q) = h(q)$. Surjectivity of $h(q)$ then gives $h(\delta)s = \text{id}$. $\square$

Proof of Theorem A(2). Corollary 5.6 gives a finite integral G -spectrum Y with a representation-orbit ghost $\Psi_Y : Y \rightarrow Y$ such that $\Psi_Y^2 \neq 0$. Theorem C gives such a pair in the p -local category as well. In either category, choose a distinguished triangle

$$Y \xrightarrow{\Psi_Y^2} Y \xrightarrow{i} E \xrightarrow{\delta} \Sigma Y.$$

Fix any of the three coefficient categories $\mathcal{C}$. Lemma 6.1 gives a section $s : h(\Sigma Y) \rightarrow h(E)$ of $h(\delta)$. To prove nonfullness, it suffices to show that s is not induced by a stable map. If s were induced by a stable map $r : \Sigma Y \rightarrow E$, then $h(\delta r) = \text{id}$, so δr would be invertible because h reflects isomorphisms. But $(\Sigma \Psi_Y^2)\delta r = 0$, forcing $\Psi_Y^2 = 0$, a contradiction. $\square$

In the notation of Lemma 6.1, the splitting already gives $h(E) \cong h(Z \vee \Sigma X)$. For Theorem B, we must also arrange that $E \not\cong Z \vee \Sigma X$. We will do so by constructing a nonzero double ghost on a finite spectrum annihilated by a power of p and comparing the orders of the resulting finite mapping groups.

Proof of Theorem B. Work integrally and choose a subgroup $C_p \leq G$. Theorem 5.5 and the proof of Theorem C give a finite integral C_p -spectrum Y on which the natural ghost of Proposition 5.1 satisfies

$$\Psi_Y^3 \neq 0, \quad p^r \Psi_Y = 0$$

for some $r \geq 1$. Set $Z = Y \wedge S/p^r$, which fits into the distinguished triangle

$$Y \xrightarrow{p^r} Y \xrightarrow{j} Z \xrightarrow{\partial} \Sigma Y.$$

We claim that $j\Psi_Y^2 \neq 0$. Otherwise, exactness of $[Y, -]^{C_p}$ would give $\Psi_Y^2 = p^r v$ for some $v : Y \rightarrow Y$, which implies

$$\Psi_Y^3 = (p^r v)\Psi_Y = v(p^r \Psi_Y) = 0,$$

a contradiction. Naturality gives

$$\Psi_Z^2 j = j\Psi_Y^2 \neq 0,$$

so the ghost Ψ_Z on Z also has nonzero square.

Since $(p^r \text{id}_Z)j = jp^r = 0$, exactness shows that $p^r \text{id}_Z$ factors through ∂ . Multiplying this factorization by p^r and using $(\Sigma p^r)\partial = 0$ gives $p^{2r} \text{id}_Z = 0$.

Induce the pair (Z, Ψ_Z) to G . By an abuse of notation, we also denote the induced pair by (Z, Ψ_Z) . By Lemma 4.1, Z is finite, Ψ_Z is a representation-orbit ghost, and $\Psi_Z^2 \neq 0$. Induction also preserves $p^{2r} \text{id}_Z = 0$. Every integer prime to p therefore acts invertibly on Z , so Z is already p -local.

Complete Ψ_Z^2 to a distinguished triangle

$$Z \xrightarrow{\Psi_Z^2} Z \xrightarrow{i} A \xrightarrow{\delta} \Sigma Z \tag{6.5}$$

and set $B = Z \vee \Sigma Z$. Both A and B are finite and p -local, since cofibers, suspensions, and finite wedges preserve these properties.

Take $h = h_{\mathcal{C}_{\text{all}}}$. Lemma 6.1 gives

$$h(A) \cong h(Z) \oplus h(\Sigma Z) \cong h(B).$$

Evaluating this module isomorphism at $G_+ \wedge_L S^{\text{Res}_L^H} W$ for $L \leq H$ gives isomorphisms

$$\pi_W^H(A) \cong \pi_W^H(B) \quad (H \leq G, W \in \text{RO}(H))$$

compatible with all stable operations.

We now show that $A \not\cong B$ by comparing the orders of $[Z, A]^G$ and $[Z, B]^G$. The tom Dieck splitting [LMS86, Theorem V.11.1] expresses the integer-graded homotopy groups of orbit spectra, at every subgroup, as finite sums of classical stable homotopy groups of spaces of finite type. These groups are finitely generated. Induction over orbit cells in the source and target, followed by passage

to retracts, therefore shows that mapping groups between finite integral G -spectra are finitely generated. Since $p^{2r}\text{id}_Z = 0$, all mapping groups below are also annihilated by p^{2r} , so they are finite.

Applying $[Z, -]^G$ to (6.5) gives

$$[Z, Z]^G \xrightarrow{(\Psi_Z^2)_*} [Z, Z]^G \longrightarrow [Z, A]^G \longrightarrow [Z, \Sigma Z]^G \xrightarrow{(\Sigma\Psi_Z^2)_*} [Z, \Sigma Z]^G.$$

Consequently,

$$\begin{aligned} |[Z, A]^G| &= |\text{coker}(\Psi_Z^2)_*| |\ker(\Sigma\Psi_Z^2)_*| \\ &< |[Z, Z]^G| |[Z, \Sigma Z]^G| = |[Z, B]^G|. \end{aligned}$$

The strict inequality holds because $(\Psi_Z^2)_*(\text{id}_Z) = \Psi_Z^2 \neq 0$. This proves that $A \not\cong B$. $\square$

7. HOMOLOGICAL CONSEQUENCES

We prove Theorem D by applying homological dimension bounds to the ghost powers of Theorem C. Throughout this section we work p -locally. Let $\mathcal{C}$ be one of the coefficient categories in (6.1), and write $h = h_{\mathcal{C}}$.

Recall that a right $\mathcal{C}$ -module M is flat if $M \otimes_{\mathcal{C}} -$ is exact on left $\mathcal{C}$ -modules, that is, additive functors $\mathcal{C} \rightarrow \text{Ab}$. The projective dimension $\text{pd}_{\mathcal{C}} M$ and flat dimension $\text{fd}_{\mathcal{C}} M$ are the least lengths of projective and flat resolutions of M , respectively. Dually, the injective dimension $\text{id}_{\mathcal{C}} M$ is the least length of an injective coresolution. A dimension is infinite if no finite resolution of the indicated kind exists. The weak global and global dimensions are

$$\begin{aligned} \text{w.gl.dim}(\text{Mod } \mathcal{C}) &= \sup_M \text{fd}_{\mathcal{C}} M, \\ \text{gl.dim}(\text{Mod } \mathcal{C}) &= \sup_M \text{pd}_{\mathcal{C}} M = \sup_M \text{id}_{\mathcal{C}} M, \end{aligned}$$

where M ranges over all right $\mathcal{C}$ -modules [Wei94].

Proof of Theorem D. We use a coefficient-category version of the bounds of Christensen [Chr98] and Hovey–Lockridge [HL11, HL13], recalled for graded rings in Ma–Xu [MX26, Proposition 6.2]. If a composite of $n \geq 1$ $\mathcal{C}$ -ghosts from a finite G -spectrum X to a G -spectrum Y is nonzero, then

$$\text{fd}_{\mathcal{C}} h(X), \quad \text{pd}_{\mathcal{C}} h(X), \quad \text{id}_{\mathcal{C}} h(Y) \geq n.$$

Each $\mathcal{C}$ is small, additive, and closed under integer suspensions, and its objects are compact and generate the ambient category. If $h(F)$ is flat, co-Yoneda and thick induction show that every map from a finite spectrum to F factors through a finite sum of objects of $\mathcal{C}$. Every ghost out of F is therefore phantom. If $\text{fd}_{\mathcal{C}} h(X) < n$, the $(n-1)$ st stage of a universal ghost tower has flat homotopy module, up to suspension. Every composite of n ghosts out of X factors through a ghost out of that stage and is zero when X is finite. This proves the flat bound, and $\text{fd} \leq \text{pd}$ gives the projective bound. The injective bound follows by dual dimension shifting, using Brown representability to realize injective $\mathcal{C}$ -modules.

For every $n \geq 1$, Theorem C with $m = 0$ gives a finite p -local G -spectrum Y_n and an endomorphism f that is a ghost for all three families, with $f^n \neq 0$. Applying the bound with $X = Y = Y_n$ to each coefficient category proves that all three homotopy modules have flat, projective, and injective dimension at least n . Taking suprema over n shows that all three module categories have infinite weak global dimension and infinite global dimension. $\square$

Remark 7.1. The four-cell examples of Theorem 2.1 already give non-injective homotopy modules. Theorem D further shows that projective and injective dimensions are unbounded among the homotopy modules of finite spectra. It does not assert that a single finite spectrum has a homotopy module of infinite injective dimension.

Remark 7.2. Infinite ghost dimension of the ordinary p -local sphere is already known [HL13, Proposition 2.1 and Corollary 2.2]. For every n , this gives a nonzero composite of n ghosts with finite source, although the intermediate spectra and the target need not be finite. Induction from the trivial subgroup preserves nonvanishing and produces ghosts for all three detector families. The bounds above already give unbounded flat and projective dimensions among the homotopy modules of finite G -spectra, and infinite weak global and global dimensions for all three coefficient categories. Our construction additionally gives arbitrarily long nonzero powers of a ghost endomorphism of a finite spectrum. These powers provide simultaneous lower bounds on the flat, projective, and injective dimensions of all three homotopy modules of the same finite spectrum.

REFERENCES

- [BC25] Mark Behrens and Jack Carlisle. Periodic phenomena in equivariant stable homotopy theory. *Quart. J. Math.*, 76(4):1033–1104, 2025.
- [BHZ26] William Balderrama, Yueshi Hou, and Shangjie Zhang. C_p^n -equivariant Mahowald invariants. *Proc. Lond. Math. Soc.*, 133(2), 2026.
- [BM12] Anna Marie Bohmann and J. P. May. A presheaf interpretation of the generalized Freyd conjecture. *Theory Appl. Categ.*, 26(16):403–411, 2012.
- [Boh10] Anna Marie Bohmann. The equivariant generating hypothesis. *Algebr. Geom. Topol.*, 10(2):1003–1016, 2010.
- [BS17] Paul Balmer and Beren Sanders. The spectrum of the equivariant stable homotopy category of a finite group. *Invent. Math.*, 208(1):283–326, 2017.
- [BXZ24] Eva Belmont, Zhouli Xu, and Shangjie Zhang. The reduced ring of the $RO(C_2)$ -graded C_2 -equivariant stable stems. *Proc. Amer. Math. Soc. Ser. B*, 11:1–14, 2024.
- [Car84] Gunnar Carlsson. Equivariant stable homotopy and Segal’s Burnside ring conjecture. *Ann. of Math. (2)*, 120(2):189–224, 1984.
- [CF17] J. Daniel Christensen and Martin Frankland. Higher Toda brackets and the Adams spectral sequence in triangulated categories. *Algebr. Geom. Topol.*, 17(5):2687–2735, 2017.
- [Chr98] J. Daniel Christensen. Ideals in triangulated categories: phantoms, ghosts and skeleta. *Adv. Math.*, 136(2):284–339, 1998.
- [Fre66] Peter Freyd. Stable homotopy. In *Proceedings of the Conference on Categorical Algebra (La Jolla, Calif., 1965)*, pages 121–172. Springer, New York, 1966.
- [GM95] J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. *Mem. Amer. Math. Soc.*, 113(543), 1995.
- [GWX21] Bogdan Gheorghe, Guozhen Wang, and Zhouli Xu. The special fiber of the motivic deformation of the stable homotopy category is algebraic. *Acta Math.*, 226(2):319–407, 2021.
- [HL11] Mark Hovey and Keir Lockridge. The ghost and weak dimensions of rings and ring spectra. *Israel J. Math.*, 182(1):31–46, 2011.
- [HL13] Mark Hovey and Keir Lockridge. Homological dimensions of ring spectra. *Homology Homotopy Appl.*, 15(2):53–71, 2013.
- [LM06] L. Gaunce Lewis, Jr. and Michael A. Mandell. Equivariant universal coefficient and Künneth spectral sequences. *Proc. Lond. Math. Soc. (3)*, 92(2):505–544, 2006.
- [LMM82] L. Gaunce Lewis, Jr., J. Peter May, and J. E. McClure. Classifying G -spaces and the Segal conjecture. In *Current trends in algebraic topology, Part 2 (London, Ont., 1981)*, volume 2 of *Canad. Math. Soc. Conf. Proc.*, pages 165–179. Amer. Math. Soc., Providence, RI, 1982.
- [LMS86] L. Gaunce Lewis, Jr., J. Peter May, and Mark Steinberger. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. with contributions by J. E. McClure.
- [MM02] Michael A. Mandell and J. Peter May. Equivariant orthogonal spectra and S -modules. *Mem. Amer. Math. Soc.*, 159(755), 2002.
- [MSZ89] J. P. May, V. P. Snaith, and P. Zelewski. A further generalization of the Segal conjecture. *Quart. J. Math. Oxford Ser. (2)*, 40(4):457–473, 1989.
- [MX26] Sihao Ma and Zhouli Xu. Generating hypotheses along the motivic deformation. preprint, 2026.
- [Oka84] Shichirô Oka. Multiplications on the Moore spectrum. *Mem. Fac. Sci. Kyushu Univ. Ser. A*, 38(2):257–276, 1984.
- [PR08] Teimuraz Pirashvili and María Julia Redondo. Universal coefficient theorem in triangulated categories. *Algebr. Represent. Theory*, 11(2):107–114, 2008.
- [SS03] Stefan Schwede and Brooke Shipley. Stable model categories are categories of modules. *Topology*, 42(1):103–153, 2003.

- [Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.
- [Wim19] Christian Wimmer. A model for genuine equivariant commutative ring spectra away from the group order. preprint, 2019.

DEPARTMENT OF MATHEMATICS, UCLA, LOS ANGELES, CA 90095-1555, USA
Email address: `masihao@math.ucla.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195, USA
Email address: `dannyshi@uw.edu`

DEPARTMENT OF MATHEMATICS, UCLA, LOS ANGELES, CA 90095-1555, USA
Email address: `xuzhouli@ucla.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195, USA
Email address: `sjzh@uw.edu`