

A Primer on Zeta Functions and Decomposition Spaces

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Abstract

Many examples of zeta functions in number theory and combinatorics are special cases of a construction in homotopy theory known as a decomposition space. This article aims to introduce number theorists to the relevant concepts in homotopy theory and lays some foundations for future applications of decomposition spaces in the theory of zeta functions.

1 Introduction

This expository article is aimed at introducing number theorists to the theory of decomposition spaces in homotopy theory. Briefly, a decomposition space is a certain simplicial space that admits an abstract notion of an incidence algebra and, in particular, an abstract zeta function. To motivate the connections to number theory, we show that most classical notions of zeta functions in number theory and algebraic geometry are special cases of the construction using decomposition spaces. These examples suggest a wider application of decomposition spaces in number theory and algebraic geometry, which we intend to explore in future work.

Some of the ideas for this article came from a reading course on 2-Segal spaces organized by Julie Bergner at the University of Virginia in spring 2020, which resulted in the survey [ABD+], as well as another reading course over the summer, co-organized by Bergner and the author. The author would like to thank the participants of these courses, and in particular Julie Bergner, Matt Feller and Bogdan Krstic, for helpful discussions on these and related topics. The present work also owes much to a similar article by Joachim Kock [Koc], which describes Riemann's zeta function from the perspective taken here. Kock notes that the techniques for posets go back at least to Stanley [Sta]. Finally, the author is grateful to Andrew Obus for helpful suggestions about the organization of the paper, especially in Section 3.

2 Classical Zeta Functions

We begin with a story that has been told countless times, about one of the most famous objects in mathematics: the *Riemann zeta function* $\zeta_{\mathbb{Q}}(s)$. However, in the spirit of the

homotopy-theoretic angle of this article, we aim to highlight certain aspects of $\zeta_{\mathbb{Q}}(s)$ which are ripe for generalization.

2.1 The Riemann Zeta Function

In his landmark 1859 manuscript “On the number of primes less than a given magnitude”, Riemann laid the foundation for modern analytic number theory with his extensive study of the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(To reserve the symbol ζ for later use, we will henceforth denote Riemann’s zeta function by $\zeta_{\mathbb{Q}}(s)$.) The series $\zeta_{\mathbb{Q}}(s)$ converges for all *complex numbers* s with $\operatorname{Re}(s) > 1$ and, amazingly, it conceals deep information about the nature of prime numbers within its analytic structure. For instance:

- (1) $\zeta_{\mathbb{Q}}(s)$ admits a meromorphic continuation to all of $\mathbb{C}$ and satisfies a functional equation $\xi(s) = \xi(1-s)$, where $\xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta_{\mathbb{Q}}(s)$ and $\Gamma(-)$ is the Gamma function.
- (2) $\zeta_{\mathbb{Q}}(s)$ has intimate ties to $\operatorname{Li}(x) = \int_2^x \frac{dt}{\log t}$, the logarithmic integral function, better known as a really good asymptotic estimate for the prime counting function $\pi(x)$.
- (3) Apart from so-called “trivial zeroes” at $s = -2, -4, \dots$, the remaining zeroes of $\zeta_{\mathbb{Q}}(s)$ lie in the *critical strip* $\{s \in \mathbb{C} \mid 0 < \operatorname{Re}(s) < 1\}$. The Riemann Hypothesis is the statement that $\operatorname{Re}(s) = \frac{1}{2}$ for any nontrivial zero of $\zeta_{\mathbb{Q}}(s)$. This would, in turn, imply the strongest possible asymptotic estimate of $\pi(x)$ using $\operatorname{Li}(x)$.

One of the fundamental properties of $\zeta_{\mathbb{Q}}(s)$ is the product formula

$$\zeta_{\mathbb{Q}}(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

which is sometimes attributed to Euler, although he only considered $\zeta_{\mathbb{Q}}(s)$ for real inputs s . Classically, the product formula is a consequence of unique factorization in $\mathbb{Z}$, but we give another proof of it in Section 3.1.

Consider any function $f : \mathbb{N} \rightarrow \mathbb{C}$, which we will refer to as an *arithmetic function*. (Note: in number theory, “arithmetic function” often refers to functions out of the natural numbers which satisfy a weak form of multiplicativity, namely $f(mn) = f(m)f(n)$ if m and n are coprime. However, we do not restrict to such functions.) To f , we associate the *Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad s \in \mathbb{C}.$$

Example 2.1. The Riemann zeta function is the Dirichlet series for the function $\zeta : \mathbb{N} \rightarrow \mathbb{C}$ defined by $\zeta(n) = 1$ for all n .

Example 2.2. The Möbius function $\mu : \mathbb{N} \rightarrow \mathbb{C}$ is defined by

$$\mu(n) = \begin{cases} 0, & p^2 \mid n \text{ for some prime } p \\ (-1)^r, & n = p_1 \cdots p_r \text{ for distinct primes } p_i. \end{cases}$$

We will write its corresponding Dirichlet series by

$$\mu_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

Proposition 2.3. For all $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, $\zeta_{\mathbb{Q}}(s) = \mu_{\mathbb{Q}}(s)^{-1}$. That is,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right)^{-1}.$$

There are classical proofs of this relation, but we will deduce it in a moment from the abstract properties of Dirichlet convolution. From now on, we ignore any considerations of convergence in our Dirichlet series.

Definition 2.4. For two arithmetic functions $f, g : \mathbb{N} \rightarrow \mathbb{C}$, their **Dirichlet convolution** is the function $f * g : \mathbb{N} \rightarrow \mathbb{C}$ defined by

$$(f * g)(n) = \sum_{ij=n} f(i)g(j)$$

for all n .

Lemma 2.5. If $f, g : \mathbb{N} \rightarrow \mathbb{C}$ have Dirichlet series $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ and $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$, respectively, then

$$F(s)G(s) = \sum_{n=1}^{\infty} \frac{(f * g)(n)}{n^s}.$$

In other words, Dirichlet convolution is a product on arithmetic functions that corresponds to ordinary multiplication of formal power series for the corresponding Dirichlet series.

Proposition 2.6. Let $A = \{f : \mathbb{N} \rightarrow \mathbb{C}\}$ be the complex vector space of arithmetic functions. Then

- (1) A is a commutative $\mathbb{C}$ -algebra via Dirichlet convolution.
- (2) The function

$$\delta : \mathbb{N} \longrightarrow \mathbb{C} \\ n \longmapsto \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}$$

is a unit for convolution, making A a unital $\mathbb{C}$ -algebra.

- (3) $\mu * \zeta = \delta = \zeta * \mu$.

Corollary 2.7 (Möbius Inversion). For any $f, g \in A$, if $f = g * \zeta$ then $g = f * \mu$. That is,

$$\text{if } f(n) = \sum_{d \mid n} g(d) \quad \text{then} \quad g(n) = \sum_{d \mid n} f(d) \mu\left(\frac{n}{d}\right).$$

2.2 Dedekind Zeta Functions

The classical story of the Riemann zeta function generalizes in several directions. First, let $K/\mathbb{Q}$ be a number field with ring of integers $\mathcal{O}_K$. By algebraic number theory, $\mathcal{O}_K$ is a Dedekind domain and therefore has unique factorization of *ideals*: any ideal $\mathfrak{a} \subset \mathcal{O}_K$ factors uniquely as a product of powers of distinct prime ideals $\mathfrak{a} = \mathfrak{p}_1^{k_1} \cdots \mathfrak{p}_r^{k_r}$. Let I_K denote the abelian group of (nonzero) *fractional ideals* of $\mathcal{O}_K$ in K , i.e. where we allow inverses of ordinary ideals $\mathfrak{a} \subset \mathcal{O}_K$ corresponding to finitely generated $\mathcal{O}_K$ -submodules of K . Let $I_K^+ \subset I_K$ be the semigroup of (nonzero) ordinary ideals $\mathfrak{a} \subset \mathcal{O}_K$. For two ideals $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_K$, we say $\mathfrak{a}$ *divides* $\mathfrak{b}$, written $\mathfrak{a} \mid \mathfrak{b}$, if $\mathfrak{b} \subseteq \mathfrak{a}$.

The *ideal norm* for $K/\mathbb{Q}$ is the function $N = N_{K/\mathbb{Q}} : I_K^+ \rightarrow \mathbb{N}$ defined by $N(\mathfrak{a}) = [\mathcal{O}_K : \mathfrak{a}]$, the index of $\mathfrak{a}$ as a sublattice of $\mathcal{O}_K$, or equivalently the cardinality of the finite ring $\mathcal{O}_K/\mathfrak{a}$. (This extends to a norm $I_K \rightarrow \mathbb{Z}$ in a natural way.)

Definition 2.8. *The Dedekind zeta function of a number field $K/\mathbb{Q}$ is the complex function*

$$\zeta_K(s) = \sum_{\mathfrak{a} \in I_K^+} \frac{1}{N(\mathfrak{a})^s} \quad \text{for } \operatorname{Re}(s) > 1.$$

As with $\zeta_{\mathbb{Q}}(s)$, Dedekind zeta functions have important properties that encode information about prime ideals in $\mathcal{O}_K$:

- (1) $\zeta_K(s)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation $\Lambda_K(s) = \Lambda_K(1-s)$ where $\Lambda_K(s) = |\operatorname{disc}(K)|^{s/2} \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{C}}(s)^{r_2} \zeta_K(s)$ is the *completed zeta function* of K , $\operatorname{disc}(K)$ is the discriminant of $K/\mathbb{Q}$ and $\Gamma_{\mathbb{R}}$ and $\Gamma_{\mathbb{C}}$ are so-called *archimedean* Gamma functions.
- (2) Apart from the trivial zeroes at negative integers (or just negative even integers if K is totally real), the remaining zeroes of $\zeta_K(s)$ lie in $\{s \in \mathbb{C} \mid 0 < \operatorname{Re}(s) < 1\}$. The Generalized Riemann Hypothesis states that $\operatorname{Re}(s) = \frac{1}{2}$ for any nontrivial zero of any Dedekind zeta function.
- (3) Beyond the zeroes, other special values of $\zeta_K(s)$ encode arithmetic information about K itself, cf. the analytic class number formula.

As with the Riemann zeta function, Dedekind zeta functions have an Euler product formula:

$$\zeta_K(s) = \prod_{\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}$$

where $\operatorname{Spec} \mathcal{O}_K$ denotes the set of prime ideals of $\mathcal{O}_K$. This follows from the unique factorization of prime ideals in $\mathcal{O}_K$, but we will give another proof of it in Section 3.1.

Following our story for the Riemann zeta function, for any arithmetic function $f : I_K^+ \rightarrow \mathbb{C}$, we associate a Dirichlet series

$$F(s) = \sum_{\mathfrak{a} \in I_K^+} \frac{f(\mathfrak{a})}{N(\mathfrak{a})^s}, \quad s \in \mathbb{C}.$$

Example 2.9. The Dedekind zeta function is the Dirichlet series for the function $\zeta : I_K^+ \rightarrow \mathbb{C}$ defined by $\zeta(\mathfrak{a}) = 1$ for all $\mathfrak{a} \in \mathcal{O}_K$.

Example 2.10. The Möbius function $\mu : I_K^+ \rightarrow \mathbb{C}$ is the function

$$\mu(\mathfrak{a}) = \begin{cases} 0, & \mathfrak{p}^2 \mid \mathfrak{a} \text{ for some prime ideal } \mathfrak{p} \\ (-1)^r, & \mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_r \text{ for distinct prime ideals } \mathfrak{p}_i. \end{cases}$$

We will write its corresponding Dirichlet series by

$$\mu_K(s) = \sum_{\mathfrak{a} \in I_K^+} \frac{\mu(\mathfrak{a})}{n^s}.$$

Proposition 2.11. For all $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, $\zeta_K(s) = \mu_K(s)^{-1}$. That is,

$$\sum_{\mathfrak{a} \in I_K^+} \frac{1}{N(\mathfrak{a})^s} = \left(\sum_{\mathfrak{a} \in I_K^+} \frac{\mu(\mathfrak{a})}{N(\mathfrak{a})^s} \right)^{-1}.$$

As we did for the Riemann zeta function, we can prove this formula using Dirichlet convolution.

Definition 2.12. For two functions $f, g : I_K^+ \rightarrow \mathbb{C}$, their **Dirichlet convolution** is the function $f * g : I_K^+ \rightarrow \mathbb{C}$ defined by

$$(f * g)(\mathfrak{a}) = \sum_{\mathfrak{b}\mathfrak{c}=\mathfrak{a}} f(\mathfrak{b})g(\mathfrak{c})$$

for all $\mathfrak{a} \in \mathcal{O}_K$.

Lemma 2.13. If $f, g : I_K^+ \rightarrow \mathbb{C}$ have Dirichlet series $F(s) = \sum_{\mathfrak{a}} \frac{f(\mathfrak{a})}{N(\mathfrak{a})^s}$ and $G(s) = \sum_{\mathfrak{a}} \frac{g(\mathfrak{a})}{N(\mathfrak{a})^s}$, respectively, then

$$F(s)G(s) = \sum_{\mathfrak{a} \in I_K^+} \frac{(f * g)(\mathfrak{a})}{N(\mathfrak{a})^s}.$$

Proposition 2.14. Let $A_K = \{f : I_K^+ \rightarrow \mathbb{C}\}$ be the complex vector space of functions on the semigroup I_K^+ . Then

- (1) A_K is a commutative $\mathbb{C}$ -algebra via Dirichlet convolution.
- (2) The function

$$\begin{aligned} \delta : I_K^+ &\longrightarrow \mathbb{C} \\ \mathfrak{a} &\longmapsto \begin{cases} 1, & \mathfrak{a} = (1) \\ 0, & \mathfrak{a} \neq (1) \end{cases} \end{aligned}$$

is a unit for convolution, making A_K a unital $\mathbb{C}$ -algebra.

(3) In A_K , we have $\mu * \zeta = \delta = \zeta * \mu$.

Corollary 2.15 (Möbius Inversion for Number Fields). *For any $f, g \in A_K$, if $f = g * \zeta$ then $g = f * \mu$. That is,*

$$\text{if } f(\mathfrak{a}) = \sum_{\mathfrak{d}|\mathfrak{a}} g(\mathfrak{d}) \quad \text{then} \quad g(\mathfrak{a}) = \sum_{\mathfrak{d}|\mathfrak{a}} f(\mathfrak{d})\mu(\mathfrak{a}\mathfrak{d}^{-1})$$

where $\mathfrak{d}^{-1}$ denotes the inverse of $\mathfrak{d}$ in the group of fractional ideals I_K .

In Section 3.1, we identify A_K with the (reduced) incidence algebra of the poset $(I_K^+, |)$, and the above results will follow from a more general version of Möbius inversion for posets.

2.3 Hasse–Weil Zeta Functions

In what at first seems to be a different direction, let X be a variety over a finite field $k = \mathbb{F}_q$. For each $n \geq 1$, let $\#X(\mathbb{F}_{q^n})$ denote the number of points of X defined over the unique field extension $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$ of degree n .

Definition 2.16. *The Hasse–Weil zeta function of $X/\mathbb{F}_q$ is the formal power series*

$$Z(X, t) = \exp \left[\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n \right]$$

where $\exp(t) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} t^n$.

The Hasse–Weil zeta function is the subject of the Weil Conjectures, all of which have been settled:

Theorem 2.17 (Weil Conjectures). *Let X be a smooth, projective, geometrically connected variety over $\mathbb{F}_q$ and let $d = \dim X$. Then*

(1) (Weil 1948, Dwork 1960) $Z(X, t)$ is a rational function.

(2) (Grothendieck 1965, Deligne 1974) $Z(X, t)$ satisfies the functional equation

$$Z(q^{-n}t^{-1}) = \varepsilon q^{nE/2} t^E Z(t)$$

where $\varepsilon = \pm 1$ and E is the self-intersection number of the diagonal $\Delta : X \hookrightarrow X \times X$.

(3) (Deligne 1974) As a rational function,

$$Z(X, t) = \frac{P_1(t)P_3(t) \cdots P_{2d-1}(t)}{P_0(t)P_2(t) \cdots P_{2d}(t)}$$

with $P_{2d}(t) = 1 - q^n t$ and all other P_i integer-valued polynomials in t with roots (α_{ij}) satisfying $|\alpha_{ij}| = q^{i/2}$.

Analogous to the product formula for Dedekind zeta functions, we have:

Proposition 2.18. *If X is a variety over $\mathbb{F}_q$, then*

$$Z(X, t) = \prod_{x \in |X|} (1 - t^{\deg(x)})^{-1}$$

where $|X|$ denotes the set of closed points of X and $\deg(x) = [k(x) : k]$ is the degree of a closed point.

Proof. For any $n \geq 1$, the set $X(\mathbb{F}_{q^n})$ may be identified with the set of closed points $x \in |X|$ with $\deg(x) \leq n$. Then

$$\#X(\mathbb{F}_{q^n}) = \sum_{d|n} d \cdot \#\{x \in |X| : \deg(x) = d\}$$

and an easy manipulation shows that

$$\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n = -\log \left(\prod_{d=1}^{\infty} (1 - t^d)^{a_d} \right)$$

where $a_d = \#\{x \in |X| : \deg(x) = d\}$ and $\log(\cdot)$ denotes the formal logarithmic power series $\log(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} t^n$. Finally, applying $\exp$ gives to this equation yields

$$Z(X, t) = \prod_{d=1}^{\infty} (1 - t^d)^{-a_d} = \prod_{x \in |X|} (1 - t^{\deg(x)})^{-1}.$$

□

Alternatively, and more in line with the generating functions defining the Riemann and Dedekind zeta functions in Subsections 2.1 and 2.2, we have the following interpretation of $Z(X, t)$.

Definition 2.19. *For a variety X over an arbitrary field k , the **group of 0-cycles** on X , denoted $Z_0(X)$, is the free abelian group generated by all closed points $x \in |X|$. Every $\alpha \in Z_0(X)$ can be written as a sum $\alpha = \sum_x a_x x$ over $x \in |X|$, where all but finitely many $a_x \in \mathbb{Z}$ are zero. An **effective 0-cycle** is a 0-cycle $\alpha = \sum_x a_x x$ such that $a_x \geq 0$ for all $x \in |X|$.*

Definition 2.20. *The **degree** of a 0-cycle $\alpha = \sum_x a_x x$ is the integer $\deg(\alpha) = \sum_x a_x \deg(x)$.*

Corollary 2.21. *For a variety X over $k = \mathbb{F}_q$, the Hasse–Weil zeta function of X can be written*

$$Z(X, t) = \sum_{\alpha \in Z_0^{\text{eff}}(X)} t^{\deg(\alpha)}$$

where $Z_0^{\text{eff}}(X) \subseteq Z_0(X)$ is the semigroup of effective 0-cycles on X .

2.4 Zeta Functions of Arithmetic Schemes

As it turns out, the number theory examples – Riemann and Dedekind zeta functions – are not wholly unrelated to the algebro-geometric examples – Hasse–Weil zeta functions. Let X be a scheme of finite type over $\operatorname{Spec} \mathbb{Z}$, sometimes called an *arithmetic scheme*.

Definition 2.22. *The zeta function of an arithmetic scheme X is the complex power series*

$$\zeta_X(s) = \prod_{x \in |X|} \left(1 - \frac{1}{N(x)^s}\right)^{-1}$$

where $N(x) = \#(\mathcal{O}_{X,x}/\mathfrak{m}_x)$ is the cardinality of the finite residue field of a closed point x .

Remark 2.23. In the literature, this is sometimes also called the *Hasse–Weil function* of an arithmetic scheme. Indeed, according to Serre [Ser, p.85], this was Hasse and Weil’s original situation, which was then later translated to varieties over finite fields.

Proposition 2.24. *For an arithmetic scheme X and a prime p , let X_p denote the reduction mod p of X , i.e. $X_p = X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{F}_p$. Then*

$$\zeta_X(s) = \prod_{p \text{ prime}} Z(X_p, p^{-s})$$

where $Z(X_p, t)$ is the Hasse–Weil zeta function of the $\mathbb{F}_p$ -variety X_p .

Example 2.25. For $X = \operatorname{Spec} \mathbb{Z}$ itself, we have $X_p = \operatorname{Spec} \mathbb{F}_p$ for all primes p . Moreover,

$$Z(\operatorname{Spec} \mathbb{F}_p, t) = \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} t^n \right] = \exp[-\log(1-t)] = (1-t)^{-1}.$$

Hence by Proposition 2.24,

$$\zeta_{\operatorname{Spec} \mathbb{Z}}(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1} = \zeta_{\mathbb{Q}}(s),$$

the Riemann zeta function.

Example 2.26. Let $K/\mathbb{Q}$ be a number field with ring of integers $\mathcal{O}_K$. Then $X = \operatorname{Spec} \mathcal{O}_K$ has reductions $X_p = \operatorname{Spec}(\mathcal{O}_K/p\mathcal{O}_K) \cong \coprod_{i=1}^r \operatorname{Spec}(\mathcal{O}_K/\mathfrak{p}_i^{e_i})$ where $p\mathcal{O}_K = \prod_{i=1}^r \mathfrak{p}_i^{e_i}$ for distinct prime ideals $\mathfrak{p}_i \subset \mathcal{O}_K$ and integers $e_i \geq 1$. Let $f_i = f(\mathfrak{p}_i | p) = \dim_{\mathbb{F}_p}(\mathcal{O}_K/\mathfrak{p}_i)$ be the inertia degree of $\mathfrak{p}_i$ over p . As a result,

$$\#X_p(\mathbb{F}_{p^n}) = \sum_{i=1}^r \# \operatorname{Spec}(\mathcal{O}_K/\mathfrak{p}_i^{e_i})(\mathbb{F}_{p^n}) = \sum_{i=1}^r \# \operatorname{Hom}(\mathcal{O}_K/\mathfrak{p}_i^{e_i}, \mathbb{F}_{p^n}).$$

For each $\mathfrak{p}_i$ lying over p , there is a $\mathbb{F}_p$ -linear map $\mathcal{O}_K/\mathfrak{p}_i^{e_i} \rightarrow \mathbb{F}_{p^n}$ exactly when $f_i | n$, and in this case $\# \operatorname{Hom}(\mathcal{O}_K/\mathfrak{p}_i^{e_i}, \mathbb{F}_{p^n}) = f_i$. Thus the local factor at p is

$$\begin{aligned} Z(X_p, t) &= \exp \left[\sum_{n=1}^{\infty} \sum_{i=1}^r \frac{\# \operatorname{Hom}(\mathcal{O}_K/\mathfrak{p}_i^{e_i}, \mathbb{F}_{p^n})}{n} t^n \right] = \prod_{i=1}^r \exp \left[\sum_{n=1}^{\infty} \frac{\# \operatorname{Hom}(\mathcal{O}_K/\mathfrak{p}_i^{e_i}, \mathbb{F}_{p^n})}{n} t^n \right] \\ &= \prod_{i=1}^r \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} t^{f_i n} \right] = \prod_{i=1}^r \exp[-\log(1 - t^{f_i})] = \prod_{i=1}^r (1 - t^{f_i})^{-1} \end{aligned}$$

Evaluating at $t = p^{-s}$ gives

$$Z(X_p, p^{-s}) = \prod_{i=1}^r (1 - (p^{-s})^{f_i})^{-1} = \prod_{i=1}^r (1 - p^{-f_i s})^{-1}.$$

Recall that for each $\mathfrak{p}_i$ lying over p , $N(\mathfrak{p}_i) = p^{f_i}$. Putting the factors together using Proposition 2.24, we have

$$\zeta_{\text{Spec } \mathcal{O}_K}(s) = \prod_{p \text{ prime}} \prod_{i=1}^r (1 - p^{-f_i s})^{-1} = \prod_{\mathfrak{p} \in \text{Spec } \mathcal{O}_K} (1 - N(\mathfrak{p})^{-s})^{-1}$$

which is precisely the Dedekind zeta function for $K/\mathbb{Q}$.

Example 2.27. Let $X = \mathbb{A}_{\mathbb{Z}}^n$ be affine n -space over the integers. Then for each prime p , $X_p = \mathbb{A}_{\mathbb{F}_p}^n$ which has $\#\mathbb{A}^n(\mathbb{F}_{p^k}) = p^{nk}$ points over any $\mathbb{F}_{p^k}$. Thus

$$Z(X_p, t) = \exp \left[\sum_{k=1}^{\infty} \frac{p^{nk}}{k} t^k \right] = \exp \left[\sum_{k=1}^{\infty} \frac{1}{k} (p^n t)^k \right] = (1 - p^n t)^{-1}.$$

By Proposition 2.24,

$$\zeta_{\mathbb{A}_{\mathbb{Z}}^n}(s) = \prod_{p \text{ prime}} (1 - p^n p^{-s})^{-1} = \zeta_{\mathbb{Q}}(s - n).$$

Lemma 2.28. If $Z \hookrightarrow X$ is a closed arithmetic subscheme with complement $U = X \setminus Z$, then $\zeta_X(s) = \zeta_Z(s) \zeta_U(s)$.

Example 2.29. For $X = \mathbb{P}_{\mathbb{Z}}^1$, we can decompose $\mathbb{P}_{\mathbb{Z}}^1 = \mathbb{A}_{\mathbb{Z}}^1 \cup \{\infty\}$ where the point ∞ is treated as a closed subscheme $\text{Spec } \mathbb{Z} \hookrightarrow \mathbb{P}_{\mathbb{Z}}^1$. Then by Proposition 2.28,

$$\zeta_{\mathbb{P}_{\mathbb{Z}}^1}(s) = \zeta_{\mathbb{Q}}(s - 1) \zeta_{\mathbb{Q}}(s).$$

More generally,

$$\zeta_{\mathbb{P}_{\mathbb{Z}}^n}(s) = \zeta_{\mathbb{Q}}(s - n) \cdots \zeta_{\mathbb{Q}}(s - 1) \zeta_{\mathbb{Q}}(s).$$

In general, it is expected that $\zeta_X(s)$ has meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation analogous to that of the Riemann and Dedekind zeta functions, at least when X is “nice” (i.e. regular and proper over $\text{Spec } \mathbb{Z}$). Further, there are deep conjectures about the values, order of poles, etc. of $\zeta_X(s)$ which we won’t spell out here. Suffice it to say that these are some of the most sought-after results in all of number theory.

3 Decomposition Spaces and Incidence Coalgebras

In this section, we discuss the theory of decomposition spaces, originating from the work of Gálvez-Carrillo, Kock and Tonks in [GKT1], [GKT2] and [GKT3]. This theory is developed in order to discuss incidence algebras in their most general form. In particular, the incidence algebra of a decomposition space has a canonical element called the *zeta functor* which generalizes all previous notions of zeta functions. As motivation, we first reinterpret the convolution products and Möbius inversion principles in Section 2.

3.1 Examples

We will see shortly that the Riemann, Dedekind and Hasse–Weil zeta functions all arise from the incidence algebras of *posets*, which are special examples of decomposition spaces. Although the full power of decomposition spaces is not needed to describe these examples, we will lay the groundwork for future applications of the general theory towards the end of the article.

Let $(\mathcal{P}, \leq)$ be a poset and for any elements $x, y \in \mathcal{P}$, define the *interval* $[x, y]$ by

$$[x, y] = \{z \in \mathcal{P} \mid x \leq z \leq y\}.$$

Example 3.1. For the poset $(\mathbb{N}, \leq)$, $[x, y]$ is the usual interval of integers between x and y . More useful to us will be the poset $(\mathbb{N}_0, \leq)$ of nonnegative integers ordered by succession, which is isomorphic to $(\mathbb{N}, \leq)$.

Example 3.2. Let $(\mathbb{N}, |)$ denote the poset of natural numbers under division. Then $[x, y] = \{d \in \mathbb{N} : x \mid d \mid y\}$.

Example 3.3. For any number field $K/\mathbb{Q}$, the set of ideals of $\mathcal{O}_K$ forms a poset $(I_K^+, |)$ with intervals $[\mathfrak{a}, \mathfrak{b}] = \{\mathfrak{d} : \mathfrak{a} \mid \mathfrak{d} \mid \mathfrak{b}\}$.

Definition 3.4. A poset $(\mathcal{P}, \leq)$ is called **locally finite** if every interval $[x, y]$ in $\mathcal{P}$ is a finite set.

Remark 3.5. All of the above posets are locally finite.

Fix a field k . The following definitions go back at least to Stanley [Sta], Rota [Rot] and their contemporaries.

Definition 3.6. The **incidence coalgebra** $C(\mathcal{P})$ of a locally finite poset $(\mathcal{P}, \leq)$ over k is the free k -vector space on the set of intervals $\{[x, y] : x, y \in \mathcal{P}\}$, together with the comultiplication and counit maps

$$\begin{aligned} \Delta : C(\mathcal{P}) &\longrightarrow C(\mathcal{P}) \otimes C(\mathcal{P}) \\ [x, y] &\longmapsto \sum_{z \in [x, y]} [x, z] \otimes [z, y] \\ \delta : C(\mathcal{P}) &\longrightarrow k \\ [x, y] &\longmapsto \delta_{xy} = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}. \end{aligned}$$

Definition 3.7. The **incidence algebra** $I(\mathcal{P})$ of $(\mathcal{P}, \leq)$ over k is the k -vector space $I(\mathcal{P}) = \text{Hom}_k(C(\mathcal{P}), k)$ together with the multiplication map

$$\begin{aligned} * : I(\mathcal{P}) \otimes I(\mathcal{P}) &\longrightarrow I(\mathcal{P}) \\ \varphi \otimes \psi &\longmapsto \left(\varphi * \psi : [x, y] \mapsto \sum_{z \in [x, y]} \varphi([x, z])\psi([z, y]) \right) \end{aligned}$$

and unit $1 \mapsto \delta$, also written δ by abuse of notation.

By [GKT1, Thm. 7.4], the (co)incidence algebra of a locally finite poset is co(associative) and (co)unital. It need not be (co)commutative in general.

In any incidence algebra $I(\mathcal{P})$ for a locally finite poset $\mathcal{P}$, there is a distinguished element $\zeta : [x, y] \mapsto 1$, called the *zeta function* for $\mathcal{P}$. If $\zeta \in I(\mathcal{P})^\times$, we denote an inverse by $\mu = \zeta^{-1}$, called the *Möbius function* for $\mathcal{P}$.

Proposition 3.8 (Möbius Inversion). *For any locally finite poset $(\mathcal{P}, \leq)$,*

(1) $\mu = \zeta^{-1}$ *exists and is defined recursively by*

$$\mu : [x, y] \mapsto \begin{cases} 1, & x = y \\ -\sum_{z \in [x, y]} \mu([x, z]), & x \neq y. \end{cases}$$

(2) (Rota's Formula) *For any $f, g \in I(\mathcal{P})$, if $f = g * \zeta$ then $g = f * \mu$. That is,*

$$\text{if } f([x, y]) = \sum_{z \in [x, y]} g([x, z]) \text{ then } g([x, y]) = \sum_{z \in [x, y]} f([x, z]) \mu([z, y]).$$

To recover classical versions such as Corollary 2.7, it is useful to pass to the *reduced incidence algebra* $\tilde{I}(\mathcal{P})$, the subalgebra of $I(\mathcal{P})$ consisting of φ that are constant on isomorphism classes of intervals (considered as subposets of $\mathcal{P}$). Alternatively, $\tilde{I}(\mathcal{P})$ is the dual of the coalgebra $\tilde{C}(\mathcal{P})$ of isomorphism classes of intervals (cf. [Koc, 2.5]).

Example 3.9. For the poset $(\mathbb{N}_0, \leq)$ and $k = \mathbb{C}$, the Möbius function is

$$\mu([x, y]) = \begin{cases} 1, & x = y \\ -1, & y = x + 1 \\ 0, & \text{otherwise.} \end{cases}$$

Here, every interval is isomorphic to one of the form $[0, y - x]$ for $y \geq x$. For an element $f \in \tilde{I}(\mathbb{N}_0, \leq)$ which is a priori a function on the intervals of $(\mathbb{N}_0, \leq)$, we write $f(n) = f([0, n])$. In this case, Möbius inversion (applied to the reduced incidence algebra $\tilde{I}(\mathbb{N}_0, \leq)$) says that

$$f(n) = \sum_{i \leq n} g(i) \implies g(n) = f(n) - f(n - 1).$$

For the poset $(\mathbb{N}_0, \leq)$, we can also interpret functions $f : \mathbb{N} \rightarrow \mathbb{C}$ in terms of their generating functions

$$F(z) = \sum_{n=0}^{\infty} f(n) z^n.$$

Ignoring questions of convergence, we have

$$\sum_{n=0}^{\infty} \zeta(n) z^n = \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z} \quad \text{and} \quad \sum_{n=0}^{\infty} \mu(n) z^n = 1 - z$$

as Möbius inversion predicts.

Example 3.10. For the poset $(\mathbb{N}, |)$ and $k = \mathbb{C}$, the Möbius function is precisely the classical μ from Example 2.2 and multiplication in $\tilde{I}(\mathbb{N}, |)$ is Dirichlet convolution. Then Proposition 2.6 and Corollary 2.7 follow directly from Proposition 3.8. For each prime p , consider the subposet $(\{p^k\}, |) \subseteq (\mathbb{N}, |)$. Then it is clear that $(\{p^k\}, |) \cong (\mathbb{N}_0, \leq)$ as posets, via $p^k \leftrightarrow k$. Further, $(\mathbb{N}, |)$ decomposes as a *restricted product* of posets

$$(\mathbb{N}, |) \cong \prod'_p (\{p^k\}, |) \cong \prod'_p (\mathbb{N}_0, \leq).$$

That is, $n \in (\mathbb{N}, |)$ corresponds to an infinite sequence $(p^k) \in \prod (\{p^k\}, |)$ where all but finitely many of the p^k are equal to 1 (equivalently, all but finitely many k are 0). Here, we use the notation of restricted direct products used, for example, to refer to the adèles and idèles of a number field. By Example 3.9, the Möbius function for each $(\{p^k\}, |)$ can be written $\mu_{(\{p^k\}, |)} = \delta - \delta_p$, where

$$\delta_p(k) = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1. \end{cases}$$

Then under the decomposition of $(\mathbb{N}, |)$ above, we have

$$\mu_{(\mathbb{N}, |)} = \bigotimes_{p \text{ prime}} \mu_{(\{p^k\}, |)} = \bigotimes_{p \text{ prime}} (\delta - \delta_p).$$

In terms of Dirichlet functions, this means

$$\mu_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_{p \text{ prime}} \left(\sum_{n=1}^{\infty} \frac{\delta(n)}{n^s} - \sum_{n=1}^{\infty} \frac{\delta_p}{n^s} \right) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s} \right).$$

Finally, since $\zeta_{\mathbb{Q}}(s) = \mu_{\mathbb{Q}}(s)^{-1}$, we recover Euler's product formula

$$\zeta_{\mathbb{Q}}(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s} \right)^{-1}.$$

Example 3.11. More generally, for a number field $K/\mathbb{Q}$, consider the poset $(I_K^+, |)$, which is locally finite by unique factorization of ideals in $\mathcal{O}_K$. The Möbius function for this poset is the function μ defined in Example 2.10 and once again, multiplication in the reduced incidence algebra $\tilde{I}(I_K^+, |)$ is Dirichlet convolution, so we recover Proposition 2.14 and Corollary 2.15 from the general case in Proposition 3.8. For each prime ideal $\mathfrak{p} \in \text{Spec } \mathcal{O}_K$, the subposet $(\{\mathfrak{p}^k\}, |) \subseteq (I_K^+, |)$ is isomorphic to $(\mathbb{N}_0, \leq)$, so $(I_K^+, |)$ decomposes as a restricted direct product

$$(I_K^+, |) \cong \prod'_{\mathfrak{p}} (\{\mathfrak{p}^k\}, |) \cong \prod'_{\mathfrak{p}} (\mathbb{N}_0, \leq).$$

Then by Example 3.9, the Möbius function for $(\{\mathfrak{p}^k\}, |)$ is $\mu_{(\{\mathfrak{p}^k\}, |)} = \delta - \delta_{\mathfrak{p}}$, where

$$\delta_{\mathfrak{p}}(\mathfrak{a}) = \begin{cases} 1, & \mathfrak{a} = \mathfrak{p} \\ 0, & \mathfrak{a} \neq \mathfrak{p}. \end{cases}$$

Then the Möbius function of $(I_K^+, |)$ decomposes as

$$\mu_{(I_K^+, |)} = \bigotimes_{\mathfrak{p} \in \text{Spec } \mathcal{O}_K} \mu_{(\{\mathfrak{p}^k\}, |)} = \bigotimes_{\mathfrak{p} \in \text{Spec } \mathcal{O}_K} (\delta - \delta_{\mathfrak{p}}).$$

Passing to Dirichlet functions, we get

$$\mu_K(s) = \sum_{\mathfrak{a} \in I_K^+} \frac{\mu(\mathfrak{a})}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p} \in \text{Spec } \mathcal{O}_K} \left(\sum_{\mathfrak{a} \in I_K^+} \frac{\delta(\mathfrak{a})}{N(\mathfrak{a})^s} - \sum_{\mathfrak{a} \in I_K^+} \frac{\delta_{\mathfrak{p}}(\mathfrak{a})}{N(\mathfrak{a})^s} \right) = \prod_{\mathfrak{p} \in \text{Spec } \mathcal{O}_K} \left(1 - \frac{1}{N(\mathfrak{p})^s} \right).$$

Therefore by Möbius inversion,

$$\zeta_K(s) = \mu_K(s)^{-1} = \prod_{\mathfrak{p} \in \text{Spec } \mathcal{O}_K} \left(1 - \frac{1}{N(\mathfrak{p})^s} \right)^{-1}.$$

Example 3.12. Let X be a variety over $\mathbb{F}_q$ and consider the poset $(Z_0^{\text{eff}}(X), \leq)$, where $\alpha \leq \beta$ if and only if $\alpha = \sum_x a_x x$, $\beta = \sum_x b_x x$ and $a_x \leq b_x$ for all $x \in |X|$. Then $(Z_0^{\text{eff}}(X), \leq)$ is a locally finite poset by definition of $Z_0^{\text{eff}}(X)$, and has Möbius function

$$\mu : \alpha = \sum_x a_x x \mapsto \begin{cases} 1, & \alpha = 0 \\ 0, & a_x > 1 \text{ for any } x \\ (-1)^r, & \alpha = x_1 + \dots + x_r \text{ for distinct closed points } x_i. \end{cases}$$

Let A_X be the complex vector space of functions $f : Z_0^{\text{eff}}(X) \rightarrow \mathbb{C}$ and define their convolution product by

$$(f * g)(\alpha) = \sum_{\beta + \gamma = \alpha} f(\beta)g(\gamma)$$

where the sum is over all “partitions” of α into effective 0-cycles, i.e. β and γ such that $\beta + \gamma = \alpha$.

Proposition 3.13. *For a variety X over $\mathbb{F}_q$,*

- (1) A_X is a commutative $\mathbb{C}$ -algebra via the convolution product defined above.
- (2) The function

$$\begin{aligned} \delta : Z_0^{\text{eff}}(X) &\longrightarrow \mathbb{C} \\ \alpha &\longmapsto \begin{cases} 1, & \alpha = 0 \\ 0, & \alpha > 0 \end{cases} \end{aligned}$$

is a unit for convolution, making A_X a unital $\mathbb{C}$ -algebra.

- (3) In A_X , we have $\mu * \zeta = \delta = \zeta * \mu$ where ζ is the zeta function sending $\alpha \mapsto 1$ for all α .
- (4) (Möbius Inversion for 0-Cycles) For any $f, g \in A_X$,

$$\text{if } f(\alpha) = \sum_{\beta \leq \alpha} g(\beta) \text{ then } g(\alpha) = \sum_{\beta \leq \alpha} f(\beta) \mu(\alpha - \beta).$$

Proof. This is purely a consequence of Proposition 3.8, after identifying A_X with the reduced incidence algebra $\tilde{I}(Z_0^{\text{eff}}(X), \leq)$. $\square$

For any function $f \in A_X$, we can form a generating series

$$F(t) = \sum_{\alpha \in Z_0^{\text{eff}}(X)} f(\alpha) t^{\deg(\alpha)},$$

analogously to forming a Dirichlet series in the number field case. The following lemma is immediate.

Lemma 3.14. *If $f, g \in A_K$ have generating series $F(t) = \sum_{\alpha} f(\alpha) t^{\deg(\alpha)}$ and $G(t) = \sum_{\alpha} g(\alpha) t^{\deg(\alpha)}$, respectively, then*

$$F(t)G(t) = \sum_{\alpha} (f * g)(\alpha) t^{\deg(\alpha)}.$$

By Corollary 2.21, the Hasse–Weil zeta function $Z(X, t)$ is the generating series for the abstract zeta function $\zeta \in A_K$. We will write the generating series for $\mu \in A_K$ by

$$M(X, t) = \sum_{\alpha \in Z_0^{\text{eff}}(X)} \mu(\alpha) t^{\deg(\alpha)}.$$

Then Proposition 3.13 shows that

$$Z(X, t) = M(X, t)^{-1} = \left(\sum_{\alpha \in Z_0^{\text{eff}}(X)} \mu(\alpha) t^{\deg(\alpha)} \right)^{-1}.$$

To see the product formula for $Z(X, t)$ from another angle, for each closed point x of X , consider the subposet $(\{ax : a \in \mathbb{N}_0\}, \leq) \subseteq (Z_0^{\text{eff}}, \leq)$ which is isomorphic to $(\mathbb{N}_0, \leq)$. Then $(Z_0^{\text{eff}}(X), \leq)$ decomposes as a restricted direct product of posets

$$(Z_0^{\text{eff}}(X), \leq) \cong \prod_{x \in |X|}' (\{ax\}, \leq) \cong \prod_{x \in |X|}' (\mathbb{N}_0, \leq).$$

By Example 3.9, the Möbius function for $(\{ax\}, \leq)$ is $\mu_{(\{ax\}, \leq)} = \delta - \delta_x$, where

$$\delta_x(\alpha) = \begin{cases} 1, & \alpha = x \\ 0, & \alpha \neq x. \end{cases}$$

Then the Möbius function for $(Z_0^{\text{eff}}(X), \leq)$ decomposes as well:

$$\mu_{(Z_0^{\text{eff}}(X), \leq)} = \bigotimes_{x \in |X|} \mu_{(\{ax\}, \leq)} = \bigotimes_{x \in |X|} (\delta - \delta_x).$$

On the level of generating series,

$$M(X, t) = \sum_{\alpha \in Z_0^{\text{eff}}(X)} \mu(\alpha) t^{\deg(\alpha)} = \prod_{x \in |X|} (\delta - \delta_x) t^{\deg(x)} = \prod_{x \in |X|} (1 - t^{\deg(x)}).$$

Applying Möbius inversion yields the product form of the Hasse–Weil zeta function:

$$Z(X, t) = M(X, t)^{-1} = \prod_{x \in |X|} (1 - t^{\deg(x)})^{-1}.$$

3.2 Discrete Decomposition Spaces

The construction of the incidence algebra of a poset in Section 3.1 works more generally for categories with certain finiteness conditions called *Möbius categories* [Ler]. The authors in [GKT1] identify situations (e.g. rooted trees with cuts) in which incidence algebras and Möbius inversion make sense despite no obvious Möbius category structure present. This suggests there is a further generalization of Möbius categories which capture the theory of incidence algebras. Indeed, this is the original motivation behind the definition of decomposition spaces in [GKT1].

Definition 3.15. Let $K : \Delta^{op} \rightarrow \mathbf{Set}$ be a simplicial set which is locally of finite length (cf. [GKT2] for a precise definition). The **incidence coalgebra** of K is the free k -vector space $C(K)$ on K_1 with comultiplication

$$\begin{aligned} \Delta : C(K) &\longrightarrow C(K) \otimes C(K) \\ f &\longmapsto \sum_{d_1\sigma=f} d_2\sigma \otimes d_0\sigma \end{aligned}$$

where the sum is over all $\sigma \in K_2$ with $d_1\sigma = f$, and counit

$$\begin{aligned} \delta : C(K) &\longrightarrow k \\ f &\longmapsto \begin{cases} 1, & \text{if } f \text{ is degenerate} \\ 0, & \text{if } f \text{ is nondegenerate.} \end{cases} \end{aligned}$$

The **incidence algebra** of K is the dual $I(K) = \mathrm{Hom}_k(C(K), k)$.

As with all previous constructions, $I(K)$ is an algebra via the convolution product:

$$\varphi * \psi : f \longmapsto \sum_{d_1\sigma=f} \varphi(d_2\sigma) \otimes \psi(d_0\sigma).$$

Let $\zeta \in I(K)$ be the zeta function of K , sending $f \mapsto 1$ for all $f \in X_1$.

Example 3.16. When $K = \mathcal{N}(\mathcal{D})$ is the nerve of a Möbius category $\mathcal{D}$, $I(\mathcal{D}) := I(\mathcal{N}(\mathcal{D}))$ agrees with the incidence algebra of $\mathcal{D}$ as defined in [Ler]. In particular, the convolution product on $I(\mathcal{D})$ is given by

$$\varphi * \psi : f \longmapsto \sum_{h \circ g = f} \varphi(g) \otimes \psi(h)$$

for any morphism $f \in \mathrm{mor}(\mathcal{D})$, where the sum is over all factorizations of f in $\mathrm{mor}(\mathcal{D})$. The axioms of a Möbius category ensure such a sum is well-defined.

In general, $I(K)$ need not be associative or unital. However, these properties hold when K is a discrete decomposition space. To state the definition, we first need the following notions.

Definition 3.17. In the category Δ , a morphism $g : [n] \rightarrow [m]$ is **active** if $g(0) = 0$ and $g(n) = m$. On the other hand, g is **inert** if $g(i+1) = g(i) + 1$ for all $0 \leq i \leq n-1$.

In other words, active morphisms “preserve endpoints” and inert morphisms “preserve distances”.

Definition 3.18. A **discrete decomposition space** is a simplicial set $K : \Delta^{op} \rightarrow \mathbf{Set}$ that takes any pushout diagram in Δ of the form

$$\begin{array}{ccc} [p] & \longleftarrow & [m] \\ \uparrow & \lrcorner & \uparrow f \\ [\ell] & \xleftarrow{g} & [n] \end{array}$$

where f is inert and g is active, to a pullback diagram in $\mathbf{Set}$:

$$\begin{array}{ccc} K_p & \longrightarrow & K_m \\ \downarrow & \lrcorner & \downarrow f^* \\ K_\ell & \xrightarrow{g^*} & K_n \end{array}$$

Example 3.19. To see things concretely, let $(\mathbb{N}, |)$ be the division poset (Examples 3.2 and 3.10), viewed as a simplicial set K with

$$K_n = \{\text{length } n+1 \text{ decompositions of intervals in } \mathbb{N}\},$$

which can be visualized as sequences of divisibilities $(a | b) : a = a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_n = b$ where some adjacent a_i ’s may repeat. If $g : [n] \rightarrow [\ell]$ is an active arrow in Δ , then $g^* : K_\ell \rightarrow K_n$ sends a length $\ell+1$ decomposition $a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_\ell$ to the length $n+1$ decomposition

$$a_0 = a_{g(0)} \rightarrow a_{g(1)} \rightarrow \cdots \rightarrow a_{g(n)} = a_\ell.$$

In other words, the interior factors $a_{g(1)}, \dots, a_{g(n-1)}$ are a possibly new way of writing the divisibility $a_0 | a_\ell$. On the other hand, if $f : [n] \rightarrow [m]$ is an inert arrow in Δ , then $f^* : K_m \rightarrow K_n$ sends a length $m+1$ decomposition $b_0 \rightarrow b_1 \rightarrow \cdots \rightarrow b_m$ to the length $n+1$ decomposition

$$b_{f(0)} \rightarrow b_{f(1)} = b_{f(0)+1} \rightarrow \cdots \rightarrow b_{f(n)} = b_{f(0)+n}.$$

This is just the decomposition of a particular divisibility $b_i | b_{i+n}$ induced by the original decomposition $b_0 | b_m$.

Let’s take $n = 1$ for simplicity. If $[p]$ is the pushout of $[\ell]$ and $[m]$ along f and g as in the diagrams above, then $p = \ell + m - 1$. To see that $K_{\ell+m-1}$ is precisely the pullback of K_ℓ and K_m along g^* and f^* , notice that for $(a_0 | a_\ell) \in K_\ell$ and $(b_0 | b_m) \in K_m$ to map to the same decomposition $(c_0 | c_1)$ in K_1 , we must have $a_0 = c_0 = b_{f(0)}$ and $a_\ell = c_1 = b_{f(0)+1}$. Then this determines a decomposition

$$(a_0 | b_{f(m)}) : a_0 \rightarrow \cdots \rightarrow a_\ell = b_{f(1)} \rightarrow b_{f(2)} \rightarrow \cdots \rightarrow b_{f(m)} \in K_{\ell+m-1}.$$

Conversely, any such decomposition in $K_{\ell+m-1}$ mapping to the same $(c_0 | c_1) \in K_1$ along g^* and f^* must split as above. Therefore $K_{\ell+m-1}$ is the desired pullback. In plain language, the inert map picks out a certain subdecomposition to preserve and the active map prescribes a further decomposition of the first term of that subdecomposition.

Example 3.20. Replacing $(\mathbb{N}, |)$ by $(I_K^+, |)$ or $(Z_0^{\text{eff}}(X), \leq)$ and the decompositions $(a | b)$ by $(\mathfrak{a} | \mathfrak{b})$ or $(\alpha \leq \beta)$, respectively, one can write an identical proof that $(I_K^+, |)$, resp. $(Z_0^{\text{eff}}(X), \leq)$, is a discrete decomposition space.

Example 3.21. More generally, for any category $\mathcal{C}$, the nerve $\mathcal{N}(\mathcal{C})$ is a discrete decomposition space. (This follows from the more general statement for decomposition spaces; cf. [GKT1, Prop. 3.7], but a direct proof can be given in a similar manner as in the example above.)

If this all feels a bit abstract, then suffice it to say that the axioms of a discrete decomposition space are in some sense the minimum requirements for the incidence algebra of a simplicial set to be well-behaved. Indeed, we have the following.

Proposition 3.22. *For a discrete decomposition space K , the incidence (co)algebra of K is (co)associative and (co)unital.*

Proof. This follows from the more general statement for decomposition spaces; cf. [GKT1, Sec. 5.3]. However, a more down-to-earth explanation for discrete decomposition spaces can be found in [ABD+, 15.3]. $\square$

As the authors in [GKT1] explain, a discrete decomposition space is *precisely* the right structure to be able to define a (co)associative, (co)unital incidence (co)algebra. Indeed, the work of Gálvez-Carrillo, Kock and Tonks in [GKT1], [GKT2] and [GKT3] subsumes the theory for posets – in particular, our examples $(\mathbb{N}, |)$, $(I_K^+, |)$, $(Z_0^{\text{eff}}(X), \leq)$ – as any (locally finite) poset can be regarded as a (Möbius) category, hence a discrete decomposition space. The three zeta functions $\zeta_{\mathbb{Q}}(s)$, $\zeta_K(s)$ and $Z(X, t)$ all arise from the canonical zeta element in the incidence algebra of the corresponding discrete decomposition space, and there is a clear theory of Möbius inversion which can be realized in the incidence algebra.

Nevertheless, an important class of zeta functions appears to be out of reach: Kapranov’s motivic zeta functions $Z_{\text{mot}}(X, t)$ for varieties X over an arbitrary field k . Indeed, there is no clear candidate for a discrete decomposition space, not to mention a locally finite poset, that naturally produces the coefficients of $Z_{\text{mot}}(X, t)$. To realize $Z_{\text{mot}}(X, t)$ as such, we would like to turn to the full power of decomposition spaces. A description of $Z_{\text{mot}}(X, t)$ in the language of incidence algebras is in progress.

3.3 Decomposition Spaces

Passing from set theory to homotopy theory, we replace the category **Set** with a suitable category $\mathcal{S}$ of spaces. In this section, we will take $\mathcal{S}$ to be either the category of simplicial sets or the category of groupoids. In [GKT1], [GKT2] and [GKT3], as well as related works, the authors jump straight to the ∞ -category of ∞ -groupoids, which are prevalent elsewhere in the literature. We elect here to keep things as concrete as possible, while noting that such generalizations are readily available.

Definition 3.23. *A simplicial space is a functor $X : \Delta^{op} \rightarrow \mathcal{S}$.*

We denote by $s\mathcal{S}$ the category of simplicial spaces. We note that $s\mathcal{S}$ has all limits and colimits and they are computed levelwise. A *discrete simplicial space* is a simplicial space that lies in the essential image of the functor $s\mathbf{Set} \rightarrow s\mathcal{S}$ induced by the embedding $\mathbf{Set} \hookrightarrow \mathcal{S}$.

Definition 3.24. A **decomposition space** is a simplicial space $X : \Delta^{op} \rightarrow \mathcal{S}$ that takes any pushout diagram in Δ of the form

$$\begin{array}{ccc} [p] & \longleftarrow & [m] \\ \uparrow & \lrcorner & \uparrow f \\ [\ell] & \xleftarrow{g} & [n] \end{array}$$

where f is inert and g is active, to a homotopy pullback diagram in $\mathcal{S}$:

$$\begin{array}{ccc} X_p & \longrightarrow & X_m \\ \downarrow & \lrcorner & \downarrow f^* \\ X_\ell & \xrightarrow{g^*} & X_n \end{array}$$

We will see that decomposition spaces are in a sense a homotopy-theoretic version of incidence coalgebras. This insight allows us to recover the algebras of arithmetic functions that played a role in Section 2 from a *purely homotopy theoretic construction*, and also should allow us in the future to study generalizations such as Kapranov’s motivic zeta functors.

Example 3.25. We now describe an important example of decomposition spaces. The notion of a *Segal space*, due to Rezk [Rez] and based on earlier work of Segal [Seg], generalizes the nerve of a (small) category in the following way. Let $\mathcal{C}$ be a small category and consider its nerve $\mathcal{N}(\mathcal{C})$ as a simplicial set whose set of n th simplices $\mathcal{N}(\mathcal{C})_n$ is the set of all strings of n composable morphisms in $\mathcal{C}$ (with obvious face and degeneracy maps). On the other hand, a simplicial set K is called a *Segal set* if for every $n \geq 1$, the so-called *Segal maps*

$$\varphi_n : K_n \longrightarrow \underbrace{K_1 \times_{K_0} \cdots \times_{K_0} K_1}_n$$

are bijections. It is an easy consequence of the definition (cf. [Rez, 4.4]) that a simplicial set is a Segal set if and only if it is isomorphic to the nerve of a small category. Rezk upgrades this definition to the context of simplicial spaces by specifying maps of spaces

$$\varphi_n : K_n \longrightarrow \underbrace{K_1 \times_{K_0}^h \cdots \times_{K_0}^h K_1}_n$$

where $\times^h$ denotes homotopy pullback, and defining a *Segal space* to be a simplicial space for which these maps are weak equivalences for all $n \geq 1$. Thus the nerve of a small category is nothing more than a discrete Segal space. Moreover, every Segal space is a decomposition space [GKT1, Prop. 3.7]. In fact, decomposition spaces are precisely the same as Dyckerhoff and Kapranov’s notion of *2-Segal spaces* [DK], a further generalization of Segal sets; by [GKT1, Rem. 3.2], a decomposition space is the same thing as a unital 2-Segal space, but the unital condition was later shown to be redundant in [FGK]. A 2-Segal space also allows one to construct categorical generalizations of *Hall algebras* (cf. [DK, Sec. 8], and it will be interesting to know to what extent these interact with the incidence algebras of Gálvez-Carrillo, Kock and Tonks.

3.4 Homotopy Linear Algebra

As we remarked above, decomposition spaces are a vast generalization of incidence coalgebras. To make this precise, we introduce the reader to the perspectives of objective linear algebra (appearing in [LM]) and homotopy linear algebra (as developed in [GKT4]). Loosely, the idea is to replace vectors, vector spaces and linear maps with spaces, slice categories and linear functors. In this setting, it is possible to define the incidence coalgebra of a decomposition space and take its homotopy linear algebraic dual to get an incidence algebra. For a detailed discussion, see [ABD+, Sec. 19].

The first step (objective linear algebra) is to relax our notions of linear algebra a bit. We take the category **Set** to be our ‘ground field of scalars’, together with the rudimentary operations of addition $S + T = S \amalg T$ and multiplication $ST = S \times T$. Notice that taking cardinality recovers ordinary addition and multiplication on our ordinary scalars, but things like subtraction and inverses, when they exist, need not lift to the realm of sets. In any case, treating **Set** as the ground field recovers enough aspects of linear algebra to be of use.

A vector can be represented by a map of sets $v : V \rightarrow S$: the ‘entries’ of v are the sets $v^{-1}(s)$ for $s \in S$. Thinking of S as a basis for some vector space (more on this in a moment), the ‘entry’ $v^{-1}(s)$ represents not just how many copies of s are in the vector, but *how they are indexed*. Taking cardinality (of finite sets) recovers our more familiar notion of a vector. Scalar multiplication then is taking a product $A \times V \rightarrow V \rightarrow S$. The sum of two vectors $v : V \rightarrow S$ and $w : W \rightarrow S$ is the vector $v + w : V \amalg W \rightarrow S$ given by the universal property of $\amalg$:

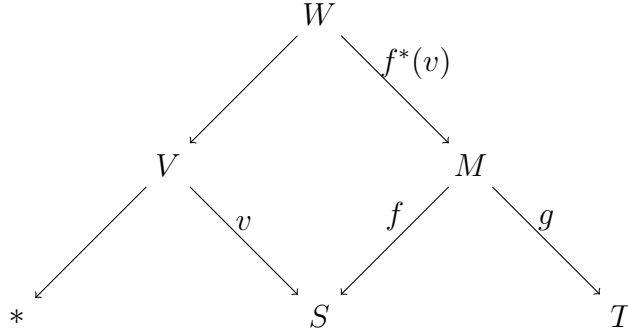
$$\begin{array}{ccc}
 & W & \\
 & \downarrow & \\
 V & \longrightarrow & V \amalg W \\
 & \searrow v & \swarrow v+w \\
 & & S
 \end{array}$$

(Note: In the original image, there is also a curved arrow from V to S labeled v , and a curved arrow from W to S labeled w . The diagram shows the universal property of the coproduct $V \amalg W$ where maps v and w from V and W respectively to S are combined into a single map $v+w$ from $V \amalg W$ to S .

Thus the slice category $\mathbf{Set}_{/S}$ of sets over S (that is, maps $v : V \rightarrow S$) should be regarded as ‘the vector space with basis S ’.

We can also translate linear maps between vector spaces to the objective setting. Suppose for the moment we are dealing with two ‘finite dimensional vector spaces’, or slice categories $\mathbf{Set}_{/S}$ and $\mathbf{Set}_{/T}$ where $|S| = n < \infty$ and $|T| = m < \infty$. A linear map $\mathbf{Set}_{/S} \rightarrow \mathbf{Set}_{/T}$ should then be an analogue of an $n \times m$ matrix of scalars. This can be represented as a map $M \rightarrow S \times T$, which in turn is the same thing as a span $* \leftarrow M \rightarrow T$. One can check (or cf. [ABD+, Sec. 19]) that the usual operations on matrices, including scalar multiplication, addition and matrix-vector and matrix-matrix multiplication, are encoded by span composition. Here, a scalar (a set) is treated as a span $* \leftarrow S \rightarrow *$ and a vector is viewed as either $* \leftarrow V \rightarrow S$ (a $1 \times n$ matrix, if $|S| = n < \infty$) or $S \leftarrow V \rightarrow *$ (an $n \times 1$ matrix), where appropriate.

So a linear map should correspond to a span $S \leftarrow M \rightarrow T$, but we'd like for such a map to actually be a functor $\mathbf{Set}_{/S} \rightarrow \mathbf{Set}_{/T}$. Given a ‘matrix’ $S \xleftarrow{f} M \xrightarrow{g} T$ and a ‘vector’ $V \xrightarrow{v} S$, applying the linear map to the vector is encoded by the composition of spans



The output vector is then $W \rightarrow T$, viewed as the larger span $* \leftarrow W \rightarrow T$. More specifically, the map $W \rightarrow T$ is the composition $g_! f^*(v)$ where $g_!$ denotes postcomposition with g and f^* denotes the pullback along f in the upper diamond (which is a pullback square). Since $g_!$ and f^* extend to the slice categories, $g_! : \mathbf{Set}_{/M} \rightarrow \mathbf{Set}_{/T}$ and $f^* : \mathbf{Set}_{/S} \rightarrow \mathbf{Set}_{/M}$, it makes sense to take this as a *definition* of a linear map. We will call a functor $a : \mathbf{Set}_{/S} \rightarrow \mathbf{Set}_{/T}$ a *linear functor* if it factors as $a = g_! f^*$ for some span $S \leftarrow M \xrightarrow{g} T$.

Other operations on vector spaces can be defined in this context as well. The tensor product of two vector spaces $\mathbf{Set}_{/S}$ and $\mathbf{Set}_{/T}$ is defined by $\mathbf{Set}_{/S} \otimes \mathbf{Set}_{/T} := \mathbf{Set}_{/S \times T}$. The vector space of linear maps from S to T is the space $\mathrm{LIN}(S, T) := \mathrm{Fun}^L(\mathbf{Set}_{/S}, \mathbf{Set}_{/T})$ of colimit-preserving functors $\mathbf{Set}_{/S} \rightarrow \mathbf{Set}_{/T}$ (the superscript L stands for left adjoint, as colimit-preserving functors are the same as left adjoints). Likewise, a vector space dual is given by $(\mathbf{Set}_{/S})^* := \mathrm{Fun}(\mathbf{Set}_{/S}, \mathbf{Set})$. From the natural equivalence $\mathbf{Set}_{/S} \simeq \mathrm{Fun}(\mathbf{Set}_{/S}, \mathbf{Set})$, we recover (cf. [GKT4, 2.10]) the formula

$$\mathrm{LIN}(S, T) \simeq (\mathbf{Set}_{/S \times T})^*.$$

In particular, $\mathrm{LIN}(S, T)$ is another vector space and the functors in $\mathrm{LIN}(S, T)$ are given by spans, so they justifiably can be called linear. Plenty more linear algebra can be translated to this objective language, but this suffices for our purposes.

To promote the above to a *homotopy linear algebra*, let $\mathcal{S}$ be the category of spaces. Following [GKT4], we think of $\mathcal{S}$ as our ground field of scalars; a space $S \in \mathcal{S}$ as a basis for the vector space $\mathcal{S}_{/S}$; a morphism $v : V \rightarrow S$, i.e. an object of $\mathcal{S}_{/S}$, as a vector; and homotopy products and coproducts as scalar multiplication and addition. Linear maps are a little more delicate to describe. Briefly, the authors in [GKT4] construct a category LIN of (∞) -categories spanned by the slice categories $\mathcal{S}_{/S}$ whose mapping spaces $\mathrm{LIN}(\mathcal{S}_{/S}, \mathcal{S}_{/T})$ behave like the spaces of linear functors constructed above. They also construct a tensor product $\mathcal{S}_{/S} \otimes \mathcal{S}_{/T} := \mathcal{S}_{/S \times T}$ and a linear dual $(\mathcal{S}_{/S})^* := \mathrm{Fun}(\mathcal{S}_{/S}, \mathcal{S})$.

Remark 3.26. As suggested by the parenthetical ∞ - in the previous paragraph, all of this can be done at the level of ∞ -categories. Indeed, this is the generality with which the authors in [GKT4] state things. Since we do not require the technology of ∞ -categories in the present article, we leave it to the reader to further explore ∞ -categorical homotopy linear algebra by reading [GKT4].

3.5 The Incidence Algebra of a Decomposition Space

Fix a simplicial space X . Homotopy linear algebra allows one to view $\mathcal{S}_{/X_1}$ as a “vector space with basis X_1 ”. Further, the face and degeneracy maps of X induce the structure of a coalgebra on $\mathcal{S}_{/X_1}$.

Definition 3.27. *The incidence coalgebra of X is the slice category $C(X) := \mathcal{S}_{/X_1}$ equipped with linear functors $\Delta : \mathcal{S}_{/X_1} \otimes \mathcal{S}_{/X_1} \rightarrow \mathcal{S}_{/X_1}$ and $\delta : \mathcal{S}_{/X_1} \rightarrow \mathcal{S}$, called **comultiplication** and **counit**, respectively, which are induced by the spans*

$$\Delta : X_1 \xleftarrow{d_1} X_1 \times X_1 \xrightarrow{(d_2, d_0)} X_2 \quad \text{and} \quad \delta : X_1 \xleftarrow{s_0} X_0 \rightarrow *.$$

In the notation above, $\Delta = (d_2, d_0)_! d_1^*$ and $\delta = t_! s_0^*$ where $t : X_0 \rightarrow *$ is the unique map to the terminal object.

Proposition 3.28 ([GKT1, Thm. 7.4]). *If X is a decomposition space, $C(X)$ is a coassociative, counital coalgebra object (homotopy comonoid) in the category $\mathbf{LIN}$, with comultiplication Δ and counit δ .*

Taking the homotopy linear algebraic dual yields a notion of incidence algebra.

Definition 3.29. *The incidence algebra of a simplicial space X is the dual $I(X) := (\mathcal{S}_{/X_1})^* = \text{Fun}(\mathcal{S}_{/X_1}, \mathcal{S})$. It is equipped with a linear functor $m : I(X) \otimes I(X) \rightarrow I(X)$ called **multiplication**. Explicitly, for objects $f, g \in I(X)$, their product $m(f, g)$ is given by*

$$m(f, g) : \mathcal{S}_{/X_1} \xrightarrow{\Delta} \mathcal{S}_{/X_1} \otimes \mathcal{S}_{/X_1} \xrightarrow{f \otimes g} \mathcal{S} \otimes \mathcal{S} \xrightarrow{\sim} \mathcal{S}.$$

Corollary 3.30. *If X is a decomposition space, $I(X)$ is an associative, unital algebra object (homotopy monoid) in the category $\mathbf{LIN}$, with multiplication m and unit δ .*

This shows that the notion of decomposition space is a good homotopy theoretic analogue to the notion of incidence algebra. More precise statements of this analogy can be found in [GKT1].

4 Future Directions

The full power of decomposition spaces are not needed to describe many of the zeta functions of interest to number theorists, as they arise directly from posets, which are discrete decomposition spaces. However, one important type of zeta function – Kapranov’s motivic zeta function $Z_{\text{mot}}(X, t)$ [Kap] – does not fall neatly into the framework of posets.

In their article [CWZ], Campbell, Wolfson and Zakharevich lift the Hasse–Weil zeta function to a map of K -theory spectra

$$\zeta : K(\text{Var}_k) \longrightarrow K(\text{Aut}(\mathbb{Z}_\ell))$$

where $\text{Aut}(\mathbb{Z}_\ell)$ denotes the exact category of finitely generated $\mathbb{Z}_\ell$ -modules with automorphism. They call this the “derived ℓ -adic zeta function” and applying π_0 recovers the Hasse–Weil zeta function via the composition

$$\begin{aligned} K_0(\mathbf{Var}_k) &\xrightarrow{\pi_0\zeta} K_0(\text{Aut}(\mathbb{Z}_\ell)) \xrightarrow{\sim} (1 + t\mathbb{Z}_\ell[[t]], \cdot) \\ [F] &\longmapsto \det(1 - tF). \end{aligned}$$

As the authors ask in [CWZ, Question 7.6], one hopes for a lift of the motivic measure

$$Z_{\text{mot}}(-, t) : K_0(\mathbf{Var}_k) \longrightarrow (1 + tK_0(\mathbf{Var}_k)[[t]], \cdot)$$

to a map of K -theory spectra, ideally in a way that is compatible with the specialization $Z_{\text{mot}}(X, t) \mapsto Z(X, t)$ via the motivic measure $\# : K_0(\mathbf{Var}_k) \rightarrow \mathbb{Z}$. We hope to address this question in future work, using the framework laid out in the present article. More specifically, starting with the simplicial space $\tilde{S}_\bullet(\mathbf{Var}_k)$ defined by Campbell in [Cam], one can perform two operations:

- (a) Take its K -theory spectrum $K(\mathbf{Var}_k)$, as considered in [Cam], [CZ] and [CWZ]. One might then construct morphisms out of $\tilde{S}_\bullet(\mathbf{Var}_k)$ which determine the various maps of ring spectra out of $K(\mathbf{Var}_k)$ in [CWZ], especially $\zeta : K(\mathbf{Var}_k) \rightarrow K(\text{Aut}(\mathbb{Z}_\ell))$. We are currently searching for such a morphism which would give a homotopy theoretic ‘motivic zeta functor’, but at present it is unclear what the target simplicial space should be.
- (b) Construct the incidence algebra of $\tilde{S}_\bullet(\mathbf{Var}_k)$ and identify its zeta function.

One question we have is:

Question 4.1. *To what extent does this abstract zeta function $\zeta \in I(\tilde{S}_\bullet(\mathbf{Var}_k))$ interact with or even determine the Hasse–Weil, derived ℓ -adic, motivic and other zeta functions?*

Despite not having answers to these questions yet, we believe there is a great deal of information hidden in the structure of $\tilde{S}_\bullet(\mathbf{Var}_k)$ that can shine a new light on structural aspects of zeta functions.

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