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Cyclotomic complexes

D. Kaledin

Abstract. We construct a triangulated category of cyclotomic complexes (homological analogues of the cyclotomic spectra of Bökstedt and Madsen) along with a version of the topological cyclic homology functor TC for cyclotomic complexes and an equivariant homology functor (commuting with TC) from cyclotomic spectra to cyclotomic complexes. We also prove that the category of cyclotomic complexes essentially coincides with the twisted 2-periodic derived category of the category of filtered Dieudonné modules, which were introduced by Fontaine and Lafaille. Under certain conditions we show that the functor TC on cyclotomic complexes is the syntomic cohomology functor.

Keywords: cyclotomic spectrum, cyclotomic complex, filtered Dieudonné module.

To I. R. Shafarevich on his 90th birthday with esteem and gratitude

Introduction

This paper is a sequel to [1]. The goal of both papers is to try to understand how some of the purely topological notions used in [2] are related to the topological cyclic homology of [3]. This has turned out to be a rather lengthy project since one has to construct the appropriate homological counterparts of several of the notions in stable homotopy theory.

In [1] we dealt with derived Mackey functors, the homological analogues of the genuine G-equivariant spectra in [4]. Although the Abelian category of Mackey functors is very well known in stable homotopy theory and plays an important role, its most naive derived generalization is not completely well behaved. Therefore a slightly different derived version of the Mackey functors was constructed and studied in [1].

In the present paper we deal with the cyclotomic spectra considered in [5] and [6]. To construct their homological counterparts, one must go beyond [1]: by definition, cyclotomic spectra are equivariant with respect to the circle group S^1 while [1] deals only with finite groups G. We cannot construct good homological analogues of all S^1 -spectra, but we construct a category $\mathcal{DMA}(\mathbb{Z})$ of 'cyclic Mackey functors'

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which captures those parts of the equivariant stable category that are relevant to topological cyclic homology. We can then introduce the triangulated category $\mathcal{D}\Lambda R(\mathbb{Z})$ of 'cyclotomic complexes'.

Ideally, the relation between cyclotomic complexes and cyclotomic spectra should be expressible in the form of a commutative diagram,

of 'brave new schemes', which can be understood, for example, as tensor triangulated categories with some enhancement. Here StHom is the stable homotopy category, $\mathcal{D}(\mathbb{Z})$ is the derived category of Abelian groups, and Cycl is the category of cyclotomic spectra. This diagram must be 'almost Cartesian'. More precisely, it must become Cartesian after replacing Cycl by the subcategory Cycl^o \subset Cycl of cyclotomic spectra T with trivial geometric fixed points $\Phi^{S^1}T$ with respect to the whole of the group S^1 .

At present, such a nice picture seems a long way beyond our reach. Besides the obvious difficulties in making the 'brave new' notions precise, it seems that no-one has yet constructed Cycl even as a triangulated category.¹ Thus in practice we confine ourselves to the following two things.

(i) For every cyclotomic spectrum T we construct an equivariant homology cyclotomic complex $C_{\bullet}(T)$. This ought to correspond to the top arrow in (0.1).

(ii) We construct a topological cyclic homology functor TC on the category $\mathcal{DAR}(\mathbb{Z})$ in such a way that for every cyclotomic spectrum T, the complex $\mathrm{TC}(C_{\bullet}(T))$ can be naturally identified with the homology of the spectrum $\mathrm{TC}(T)$.

We can then give another and very simple description of the category $\mathcal{D}\Lambda R(\mathbb{Z})$. It turns out that cyclotomic complexes are essentially equivalent to the 'filtered Dieudonné modules' in [7]. Filtered Dieudonné modules are rather simple linear-algebraic gadgets with a deep meaning: they supply a *p*-adic counterpart of Deligne's notion of a mixed Hodge structure. Hence the whole story acquires a distinctly motivic flavour. This is discussed in more detail in [8].

Filtered Dieudonné modules arise naturally as the crystalline cohomology of algebraic varieties over \mathbb{Z}_p , while cyclotomic spectra appear as topological Hochschild homology spectra of ring spectra A. In view of the equivalence established by us, there are many situations in which one can compare these two constructions. We make no attempt to do this here. We focus instead on pure linear algebra and leave the geometric applications for future research. The only comparison result that we prove says that for profinitely complete cyclotomic complexes, the topological cyclic homology TC coincides with the syntomic cohomology of [9], familiar in the theory of Dieudonné modules.

The paper is organized as follows. To begin the story, we need a model for S^1 -equivariant spaces and their homology; for better or for worse, we have chosen a combinatorial approach using Connes' category Λ . The basic facts about Λ and related categories are contained in §1. In §2 we construct the cyclic Mackey

¹After submission of our paper, this was done by Blumberg and Mandell, arXiv:1303.1694.

functors. §3 deals with cyclotomic complexes. In §4 we construct equivariant homology functors from S^1 -spectra to $\mathcal{DMA}(\mathbb{Z})$ and from cyclotomic spectra to $\mathcal{DAR}(\mathbb{Z})$. A theorem comparing cyclotomic spectra and filtered Dieudonné modules is proved in §5. Finally, in §6 we briefly discuss topological cyclic homology and prove comparison theorems for TC. The appendix, §7, contains some technicalities, mostly from [1].

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§1. Cyclic categories

1.1. Connes' cyclic category. We recall that *Connes' cyclic category* Λ is a small category whose objects [n] are indexed by positive integers n. Maps from [n] to [m] can be defined in various equivalent ways; for the convenience of the reader, we recall two of them.

Topological description. The object [n] is to be thought of as a 'wheel': a cellular decomposition of the circle S^1 with n 0-cells, called *vertices*, and n 1-cells, called *edges*. A continuous map $f: S^1 \to S^1$ induces a map $\tilde{f}: \mathbb{R} \to \mathbb{R}$ between the universal coverings. We say that f is *monotone* if $\tilde{f}(a) \geq \tilde{f}(b)$ for all $a, b \in \mathbb{R}$, $a \geq b$. Then the morphisms from [n] to [m] in Λ are the homotopy classes of monotone continuous maps $f: [n] \to [m]$ of degree 1 that send vertices to vertices. *Combinatorial description*. Consider the category Λ_{big} of totally ordered sets equipped with an order-preserving endomorphism τ . Let $[n] \in \Lambda_{\text{big}}$ be the set \mathbb{Z} with its natural total order and the endomorphism $\tau: \mathbb{Z} \to \mathbb{Z}, \tau(a) = a + n$. Let $\Lambda_{\infty} \subset \Lambda_{\text{big}}$ be the full subcategory spanned by $[n], n \geq 1$. For any $[n], [m] \in \Lambda_{\infty},$ τ acts on the set $\Lambda_{\infty}([n], [m])$ (on the left or on the right; by definition it does not matter). We define the set $\Lambda([n], [m])$ of maps in the category Λ by putting

$$\Lambda([n], [m]) = \Lambda_{\infty}([n], [m])/\tau.$$
(1.1)

There are other descriptions (see [10], Pt. 6, and [11], Appendix).

For every $[n] \in \Lambda$, the set V([n]) of vertices of the corresponding decomposition of the circle can be naturally identified with the set $\Lambda([1], [n])$ of maps from [1]to [n], and the set E([n]) of edges can be identified with $\Lambda([n], [1])$. In particular, E(-) is a contravariant functor (geometrically, the pre-image of an edge is contained in exactly one edge). The automorphism group $\operatorname{Aut}([n])$ is the cyclic group $\mathbb{Z}/n\mathbb{Z}$ generated by the clockwise rotation. We denote the generator by σ . In the combinatorial description, σ corresponds to the map $\mathbb{Z} \to \mathbb{Z}$, $a \mapsto a + 1$.

Given any integer $p \ge 2$, we define a category Λ_p by taking the same set of objects $[n], n \ge 1$, and putting

$$\Lambda_p([n], [m]) = \Lambda_\infty([n], [m]) / \tau^p$$

The category Λ_p is intermediate between Λ_{∞} and Λ . In particular, the obvious projections $\Lambda_{\infty}([n], [m])/\tau^p \to \Lambda_{\infty}([n], [m])/\tau$ together determine a functor

$$\pi_p \colon \Lambda_p \to \Lambda.$$

The functor π_p is a bifibration whose fibre $\operatorname{pt}_p = [1/(\mathbb{Z}/p\mathbb{Z})]$ is the groupoid with one object and automorphism group $\mathbb{Z}/p\mathbb{Z}$. On the other hand, one can identify $\Lambda_p([n], [m])$ with the set of maps $f: [np] \to [mp]$ in Λ such that $f \circ \sigma^n = \sigma^m \circ f$. This determines a canonical functor

$$i_p \colon \Lambda_p \to \Lambda$$

such that $i_p([n]) = [np]$ on the objects. We denote by

$$\Lambda I = \coprod_{p \ge 1} \Lambda_p \tag{1.2}$$

the disjoint union of all the categories Λ_p , $p \ge 1$. Then all functors i_p and π_p together may be regarded as two functors

$$i, \pi \colon \Lambda I \to \Lambda.$$
 (1.3)

The category Λ is self-dual: there is an equivalence $\Lambda \cong \Lambda^{\text{opp}}$ sending every object to itself, and a morphism $[n] \to [m]$ represented by a map $f: \mathbb{Z} \to \mathbb{Z}$ goes to the morphism represented by $f_{\sharp}: \mathbb{Z} \to \mathbb{Z}$, where

$$f_{\sharp}(a) = \max\{b \in \mathbb{Z} \mid f(b) \leqslant a\}.$$
(1.4)

In the topological description, the duality interchanges edges and vertices and corresponds to taking the dual cellular decomposition.

For any $[n], [m] \in \Lambda$, the set $\Lambda([n], [m])$ of maps is finite. The groups $\operatorname{Aut}([n])$ and $\operatorname{Aut}([m])$ act on $\Lambda([n], [m])$ by composition, and both these actions are stabilizer-free. We shall use the following slightly more general fact.

Lemma 1.1. Let m, n, l be integers such that $m, l \ge 1, n \ge 2$, and let $f: [nl] \rightarrow [m]$ be a map in Λ such that

$$f \circ \sigma^l = \sigma^{l_1} \circ f$$

for some integer l_1 , $0 \leq l_1 < m$. Then $m = nl_1$.

Proof. We use the combinatorial description of Λ . Then f is represented by a monotone map $\tilde{f}: \mathbb{Z} \to \mathbb{Z}$ such that

$$\widetilde{f}(a+nl) = \widetilde{f}(a) + m, \qquad \widetilde{f}(a+l) = \widetilde{f}(a) + l_1 + bm$$
 (1.5)

for all $a \in \mathbb{Z}$, where b is a fixed integer independent of a. Since $a \leq a+l \leq a+nl$, it follows from (1.5) that $0 \leq l_1 + bm \leq m$, whence either $l_1 = 0$ and b = 1, or b = 0. The first case is impossible since σ^l acts on $\Lambda([nl], [m])$ without fixed points. Thus b = 0, and the desired assertion follows from (1.5). \Box

The category $\Lambda/[1]$ of objects $[n] \in \Lambda$ equipped with a map $[n] \to [1]$ is naturally equivalent to the category Δ of non-empty finite totally ordered sets. (Geometrically, $\Lambda/[1]$ is the category of wheels with a fixed edge. Removing this edge, we get a canonical 'clockwise' total order on the set of vertices of the wheel.) Thus we have a natural discrete fibration $\Delta \cong \Lambda/[1] \to \Lambda$, which induces a cofibration $j^{\circ} \colon \Delta^{\text{opp}} \to \Lambda^{\text{opp}}$. Dually, the category $[1] \setminus \Lambda$ of objects $[n] \in \Lambda$ equipped with a map $[1] \to [n]$ is equivalent to Δ^{opp} , whence we get a natural discrete fibration $j \colon \Delta \to \Lambda^{\text{opp}}$ (geometrically, Δ^{opp} is the category of wheels with a fixed vertex). The same construction works for the categories Λ_n , $n \ge 2$. In particular, we obtain a canonical functor

$$j_n \colon \Delta \to \Lambda_n^{\mathrm{opp}}$$

and we have $\pi \circ j_n = j$ for any $n \ge 2$. Let $\Delta_n \to \Delta$ be the bifibration given by the Cartesian square



Then the functor j_n gives a splitting $\Delta \to \Delta_n$ of this bifibration, so that we have $\Delta_n \cong \Delta \times \operatorname{pt}_n$. In more down-to-earth terms, this means that the group $\mathbb{Z}/n\mathbb{Z}$ acts on j_n . Taking the composite of j_n with the embedding i_n , we get a commutative diagram



where $r_n: \Delta \to \Delta$ is the *edgewise subdivision functor* given by the formula

$$r_n([m]) = [n] \times [m].$$
 (1.6)

Here [m] (resp. [n]) is a totally ordered set of m (resp. n) elements and $[n] \times [m]$ is endowed with the left-to-right lexicographic order.

1.2. The cyclotomic category. We now introduce the following definition, which is based on the topological description of the category Λ .

Definition 1.2. The cyclotomic category ΛR is a small category with the same objects [n], $n \ge 1$, as Λ . If we regard [n] as an *n*-tuple of marked points on the circle S^1 , then the morphisms $[n] \to [m]$ in ΛR are homotopy classes of monotone continuous maps $f: [n] \to [m]$ that send marked points to marked points and have positive degree, deg $f \ge 1$.

The only difference between ΛR and Λ is that the morphisms are allowed to have degree greater than 1. A typical new map is obtained as follows. For every *n*-tuple of points on the circle and every positive integer $l \ge 1$ we consider the *l*-fold étale covering $\pi_l \colon S^1 \to S^1$ and the *nl*-tuple of pre-images of the *n* marked points. Then π determines a well-defined morphism $\pi_{n,l} \colon [nl] \to [n]$ in ΛR . Moreover, every morphism $f \colon [m] \to [m]$ of degree l in ΛR factors as a composite $f = \pi_{n,l} \circ f'$ for some $f': [m] \to [nl]$ of degree 1, and such a factorization is unique up to the action of the group $\mathbb{Z}/l\mathbb{Z}$ of deck transformations of the covering $\pi_l: S^1 \to S^1$. Thus the set $\Lambda R_l([m], [n])$ of morphisms of degree l from [m] to [n] can be naturally identified with the quotient

$$\Lambda R_l([m], [n]) = \Lambda([m], [nl]) / (\mathbb{Z}/l\mathbb{Z})$$
(1.7)

by the action of the group $\mathbb{Z}/l\mathbb{Z}$ generated by $\sigma^n \colon [nl] \to [nl]$. In particular, $\Lambda R_l([m], [n])$ is finite for all [m], [n] and l.

Definition 1.3. A map $f: [m] \to [n]$ in ΛR is *horizontal* if it is of degree 1. A map $f: [m] \to [n]$ of degree $l \ge 1$ is *vertical* if the map $f': [m] \to [nl]$ in the decomposition $f = \pi_{n,l} \circ f'$ is invertible.

It follows from the discussion above that the vertical and horizontal morphisms form a factorization system on ΛR in the sense of Definition A.2 (see § 7). The subcategory ΛR_h formed by the horizontal maps is by definition equivalent to Λ . Moreover, for any group G, let O_G be the category of *finite G-orbits*, that is, finite sets equipped with a transitive *G*-action. Then the subcategory $\Lambda R_v \subset \Lambda R$ formed by the vertical maps is obviously equivalent to the orbit category $O_{\mathbb{Z}}$. The equivalence sends $[n] \in \Lambda R$ to the orbit $\mathbb{Z}/n\mathbb{Z}$ (all finite \mathbb{Z} -orbits are of this form).

Lemma 1.4. For any pair consisting of a horizontal map $h: [m_1] \to [m]$ and a vertical map $v: [m_2] \to [m]$ in ΛR there is a Cartesian square

$$\begin{bmatrix} m_{12} \end{bmatrix} \xrightarrow{h_1} \begin{bmatrix} m_2 \end{bmatrix} \\ \downarrow^{v_1} & \downarrow^{v} \\ \llbracket m_1 \end{bmatrix} \xrightarrow{h} \begin{bmatrix} m \end{bmatrix}$$

with horizontal h_1 and vertical v_1 .

Proof. This is clear. \Box

Composing the functors i and π in (1.3) with the natural embeddings $\Lambda \cong \Lambda R_h \hookrightarrow \Lambda R$, we obtain functors

$$\widetilde{i}, \widetilde{\pi} \colon \Lambda I \to \Lambda R.$$

Moreover, the factorization maps $\pi_{n,l}$, $n, l \ge 1$, together determine a vertical map

$$\widetilde{v} \colon \widetilde{i} \to \widetilde{\pi}. \tag{1.8}$$

Let ΛI be the category of vertical maps $v \colon [m] \to [m']$ in ΛR , where the morphisms from $v_1 \colon [m_1] \to [m'_1]$ to $v_2 \colon [m_2] \to [m'_2]$ are given by commutative squares

with horizontal f, f'. Sending $a \in \Lambda I$ to $\tilde{v} \colon \tilde{i}(a) \to \tilde{\pi}(a)$, we get a functor

$$\Lambda I \to \Lambda I. \tag{1.10}$$

Lemma 1.5. The functor (1.10) is an equivalence of categories.

Proof. This is clear. \Box

Sending a wheel $[n] \in \Lambda R$ to its vertex set V([n]), we get a functor $\Lambda R \to \text{Sets.}$ Let \tilde{i}

$$\Delta R \xrightarrow{j} \Lambda R^{\text{opp}} \tag{1.11}$$

be the discrete fibration corresponding to V in the Grothendieck construction.

Lemma 1.6. The functor $\delta = \deg \circ j \colon \Delta R \to [1/\mathbb{N}^*]$ is a cofibration with fibre Δ . The transition functor r_m corresponding to $m \in \mathbb{N}^*$ is given by the edgewise subdivision functor.

Proof. By definition, ΔR is opposite to the full subcategory in the slice category $[1] \setminus \Lambda R$ spanned by the horizontal maps $h: [1] \to [n], [n] \in \Lambda R$. Moreover, ΔR inherits the vertical/horizontal system from ΛR . It follows that vertical maps are Cartesian with respect to δ , and the fibre of δ is spanned by horizontal maps. \Box

1.3. Extended categories. Let \mathbb{N}^* be the monoid of positive integers $l \ge 1$ with respect to multiplication, and let $[1/\mathbb{N}^*]$ be the category with one object 1 and

$$\text{Hom}_{[1/\mathbb{N}^*]}(1,1) = \mathbb{N}^*.$$

Sending every morphism to its degree, we get a functor

$$\deg \colon \Lambda R \to [1/\mathbb{N}^*]. \tag{1.12}$$

This functor has a section: the fully faithful embedding $\alpha \colon [1/\mathbb{N}^*] \to \Lambda R$ which sends 1 to $[1] \in \Lambda R$ (any map $[1] \to [1]$ is uniquely determined by its degree). Moreover, let

 $I = 1 \setminus [1/\mathbb{N}*]$

be the category of objects $a \in [1/\mathbb{N}^*]$ equipped with a map $1 \to a$ (the slice category). Equivalently, I is \mathbb{N}^* regarded as a partially ordered set (with order given by divisibility) and made into a small category in the standard way. Then we have a natural cofibration

$$I \to [1/\mathbb{N}^*],\tag{1.13}$$

whose fibre is the set \mathbb{N}^* regarded as a discrete category. By the Grothendieck construction, this corresponds to a functor $[1/\mathbb{N}^*] \to$ Sets that sends 1 to \mathbb{N}^* or, in other words, to an action of the monoid \mathbb{N}^* on itself. That action is by right multiplication.

We now suppose that \mathbb{N}^* acts on itself both on the right and on the left, and let \mathbb{I} be the corresponding category cofibred over $[1/\mathbb{N}^*] \times [1/\mathbb{N}^*]$ with fibre \mathbb{N}^* . Equivalently, \mathbb{I} is given by the Cartesian square

$$\begin{array}{c} \mathbb{I} \longrightarrow I \\ \downarrow & \downarrow \\ [1/\mathbb{N}^*] \times [1/\mathbb{N}^*] \longrightarrow [1/\mathbb{N}^*] \end{array}$$

where the bottom map is induced by the product map $\mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^*$. Composing the cofibration $\mathbb{I} \to [1/\mathbb{N}^*] \times [1/\mathbb{N}^*]$ with the projection onto the right factor $[1/\mathbb{N}^*]$, we get a cofibration

$$\mathbb{I} \to [1/\mathbb{N}^*] \tag{1.14}$$

with fibre I. We also have a natural Cartesian functor $I \to \mathbb{I}$. On fibres, it is given by the inclusion of the discrete category \mathbb{N}^* in I (which is nothing but \mathbb{N}^* regarded as a partially ordered set). Explicitly, the objects of \mathbb{I} are positive integers $n \ge 1$ and the morphisms are generated by the morphisms

$$F_l, R_l \colon n \to nl \tag{1.15}$$

for all $n, l \ge 1$ modulo the relations

$$F_n \circ F_m = F_{nm}, \quad R_n \circ R_m = R_{nm}, \quad F_n \circ R_m = R_m \circ F_m$$

for all $n, m \ge 1$. The category I is opposite to the category introduced by Good-willie [12].

Definition 1.7. The extended cyclic category $\Lambda \mathbb{Z}$ is the fibre product

$$\Lambda \mathbb{Z} = \Lambda R \times_{[1/\mathbb{N}^*]} I,$$

and the extended cyclotomic category $\Lambda \overline{R}$ is the fibre product

$$\widetilde{\Lambda R} = \Lambda R \times_{[1/\mathbb{N}^*]} \mathbb{I},$$

where $\Lambda R \to [1/\mathbb{N}^*]$ is the degree functor deg in (1.12), and $I \to [1/\mathbb{N}^*]$ (resp. $\mathbb{I} \to [1/\mathbb{N}^*]$) is the cofibration (1.13) (resp. (1.14)).

The section $\alpha \colon [1/\mathbb{N}^*] \to \Lambda R$ of the degree functor deg: $\Lambda R \to [1/\mathbb{N}^*]$ induces a functor

$$\widetilde{\alpha} \colon \mathbb{I} \to \Lambda R. \tag{1.16}$$

By definition, we have cofibrations

$$\lambda \colon \Lambda \mathbb{Z} \to \Lambda R, \qquad \widehat{\lambda} \colon \widehat{\Lambda R} \to \Lambda R,$$
 (1.17)

whose fibres are identified with \mathbb{N}^* and I respectively. The objects in $\Lambda\mathbb{Z}$ and ΛR are explicitly given by pairs $\langle [n] \in \Lambda R, m \in \mathbb{N}^* \rangle$. We denote such a pair by [n|m]. A morphism from [n|m] to [n'|m'] in $\Lambda\mathbb{Z}$ (resp. ΛR) is a morphism $f: [n] \to [n']$ in ΛR such that $m' = m \deg f$ (resp. $m' = lm \deg f$ for some integer $l \ge 1$). Then the vertical/horizontal factorization system induces analogous vertical/horizontal factorization systems on $\Lambda\mathbb{Z}$ and ΛR . We use the symbols $\Lambda\mathbb{Z}_v, \Lambda\mathbb{Z}_h \subset \Lambda\mathbb{Z}$ and ΛR_v , $\Lambda R_h \subset \Lambda R$ for the subcategories spanned by the vertical and horizontal maps respectively. Then $\Lambda\mathbb{Z}_h$ and ΛR_h decompose as follows:

$$\Lambda \mathbb{Z}_h = \prod_{m \ge 1} \Lambda \mathbb{Z}_h^m \cong \mathbb{N}^* \times \Lambda, \qquad \qquad \widetilde{\Lambda R}_h = I \times \Lambda, \tag{1.18}$$

where $\Lambda \mathbb{Z}_{h}^{m}$ is the full subcategory spanned by the objects [n|m], $n \ge 1$. For every m, the category $\Lambda \mathbb{Z}_{h}^{m}$ is naturally equivalent to Λ . On the other hand, the category $\Lambda \mathbb{Z}_{v}$ decomposes as follows:

$$\Lambda \mathbb{Z}_{v} = \prod_{n \ge 1} \Lambda \mathbb{Z}_{v}^{n} \cong \prod_{n \ge 1} O_{\mathbb{Z}/n\mathbb{Z}}, \qquad (1.19)$$

where $\Lambda \mathbb{Z}_v^n$ is the full subcategory spanned by the objects [n'|m'] with n = n'm'. For every n, the category $\Lambda \mathbb{Z}_v^n$ is naturally equivalent to $O_{\mathbb{Z}/n\mathbb{Z}}$. These decompositions are induced by the identifications $\Lambda R_h \cong \Lambda$, $\Lambda R_v \cong O_{\mathbb{Z}}$.

1.4. More on the extended cyclic category. We state some further simple properties of the category $\Lambda \mathbb{Z}$. By Yoneda's lemma, the category $\Lambda \mathbb{Z}$ is fully and faithfully embedded in the category Fun($\Lambda \mathbb{Z}^{opp}$, Sets). Restricting this embedding to $\Lambda = \Lambda \mathbb{Z}_h^1 \subset \Lambda \mathbb{Z}$, we obtain a functor

$$Y \colon \Lambda \mathbb{Z} \to \operatorname{Fun}(\Lambda^{\operatorname{opp}}, \operatorname{Sets}).$$

This gives an alternative purely combinatorial description of the category $\Lambda \mathbb{Z}$ because of the following lemma.

Lemma 1.8. The functor Y is fully faithful.

Proof. By definition, Y([n|1]) is the functor $h_{[n]}: \Lambda^{\text{opp}} \to \text{Sets}$ represented by $[n] \in \Lambda$. On the other hand, (1.7) shows that Y([n|m]) for $m \ge 2$ is the quotient $h_{[nm]}/(\mathbb{Z}/m\mathbb{Z})$ of the functor $h_{[nm]}$ by the action of the cyclic group $\mathbb{Z}/m\mathbb{Z} \subset \text{Aut}([nm])$. Thus we have

$$\operatorname{Hom}(Y([n|m]), Y([n'|m'])) = \operatorname{Hom}(h_{[nm]}, h_{[n'm']}/(\mathbb{Z}/m'\mathbb{Z}))^{\mathbb{Z}/m\mathbb{Z}}$$
$$= \left(\Lambda([nm], [n'm'])/(\mathbb{Z}/m'\mathbb{Z})\right)^{\mathbb{Z}/m\mathbb{Z}},$$

and, by Lemma 1.1, this set is non-empty if and only if m' = lm for some l and coincides with the set $\Lambda([n, n'l])/(\mathbb{Z}/l\mathbb{Z})$. Again using (1.7), we conclude that the last set coincides with $\Lambda\mathbb{Z}([n|m], [n'|m']) = \Lambda R_l([n], [n'l])$. \Box

Lemma 1.9. For every pair consisting of a horizontal map $h: [n_1|m] \to [n|m]$ and a vertical map $v: [n'|m'] \to [n|m]$ there is a Cartesian square

$$\begin{array}{c|c} [n'_1|m'] & \xrightarrow{h'} & [n'|m'] \\ \downarrow & & \downarrow v \\ [n_1|m] & \xrightarrow{h} & [n|m] \end{array}$$

$$(1.20)$$

in $\Lambda \mathbb{Z}$ with horizontal h' and vertical v'.

Proof. Use Lemma 1.4 and notice that deg $v_1 = \deg v$. \Box

Lemma 1.9 enables us to define a new category $\widehat{\Lambda \mathbb{Z}}$ in the following way. The objects are the same as in $\Lambda \mathbb{Z}$, and the morphisms from c to c' are the isomorphism classes of the diagrams

$$c \xleftarrow{v} c_1 \xrightarrow{h} c' \tag{1.21}$$

with vertical v and horizontal h. Composition is given by the fibre product. Note that the diagrams (1.21) have no non-trivial automorphisms. Therefore $\widehat{\Lambda \mathbb{Z}}$ has a natural factorization system whose horizontal (resp. vertical) maps are given by the diagrams with invertible v (resp. h). As above, we denote the corresponding subcategories by $\widehat{\Lambda \mathbb{Z}}_h, \widehat{\Lambda \mathbb{Z}}_v \subset \widehat{\Lambda \mathbb{Z}}$. We have decompositions

$$\widehat{\Lambda \mathbb{Z}}_{v} = \coprod_{n \ge 1} O_{\mathbb{Z}/n\mathbb{Z}}^{\text{opp}}, \qquad \widehat{\Lambda \mathbb{Z}}_{h} = \mathbb{N}^{*} \times \Lambda,$$

induced by (1.19) and (1.18). Hence $\widehat{\Lambda \mathbb{Z}}_v \cong \Lambda \mathbb{Z}_v^{\text{opp}}$ and $\widehat{\Lambda \mathbb{Z}}_h \cong \Lambda \mathbb{Z}_h$. Note that the equivalence $\Lambda \cong \Lambda^{\text{opp}}$ in (1.4) gives an equivalence

$$\Lambda \mathbb{Z}_h \cong \mathbb{N}^* \times \Lambda \to \Lambda \mathbb{Z}_h^{\text{opp}} \cong \mathbb{N}^* \times \Lambda^{\text{opp}}.$$
(1.22)

Lemma 1.10. There is an equivalence of categories

$$\widehat{\Lambda \mathbb{Z}} \cong \Lambda \mathbb{Z}^{\mathrm{opp}}$$

which restricts to the identity functor $\widehat{\Lambda \mathbb{Z}}_v \cong \Lambda \mathbb{Z}_v^{\text{opp}} \to \Lambda \mathbb{Z}_v^{\text{opp}} \subset \Lambda \mathbb{Z}^{\text{opp}}$ and induces the equivalence $\widehat{\Lambda \mathbb{Z}}_h \cong \Lambda \mathbb{Z}_h \to \Lambda \mathbb{Z}_h^{\text{opp}} \subset \Lambda \mathbb{Z}^{\text{opp}}$ in (1.22).

Proof. The form of the equivalence $\widehat{\Lambda \mathbb{Z}} \cong \Lambda \mathbb{Z}^{\text{opp}}$ is prescribed by the conditions: it is identical on the objects and vertical morphisms and sends every horizontal morphism h lying in some category $\Lambda \mathbb{Z}_h^m \subset \Lambda \mathbb{Z}, \Lambda \mathbb{Z}_h^m \cong \Lambda$ to h_{\sharp} . To see that this definition is consistent, we must check that for every Cartesian square (1.20) the diagram

$$\begin{array}{c|c} [n'_1|m'] \xrightarrow{v'} [n_1|m] \\ h'_{\sharp} & & \uparrow h_{\sharp} \\ [n'|m'] \xrightarrow{v} [n|m] \end{array}$$

is commutative. This follows immediately from the construction: the maps h and h' can be represented by the same map $[n_1m] \to [nm]$. \Box

1.5. Homological properties. The geometric realization $|\Lambda|$ of the nerve of the category Λ is homotopy equivalent to BU(1), the classifying space of the unit circle group U(1). In particular, the cohomology of the category $\Lambda \cong \Lambda^{\text{opp}}$ with coefficients in the constant functor \mathbb{Z} is given by

$$H^{\bullet}(\Lambda^{\mathrm{opp}},\mathbb{Z}) = H^{\bullet}(\Lambda,\mathbb{Z}) \cong \mathbb{Z}[u],$$

where u is a generator of degree 2. To represent $u \in H^2(\Lambda, \mathbb{Z})$ in an explicit way, we associate with a wheel $[n] \in \Lambda$ its cellular cohomology complex $C^{\bullet}([n], \mathbb{Z})$. Topologically, every wheel is the circle $S^1 \cong U(1)$. Therefore we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow C^0([n], \mathbb{Z}) \longrightarrow C^1([n], \mathbb{Z}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

for every $[n] \in \Lambda$. Since this sequence depends functorially on [n], we get an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{b_0} j_* \mathbb{Z} \xrightarrow{B} j_!^{\circ} \mathbb{Z} \xrightarrow{b_1} \mathbb{Z} \longrightarrow 0$$
 (1.23)

of functors in $\operatorname{Fun}(\Lambda^{\operatorname{opp}},\mathbb{Z})$, where we have made the identifications

$$C^{1}([n],\mathbb{Z}) \cong \mathbb{Z}[E([n])] \cong (j_{!}^{o}\mathbb{Z})([n]), \qquad C^{0}([n],\mathbb{Z}) \cong \mathbb{Z}[V([n])]^{*} \cong (j_{*}\mathbb{Z})([n]).$$

Here $j: \Delta \to \Lambda^{\text{opp}}$ and $j^{\circ}: \Delta^{\text{opp}} \to \Lambda^{\text{opp}}$ are as in § 1.1. The exact sequence (1.23) represents by Yoneda an element $u \in \text{Ext}^2(\mathbb{Z}, \mathbb{Z})$. This is the generator of the polynomial algebra $\text{Ext}^{\bullet}(\mathbb{Z}, \mathbb{Z}) = H^{\bullet}(\Lambda^{\text{opp}}, \mathbb{Z})$.

The cellular cohomology complexes are also functorial with respect to maps of degree greater than 1. Hence the exact sequence (1.23) extends to the category $\operatorname{Fun}(\Lambda R, \mathbb{Z})$. The extended sequence takes the form

$$0 \longrightarrow \mathbb{Z} \xrightarrow{b_0} \widetilde{j}_* \mathbb{Z} \xrightarrow{B} E \xrightarrow{b_1} \deg^* \mathbb{Z}(1) \longrightarrow 0, \qquad (1.24)$$

where \tilde{j} is as in (1.11), $E \in \operatorname{Fun}(\Lambda R^{\operatorname{opp}}, \mathbb{Z})$ stands for the functor $[n] \mapsto C^1([n], \mathbb{Z})$, and $\mathbb{Z}(1) \in \operatorname{Fun}([1/\mathbb{N}^*], \mathbb{Z})$ is the functor corresponding to \mathbb{Z} , where every $n \in \mathbb{N}^*$ acts as multiplication by n.

Using the functors i, π in (1.3), we can pull back the sequence (1.23) to the category ΛI^{opp} in two different ways, and the map (1.8) induces a map between these pullbacks. Fixing a positive integer $n \ge 2$ and restricting all sequences to $\Lambda_n^{\text{opp}} \subset \Lambda I^{\text{opp}}$, we obtain a commutative diagram

where the vertical maps η_n are isomorphisms. The geometric realization $|\Lambda_n|$ of the category Λ_n has the same homotopy type as $|\Lambda|$, and the functor $i_n \colon \Lambda_n \to \Lambda$ induces a homotopy equivalence between the realizations. The first row in (1.25) represents by Yoneda the generator $u \in H^2(\Lambda_n^{\text{opp}}, \mathbb{Z})$ of the cohomology algebra

$$H^{\bullet}(\Lambda_n^{\mathrm{opp}}, \mathbb{Z}) \cong H^{\bullet}(\Lambda^{\mathrm{opp}}, \mathbb{Z}) \cong \mathbb{Z}[u].$$

On the other hand, the functor $\pi_n \colon \Lambda_n \to \Lambda$ does not induce a homotopy equivalence of realizations: at the level of the realizations $|\Lambda_n| \cong |\Lambda| \cong BU(1)$, the map induced by π_n corresponds to the *n*-fold covering $U(1) \to U(1)$. The second row in (1.25) represents the element $\pi_n^* u = \nu \in H^2(\Lambda_n^{\text{opp}}, \mathbb{Z})$.

1.6. Homological vanishing. We shall use some results on the vanishing of cohomology for the small categories ΔR and ΛR^{opp} . We first recall that if we take any $p \ge 1$ and let $r_p: \Delta \to \Delta$ be the edgewise subdivision functor (1.6), then the natural map

$$H^{\bullet}(\Delta, r_p^*M) \to H^{\bullet}(\Delta, M)$$

is an isomorphism for every $M \in \mathcal{D}(\Delta, k)$ (see, for example, [2], Lemma 1.14, although this fact is very well known). By adjunction, this means that the natural map

$$r_{p!}k \to k \tag{1.26}$$

is a quasi-isomorphism for every $p \ge 1$ (here k stands for the constant functors).

Lemma 1.11. Let $h: \Delta \cong \Delta R_h \to \Delta R$ be the natural embedding and let $M \in \mathcal{D}(\Delta R^{\text{opp}}, \mathbb{Z})$ be an object such that $H^{\bullet}(\Delta, M) = 0$. Then $H^{\bullet}(\Delta R, M) = 0$.

Proof. Let $\delta: \Delta R \to [1/\mathbb{N}^*]$ be the cofibration in Lemma 1.6. It suffices to prove that $R^{\bullet}\delta_*M = 0$. Equivalently,

$$H^{\bullet}(\widetilde{\Delta R}, \kappa^* M) = 0,$$

where ΔR is the category of objects $[n] \in \Delta R$ equipped with a map $\delta([n]) \to 1$ (that is, with a number $m \in \mathbb{N}^*$) and $\kappa \colon \widetilde{\Delta R} \to \Delta R$ is the forgetful functor. For every $l \ge 1$ let $\iota_l \colon \Delta \to \widetilde{\Delta R}$ be the embedding sending [n] to itself equipped with the number l. We also have the forgetful functor $\delta \colon \widetilde{\Delta R} \to I$ induced by $\delta \colon \Delta R \to$ $[1/\mathbb{N}^*]$. By Lemma 1.6, this is a cofibration with transition functors r_n . Therefore the functor $i_l^* i_{m!} \colon \mathcal{D}(\Delta, k) \to \mathcal{D}(\Delta, k)$ is trivial unless m = lp for some p and is isomorphic to $r_{p!}$ in that case. Then, by (1.26), the conjugation map $i_{l!}k \to k$ is an isomorphism at $\iota_p(\Delta) \subset \widetilde{\Delta R}$ for all p dividing l. Therefore,

$$k \cong \lim i_{l!}k$$

and it suffices to prove that

 $\operatorname{RHom}^{\bullet}(i_{l!}k, \kappa^*M) = 0$

for any *l*. By adjunction, this means that $H^{\bullet}(\Delta, \iota_l^* \kappa^* M) = 0$, while we have $\kappa \circ \iota_l = \text{id.} \square$

Corollary 1.12. For every $M \in \operatorname{Fun}(\Lambda R^{\operatorname{opp}}, k)$ we have

$$H^{\bullet}(\Lambda R^{\mathrm{opp}}, M \otimes E) = 0,$$

where $E \in \operatorname{Fun}(\Lambda R^{\operatorname{opp}}, k)$ is as in (1.24).

Proof. As in Lemma 1.10 of [2], (1.24) yields by dévissage that

$$H^{\bullet}(\Delta R, \tilde{j}^*(M \otimes E)) = 0,$$

and by Lemma 1.11 it suffices to prove that $H^{\bullet}(\Delta, j^*h^*(M \otimes E)) = 0$. This is dual to Lemma 1.10 in [2]. \Box

By Corollary 1.12, (1.24) induces a long exact cohomology sequence

$$H^{\bullet}(\Lambda R^{\operatorname{opp}}, M) \longrightarrow H^{\bullet}(\Delta R, \widetilde{j}^*M) \longrightarrow H^{\bullet-1}(\Lambda R^{\operatorname{opp}}, M \otimes \operatorname{deg}^* \mathbb{Z}(1)) \longrightarrow$$
for every $M \in \operatorname{Fun}(\Lambda R^{\operatorname{opp}}, \mathbb{Z}).$

$$(1.27)$$

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Proposition 1.13. Let $M \in \operatorname{Fun}(\Lambda R^{\operatorname{opp}}, \mathbb{Z})$ be a profinitely complete functor. Then the natural map

$$H^{\bullet}(\Lambda R^{\mathrm{opp}}, M) \to H^{\bullet}(\Delta R, \tilde{j}^*M)$$
 (1.28)

is an isomorphism.

Proof. By the projection formula we have

$$H^{\bullet}(\Lambda R^{\mathrm{opp}}, M \otimes \deg^* \mathbb{Z}(1)) \cong H^{\bullet}([1/\mathbb{N}^*], \mathbb{Z}(1) \otimes R^{\bullet} \deg_* M)$$

and, by (1.27), it suffices to prove that (1.28) is tautologically equal to 0. Since $R^{\bullet} \deg_*$ commutes with profinite completion, it suffices to prove the following lemma. \Box

Lemma 1.14. For any profinitely complete $M \in \operatorname{Fun}([1/\mathbb{N}^*], \mathbb{Z})$ we have

$$H^{\bullet}([1/\mathbb{N}^*], M \otimes \mathbb{Z}(1)) = 0.$$

Proof. Every profinitely complete Abelian group M can be decomposed as follows:

$$M = \prod_{p} M_{p},$$

where the product is taken over all primes p, and M_p is pro-p-complete. Therefore we can assume that M is pro-p-complete for some prime p. We decompose the monoid \mathbb{N}^* into the product $\mathbb{N}^* = \mathbb{N} \times \mathbb{N}^*_{\{p\}}$, where $\mathbb{N}_p \subset \mathbb{N}$ consists of all integers prime to p, and $\mathbb{N} \subset \mathbb{N}^*$ is the monoid of all powers p^n , $n \ge 0$. Then, by the Künneth formula, it suffices to prove that

$$H^{\bullet}([1/\mathbb{N}],\mathbb{Z}(1)\otimes M)=0.$$

But $H^{\bullet}([1/\mathbb{N}], M')$ can be computed for every $M' \in \operatorname{Fun}([1/\mathbb{N}], \mathbb{Z})$ by means of the two-term complex

$$M' \xrightarrow{\operatorname{id} - t} M',$$

where $t: M' \to M'$ is the action of the generator $1 \in \mathbb{N}$. In our case we have $M' = M \otimes \mathbb{Z}(1), M'$ is pro-*p*-complete and *t* is divisible by *p*. Therefore the map id-*t* is invertible. \Box

§ 2. Cyclic Mackey functors

2.1. Description in terms of the quotient category. We consider the wreath product $\Lambda \mathbb{Z} \wr \Gamma$ of the enhanced cyclic category $\Lambda \mathbb{Z}$ and the category Γ of finite sets. Explicitly, $\Lambda \mathbb{Z} \wr \Gamma$ can be identified with the full subcategory of Fun $(\Lambda^{\text{opp}}, \text{Sets})$ spanned by the finite disjoint unions of objects $[n|m] \in \Lambda \mathbb{Z} \subset \text{Fun}(\Lambda^{\text{opp}}, \text{Sets})$. (In particular, we have a natural full embedding $\Lambda \mathbb{Z} \subset \Lambda \mathbb{Z} \wr \Gamma$).) A morphism

$$\coprod_{s\in S} [n_s|m_s] \to \coprod_{s'\in S'} [n_{s'}|m_{s'}]$$

between two such unions indexed by finite sets S, S' consists of a map $f: S \to S'$ and a map $f_s: [n_s|m_s] \to [n_{f(s)}|m_{f(s)}]$ for every $s \in S$. A map $\langle f, \{f_s\} \rangle$ is said to be *vertical* (resp. *horizontal*) if f_s is vertical for every s (resp. f is invertible and f_s is horizontal for every s). Clearly, the vertical and horizontal maps form a factorization system on $\Lambda \mathbb{Z} \wr \Gamma$ and we have the following result.

Lemma 2.1. For any vertical map $v: [a] \to [b]$ in $\Lambda \mathbb{Z} \wr \Gamma$ and any map $f: [b'] \to [b]$ there is a Cartesian square



with vertical v'.

Proof. Since the vertical and horizontal maps form a factorization system on $\Lambda \mathbb{Z} \wr \Gamma$, we can assume that f is either horizontal or vertical. In the first case, the desired assertion follows from Lemma 1.9, and in the second it suffices to prove that the category $(\Lambda \mathbb{Z} \wr \Gamma)_v = \Lambda \mathbb{Z}_v \wr \Gamma$ has pullbacks. This follows from (1.19) since, for every m, the category $O_{\mathbb{Z}/m\mathbb{Z}} \wr \Gamma$ has pullbacks (it is equivalent to the category of finite sets equipped with an action of the group $\mathbb{Z}/m\mathbb{Z}$). \Box

Our definition of (derived) cyclic Mackey functors mimicks that of derived Mackey functors given in [1], § 3.4. For any two objects $c, c' \in \Lambda \mathbb{Z}$ we let $Q^{\wr} \Lambda \mathbb{Z}(c, c')$ be the category of diagrams

$$c \stackrel{v}{\longleftrightarrow} c_1 \stackrel{f}{\longrightarrow} c' \tag{2.1}$$

in $\Lambda \mathbb{Z} \wr \Gamma$ with vertical v. Morphisms from a diagram $c \leftarrow c_1 \rightarrow c'$ to a diagram $c \leftarrow c_2 \rightarrow c'$ are given by maps $g = \langle g, \{g_s\} \rangle : c_1 \rightarrow c_2$ such that g commutes with f and v, and each of the components g_s is invertible.

We have $Q^{l}\Lambda\mathbb{Z}(c,c') = Q(\Lambda\mathbb{Z}(c,c'))\wr\Gamma$, where $Q\Lambda\mathbb{Z}(c,c') \subset Q^{l}\Lambda\mathbb{Z}(c,c')$ is the subcategory of diagrams (2.1) with $c_1 \in \Lambda\mathbb{Z} \subset \Lambda\mathbb{Z}\wr\Gamma$ and invertible maps between such diagrams. This identification determines the projection functor $\rho_{c,c'} : Q^{l}(c,c') \to \Gamma$ that sends a diagram $c \leftarrow c_1 \to c'$ to the finite set S of components of the object $c_1 = \coprod_{s \in S} [n_s | m_s] \in \Lambda\mathbb{Z}\wr\Gamma$. Let

$$T_{c,c'} = \rho^* T \in \operatorname{Fun}(Q^{\wr}(c,c'),\mathbb{Z})$$

be the functor from $\mathcal{Q}^{\wr}(c,c')$ to the category of Abelian groups obtained by taking the pullback of the functor $T \in \operatorname{Fun}(\Gamma,\mathbb{Z}), T(S) = \mathbb{Z}[S]$.

By Lemma 2.1, for any $c, c', c'' \in \Lambda \mathbb{Z}$ we have a natural functor

$$m_{c,c',c''} \colon Q^{\mathfrak{d}}(c,c') \times Q^{\mathfrak{d}}(c',c'') \to Q^{\mathfrak{d}}(c,c'')$$

given by pullback. This operation is associative, so that we have a 2-category $Q\Lambda\mathbb{Z}$ with the same objects as $\Lambda\mathbb{Z}$, and with morphism categories $Q^{\ell}(-,-)$. As in (3.7) in [1], we have natural maps

$$\mu_{c,c',c''} \colon T_{c,c'} \boxtimes T_{c',c''} \to m^*_{c,c',c''} T_{c,c''},$$

and these maps are associative on triple products. Therefore, by § 1.6 in [1] we have an A_{∞} -category \mathcal{B}_{\bullet} with the same objects as $\Lambda \mathbb{Z}$, with morphisms given by the bar complexes of the categories $Q^{\ell}(c, c')$ with coefficients in the functors $T_{c,c'}$:

$$\mathcal{B}_{\bullet}(c,c') = C_{\bullet}(Q^{\wr}(c,c'), T_{c,c'}),$$

and with compositions induced by the functors $m_{c,c'}$ and the maps $\mu_{c,c',c''}$.

For any Abelian category Ab we consider the derived category $\mathcal{D}(\mathcal{B}^{\text{opp}}_{\bullet}, Ab)$ of A_{∞} -functors from the opposite category $\mathcal{B}^{\text{opp}}_{\bullet}$ to the category of complexes of objects of Ab. By definition, the category $\Lambda \mathbb{Z}$ is embedded in the 2-category $\mathcal{Q}\Lambda \mathbb{Z}$. The embedding functor $q: \Lambda \mathbb{Z} \to \mathcal{Q}\Lambda \mathbb{Z}$ is the identity on objects and sends morphisms to diagrams (2.1) with v = id. For any $c, c' \in \Lambda \mathbb{Z}$ the restriction $q^*T_{q(c),q(c')}$ is the constant functor \mathbb{Z} . Hence restriction determines a natural functor

$$q^*: \mathcal{D}(\mathcal{B}^{\mathrm{opp}}_{\bullet}, \mathrm{Ab}) \to \mathcal{D}(\Lambda \mathbb{Z}^{\mathrm{opp}}, \mathrm{Ab}).$$
 (2.2)

Let $h: \Lambda \mathbb{Z}_h \to \Lambda \mathbb{Z}$ be the natural embedding. Composing q^* with h^* , we obtain a restriction functor

$$\mathcal{D}(\mathcal{B}^{\mathrm{opp}}_{\bullet}, \mathrm{Ab}) \to \mathcal{D}(\Lambda \mathbb{Z}_h^{\mathrm{opp}}, \mathrm{Ab}).$$

Definition 2.2. A cyclic Mackey functor with values in an Abelian category Ab is an A_{∞} -functor $M \in \mathcal{D}(\mathcal{B}^{\text{opp}}_{\bullet}, \operatorname{Ab})$ whose restriction $h^*q^*M \in \mathcal{D}(\Lambda \mathbb{Z}^{\text{opp}}_h, \operatorname{Ab})$ is locally constant in the sense of Definition A.1 in § 7.

The cyclic Mackey functors form a full triangulated subcategory of $\mathcal{D}(\mathcal{B}^{\mathrm{opp}}_{\bullet}, \mathrm{Ab})$. We denote it by $\mathcal{DMA}(\mathrm{Ab}) \subset \mathcal{D}(\mathcal{B}^{\mathrm{opp}}_{\bullet}, \mathrm{Ab})$. In this paper we shall use only the case when $\mathrm{Ab} = k$ -mod is the category of modules over a commutative ring k. To simplify the notation, we put $\mathcal{DMA}(k\text{-mod}) = \mathcal{DMA}(k)$.

By definition, for every $m \ge 1$ the embedding $O_{\mathbb{Z}/m\mathbb{Z}} \cong \Lambda \mathbb{Z}_v^m \subset \Lambda \mathbb{Z}$ (see (1.19)) extends to a 2-functor

$$i_m : \overline{\mathcal{Q}}^l O_{\mathbb{Z}/m\mathbb{Z}} \to \mathcal{Q}\Lambda\mathbb{Z},$$
 (2.3)

where $\mathcal{Q}^{l}O_{\mathbb{Z}/m\mathbb{Z}}$ is the 2-category in [1], §3.4, and $\overline{\mathcal{Q}}^{l}O_{\mathbb{Z}/m\mathbb{Z}} \subset \mathcal{Q}^{l}O_{\mathbb{Z}/m\mathbb{Z}}$ is the full subcategory spanned by the $\mathbb{Z}/m\mathbb{Z}$ -orbits. This 2-functor is compatible with the coefficients $T_{c,c'} \in \operatorname{Fun}(\mathcal{Q}^{l}(c,c'),\mathbb{Z})$ and, therefore, extends to an A_{∞} -functor

$$\widetilde{\mathcal{B}}^{O_{\mathbb{Z}/m\mathbb{Z}}}_{\bullet} \to \mathcal{B}_{\bullet},$$

where $\widetilde{\mathcal{B}}_{\bullet}^{O_{\mathbb{Z}/m\mathbb{Z}}}$ is as in [1], §3.5. The category $\widetilde{\mathcal{B}}_{\bullet}^{O_{\mathbb{Z}/m\mathbb{Z}}}$ is self-dual by definition. Hence, for any commutative ring k, restriction determines a natural functor

$$i_m^*: \mathcal{D}(B^{\mathrm{opp}}_{\bullet}, k) \to \mathcal{DM}(\mathbb{Z}/m\mathbb{Z}, k)$$

where $\mathcal{DM}(\mathbb{Z}/m\mathbb{Z}, k)$ is the category of derived $\mathbb{Z}/m\mathbb{Z}$ -Mackey functors constructed in [1]. We also have the left-adjoint functor

$$i_{m!} \colon \mathcal{DM}(\mathbb{Z}/m\mathbb{Z},k) \to \mathcal{D}(B^{\mathrm{opp}}_{\bullet},k).$$

2.2. Geometric fixed points. We now consider the category $\widehat{\Lambda \mathbb{Z}}$ in §1.4. It is also embedded in $\mathcal{Q}\Lambda\mathbb{Z}$. The embedding functor $\widehat{q}: \widehat{\Lambda \mathbb{Z}} \to \mathcal{Q}\Lambda\mathbb{Z}$ is the identity on objects and sends a diagram (1.21) to the corresponding diagram (2.1) (we recall that the diagrams (1.21) have no automorphisms and, therefore, this construction involves no choices). As in the case of the functor q in (2.2), the functor \widehat{q} is compatible with the coefficients $T_{c,c'}$, whence we obtain the restriction functor

$$\widehat{q}^*: \mathcal{D}(B^{\mathrm{opp}}_{\bullet}, k) \to \mathcal{D}(\widehat{\Lambda \mathbb{Z}}^{\mathrm{opp}}, k)$$

For every positive integer $m \ge 1$, the 2-functor i_m restricts to the embedding $O_{\mathbb{Z}/m\mathbb{Z}} \to \widehat{\Lambda\mathbb{Z}}$. Hence we have a functorial isomorphism

$$i_m^* \circ \widehat{q}^* \cong \widehat{q}_m^* \circ i_m^*, \tag{2.4}$$

where \widehat{q}_m^* is the restriction with respect to the 2-functor $\widehat{q}_m : O_{\mathbb{Z}/m\mathbb{Z}} \to \mathcal{Q}^{\ell}(O_{\mathbb{Z}/m\mathbb{Z}})$ considered in [1], § 5.3 (and denoted by q there).

We now recall that we have decompositions (1.19), (1.18) and the identification $\Lambda \mathbb{Z}_h \cong \widehat{\Lambda \mathbb{Z}}_h \cong \mathbb{N}^* \times \Lambda$. Define a functor

$$\nu \colon \operatorname{Fun}(\Lambda \mathbb{Z}^{\operatorname{opp}},k) = \operatorname{Fun}(\widehat{\Lambda \mathbb{Z}}_h^{\operatorname{opp}},k) \to \operatorname{Fun}(\widehat{\Lambda \mathbb{Z}}^{\operatorname{opp}},k)$$

by putting $\nu(M)([n|m]) = M([n|m])$ for any $M \in \operatorname{Fun}(\widehat{\Lambda \mathbb{Z}}_h^{\operatorname{opp}}, k), \ [n|m] \in \widehat{\Lambda \mathbb{Z}}$, and

$$\nu(M)(f) = \begin{cases} M(v \circ h), & v \text{ is invertible,} \\ 0, & v \text{ is not invertible,} \end{cases}$$
(2.5)

where $f = v \circ h$ is the horizontal/vertical factorization of a morphism f in $\widehat{\Lambda \mathbb{Z}}_h$. This construction is easily seen to be well defined and yields an exact functor ν . We denote its extension to the derived categories again by ν . There is a left-adjoint functor φ : Fun $(\widehat{\Lambda \mathbb{Z}}^{opp}, k) \to \operatorname{Fun}(\Lambda \mathbb{Z}^{opp}_h, k)$, and its derived functor

$$L^{\bullet}\varphi \colon \mathcal{D}(\widehat{\Lambda}\mathbb{Z}^{\mathrm{opp}},k) \to \mathcal{D}(\Lambda\mathbb{Z}_{h}^{\mathrm{opp}},k)$$

is left adjoint to $\nu \colon \mathcal{D}(\Lambda \mathbb{Z}_h^{\mathrm{opp}}, k) \to \mathcal{D}(\widehat{\Lambda \mathbb{Z}}^{\mathrm{opp}}, k).$

Definition 2.3. The geometric fixed-point functor

$$\Phi\colon \mathcal{D}(B^{\mathrm{opp}}_{\bullet},k) \to \mathcal{D}(\Lambda \mathbb{Z}_{h}^{\mathrm{opp}},k)$$

is given by

$$\Phi = L^{\bullet} \varphi \circ \widehat{q}^*.$$

We note that, by definition, the fixed-point functor Φ splits into components

$$\Phi_m \colon \mathcal{D}(B^{\mathrm{opp}}_{\bullet}, k) \to \mathcal{D}((\Lambda \mathbb{Z}_h^m)^{\mathrm{opp}}, k) \cong \mathcal{D}(\Lambda^{\mathrm{opp}}, k)$$

labelled by positive integers m. We will need some results on the compatibility of Φ with the geometric fixed-point functors constructed in [1]. Namely, we fix a positive integer $m \ge 1$ and consider the embedding i_m in (2.3) and the corresponding restriction functor i_m^* . Let $\overline{O}_{\mathbb{Z}/m\mathbb{Z}}$ be the category of $\mathbb{Z}/m\mathbb{Z}$ -orbits and their isomorphisms. We define a functor

$$\nu_m \colon \operatorname{Fun}(\overline{O}_{\mathbb{Z}/m\mathbb{Z}}, k) \to \operatorname{Fun}(O_{\mathbb{Z}/m\mathbb{Z}}, k)$$

by the formula (2.5) for the functor ν (that is, $\nu_m(M)(v)$ is equal to M(v) for invertible v and is 0 otherwise). Let $\varphi_m \colon \operatorname{Fun}(O_{\mathbb{Z}/m\mathbb{Z}}, k) \to \operatorname{Fun}(\overline{O}_{\mathbb{Z}/m\mathbb{Z}}, k)$ be the left-adjoint functor for ν_m , and let $L^{\bullet}\varphi_m$ be the derived functor of φ_m . Then we have an obvious isomorphism

$$i_m^* \circ \nu \cong \nu_m \circ i_m^*. \tag{2.6}$$

By adjunction, the isomorphisms (2.4) and (2.6) induce base-change maps

$$i_m^* \circ L^{\bullet} \varphi \to L^{\bullet} \varphi_m \circ i_m^*, \qquad i_{m!} \circ \widehat{q}_m^* \to \widehat{q}^* \circ i_{m!}, \tag{2.7}$$

where i_m stands for the 2-functor (2.3) and its restrictions

$$i_m : O_{\mathbb{Z}/m\mathbb{Z}} \to \widehat{\Lambda \mathbb{Z}}^{\mathrm{opp}}, \qquad i_m : \overline{O}_{\mathbb{Z}/m\mathbb{Z}} \to \widehat{\Lambda \mathbb{Z}}_h^{\mathrm{opp}}$$

to $\widehat{q}_m(O_{\mathbb{Z}/m\mathbb{Z}})$ and $\widehat{q}_m(\overline{O}_{\mathbb{Z}/m\mathbb{Z}})$ respectively.

Lemma 2.4. The base-change maps (2.7) are invertible for all integers $m \ge 1$.

Proof. Let $\hat{h}: \widehat{\Lambda \mathbb{Z}}_{h}^{\text{opp}} \to \widehat{\Lambda \mathbb{Z}}^{\text{opp}}$ and $h_m: \overline{O}_{\mathbb{Z}/m\mathbb{Z}} \to O_{\mathbb{Z}/m\mathbb{Z}}$ be the tautological embeddings. Then we obviously have $\hat{h}^* \circ \nu \cong$ id and $h_m^* \circ \nu_m^* \cong$ id, whence by adjunction,

$$L^{\bullet}\varphi \circ h_! \cong \mathrm{id}, \qquad L^{\bullet}\varphi_m \circ h_{m!} \cong \mathrm{id}.$$

On the other hand, we have the horizontal/vertical factorization system on $\widehat{\Lambda \mathbb{Z}}^{\text{opp}} \cong \Lambda \mathbb{Z}$, and Lemma A.4 (see § 7) yields that

$$i_m^* \circ \widehat{h}_! \cong h_{m!} \circ i_m^*.$$

Hence the map

$$i_m^* \circ L^{\bullet} \varphi \circ h_! \to L^{\bullet} \varphi_m \circ i_m^* \circ h_!$$

is invertible and, therefore, the first of the maps (2.7) becomes an isomorphism after evaluation at any object $E \in \mathcal{D}(\widehat{\Lambda \mathbb{Z}}^{\text{opp}}, k)$ of the form $E = \widehat{h}_{!}E', E' \in \mathcal{D}(\widehat{\Lambda \mathbb{Z}}_{h}^{\text{opp}}, k)$. Since the category $\mathcal{D}(\widehat{\Lambda \mathbb{Z}}^{\text{opp}}, k)$ is generated by objects of this form, the map $i_{m}^{*} \circ L^{\bullet} \varphi \to L^{\bullet} \varphi_{m} \circ i_{m}^{*}$ is itself an isomorphism.

For the second of the maps (2.7), we put $\hat{h}' = \hat{q} \circ \hat{h}$, $h'_m = \hat{q}_m \circ h_m$. Then the horizontal/vertical factorization system on $\widehat{\Lambda \mathbb{Z}} \wr \Gamma$ also shows that every diagram of the form (2.1) decomposes as

$$c \stackrel{v}{\longleftrightarrow} c_1 \stackrel{v_1}{\longrightarrow} c_2 \stackrel{h}{\longrightarrow} c'$$

with vertical v, v_1 and horizontal h. It follows that

$$i_m^* \circ h'_* \cong h'_{m*} \circ i_m^*.$$

The proof is the same as in Lemma A.4 except that the Hom-sets $\widehat{\Lambda \mathbb{Z}}^{\text{opp}}(-,-)$, $\widehat{\Lambda \mathbb{Z}}^{\text{opp}}_{v}(-,-)$ in (A.3) must be replaced by the Hom-categories $Q^{i}\widehat{\Lambda \mathbb{Z}}(-,-)$, $Q^{i}(O_{\mathbb{Z}/m\mathbb{Z}})(-,-)$ respectively. Therefore the base-change map

$$i_m^*(\widehat{q}^*E) \to \widehat{q}_m^*(i_m^*E)$$

is an isomorphism for every $E \in \mathcal{D}(\widehat{\Lambda \mathbb{Z}}^{\text{opp}}, k)$ of the form $E = \widehat{h}'_*E', E' \in \mathcal{D}(\widehat{\Lambda \mathbb{Z}}^{\text{opp}}_h, k)$. Since the objects of this form generate the derived category, we have $i_m^* \circ \widehat{q}^* \cong \widehat{q}_m^* \cong i_m^*$. By adjunction, this gives the desired assertion. \Box

Corollary 2.5. For any $M \in \mathcal{D}(B^{\text{opp}}_{\bullet}, k)$ and all positive integers $m, n \ge 1$ we have a natural isomorphism

$$\Phi_m(M)([n]) = \Phi(M)([n|m]) \cong \Phi^{[(\mathbb{Z}/nm\mathbb{Z})/(\mathbb{Z}/m\mathbb{Z})]} i_{mn}^* M,$$

where $[(\mathbb{Z}/n\mathbb{Z})/(\mathbb{Z}/m\mathbb{Z})] \in O_{\mathbb{Z}/n\mathbb{Z}}$ is understood as a $\mathbb{Z}/n\mathbb{Z}$ -orbit, and Φ^c , $c \in O_{\mathbb{Z}/n\mathbb{Z}}$, is the geometric fixed-point functor in [1], § 5.1. Moreover, for every derived Mackey functor $M \in \mathcal{DM}(\mathbb{Z}/m\mathbb{Z}, k)$ we have a natural isomorphism

$$\Phi_m(i_{mn!}M) \cong i_{mn!}(\Phi^{[(\mathbb{Z}/nm\mathbb{Z})/(\mathbb{Z}/m\mathbb{Z})]}M).$$
(2.8)

Proof. This follows immediately from Lemma 2.4 and [1], Proposition 6.5. \Box

2.3. Description in terms of coalgebras. We now adopt the constructions in [1], § 6, and use the fixed-point functor Φ to obtain another description of the category $\mathcal{DMA}(k)$ of cyclic Massey functors.

2.3.1. A_{∞} -coalgebras. Consider the cyclotomic category ΛR in § 1.2 with its vertical/horizontal factorization system. For any object $[m] \in \Lambda R$ let $[m] \setminus \Lambda R$ be the category of objects $[m'] \in \Lambda R$ equipped with a map $[m] \to [m']$. The factorization system on ΛR induces a factorization system on $[m] \setminus \Lambda R$ for every [m].

Lemma 2.6. For any vertical map $v : [a] \to [b]$ in $[m] \setminus \Lambda R$ and any map $f : [b'] \to [b]$ there is a Cartesian square



with vertical v'.

Proof. Arguing as in the proof of Lemma 2.1, we see that the wreath product category $\Lambda R \wr \Gamma$ has fibre products. Therefore so does the category $[m] \setminus (\Lambda R \wr \Gamma)$. It remains to notice that the embedding $[m] \setminus \Lambda R \to [m] \setminus (\Lambda R \wr \Gamma)$ has a left adjoint (any $S \in \Lambda R \wr \Gamma$ is a disjoint union of objects in ΛR , and any map $[m] \to S$ factors uniquely through one of these objects). \Box

Given any morphism $f: [m'] \to [m]$ in ΛR and any integer $n \ge 0$, we consider all diagrams

$$[m'] \xrightarrow{g} [m_0] \xrightarrow{v_0} \cdots \xrightarrow{v_{n-1}} [m_n] \xrightarrow{v_n} [m]$$
(2.9)

in ΛR with vertical $v_i, 0 \leq i \leq n$, non-invertible v_n , and $f = v_n \circ \cdots \circ v_0 \circ g$. Let $V_n(f)$ be the groupoid of all such diagrams and isomorphisms between them. Since the category ΛR is small, the groupoid $V_n(f)$ is small for all f, n.

Suppose that $f = f^{(1)} \circ f^{(2)}$ for some $[m''] \in \Lambda R$ and some morphisms $f^{(1)}$: $[m''] \to [m], f^{(2)}: [m'] \to [m'']$. Then for every $l \ge 1$ and any diagram $\alpha \in V_n(f)$ of the form (2.9) we can use Lemma 2.6 to construct a commutative diagram

$$\begin{array}{c} [m'] \xrightarrow{g} [m_0] \xrightarrow{v_0} \cdots \xrightarrow{v_{n-1}} [m_n] \xrightarrow{v_n} [m] \\ \\ \parallel & f_0^{(1)} \uparrow & f_n^{(1)} \uparrow & f^{(1)} \uparrow \\ [m'] \xrightarrow{g'} [m'_0] \xrightarrow{v'_0} \cdots \xrightarrow{v'_{n-1}} [m'_n] \xrightarrow{v'_n} [m''] \end{array}$$

where $f^{(2)} = v'_n \circ \cdots \circ v'_0 \circ g'$ and all the commutative squares are Cartesian squares in the category $[m'] \setminus \Lambda R$. For any $i, 0 \leq i \leq n$, we have a natural vertical map $\nu_i = v'_n \circ \cdots \circ v_i \colon [m'_i] \to [m'']$. Take the minimal i such that ν_i is an isomorphism. Let $\alpha^{(2)}$ be the diagram

$$[m'] \xrightarrow{g'} [m'_0] \xrightarrow{v'_0} \cdots \xrightarrow{v'_{i-2}} [m'_{i-1}] \xrightarrow{\nu_i \circ v'_{i-1}} [m'']$$

in $V_i(f^{(2)})$, and let $\alpha^{(1)}$ be the diagram

$$[m''] \xrightarrow{f_i^{(1)} \circ \nu_i^{-1}} [m_i] \xrightarrow{v_i} \cdots \xrightarrow{v_n} [m]$$

in $V_{n-i}(f^{(1)})$. Sending α to $\alpha^{(1)} \times \alpha^{(2)}$, we get a well-defined functor

$$V_n(f) \to \prod_{0 \le i \le n} V_i(f^{(1)}) \times V_{n-i}(f^{(2)}).$$
 (2.10)

This construction is obviously associative: for every *l*-tuple f_1, \ldots, f_l of composable morphisms in ΛR we can compose the functors (2.10) and obtain a functor

$$V_n(f_1 \circ \dots \circ f_l) \times I_l \to \coprod_{n_1 + \dots + n_l = n} V_{n_1}(f_1) \times \dots \times V_{n_l}(f_l), \qquad (2.11)$$

where I_l is the *l*th groupoid of the monoidal category operad in Definition A.8. These functors are compatible with the natural operad structure on I_{\bullet} in the obvious sense.

For every $i, 1 \leq i \leq n$, forgetting the object $[m_i]$ in the diagrams (2.9) yields a functor $\delta_i \colon V_n(f) \to V_{n-1}(f)$, and these functors satisfy the relations between simplicial face maps (not only up to an isomorphism, but on the nose). Therefore we can define a bicomplex $\mathcal{T}_{\bullet,\bullet}(f)$ by setting

$$\mathcal{T}_{\bullet,\bullet}(f) = C_{\bullet}(V_{\bullet}(f), \mathbb{Z}), \qquad (2.12)$$

where $C_{\bullet}(V_{\bullet}(f),\mathbb{Z})$ is the bar complex of the groupoid $V_{\bullet}(f)$ with coefficients in the constant functor \mathbb{Z} . One of the differentials in the bicomplex (2.12) comes from the bar complex, and the other is given by $d = d_1 - d_2 + \cdots \pm d_n$, where d_i is the map induced by the functor δ_i . The coproduct operations (2.11) strictly commute with the functors δ_i , whence we obtain canonical operations

$$C_{\bullet}(I_l,\mathbb{Z})\otimes \mathcal{T}_{\bullet}(f_1\circ\cdots\circ f_l)\to \mathcal{T}_{\bullet}(f_1)\otimes\cdots\otimes \mathcal{T}_{\bullet}(f_n),$$

which are also compatible with the asymmetric operad structure on $C_{\bullet}(I_{\bullet},\mathbb{Z})$. Fixing a map $\operatorname{Ass}_{\infty} \to C_{\bullet}(I_{\bullet},\mathbb{Z})$ as in §7, we equip $\mathcal{T}_{\bullet}(-)$ with the structure of a ΛR -graded A_{∞} -coalgebra in the sense of [1], §1.5.4.

In what follows we shall use some elementary properties (given in the following lemma) of the ΛR -graded A_{∞} -coalgebra \mathcal{T}_{\bullet} .

Lemma 2.7. (i) The A_{∞} -coalgebra \mathcal{T}_{\bullet} is augmented and we have $\mathcal{T}_{l}(f) = 0$ for every horizontal map f and all $l \ge 1$.

(ii) For any composable maps f_1 , f_2 with f_1 horizontal, the coproduct map

$$b_2 \colon \mathcal{T}_{\bullet}(f_2 \circ f_1) \to \mathcal{T}_{\bullet}(f_1) \otimes \mathcal{T}_{\bullet}(f_2) \cong \mathcal{T}_{\bullet}(f_2)$$

is an isomorphism.

(iii) Let f_1, \ldots, f_n be an n-tuple of composable maps in ΛR , where $n \ge 3$ and f_i is horizontal for $i \ge 2$. Then the corresponding A_{∞} -operation

$$b_n: \mathcal{T}_{\bullet}(f_n \circ f_1) \to \mathcal{T}_{\bullet}(f_1) \otimes \cdots \otimes \mathcal{T}_{\bullet}(f_n) \cong \mathcal{T}_{\bullet}(f_n)$$

is equal to zero.

Proof. Part (i) is obvious: for every f, the set $V_0(f)$ is by definition a single point, and if f is horizontal, then $V_l(f)$ is empty for $l \ge 1$. To verify (ii), we note that since f_1 is horizontal, the only non-trivial term in the coproduct (2.10) is the map

$$V_l(f_2 \circ f_1) \to V_l(f_2) \times V_0(f_1) = V_l(f_2).$$
 (2.13)

Sending a diagram $\alpha \in V_n(f_2)$ of the form (2.9) to the diagram

$$[m'] \xrightarrow{g \circ f_1} [m_0] \xrightarrow{v_0} \cdots \xrightarrow{v_{n-1}} [m_n] \xrightarrow{v_n} [m],$$

we get a map $V_n(f_2) \to V_n(f_2 \circ f_1)$ which is strictly inverse to (2.13). Moreover, this inverse map construction is obviously strictly associative, so that for any *n*-tuple of composable maps f_1, \ldots, f_n with horizontal f_2, \ldots, f_n for $n \ge 3$ we obtain a single map

$$V_l(f_n) \to V_n(f_n \circ \cdots \circ f_1).$$

Therefore the coproduct map (2.10) is also strictly associative, that is, the map (2.11) factors through the map $I_n \to \text{pt.}$ By definition, this means that the A_{∞} -operation b_n vanishes. This proves (iii). \Box

Lemma 2.8. Let $h: \Lambda R_h \cong \Lambda \to \Lambda \mathbb{Z}$ and $i: \Lambda R_v \cong O_{\mathbb{Z}} \to \Lambda \mathbb{Z}$ be the tautological embeddings. Then $h^*\mathcal{T}_{\bullet}$ is the trivial Λ -graded A_{∞} -coalgebra, $h^*\mathcal{T}_{\bullet}(f) \cong \mathbb{Z}$ for any morphism f in Λ , and $i_m^*\mathcal{T}_{\bullet}$ is isomorphic to the $O_{\mathbb{Z}}$ -graded A_{∞} -coalgebra in [1], § 6.3.3.

Proof. The first assertion follows from Lemma 2.7. To prove the second, we note that if f is vertical, then all the diagrams (2.9) consist of vertical maps and, therefore, coincide with the diagrams used in [1], § 6.3.3. \Box

2.3.2. The comparison theorem. We now consider the natural cofibration λ : $\Lambda \mathbb{Z} \to \Lambda R$ in (1.17), and let $\lambda^* \mathcal{T}_{\bullet}$ be the $\Lambda \mathbb{Z}$ -graded coalgebra obtained by pulling back. For any ring k we consider the derived category $\mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k)$ of A_{∞} comodules over $\lambda^* \mathcal{T}_{\bullet}$. By Lemma 2.8, the pullback $h^* \lambda^* \mathcal{T}_{\bullet}$ with respect to the tautological embedding $h: \Lambda \mathbb{Z}_h \to \Lambda \mathbb{Z}$ is the trivial $\Lambda \mathbb{Z}_h$ -graded A_{∞} -coalgebra. Hence we have a natural pullback functor

$$h^*: \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k) \to \mathcal{D}(\Lambda \mathbb{Z}_h^{\mathrm{opp}}, k).$$

Let $\mathcal{DMA}_{\mathcal{T}}(k) \subset \mathcal{D}(\Lambda\mathbb{Z}, \lambda^*\mathcal{T}, k)$ be the full subcategory spanned by the objects M whose restriction h^*M is locally constant in the sense of Definition A.1. We want to show that the category $\mathcal{DMA}_{\mathcal{T}}(k)$ is naturally equivalent to the category $\mathcal{DMA}(k)$ of k-mod-valued cyclic Mackey functors.

To construct a comparison functor between these two categories, let $V_l([n|m])$, $[n|m] \in \Lambda \mathbb{Z}, \ l \ge 0$, be the groupoid of diagrams

$$[n_1|m_1] \xrightarrow{v_1} \cdots \xrightarrow{v_{n-1}} [n_l|m_l] \xrightarrow{v_n} [n|m]$$

in $\Lambda \mathbb{Z}$ with v_1, \ldots, v_n vertical and v_n non-invertible. Let $\sigma_l \colon V_l([n|m]) \to \widehat{\Lambda \mathbb{Z}}$ be the functor sending such a diagram to $[n_1|m_1] \in \widehat{\Lambda \mathbb{Z}}$ or to [n|m] if l = 0. For any A_{∞} -functor E_{\bullet} from \mathcal{B}_{\bullet} to the category of complexes of k-modules, let $\Phi_{\bullet}^{[n|m]}(E_{\bullet})$ be the total complex of the triple complex

$$C_{\bullet}(V_{\bullet}([n|m]), \sigma_l^* \widehat{q}^* E_{\bullet}),$$

where two of the differentials are induced by the differentials in E_{\bullet} and in the bar complex, and the third is as in [1], §6.3.1. Then the construction in [1], §6.3.2, shows that $\Phi_{\bullet}^{[n|m]}(E_{\bullet})$ has the natural structure of an A_{∞} -comodule over $\lambda^* \mathcal{T}_{\bullet}$. Hence we obtain a functor

$$\Phi_{\bullet}: \mathcal{D}(B^{\mathrm{opp}}_{\bullet}, k) \to \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k).$$
(2.14)

Choose an integer $m \ge 1$ and consider the embedding $i_m: O_{\mathbb{Z}/m\mathbb{Z}} \cong \Lambda \mathbb{Z}_v^m \to \Lambda \mathbb{Z}$. The composite $\lambda \circ i_m: O_{\mathbb{Z}/m\mathbb{Z}} \to \Lambda R$ factors through the natural embedding $O_{\mathbb{Z}/m\mathbb{Z}} \subset O_{\mathbb{Z}} = \Lambda R_v \to \Lambda R$. Therefore, by Lemma 2.8, the $O_{\mathbb{Z}/m\mathbb{Z}}$ -graded A_{∞} -coalgebra $i_m^* \lambda^* \mathcal{T}_{\bullet}$ is isomorphic to the A_{∞} -coalgebra in [1], §6.3.3, and we have a natural pullback functor

$$i_m^*: \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k) \to \mathcal{DM}(\mathbb{Z}/m\mathbb{Z}, k),$$

where $\mathcal{DM}(\mathbb{Z}/m\mathbb{Z}, k)$ is the category of k-valued derived Mackey functors in [1] for the group $\mathbb{Z}/m\mathbb{Z}$. We have an obvious isomorphism

$$i_m^* \circ h^* \cong h_m^* \circ i_m^*, \tag{2.15}$$

where $h_m: \overline{O}_{\mathbb{Z}/m\mathbb{Z}} \to O_{\mathbb{Z}/m\mathbb{Z}}$ is the tautological embedding and $h_m^*: \mathcal{DM}(\mathbb{Z}/m\mathbb{Z}, k) \to \mathcal{D}(\overline{O}_{\mathbb{Z}/m\mathbb{Z}}, k)$ is the corresponding pullback functor. We note that h_m^* admits

a right-adjoint functor h_{m*} by Lemma 6.18 in [1]. Moreover, by construction, the comparison functor Φ_{\bullet} in (2.14) restricts to give the corresponding functor for $O_{\mathbb{Z}/m\mathbb{Z}}$. In other words, we have a canonical isomorphism

$$\Phi^{[m]}_{\bullet} \circ i_m^* E \cong i_m^* \circ \Phi_{\bullet}, \qquad (2.16)$$

where $\Phi_{\bullet}^{[m]}$ is the functor Φ_{\bullet} in [1], Theorem 6.17, for $\mathcal{C} = O_{\mathbb{Z}/m\mathbb{Z}}$.

Lemma 2.9. The functor $h^*: \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k) \to \mathcal{D}(\Lambda \mathbb{Z}_h^{\mathrm{opp}}, k)$ admits a right adjoint

$$h_*: \mathcal{D}(\Lambda \mathbb{Z}_h^{\mathrm{opp}}, k) \to \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k).$$

For every $m \ge 1$, the base-change map $i_m^* \circ h_* \to h_{m*} \circ i_m^*$ induced by (2.15) is an isomorphism.

Proof. This follows immediately from Lemmas A.6, A.7. \Box

Lemma 2.10. Let \mathcal{D}' be a triangulated subcategory of $\mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k)$. Suppose that \mathcal{D}' is closed with respect to arbitrary products and contains all the objects h_*M , $M \in \mathcal{D}(\Lambda \mathbb{Z}_h^{\text{opp}}, k)$. Then $\mathcal{D}' = \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k)$.

Proof. For any integer $n \ge 1$ let $h_n \colon \Lambda \cong \Lambda \mathbb{Z}_h^n \subset \Lambda \mathbb{Z}_h$ be the embedding of the *n*th component of the decomposition (1.18). For every $M \in \mathcal{D}(\Lambda \mathbb{Z}_h^{\text{opp}}, k)$ we define the support $\operatorname{Supp}(M) \subset \mathbb{N}^*$ as the set of all $n \ge 1$ such that $h_l^* M$ is non-zero for some l dividing n. For every $M \in \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k)$ we put $\operatorname{Supp} M = \operatorname{Supp} h^* M$. Note that $h_{n'}^* \circ h_* \circ h_{n*} = 0$ for all integers $n, n' \ge 1$ unless n divides n', and $h_n^* \circ h_* \circ h_{n*} \cong \operatorname{id}$ (by Lemma 2.9 it suffices to verify both statements after applying the functors i_m^* , $m \ge 1$, and then they follow immediately from Lemma 6.18 in [1]). Therefore, in particular, for every $n \ge 1$ and all $M \in \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k)$ we have

$$\operatorname{Supp}(h_*h_{n*}h_n^*h^*M) \subset \operatorname{Supp}(M).$$

Moreover, given an object $M \in \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k)$, let $M^{[1]}$ be the cone of the adjunction map

$$M \to h_*(h^*M)_m,$$

where m is the smallest integer in Supp(M). Then

$$\operatorname{Supp}(M^{[1]}) = \operatorname{Supp}(M) \setminus \{m\}.$$

By induction, we put $M^{[n]} = (M^{[n-1]})^{[1]}$ for all n > 1. Then we have natural maps $M^{[n]} \to M$ whose comes $\widetilde{M}^{[n]}$ lie in \mathcal{D}' and form an inverse system. We have a compatible system of maps $\eta_n \colon M \to \widetilde{M}^{[n]}$, $n \ge 1$. By induction, for all $n > n' \ge 1$, the transition map $\widetilde{M}^{[n+1]} \to \widetilde{M}^{[n]}$ becomes an isomorphism after applying $h_{n'}^*h^*$, and so does the map η_n . We put

$$\widetilde{M} = \operatorname{holim} \widetilde{M}^{[n]},$$

where holim is defined by the telescope construction, and let $\eta: M \to \widetilde{M}$ be the natural map. Then, for every $n \ge 1$, the inverse system $h_n^* h^* \widetilde{M}^{[n']}$ stabilizes for n' > n, and $h_n^* h^*(\eta)$ is an isomorphism. Thus η is an isomorphism. But, by construction, \widetilde{M} lies in the category \mathcal{D}' . \Box

Lemma 2.11. The composite

$$h^* \circ \Phi_{\bullet} \colon \mathcal{D}(B^{\mathrm{opp}}_{\bullet}, k) \to \mathcal{D}(\Lambda \mathbb{Z}_h^{\mathrm{opp}}, k)$$

is isomorphic to the fixed-point functor Φ in Definition 2.3.

Proof. By construction, the image $h^*\Phi_{\bullet}(E)$ of any $E \in \mathcal{D}(B^{\mathrm{opp}}_{\bullet}, k)$ depends only on the restriction $\widehat{q}^*E \in \mathcal{D}(\widehat{\Lambda\mathbb{Z}}^{\mathrm{opp}}, k)$: the construction of the functor Φ_{\bullet} also gives a functor $\varphi_{\bullet}: \mathcal{D}(\widehat{\Lambda\mathbb{Z}}^{\mathrm{opp}}, k) \to \mathcal{D}(\Lambda\mathbb{Z}_h^{\mathrm{opp}}, k)$, and we have $h^* \circ \Phi_{\bullet} \cong \varphi_{\bullet} \circ \widehat{q}^*$. For every object $E \in \mathrm{Fun}(\widehat{\Lambda\mathbb{Z}}^{\mathrm{opp}}, k)$, the degree-0 homology of the complex $\varphi_{\bullet}(E)$ is easily seen to be isomorphic to $\varphi(E)$, and this isomorphism is functorial in E. Thus, by the universal property of the derived functor, it extends to a map

$$e\colon \varphi_{\bullet}\to L^{\bullet}\varphi.$$

We claim that $e: \varphi_{\bullet}(E) \to L^{\bullet}\varphi(E)$ is an isomorphism for every $E \in \mathcal{D}(\widehat{\Lambda \mathbb{Z}}^{\text{opp}}, k)$ of the form $E = \widehat{q}^* E', E' \in \mathcal{D}(B^{\text{opp}}_{\bullet}, k)$. Indeed, it suffices to prove that

$$i_m^*(e) \colon i_m^* \circ h^* \circ \Phi_{\bullet} \to i_m \circ \Phi \tag{2.17}$$

is an isomorphism for all $m \ge 1$. By (2.16) and (2.15), the left-hand side of (2.17) is isomorphic to $h_m^* \circ \Phi_{\bullet}^{[m]} \circ i_m^*$, and by Lemma 6.15 in [1], the functor $h_m^* \circ \Phi^{[m]}$ is isomorphic to the direct sum of the functors $\Phi^{[(\mathbb{Z}/nm\mathbb{Z})/(\mathbb{Z}/m\mathbb{Z})]}$ in Corollary 2.5. To complete the proof, it suffices to use Corollary 2.5. \Box

Proposition 2.12. The functor Φ_{\bullet} in (2.14) is an equivalence of categories, and it identifies $\mathcal{DMA}(k) \subset \mathcal{D}(\mathcal{B}^{opp}_{\bullet}, k)$ with $\mathcal{DMA}_{\mathcal{T}} \subset \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k)$.

Proof. As in Lemma A.6, we write $\mathcal{D}' \subset \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k)$ for the subcategory of objects M in the category $\mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k)$ such that the functor $\operatorname{Hom}(\Phi_{\bullet}(-), M)$ from $\mathcal{D}(B^{\operatorname{opp}}_{\bullet}, k)$ to $\mathcal{D}(k)$ is representable. The geometric fixed-point functor Φ obviously has a right adjoint. Therefore, by Lemma 2.11, the subcategory \mathcal{D}' satisfies all the hypotheses of Lemma 2.10. Thus \mathcal{D}' is the whole of the category $\mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k)$ and Φ_{\bullet} admits a right-adjoint functor

$$\Phi_{\bullet}^{-1}: \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k) \to \mathcal{D}(B_{\bullet}^{\mathrm{opp}}, k).$$

By Lemma 2.11, the composite $\Phi_{\bullet}^{-1} \circ h_*$ is right adjoint to the geometric fixedpoint functor Φ in Definition 2.3. Moreover, for every $m \ge 1$, the functor $\Phi_{\bullet}^{[m]}$ in (2.16) is an equivalence of categories by Theorem 6.17 in [1]. Let Φ_m^{-1} be the inverse equivalence. Then, by Corollary 2.5, the base-change map

$$i_m^* \circ \Phi_{\bullet}^{-1} \circ h_* \to \Phi_m^{-1} \circ h_{m*} \circ i_m^*,$$

being adjoint to the direct product of the isomorphisms (2.8), is itself an isomorphism. Hence Lemma 2.9 yields that the base-change map

$$i_m^*(\Phi_{\:\raisebox{1pt}{\text{\circle*{1.5}}}}^{-1}(M)) \to \Phi_m^{-1}(i_m^*(M))$$

is an isomorphism for every $M \in \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k)$ of the form $M = h_* M', M' \in \mathcal{D}(\Lambda \mathbb{Z}^{\text{opp}}, k)$. By Lemma 2.10, this implies that it is an isomorphism for all M. Thus we have

$$i_m^* \circ \Phi_{{\scriptscriptstyle\bullet}} \circ \Phi^{-1} \cong \Phi^{[m]} \circ \Phi_m^{-1} \circ i_m^*, \qquad i_m^* \circ \Phi^{-1} \circ \Phi_{{\scriptscriptstyle\bullet}} \cong \Phi_m^{-1} \circ \Phi^{[m]} \circ i_m^*$$

for all $m \ge 1$. Since $\Phi_{\bullet}^{[m]}$ and Φ_{m}^{-1} are mutually inverse equivalences of categories, it follows that the adjunction maps $\mathrm{Id} \to \Phi^{-1} \circ \Phi_{\bullet}, \Phi^{-1} \circ \Phi_{\bullet} \to \mathrm{Id}$ become isomorphisms after restricting to $\Lambda \mathbb{Z}_{v}$. Since the restriction functor is obviously conservative, we conclude that Φ_{\bullet} and Φ^{-1} are mutually inverse equivalences of categories. To prove that Φ_{\bullet} identifies $\mathcal{DMA}(k)$ and $\mathcal{DMA}_{\mathcal{T}}(k)$, note that $h^{*} \circ \Phi_{\bullet} \cong h^{*}$. \Box

2.4. Restriction and corestriction. We make some further observations on cyclic Massey functors for later use. Recall that there are 2-functors $q: \Lambda \mathbb{Z} \to Q\Lambda \mathbb{Z}$ and $\hat{q}: \widehat{\Lambda \mathbb{Z}} \to Q\Lambda \mathbb{Z}$, which agree on horizontal maps: $q \circ h \cong \hat{q} \circ \hat{h}$. Hence we have an isomorphism $h^* \circ q^* \cong \hat{h}^* \circ \hat{q}^*$.

Lemma 2.13. The base-change map

$$\widehat{h}_! \circ h^* \to \widehat{q}^* \circ q_!$$

induced by the isomorphism $h^* \circ q^* \cong \widehat{h}^* \circ \widehat{q}^*$ is itself an isomorphism.

Proof. For reasons explained in the proof of Lemma 2.4, this can be proved in the same way as in the proof of Lemma A.4. \Box

In particular, Lemma 2.13 shows that we have natural identifications

$$h^* \circ \Phi_{\bullet} \circ q_! \cong \Phi \circ q_! \cong L^{\bullet} \varphi \circ \widehat{q}^* \circ q_! \cong L^{\bullet} \varphi \circ \widehat{h}_! \circ h^*.$$

$$(2.18)$$

Since $L^{\bullet}\varphi \circ \hat{h}_{!}$ is adjoint to $\hat{h}^{*} \circ \nu = \text{Id}$, the right-hand side of (2.18) is just h^{*} . One can actually say more: the composite

$$\Phi_{\bullet} \circ q \colon \mathcal{D}(\Lambda \mathbb{Z}^{\mathrm{opp}}, k) \to \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k)$$

is naturally isomorphic to the corestriction functor ξ^* with respect to the augmentation map of the augmented $\Lambda\mathbb{Z}$ -graded A_{∞} -coalgebra $\lambda^*\mathcal{T}_{\bullet}$ (to construct an isomorphism, we resolve a functor $E \in \operatorname{Fun}(\Lambda\mathbb{Z}^{\operatorname{opp}}, k)$ by functors of the form $i_{m!}E_m$, $E_m \in \operatorname{Fun}(O_{\mathbb{Z}/m\mathbb{Z}}^{\operatorname{opp}}, k)$ and use Lemma 6.20 in [1]). Therefore the corestriction functor ξ^* admits a right adjoint ξ_* . It is given by

$$\xi_* = q^* \circ \Phi_{\bullet}^{-1} \colon \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{T}_{\bullet}, k) \to \mathcal{D}(\Lambda \mathbb{Z}^{\mathrm{opp}}, k).$$

We also have

$$h^* \circ \xi_* \circ \xi^* \cong h^* \circ q^* \circ q_! \cong \widehat{h}^* \circ \widehat{q}^* \circ q_! \cong \widehat{h}^* \circ \widehat{h}_! \circ h^*.$$
(2.19)

To compute the right-hand side of (2.17) more effectively, it is useful to consider the category $\Lambda I \cong \widetilde{\Lambda I}$ in Lemma 1.5. We also consider the product $\Lambda I \times \mathbb{N}^*$ and define projections $i, \pi \colon \Lambda I \times \mathbb{N}^* \to \widehat{\Lambda \mathbb{Z}_h^{\mathrm{opp}}} \cong \Lambda \mathbb{Z}_h \cong \Lambda \times \mathbb{N}^*$ by putting

$$i = i \times \mathrm{id}, \qquad \pi = \pi \times \rho_m \quad \mathrm{on} \quad \Lambda_m \times \mathbb{N}^* \subset \Lambda I \times \mathbb{N}^*, \tag{2.20}$$

where $\rho_m \colon \mathbb{N}^* \to \mathbb{N}^*$ is the map of multiplication by the integer $m \ge 1$.

Lemma 2.14. We have a natural isomorphism

$$\widehat{h}^* \circ \widehat{h}_! \cong \pi_! \circ i^*.$$

Proof. Under the identification $\widehat{\Lambda \mathbb{Z}}^{\text{opp}} \cong \Lambda \mathbb{Z}$, the functor \widehat{h} becomes the tautological embedding $h: \Lambda \mathbb{Z}_h \to \Lambda \mathbb{Z}$. Let $\overline{\Lambda I}$ be the category of vertical maps $v: a \to a'$ in $\Lambda \mathbb{Z}$ whose morphisms are given by commutative squares

$$\begin{array}{c|c} a_1 & \xrightarrow{f} & a_2 \\ v_1 & & & \downarrow v_2 \\ a_1' & \xrightarrow{f'} & a_2' \end{array}$$

with horizontal f (and arbitrary f'). Let $s: \overline{\Lambda I} \to \Lambda \mathbb{Z}_h$ (resp. $t: \overline{\Lambda I} \to \Lambda \mathbb{Z}$) be the functor sending a map to its source (resp. target). Then t is a cofibration and s has a left adjoint $\iota: \Lambda \mathbb{Z}_h \to \overline{\Lambda I}$ that sends every object $a \in \Lambda \mathbb{Z}$ to its identity endomorphism. We have $\hat{h} = t \circ \iota$, whence

$$h_! \cong t_! \circ \iota_! \cong t_! \circ s^*.$$

It remains to note that we have a natural Cartesian square

Hence $h^* \circ t_! \cong \pi_! \circ h^*$ by base change, and we have $s \circ h = i$. \Box

§3. Cyclotomic complexes

We can now introduce the main subject of this paper: the notion of a cyclotomic complex. Fix a commutative ring k. Consider the cyclotomic category ΛR and the ΛR -graded A_{∞} -coalgebra \mathcal{T}_{\bullet} in § 2.3. Let $\widetilde{\lambda} \colon \Lambda R \to \Lambda R$ be the cofibration in (1.17), and let $h \colon \Lambda R_h \cong \Lambda \times I \to \Lambda R$ be the natural embedding. By Lemma 2.7, the restriction $h^* \widetilde{\lambda}^* \mathcal{T}_{\bullet}$ is the trivial ΛR_h -graded A_{∞} -coalgebra, $h^* \widetilde{\lambda}^* \mathcal{T}_{\bullet}(f) = \mathbb{Z}$ for any morphism f in ΛR_h , and we have a restriction functor

$$h^*: \mathcal{D}(\Lambda R, \lambda^* \mathcal{T}_{\bullet}, k) \to \mathcal{D}(\Lambda R_h, k).$$

Definition 3.1. A cyclotomic complex over k is an A_{∞} -comodule M_{\bullet} over $\tilde{\lambda}^* \mathcal{T}_{\bullet}$ with values in the category k-mod such that the restriction $h^*M_{\bullet} \in \mathcal{D}(\Lambda R_h, k)$ is locally constant in the sense of Definition A.1.

The derived category of cyclotomic complexes over k will be denoted by $\mathcal{D}\Lambda R(k)$.

3.1. Normalized ΛR -graded coalgebras. Definition 3.1 is short, but neither explicit nor convenient for computations since the A_{∞} -coalgebra \mathcal{T}_{\bullet} is given by an implicit and rather complicated construction. In this section we provide more explicit descriptions of the categories $\mathcal{D}\Lambda R(k)$. This will also result in a more convenient description of the category $\mathcal{D}M\Lambda(k)$ in § 2. We start with the following reduction, which is similar to the results in [1], §§ 7.5, 7.6.

Definition 3.2. Let G be a finite group. A complex E_{\bullet} of $\mathbb{Z}[G]$ -modules is *strongly acyclic* with respect to G if, for any subgroup $H \subset G$, $H \neq G$, and any $\mathbb{Z}[H]$ -module V we have

$$\lim_{n} H^{\bullet}(H, V \otimes F^{n}E_{\bullet}) = 0,$$

where $F^{\bullet}E_{\bullet}$ is the stupid filtration on E_{\bullet} . A map $f: E_{\bullet} \to E'_{\bullet}$ between $\mathbb{Z}[G]$ -modules is a *strong quasi-isomorphism* with respect to G if its cone is strongly acyclic.

Definition 3.3. A ΛR -graded A_{∞} -coalgebra \mathcal{R}_{\bullet} is said to be *normalized* if it possesses properties (i)–(iii) in Lemma 2.7.

In particular, the coalgebra \mathcal{T}_{\bullet} is normalized (by Lemma 2.7). For every normalized ΛR -graded A_{∞} -coalgebra \mathcal{R}_{\bullet} and any map $f: [m] \to [m']$ in the category ΛR , the complex $\mathcal{R}_{\bullet}(f)$ is by definition equipped with an action of the cyclic group $\operatorname{Aut}([m])$.

Definition 3.4. An A_{∞} -map $\xi \colon \mathcal{R}_{\bullet} \to \mathcal{R}'_{\bullet}$ between normalized ΛR -graded A_{∞} coalgebras is a *strong quasi-isomorphism* if, for any map $f \colon [m] \to [m']$, the corresponding map $\xi \colon \mathcal{R}_{\bullet}(f) \to \mathcal{R}'_{\bullet}(f)$ is a strong quasi-isomorphism with respect to
the subgroup $\operatorname{Aut}(f) \subset \operatorname{Aut}([m])$ consisting of all $g \in \operatorname{Aut}([m])$ such that $f \circ g = f$.

Proposition 3.5. For every strong A_{∞} -quasi-isomorphism $\xi \colon \mathcal{R}_{\bullet} \to \mathcal{R}'_{\bullet}$ between normalized ΛR -graded A_{∞} -coalgebras and any commutative ring k, the corestriction functors

$$\begin{split} \xi^* &: \mathcal{D}(\widetilde{\Lambda R}, \widetilde{\lambda}^* \mathcal{R}_{\bullet}, k) \to \mathcal{D}(\widetilde{\Lambda R}, \widetilde{\lambda}^* \mathcal{R}'_{\bullet}, k), \\ \xi^* &: \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{R}_{\bullet}, k) \to \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{R}'_{\bullet}, k) \end{split}$$

between the derived categories of A_{∞} -comodules are equivalences of categories.

Proof. Lemmas 2.9 and 2.10 hold with any normalized ΛR -graded A_{∞} -coalgebra instead of \mathcal{T}_{\bullet} and with the same proof. They also hold for $\Lambda \widetilde{R}$ instead of $\Lambda \mathbb{Z}$ (again with the same proof). Thus, as in the proof of Proposition 2.12, the functors ξ^* admit right adjoints

$$\begin{split} \xi_* \colon \mathcal{D}(\widetilde{\Lambda R}, \widetilde{\lambda}^* \mathcal{R}'_{\bullet}, k) &\to \mathcal{D}(\widetilde{\Lambda R}, \widetilde{\lambda}^* \mathcal{R}_{\bullet}, k), \\ \xi_* \colon \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{R}'_{\bullet}, k) &\to \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{R}_{\bullet}, k). \end{split}$$

Moreover, for every integer $m \ge 1$ we have a natural embedding $i_m: O_{\mathbb{Z}/m\mathbb{Z}} \cong \Lambda R_n^m \to \Lambda R$ and the corresponding restriction functors

$$\begin{split} &i_m^* \colon \mathcal{D}(\Lambda \overline{R}, \widetilde{\lambda}^* \mathcal{R}_{\bullet}, k) \to \mathcal{D}(O_{\mathbb{Z}/m\mathbb{Z}}, i_m^* \widetilde{\lambda}^* \mathcal{R}_{\bullet}, k), \\ &i_m^* \colon \mathcal{D}(\Lambda \overline{R}, \widetilde{\lambda}^* \mathcal{R}_{\bullet}', k) \to \mathcal{D}(O_{\mathbb{Z}/m\mathbb{Z}}, i_m^* \widetilde{\lambda}^* \mathcal{R}_{\bullet}', k). \end{split}$$

For $\Lambda\mathbb{Z}$, these functors were already considered in §2. For either $\Lambda\mathbb{Z}$ or ΛR , we have an obvious isomorphism $\xi_m^* \circ i_m^* \cong i_m^* \circ \xi^*$, where $\xi_m = i_m^*(\xi)$, and the corresponding base-change map $i_m^* \circ \xi_* \to \xi_{m*} \circ i_m^*$ is also an isomorphism, as in Lemma 2.9. Thus it suffices to prove that ξ_m^* and ξ_{m*} are mutually inverse equivalences of categories for every m. This is done in [1], Lemma 7.14. \Box **3.2. Reduced \Lambda R-graded coalgebras.** The first corollary of Proposition 3.5 is analogous to the reduction made in [1], § 7.5.

Definition 3.6. A normalized ΛR -graded A_{∞} -coalgebra \mathcal{R}_{\bullet} is said to be *reduced* if, for any map f in ΛR of degree n > 1, we have $\mathcal{R}(f) = 0$ unless n is prime. The *reduction* $\mathcal{R}_{\bullet}^{\text{red}}$ of a normalized ΛR -graded A_{∞} -coalgebra R_{\bullet} is defined by setting

$$\mathcal{R}_{\bullet}^{\mathrm{red}}(f) = \begin{cases} \mathcal{R}_{\bullet}(f) & \text{if the degree of } f \text{ is 1 or a prime,} \\ 0 & \text{otherwise,} \end{cases}$$

with the same A_{∞} -operations as in \mathcal{R}_{\bullet} when it makes sense, and 0 otherwise.

For every normalized ΛR -graded A_{∞} -coalgebra \mathcal{R}_{\bullet} , we obviously have a canonical map $\mathcal{R}_{\bullet}^{\mathrm{red}} \to \mathcal{R}_{\bullet}$.

Lemma 3.7. For every normalized ΛR -graded A_{∞} -coalgebra \mathcal{R}_{\bullet} , the canonical map $\mathcal{R}_{\bullet}^{\mathrm{red}} \to \mathcal{R}_{\bullet}$ induces equivalences of categories

$$\mathcal{D}(\widetilde{\Lambda R}, \widetilde{\lambda}^* \mathcal{R}^{\mathrm{red}}_{\bullet}, k) \cong \mathcal{D}(\widetilde{\Lambda R}, \widetilde{\lambda}^* \mathcal{R}_{\bullet}, k), \quad \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{R}^{\mathrm{red}}_{\bullet}, k) \cong \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{R}_{\bullet}, k).$$

Proof. By Lemma 7.15(ii) in [1], the map $\mathcal{R}^{\text{red}}_{\bullet} \to \mathcal{R}_{\bullet}$ is a strong quasi-isomorphism in the sense of Definition 3.4. Hence the desired assertion follows from Proposition 3.5. \Box

We now observe that reduced ΛR -graded A_{∞} -coalgebras are essentially linear objects: all potentially non-linear comultiplication maps are 0 by definition. To make this assertion precise, we put

$$\Lambda I_{\mathrm{red}} = \prod_{p \text{ is prime}} \Lambda_p \subset \Lambda I,$$

where ΛI is the category (1.2).

Definition 3.8. An A_{∞} -functor from a small category C to an Abelian category Ab is said to be *normalized* if, for any *n*-tuple of composable invertible morphisms f_1, \ldots, f_n in C with $n \ge 3$, the corresponding A_{∞} -operation b_n is equal to 0.

Lemma 3.9. The category of reduced normalized ΛR -graded A_{∞} -coalgebras and A_{∞} -maps between them is equivalent to the category of normalized A_{∞} -functors from $\Lambda I_{\text{red}}^{\text{opp}}$ to \mathbb{Z} -mod and A_{∞} -maps between them.

Proof. Let \mathcal{R}_{\bullet} be a reduced normalized ΛR -graded A_{∞} -coalgebra. For every object $[m] \in \Lambda R$ we have the category $\Lambda R_h/[m]$ of horizontal maps $f: [n] \to [m], [n] \in \Lambda R$. Since \mathcal{R}_{\bullet} is normalized, for every map $g: [m] \to [m']$ we can define a functor \mathcal{R}_{\bullet}^g from $(\Lambda R_h/[m])^{\text{opp}}$ to complexes of Abelian groups by putting

$$\mathcal{R}^g_{\bullet}(f) = \mathcal{R}_{\bullet}(g \circ f).$$

This functor is constant (all transition maps are isomorphisms). We put

$$P_{\bullet}(g) = \lim \mathcal{R}^g_{\bullet}(f),$$

where the limit is taken over the category $(\Lambda R_h/[m])^{\text{opp}}$. Moreover, every horizontal map $h: [m] \to [m']$ induces a functor $\Lambda R_h/[m] \to \Lambda R_h/[m']$, $f \mapsto h \circ f$. For every map $g: [m'] \to [m'']$, restriction with respect to this functor gives a natural map

$$h^*: P_{\bullet}(g) \to P_{\bullet}(g \circ h),$$

and this construction is associative.

Recall that we have the equivalence (1.10). Restrict it to $\Lambda I_{\text{red}} \subset \Lambda I$. For every object $a \in \Lambda I_{\text{red}}$ we put

$$P_{\bullet}(a) = P_{\bullet}(v(a)).$$

Then $P_{\bullet}(-)$ has the natural structure of a normalized A_{∞} -functor from $\Lambda I_{\text{red}}^{\text{opp}}$ to complexes of Abelian groups: for every *n*-tuple

$$\begin{array}{c|c} i_p(a_0) & \xrightarrow{f_1} i_p(a_1) & \xrightarrow{f_2} \cdots & \xrightarrow{f_n} i_p(a_n) \\ v_0 & & & & \downarrow \\ v_0 & & & & \downarrow \\ v_1 & & & & \downarrow \\ \pi_p(a_0) & \xrightarrow{f_1'} \pi_p(a_1) & \xrightarrow{f_2'} \cdots & \xrightarrow{f_n'} \pi_p(a_n) \end{array}$$

of maps of the form (1.9), the A_{∞} -operation $P_{\bullet}(a_n) \to P_{\bullet}(a_0)$ is the composite of the map

$$(f_n \circ \cdots \circ f_1)^* \colon P_{\bullet}(a_n) \to P_{\bullet}(v_n \circ f_n \circ \cdots \circ f_1) = P_{\bullet}(f'_n \circ \cdots \circ f'_1 \circ v_0)$$

and the map

$$P_{\bullet}(f'_n \circ \cdots \circ f'_1 \circ v_0) \to P_{\bullet}(v_0) = P_{\bullet}(a_0)$$

induced by the A_{∞} -operation on \mathcal{R}_{\bullet} .

Conversely, let P_{\bullet} be a normalized A_{∞} -functor from $\Lambda I_{\text{red}}^{\text{opp}}$ to complexes of Abelian groups. For every morphism $f: [m] \to [n]$ of prime degree in ΛR , let $\mathcal{C}(f)$ be the category of diagrams

$$[m] \stackrel{h}{\longrightarrow} [m'] \stackrel{v}{\longrightarrow} [n]$$

with vertical v, horizontal h and $f = v \circ h$. Sending such a diagram to $P_{\bullet}(v)$, we get a functor $P_{\bullet}(f)$ from $\mathcal{C}(f)$ to complexes of Abelian groups. We put

$$\mathcal{R}_{\bullet}(f) = \lim P_{\bullet}(f),$$

where the limit is taken over C_f . It remains to verify that the structure of an A_{∞} -functor on P_{\bullet} induces the structure of a reduced normalized A_{∞} -coalgebra on $\mathcal{R}_{\bullet}(-)$ and that both constructions are mutually inverse. \Box

We can now make our final reduction. A normalized A_{∞} -functor M_{\bullet} from $\Lambda I_{\text{red}}^{\text{opp}}$ to \mathbb{Z} -mod is said to be *admissible* if, for every object $[a] \in \Lambda_p \subset \Lambda I_{\text{red}}$, we have

$$M_i([a]) = \begin{cases} 0 & \text{for } i < 0, \\ \mathbb{Z} & \text{for } i = 0, \end{cases}$$

and $M_i([a])$ is a free $\mathbb{Z}[\mathbb{Z}/pZ]$ -module for $i \ge 1$. A reduced normalized ΛR -graded A_∞ -coalgebra \mathcal{R}_{\bullet} is said to be *admissible* if the A_∞ -functor E_{\bullet} that corresponds to \mathcal{R}_{\bullet} under the equivalence in Lemma 3.9 is admissible. Note that the A_∞ -coalgebra \mathcal{T}_{\bullet} is admissible by Proposition 7.8 in [1].

Lemma 3.10. For all admissible reduced normalized ΛR -graded A_{∞} -coalgebras $\mathcal{R}_{\bullet}, \mathcal{R}'_{\bullet}$ and any ring k we have canonical equivalences

$$\mathcal{D}(\Lambda \overline{R}, \widetilde{\lambda}^* \mathcal{R}'_{\bullet}, k) \cong \mathcal{D}(\Lambda \overline{R}, \widetilde{\lambda}^* \mathcal{R}_{\bullet}, k), \qquad \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{R}'_{\bullet}, k) \cong \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{R}_{\bullet}, k).$$

Proof. We choose a projective resolution \widetilde{P}_{\bullet} of the constant functor $\mathbb{Z} \in \operatorname{Fun}(\Lambda I_{\operatorname{red}}^{\operatorname{opp}}, \mathbb{Z})$ and put $P_0 = \mathbb{Z}$, $P_i = \widetilde{P}_{i-1}$ for $i \ge 1$. Then P_{\bullet} is an admissible normalized A_{∞} -functor from $\Lambda I_{\operatorname{red}}^{\operatorname{opp}}$ to \mathbb{Z} -mod, and \widetilde{P}_{\bullet} is *h*-projective by Lemma A.5. Therefore for any other admissible normalized A_{∞} -functor P'_{\bullet} we have an A_{∞} -map

$$\xi \colon P_{\bullet} \to P'_{\bullet}.$$

Moreover, for every $[m] \in \Lambda_p \subset \Lambda I_{\text{red}}$, the map $\xi([m])$ is a strong quasi-isomorphism with respect to $\mathbb{Z}/p\mathbb{Z} \subset \text{Aut}([m])$. Let P'_{\bullet} be the A_{∞} -functor corresponding to the A_{∞} -coalgebra R_{\bullet} , and let $\widehat{\mathcal{R}}_{\bullet}$ be the A_{∞} -coalgebra corresponding to the A_{∞} functor P_{\bullet} . Then ξ induces a map

$$\xi: \widehat{\mathcal{R}}_{\bullet} \to \mathcal{R}_{\bullet}$$

which is a strong quasi-isomorphism in the sense of Definition 3.4. Therefore, by Proposition 3.5, the corestriction functors corresponding to $\tilde{\lambda}^* \xi$ and $\lambda^* \xi$ are equivalences of categories (and similarly for \mathcal{R}'_{\bullet}). \Box

3.3. Comodules. We now consider the category $\Lambda R_h \cong \Lambda \times I$ and the products $\Lambda I_{\text{red}} \times \mathbb{N}^* \subset \Lambda I_{\text{red}} \times I$. Define the functors

$$i, \pi \colon \Lambda I_{\mathrm{red}} \times I \to \Lambda R_h$$

by the formulae (2.20). Moreover, let $\tau \colon \Lambda I_{\text{red}} \times I \to \Lambda I_{\text{red}}$ be the tautological projection.

Any complex P_{\bullet} of functors in $\operatorname{Fun}(\Lambda I_{\operatorname{red}}^{\operatorname{opp}}, \mathbb{Z})$ is in particular a normalized A_{∞} -functor from $\Lambda I_{\operatorname{red}}^{\operatorname{opp}}$ to \mathbb{Z} -mod. We fix the complex P_{\bullet} in such a way that it is admissible in the sense of Lemma 3.10. For every commutative ring k we consider the category of pairs $\langle V_{\bullet}, \varphi \rangle$ formed by a complex V_{\bullet} in $\operatorname{Fun}(\widehat{\Lambda R}_{h}^{\operatorname{opp}}, k)$ and a map

$$\varphi \colon \pi^* V_{\bullet} \to i^* V_{\bullet} \otimes \tau^* P_{\bullet}.$$

Inverting quasi-isomorphisms in this category, we obtain a triangulated category and denote it by $\mathcal{D}(\widetilde{AR}_h, P_{\bullet}, k)$. Forgetting the map φ , we get a functor

$$\widetilde{h}^*: \mathcal{D}(\widetilde{\Lambda R}_h, P_{\bullet}, k) \to \mathcal{D}(\widetilde{\Lambda R}_h^{\mathrm{opp}}, k).$$

In a similar vein, let $\mathcal{D}(\Lambda \mathbb{Z}, P_{\bullet}, k)$ be the category of pairs $\langle V_{\bullet}, \varphi \rangle$ formed by a complex V_{\bullet} in Fun $(\Lambda \mathbb{Z}_{h}^{\text{opp}}, k)$ and a map

$$\varphi \colon \pi^* V_{\bullet} \to i^* V_{\bullet} \otimes \tau^* P_{\bullet},$$

with inverted quasi-isomorphisms, where i and π are as in (2.20). Restricting from $\widetilde{\Lambda R_h}$ to $\Lambda \mathbb{Z}_h$ gives a forgetful functor

$$\mathcal{D}(\Lambda R_h, P_{\bullet}, k) \to \mathcal{D}(\Lambda \mathbb{Z}_h, P_{\bullet}, k), \tag{3.1}$$

and we have the restriction functor

$$h^*: \mathcal{D}(\Lambda \mathbb{Z}_h, P_{\bullet}, k) \to \mathcal{D}(\Lambda \mathbb{Z}_h^{\mathrm{opp}}, k).$$

Let \mathcal{R}_{\bullet} be the reduced normalized $\Lambda \overline{R}$ -graded A_{∞} -coalgebra corresponding to P_{\bullet} under the equivalence in Lemma 3.9. We consider the derived categories $\mathcal{D}(\Lambda \mathbb{Z}, \widetilde{\lambda}^* \mathcal{R}_{\bullet}, k), \mathcal{D}(\widetilde{\Lambda R}, \widetilde{\lambda}^* \mathcal{R}_{\bullet}, k).$

Lemma 3.11. There are canonical equivalences of categories

$$\mathcal{D}(\Lambda \mathbb{Z}_h, P_{\bullet}, k) \cong \mathcal{D}(\Lambda \mathbb{Z}, \lambda^* \mathcal{R}_{\bullet}, k), \quad \mathcal{D}(\widetilde{\Lambda R}_h, P_{\bullet}, k) \cong \mathcal{D}(\widetilde{\Lambda R}, \widetilde{\lambda}^* \mathcal{R}_{\bullet}, k)$$

commuting with the restriction functors h^* and \tilde{h}^* respectively.

Proof. By Lemma A.5 we can modify the definition of the derived category $\mathcal{D}(\Lambda R_h, P_{\bullet}, k)$: replace the complexes in Fun $(\Lambda R_h^{\text{opp}}, k)$ by A_{∞} -comodules over the trivial ΛR_h^{opp} -graded A_{∞} -coalgebra (and similarly for $\Lambda \mathbb{Z}$). The resulting category of complexes is tautologically equivalent to the category of A_{∞} -comodules over $\tilde{\lambda}^* \mathcal{R}_{\bullet}$ (resp. $\lambda^* \mathcal{R}_{\bullet}$), and the equivalence commutes with \tilde{h}^* (resp. h^*). In particular, the equivalence preserves quasi-isomorphisms and hence descends to the derived categories. \Box

We write $\mathcal{D}_c(\Lambda R_h, P_{\bullet}, k) \subset \mathcal{D}(\Lambda R_h, P_{\bullet}, k)$ for the full subcategory spanned by all $M \in \mathcal{D}(\Lambda R_h, P_{\bullet}, k)$ such that $h^*M \in \mathcal{D}(\Lambda R_h^{\text{opp}}, k)$ is locally constant in the sense of Definition A.1. Let $\mathcal{D}_c(\Lambda \mathbb{Z}_h, P_{\bullet}, k) \subset \mathcal{D}(\Lambda \mathbb{Z}_h, P_{\bullet}, k)$ be the full subcategory spanned by all $M \in \mathcal{D}(\Lambda \mathbb{Z}_h, P_{\bullet}, k)$ such that h^*M is locally constant. Then Lemmas 3.11, 3.7, 3.10 together yield the following effective description of the categories $\mathcal{D}\mathcal{M}\Lambda(k), \mathcal{D}\Lambda R(k)$ of k-valued cyclic Mackey functors and k-valued cyclotomic complexes.

Proposition 3.12. For every admissible complex P_{\bullet} of functors from ΛI_{red} to \mathbb{Z} -mod there are canonical equivalences

$$\mathcal{DMA}(k) \cong \mathcal{D}_c(\Lambda \mathbb{Z}_h, P_{\bullet}, k), \qquad \mathcal{DAR}(k) \cong \mathcal{D}_c(\Lambda R_h, P_{\bullet}, k)$$

of triangulated categories.

Moreover, restricting from ΛR to $\Lambda \mathbb{Z}$ as in (3.1), we get restriction functors

$$h^*: \mathcal{D}(\Lambda \overline{R}^{\mathrm{opp}}, k) \to \mathcal{D}(\Lambda \mathbb{Z}^{\mathrm{opp}}, k), \qquad h^*: \mathcal{D}(\Lambda \overline{R}_h, P_{\bullet}, k) \to \mathcal{D}(\Lambda \mathbb{Z}^{\mathrm{opp}}, k).$$

Let $\mathcal{D}_w(\widetilde{\Lambda R}_h^{\mathrm{opp}}, k) \subset \mathcal{D}(\widetilde{\Lambda R}_h^{\mathrm{opp}}, k)$ and $\mathcal{D}_w(\widetilde{\Lambda R}_h, P_{\bullet}, k) \subset \mathcal{D}(\widetilde{\Lambda R}_h, P_{\bullet}, k)$ be the full subcategories spanned by all M such that $h^*M \in \mathcal{D}(\Lambda \mathbb{Z}^{\mathrm{opp}}, k)$ is locally constant.

Proposition 3.13. (i) The restriction functor \tilde{h}^* has a right adjoint

$$\widetilde{h}_*: \mathcal{D}(\Lambda \widetilde{R}_h^{\mathrm{opp}}, k) \to \mathcal{D}(\Lambda \widetilde{R}_h, P_{\bullet}, k),$$

which sends the subcategory $\mathcal{D}_w(\widetilde{\Lambda R}_h^{\text{opp}}, k) \subset \mathcal{D}(\widetilde{\Lambda R}_h^{\text{opp}}, k)$ into the subcategory $\mathcal{D}_w(\widetilde{\Lambda R}_h, P_{\bullet}, k) \subset \mathcal{D}(\widetilde{\Lambda R}_h, P_{\bullet}, k).$

(ii) Let $\mathcal{D}' \subset \mathcal{D}_w(\Lambda \overline{R}_h, P_{\bullet}, k)$ be a triangulated subcategory such that \mathcal{D}' is closed with respect to arbitrary products and contains all the objects h_*M , $M \in \mathcal{D}(\Lambda \overline{Z}_h^{\text{opp}}, k)$. Then $\mathcal{D}' = \mathcal{D}_w(\Lambda \overline{R}_h, P_{\bullet}, k)$.

Proof. At the level of categories of complexes, the forgetful functor \widetilde{h}^* has an obvious adjoint given by

$$\widetilde{h}_*V_{\bullet} = \pi_*(i^*V_{\bullet} \otimes \tau^*P_{\bullet})$$

with the tautological map φ . Since every complex in Fun $(\Lambda R_h^{\text{opp}}, k)$ has an *h*-injective replacement, this functor descends to the derived categories.

Lemma 3.14. For any $l \ge 0$ let $F^l P_{\bullet} \subset P_{\bullet}$ be the *l*th term of the stupid filtration on P_{\bullet} (in other words, $(F^l P_{\bullet})_m$ is equal to P_m for $m \le l$ and zero otherwise). Let V_{\bullet} be any *h*-injective complex in $\operatorname{Fun}(\widetilde{\Lambda R}_h^{\operatorname{opp}}, k)$. Then the natural map

$$\pi_*(i^*V_{\bullet} \otimes \tau^*P_{\bullet}) \to \lim_{l \to l} R^{\bullet} \pi_*(i^*V_{\bullet} \otimes \tau^*F^lP_{\bullet})$$
(3.2)

is a quasi-isomorphism.

Proof. Clearly, we have

$$i^*V_{\bullet} \otimes \tau^*P_{\bullet} \cong \lim_{\stackrel{l}{\to}} i^*V_{\bullet} \otimes \tau^*F^lP_{\bullet}.$$

Hence it suffices to prove that the map

$$\pi_*(i^*V_{\bullet}\otimes\tau^*F^lP_{\bullet})\to R^{\bullet}\pi_*(i^*V_{\bullet}\otimes\tau^*F^lP_{\bullet})$$

is a quasi-isomorphism for every $l \ge 0$. This reduces by induction to proving that for all integers $m, l \ge 0, n \ge 1$ we have

$$R^n \pi_*(i^* V_m \otimes \tau^* P_l) = 0.$$

This can be checked after evaluating at any object $a \in \Lambda R_h$. By the base-change theorem, it suffices to prove that

$$H^n(\mathbb{Z}/p\mathbb{Z}, i^*V_m(b) \otimes P_l(\tau(b))) = 0$$

for every $b \in \Lambda_p \times I \subset \Lambda I_{\text{red}} \times I$. Since V_{\bullet} is *h*-injective, $V_{\bullet}(c)$ is an *h*-injective complex of $\mathbb{Z}[\text{Aut}(c)]$ -modules for any $c \in \widetilde{\Lambda R_h}$ and, therefore, $i^*V_m(b)$ is a free $\mathbb{Z}/p\mathbb{Z}$ -module. Hence so is the product $i^*V_m(b) \otimes \tau^*P_l(b)$. \Box

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Now, by Proposition 3.12, the category $\mathcal{D}(\Lambda R_h, P_{\bullet}, k)$ is independent of the choice of the admissible complex P_{\bullet} . Choose P_{\bullet} in such a way that $F^l P_{\bullet}$ is quasiisomorphic to the shift $\mathbb{Z}[l]$ of the constant functor $\mathbb{Z} \in \operatorname{Fun}(\Lambda I_{\operatorname{red}}^{\operatorname{opp}}, k)$ for every even integer $l \ge 0$. Let $\overline{\pi} \colon \Lambda I_{\operatorname{red}} \times \mathbb{N}^* \to \Lambda \mathbb{Z}_h$ be the restriction of the functor π to $\Lambda I_{\operatorname{red}} \times \mathbb{N}^*$. Then $h^* \circ R^{\bullet} \pi_* \cong R^{\bullet} \overline{\pi}_* \circ h^*$ by base change, and

$$R^{\bullet}\overline{\pi}_* : \mathcal{D}(\Lambda I_{\mathrm{red}}^{\mathrm{opp}}, k) \to \mathcal{D}(\Lambda \mathbb{Z}_h^{\mathrm{opp}}, k)$$

obviously sends $\mathcal{D}_c(\Lambda I_{\mathrm{red}}^{\mathrm{opp}}, k)$ to $\mathcal{D}_c(\Lambda \mathbb{Z}_h^{\mathrm{opp}}, k)$. Therefore, restricting to even l on the right-hand side of (3.2), we see that, for every $V_{\bullet} \in \mathcal{D}_w(\Lambda R_h^{\mathrm{opp}}, k)$,

$$\widetilde{h}^*\widetilde{h}_*V_{\bullet} \cong \lim_{\stackrel{l}{\longrightarrow}} R^{\bullet}\pi_*i^*V_{\bullet}[2l]$$

does indeed lie in $\mathcal{D}_w(\widetilde{\Lambda R}_h^{\text{opp}}, k)$. This proves part (i).

Part (ii) can now be proved by arguing as in the proof of Lemma 2.10. \Box

§4. Equivariant homology

4.1. Generalities on equivariant homotopy. To fix notation, we start by recalling some general facts from equivariant stable homotopy theory [4] (we mostly follow the exposition in [6], which contains everything we will need in a concise form).

Let G be a compact Lie group. A G-CW-complex X is a pointed CW-complex X equipped with a continuous action of G preserving the distinguished point and satisfying the following condition: for every $g \in G$, the subset $X^g \subset X$ of fixed points is a subcomplex. Consider the category of pointed G-topological spaces and G-equivariant maps between them modulo G-equivariant homotopy. Let G-Top be the full subcategory spanned by spaces homotopy equivalent to G-CW-complexes, and let $H \subset G$ be a closed Lie subgroup. Sending a space X to the set $X^H \subset X$ of fixed points, we get a well-defined functor

$$G\text{-}\mathrm{Top} \to W_H\text{-}\mathrm{Top},$$

where $W_H = N_H/H$, with $N_H \subset G$ being the normalizer of H.

Given any finite-dimensional representation V of G over \mathbb{R} , we shall write S^V for the one-point compactification of V, with infinity being the distinguished point. For any $X \in G$ -Top we put $\Sigma^V X = X \wedge S^V$ and denote the set of pointed continuous maps from S^V to X by $\Omega^V X$. Clearly, the functors $\Sigma^V, \Omega^V : G$ -Top $\to G$ -Top are adjoint.

A *G*-universe is an \mathbb{R} -vector space *U* equipped with a continuous linear *G*-action and a *G*-invariant positive definite scalar product. For any *G*-universe *U*, a *G*prespectrum *X* indexed by *U* is a set of *G*-CW-complexes *X*(*V*), one for each finitedimensional *G*-invariant subspace $V \subset U$, and *G*-equivariant continuous maps

$$X(V) \to \Omega^W X(V \oplus W), \tag{4.1}$$

one for each pair of transversal mutually orthogonal finite-dimensional G-invariant subspaces $V, W \subset U$, subject to an obvious associativity condition. The category of G-prespectra indexed by U and homotopy classes of maps between them is denoted by G-sp(U). A *G*-prespectrum is a *spectrum* if the maps (4.1) are homeomorphisms. The category of *G*-spectra indexed by *U* and homotopy classes of maps between them is denoted by G-Sp(*U*). We have the tautological embedding G-Sp(*U*) $\rightarrow G$ -sp(*U*). It admits a left-adjoint *spectrification functor L* given by

$$Lt(V) = \lim_{W \to W} \Omega^W \Sigma^W X(V \oplus W), \tag{4.2}$$

where the limit is taken over all finite-dimensional G-invariant subspaces $W \subset U$ orthogonal to V.

For every inclusion $u: U_1 \subset U_2$ of *G*-universes we have an obvious restriction functor $\rho^{\#}(u): G\operatorname{-sp}(U_2) \to G\operatorname{-sp}(U_1)$, called *change of universe*. This functor has a left adjoint $\rho_{\#}(u): G\operatorname{-sp}(U_1) \to G\operatorname{-sp}(U_2)$ given by

$$\tilde{\rho}_{\#}(u)(t)(V) = \Sigma^{V - (V \cap u(U_1))} t(u^{-1}(V)),$$

where $V - (V \cap u(U_1)) \subset V$ is the orthogonal complement of the intersection $V \cap u(U_1) \subset V$. The functor $\rho^{\#}(u)$ sends spectra to spectra. The corresponding functor $\rho^{\#}(u)$: G-Sp $(U_2) \to G$ -Sp (U_1) has a left adjoint $\rho_{\#} : G$ -Sp $(U_1) \to G$ -Sp (U_2) given by

$$\rho_{\#}(u) = L\widetilde{\rho}_{\#}(u),$$

where L is the spectrification functor (4.2).

In particular, spectra indexed by the trivial universe U = 0 are just *G*-spaces, and the restriction *G*-Sp \rightarrow *G*-Top with respect to the inclusion $0 \hookrightarrow U$ is the forgetful functor sending a *G*-spectrum to its value at 0. Its right adjoint is called the *suspension spectrum functor* and is denoted by Σ^{∞} : *G*-Top \rightarrow *G*-Sp(*U*). Explicitly, $\Sigma^{\infty}X = L\tilde{\Sigma}^{\infty}X$, where $\tilde{\Sigma}^{\infty}X \in G$ -sp(*U*) is given by

$$\widetilde{\Sigma}^{\infty} X(V) = \Sigma^V X.$$

Among the non-trivial G-universes, there are two particularly important types.

(i) $U = \mathbb{R}^{\infty}$ with the trivial *G*-action (where ∞ is assumed to be countable). The *G*-spectra indexed by *U* are called *naive* equivariant *G*-spectra. We denote the category G-Sp(*U*) by *G*-Sp^{naive}.

(ii) A *G*-universe is said to be *complete* if every finite-dimensional representation V of the compact Lie group G appears in U countably many times. All complete *G*-universes are isomorphic. A *G*-spectrum indexed by a complete *G*-universe U is called a *genuine* equivariant *G*-spectrum. We denote G-Sp(U) simply by *G*-Sp.

Note that G-Sp^{naive} and G-Sp are triangulated categories, with shifts given by

$$X[n](W) = X(W \oplus \mathbb{R}^n).$$

In addition, for every finite-dimensional representation V, the category G-Sp has an auto-equivalence $\Sigma^V : G$ -Sp $\to G$ -Sp given by the formula $\Sigma^V X(W) = X(W \oplus V)$ (to make this definition precise, one must fix an isomorphism $U \oplus V \cong U$, or $U \oplus \mathbb{R}^n \cong U$ in the naive case, and apply the functor of change of universe).

Let $H \subset G$ be a closed Lie subgroup. Then U^H is a W_H -universe for every G-universe U, and U^H is complete if U is complete. For every G-spectrum $X \in G$ -Sp(U), the Lewis-May fixed point spectrum $X^H \in W_H$ -Sp (U^H) is given by putting

$$X^H(V) = X(V)^H$$

for all finite-dimensional $V \subset U^H \subset U$. There is another fixed point functor $\Phi^H: G\operatorname{-Sp}(U) \to H\operatorname{-Sp}(U^H)$ called the *geometric fixed-point functor*. To define it, we choose a finite-dimensional G-invariant subspace $W(V) \subset U$ for every finite-dimensional W_H -invariant subspace $V \subset U^H$ in such a way that $V = W(V)^H$ and

$$\bigcup_{V \subset U^H} W(V) = U.$$

Then we put

$$\varphi^H t(V) = t(W(V))^H \tag{4.3}$$

for all $t \in G$ -sp(U) and

 $\Phi^H X = L\varphi^H X$

for any G-spectrum $X \in G$ -Sp(U). Here φ^H depends on the choice of the subspaces W(V), but the spectrification Φ^H is independent of this choice (a more invariant description of Φ^H , which is manifestly independent of this choice, can be found in [6], Lemma 1.1). For every $X \in G$ -Sp there is a natural map

$$X^H \to \Phi^H X,$$

and this map is functorial in X.

For naive *G*-spectra, the two fixed-point functors coincide. In the genuine case we fix a complete *G*-universe *U*. Then $U^G \subset U$ is isomorphic to \mathbb{R}^{∞} , whence G-Sp (U^G) is *G*-Sp^{naive}, and the inclusion $u: U^G \to U$ induces a pair of adjoint functors $\rho^{\#}(u): G$ -Sp $\to G$ -Sp^{naive}, $\rho_{\#}(u): G$ -Sp^{naive} $\to G$ -Sp. We have commutative diagrams

$$G\text{-Top} \xrightarrow{\Sigma^{\infty}} G\text{-Sp}^{\text{naive}} \xrightarrow{\rho_{\#}(u)} G\text{-Sp}$$

$$(-)^{H} \bigvee (-)^{H} \bigvee \phi^{H}$$

$$W_{H}\text{-Top} \xrightarrow{\Sigma^{\infty}} W_{H}\text{-Sp}^{\text{naive}} \xrightarrow{\rho_{\#}(u')} W_{H}\text{-Sp}$$

$$G\text{-Sp}^{\text{naive}} \xleftarrow{\rho^{\#}(u)} G\text{-Sp}$$

$$(-)^{H} \bigvee (-)^{H} \bigvee (-)^{H}$$

$$W_{H}\text{-Sp}^{\text{naive}} \xleftarrow{\rho^{\#}(u')} W_{H}\text{-Sp}$$

$$(4.4)$$

where u' is the embedding $U^G \subset U^H$ (and the W_H -universe U^H is obviously complete).

4.2. Cyclic sets. From now on, let $G = S^1 = U(1)$ be the unit circle. Then it is well known that G-spaces are related to cyclic sets. We recall this relation (see, for example, [10] for details and references).

For any object $[n] \in \Lambda$ let |[n]| be its geometric realization: the union of points labelled by the vertices $v \in V([n])$ and open intervals I_e labelled by the edges $e \in E([n])$, with the natural topology making |[n]| a circle. Given a function $a: E([n]) \to \mathbb{R}$ with a > 0, we can endow |[n]| with the structure of a metric space |[n]|(a) by declaring a(e) to be the length of the interval I_e . Let $\mathbb{R}([n])^\circ$ be the space of pairs $\langle a, b \rangle$ formed by a function $a: E([n]) \to \mathbb{R}$, a > 0, and a metricpreserving monotone continuous map $b: |[n]|(a) \to S^1$ to the unit circle $S^1 \subset \mathbb{C}$. Such a map b exists if and only if $a_1 + \cdots + a_n = 2\pi$, and the space of all such maps can be identified non-canonically with S^1 . Hence we have a non-canonical homeomorphism

$$\mathsf{R}([n])^{\mathrm{o}} \cong S^1 \times \mathsf{T}_{n-1}^{\mathrm{o}},$$

where $\mathsf{T}_m^{\circ} \subset \mathsf{T}_m$ is the interior of the standard *m*-simplex T_m for every $m \ge 0$. We embed $\mathsf{R}([n])^{\circ}$ in a compact space $\mathsf{R}([n])$ by allowing *a* to take zero values (and admitting degenerate metrics on |[n]|). Then we have

$$\mathsf{R}([n]) \cong S^1 \times \mathsf{T}_{n-1}.$$

This decomposition is non-canonical, but the space $\mathsf{R}([n])$ is completely canonical and, by construction, carries a continuous *G*-action.

For every map $f: [n] \to [m]$ and any pair $\langle a, b \rangle \in \mathsf{R}([m])$ we put

$$f(a)(e) = \sum_{f(e')=e} s(e')$$

Then there is an obvious metric-preserving map $g: |[n]|(f(a)) \to |[m]|(a)$. We denote the composite of g and b by f(b). This endows R with the structure of a contravariant functor from Λ to G-Top. We make this functor covariant by using the duality $\Lambda^{\text{opp}} \cong \Lambda$. The result is a functor

$$\mathsf{R} \colon \Lambda \to G\text{-}\mathrm{Top.} \tag{4.5}$$

By the standard Kan extension procedure, (4.5) extends uniquely to a colimitpreserving *realization functor*

Real:
$$\Lambda^{\mathrm{opp}}$$
 Sets $\to G$ -Top

such that Real $\circ Y \cong \mathsf{R}$, where $Y \colon \Lambda \to \Lambda^{\mathrm{opp}}$ Sets is the Yoneda embedding. We also have the right-adjoint functor

$$S: G\text{-}\mathrm{Top} \to \Lambda^{\mathrm{opp}} \operatorname{Sets}$$

which is such that

$$S(X)([n]) = \operatorname{Maps}_G(\mathsf{R}([n]), X)$$

for every $X \in G$ -Top, where $Maps_G(-, -)$ is the space of G-equivariant non-based maps.

It is well known that the realization Real(A) of any $A \in \Lambda^{\text{opp}}$ Sets is homeomorphic to the ordinary geometric realization of the simplicial set $j^*A \in \Delta^{\text{opp}}$ Sets and, for every $X \in G$ -Top, the adjunction map

$$\operatorname{Real}(S(X)) \to X$$

is a homotopy equivalence (see [10], or [13] for a modern treatment). In particular, writing $\mathbb{Z}[A]$ for the free Abelian group spanned by an arbitrary set A and applying this construction pointwise to $S(X): \Lambda \to \text{Sets}$, we get

$$H_{\bullet}(\Lambda^{\mathrm{opp}}, \mathbb{Z}[S(X)]) \cong H_{\bullet}(X_{hG}, \mathbb{Z}), \tag{4.6}$$

where X_{hG} is the homotopy quotient of X by the G-action.

The functors $\mathbb{Z}[S(X)] \in \operatorname{Fun}(\Lambda^{\operatorname{opp}}, \mathbb{Z})$ are inconvenient for our purposes since they are not locally constant in the sense of Definition A.1. To remedy this, we note that all the sets S(X)([n]) carry a natural topology.

Definition 4.1. For every $X \in G$ -Top, the *chain complex* $C_{\bullet}(X)$ is the complex in Fun $(\Lambda^{\text{opp}}, \mathbb{Z})$ given by

$$C_{\bullet}(X)([n]) = C_{\bullet}(S(X)([n]), \mathbb{Z}),$$

where $C_{\bullet}(-,\mathbb{Z})$ is the normalized singular chain complex with coefficients in \mathbb{Z} .

Since all the maps $\mathsf{R}([n]) \to \mathsf{R}([n'])$ are homotopy equivalences, the chain complex $C_{\bullet}(X)$ (unlike $\mathbb{Z}[S(X)]$) is locally constant in the sense of Definition A.1. More explicitly, $C_{\bullet}(X)$ is obtained as follows. Define a functor $\widetilde{\mathsf{R}} \colon \Lambda \times \Delta \to G$ -Top by the formula

$$\widehat{\mathsf{R}}([n] \times [m]) = \mathsf{R}([n]) \times \mathsf{T}_{m-1},$$

where $[m] \in \Delta$ is the totally ordered set of m elements. The functor $\hat{\mathsf{R}}$ extends to a pair of adjoint functors

$$\widetilde{\text{Real}}: \operatorname{Fun}(\Lambda^{\operatorname{opp}} \times \Delta^{\operatorname{opp}}, \operatorname{Sets}) \to G\text{-}\operatorname{Top},$$
$$\widetilde{S}: G\text{-}\operatorname{Top} \to \operatorname{Fun}(\Lambda^{\operatorname{opp}} \times \Delta^{\operatorname{opp}}, \operatorname{Sets}).$$

To obtain $C_{\bullet}(S(X))$, we take $\mathbb{Z}[\widetilde{S}(X)] \in \operatorname{Fun}(\Lambda^{\operatorname{opp}} \times \Delta^{\operatorname{opp}}, \mathbb{Z})$ and apply the construction of the normalized chain complex fibrewise with respect to the projection $\tau \colon \Lambda \times \Delta \to \Lambda$.

To explain the relation between $\mathbb{Z}[S(X)]$ and $C_{\bullet}(X)$, we define an embedding $\iota_m \colon \Lambda \to \Lambda \times \Delta$ for every $m \ge 1$ by putting $\iota([n]) = [n] \times [m]$. Then ι_1 is adjoint to τ . Hence for every $A \in \operatorname{Fun}(\Lambda^{\operatorname{opp}} \times \Delta^{\operatorname{opp}}, \operatorname{Sets})$ there is a natural adjunction map

$$\tau^* \iota_1^* A \to A. \tag{4.7}$$

We apply (4.7) to $\widetilde{S}(X)$ for some $X \in G$ -Top. By definition, we have $\iota_1^* \widetilde{S}(X) \cong S(X)$. Taking the free Abelian groups spanned by these sets and passing to normalized chain complexes, we obtain a natural map

$$\mathbb{Z}[S(X)] \to C_{\bullet}(X). \tag{4.8}$$

Lemma 4.2. Let $A \in \operatorname{Fun}(\Lambda^{\operatorname{opp}} \times \Delta^{\operatorname{opp}}, \operatorname{Sets})$ be such that, for every morphism $f : [m] \to [n]$ in Δ , the corresponding map

$$\iota_n^* A \to \iota_m^* A$$

induces a homotopy equivalence of geometric realizations. Then the map

$$\operatorname{Real}(\iota^*A) \cong \operatorname{Real}(\tau^*\iota^*A) \to \operatorname{Real}(A)$$

induced by (4.7) is a homotopy equivalence.

Proof. Let

$$\operatorname{Real}_{\Lambda}$$
: $\operatorname{Fun}(\Lambda^{\operatorname{opp}} \times \Delta^{\operatorname{opp}}, \operatorname{Sets}) \to \Delta^{\operatorname{opp}}$ -Top

be the functor of fibrewise geometric realization with respect to the projection $\Lambda \times \Delta \to \Delta$. Then the assumption on A means that (4.7) already becomes a homotopy equivalence after applying Real_{Λ}. \Box

For every $X \in G$ -Top we have $\iota_m^* \widetilde{S}(X) \cong S(\operatorname{Maps}(\Delta_{m-1}, X))$, whence $\widetilde{S}(X)$ automatically satisfies the hypotheses of Lemma 4.2. In particular, the natural map $\operatorname{Real}(\widetilde{S}(X)) \to \operatorname{Real}(\widetilde{S}(X))$ is a homotopy equivalence, and the map

$$H_{\bullet}(\Lambda, \mathbb{Z}[S(X)]) \to H_{\bullet}(\Lambda, C_{\bullet}(X))$$

$$(4.9)$$

induced by (4.8) is an isomorphism. More generally, if we denote the functor left adjoint to the embedding $\mathcal{D}_c(\Lambda, \mathbb{Z}) \subset \mathcal{D}(\Lambda, \mathbb{Z})$ by $C: \mathcal{D}(\Lambda, \mathbb{Z}) \to \mathcal{D}_c(\Lambda, \mathbb{Z})$, then $C(\mathbb{Z}[S(X)]) \cong C_{\bullet}(X)$.

The complexes $C_{\bullet}(X)$ are usually rather big. We shall use smaller models for the standard *G*-orbit spaces G/C_n , where $C_n = \mathbb{Z}/n\mathbb{Z} \subset G$ is the group of *n*th roots of unity. To obtain these models, we consider the functors $j_n \colon \Delta \to \Lambda_n^{\text{opp}}$ in §1.1 with the natural C_n -action on them. Define functors $J_n \colon \Lambda^{\text{opp}} \times \Delta^{\text{opp}} \to \text{Sets}$ by putting

$$J_n([m] \times [l]) = \Lambda(i_n j_n([l]), [m]) / C_n,$$
(4.10)

where C_n acts through its action on j_n .

Lemma 4.3. For every $n \ge 1$ we have a natural homotopy equivalence

$$\operatorname{Real}(J_n) \cong G/C_n.$$

Proof. By Lemma 4.2, it suffices to construct equivalences $\operatorname{Real}(\iota_l^* J_n) \cong G/C_n$ for any $[l] \in \Delta$. By definition, we have

$$\iota_l^* J_n \cong Y(j_n([l]))/C_n = Y([nl])/C_n,$$

where $Y([nl]) \subset \Lambda^{\text{opp}}$ Sets is the Yoneda image of $[nl] \in \Lambda$. Since Real preserves colimits, we have homeomorphisms

$$\operatorname{Real}(Y([nl])/C_n) \cong \operatorname{Real}(Y([nl]))/C_n \cong \mathsf{R}([nl])/C_n,$$

and $\mathsf{R}([nl])$ is indeed homotopy equivalent to G. \Box

We now consider the subsets of fixed points. By Lemma 1.8, the functor R in (4.5) extends to a functor

$$\mathsf{R} = \operatorname{Real} \circ Y \colon \Lambda \mathbb{Z} \to G\text{-}\operatorname{Top},$$

whence we obtain a pair of adjoint functors

Real: Fun($\Lambda \mathbb{Z}^{\text{opp}}$, Sets) $\rightarrow G$ -Top, S: G-Top \rightarrow Fun($\Lambda \mathbb{Z}^{\text{opp}}$, Sets).

We keep the same notation because these functors are direct extensions of the functors Real and S considered above. Indeed, the realization functor Λ^{opp} Sets \rightarrow Fun($\Lambda \mathbb{Z}^{\text{opp}}$, Sets) admits a fully faithful left adjoint

$$L: \Lambda^{\mathrm{opp}} \operatorname{Sets} \to \operatorname{Fun}(\Lambda \mathbb{Z}^{\mathrm{opp}}, \operatorname{Sets}),$$

and we have $S \cong L \circ S$, Real \cong Real $\circ L$. Restricting L(A) to $\Lambda \cong \Lambda \mathbb{Z}_h^m \subset \Lambda \mathbb{Z}$, we explicitly have

$$L(A)|_{\Lambda\mathbb{Z}_h^m} = \pi_{m*} i_m^* A, \tag{4.11}$$

where the adjoint π_{m*} : Λ_m^{opp} Sets $\to \Lambda^{\text{opp}}$ Sets to the pullback functor π_m^* is obtained by taking the $\mathbb{Z}/m\mathbb{Z}$ -fixed points fibrewise, and the vertical morphisms in $\Lambda\mathbb{Z}$ act by natural inclusions of fixed points.

As above, we extend the functors S and Real to the product category $\Lambda \mathbb{Z} \times \Delta$. For any $X \in G$ -Top we define its *extended chain complex* $\widetilde{C}_{\bullet}(X)$ of functors in Fun $(\Lambda \mathbb{Z}^{\text{opp}}, \mathbb{Z})$ by putting

$$\widetilde{C}_{\:\raisebox{1pt}{\text{\circle*{1.5}}}}(X) = C_{\:\raisebox{1pt}{\text{\circle*{1.5}}}}(S(X)) = N(\mathbb{Z}[\widetilde{S}(X)]),$$

where N is the normalized chain complex functor applied fibrewise to the projection $\Lambda \mathbb{Z} \times \Delta \to \Lambda \mathbb{Z}$. We also define the *reduced chain complex* $\overline{C}_{\bullet}(X)$ by putting

$$\overline{C}_{\:\raisebox{1pt}{\text{\circle*{1.5}}}}(X)([n|m]) = \widetilde{C}_{\:\raisebox{1pt}{\text{\circle*{1.5}}}}(X)([n|m])/\mathbb{Z}[o],$$

where $o \in \operatorname{Maps}_{S^1}(\mathsf{R}([m|n]), X)$ is the distinguished point. Then the standard shuffle map induces a natural quasi-isomorphism

$$\overline{C}_{\bullet}(X) \otimes \overline{C}_{\bullet}(Y) \to \overline{C}_{\bullet}(X \wedge Y)$$
(4.12)

for all $X, Y \in G$ -Top, where $X \wedge Y$ is the smash product, and the tensor product on the right-hand side of (4.12) is the pointwise tensor product in the category $\operatorname{Fun}(\Lambda \mathbb{Z}^{\operatorname{opp}}, \mathbb{Z}).$

The reduced chain complex functor $\overline{C}_{\bullet}(-)$ easily extends to the category G-Sp^{naive} of naive G-equivariant spectra. Namely, for any $X \in G$ -Sp^{naive} and all integers $i, j \ge 0$, the transition maps (4.1) induce (by adjunction) maps

$$\Sigma^i X(\mathbb{R}^{\oplus j}) \to X(\mathbb{R}^{\oplus i+j}),$$

which give rise to maps

$$\overline{C}_{\bullet}(X(\mathbb{R}^{\oplus i}))[-i] \cong \overline{C}_{\bullet}(\Sigma^{i}X(\mathbb{R}^{\oplus j})) \to \overline{C}_{\bullet}(X(\mathbb{R}^{\oplus i+j})).$$
(4.13)

Definition 4.4. The equivariant homology complex $C^{\text{naive}}_{\bullet}(X)$ of a naive G-spectrum $X \in G$ -Sp^{naive} is given by

$$C^{\text{naive}}_{\bullet}(X) = \lim_{\stackrel{i}{\to}} \overline{C}_{\bullet}(X(\mathbb{R}^{\oplus i}))[i] \in \mathcal{D}(\Lambda \mathbb{Z}^{\text{opp}}, \mathbb{Z}),$$

where the limit is taken with respect to the maps (4.13).

4.3. Cyclic Massey functors. To extend the equivariant homology complex in Definition 4.4 to genuine *G*-spectra, we must pass to the category $\mathcal{DMA}(\mathbb{Z})$ of cyclic Massey functors in §2. We shall use the model given in Proposition 3.12, with an appropriately chosen complex P_{\bullet} .

We begin with the following observation. For every $n \ge 1$ let P^n_{\bullet} be the complex in Fun $(\Lambda \mathbb{Z}^{\text{opp}}, \mathbb{Z})$ obtained as the cone of the natural augmentation map

$$N(L(J_n)) \to \mathbb{Z},$$

where $\mathbb{Z} \in \operatorname{Fun}(\Lambda \mathbb{Z}^{\operatorname{opp}}, \mathbb{Z})$ is the constant functor, and $J_n \colon \Lambda^{\operatorname{opp}} \times \Delta^{\operatorname{opp}} \to \operatorname{Sets}$ is as in (4.10). We note that $P^n_{\bullet}([m|l])$ is a finite-length complex of finitely generated free Abelian groups for every $[m|l] \in \Lambda \mathbb{Z}$. Let $\mathbb{C}(1)$ be the standard complex representation of the group G = U(1). We put $\mathbb{C}(n) = \mathbb{C}(1)^{\otimes n}$ and regard $\mathbb{C}(n)$ as a 2-dimensional real representation by restriction of scalars. We also put

$$\Sigma_n = \Sigma^{\mathbb{C}(n)} \colon G\text{-}\mathrm{Top} \to G\text{-}\mathrm{Top}.$$

Lemma 4.5. For every $X \in G$ -Top we have a natural functorial quasi-isomorphism

$$P^n_{\bullet} \otimes \overline{C}_{\bullet}(X) \to \overline{C}_{\bullet}(\Sigma_n X).$$

Proof. Note that $S^{\mathbb{C}(n)}$ is homeomorphic to the non-equivariant suspension $\Sigma(G/C_n)$ of the standard orbit G/C_n : $S^{\mathbb{C}(n)} \cong \Sigma(G/C_n)$. The homotopy equivalence in Lemma 4.3 induces by adjunction a map

$$J_n \to \widetilde{S}(G/C_n),$$

which in turn induces a quasi-isomorphism

$$P^n_{\bullet} \to \overline{C}_{\bullet}(S^{\mathbb{C}(n)}).$$

Combining this with (4.12), we get the desired assertion. \Box

We now fix a complete G-universe U by taking

$$U = \bigoplus_{n \ge 0} \mathbb{C}(n)^{\infty}, \tag{4.14}$$

where ∞ means the sum of countably many copies (this universe is complete since $\mathcal{C}(-n) \cong \mathcal{C}(n)$ as real representations).

A map $\nu \colon \mathbb{N} \to \mathbb{N}$ is said to be *admissible* if $\nu(i) \neq 0$ for at most finitely many values of *i*. Given any admissible sequence ν , we put

$$V(\nu) = \bigoplus_{i} \mathbb{C}(i)^{\oplus \nu(i)} \subset U$$

and

$$\Sigma_{\nu} = \Sigma^{V(\nu)} = (\Sigma_{i_1})^{\nu(i_1)} \circ \dots \circ (\Sigma_{i_n})^{\nu(i_n)}, \qquad P_{\bullet}^{\nu} = \bigotimes_{1 \le l \le n} (P_{\bullet}^{i_l})^{\otimes \nu(i_l)}$$

where $i_1 \leq \cdots \leq i_n$ are all integers such that $\nu(i_l) \neq 0$. Then Lemma 4.5 immediately gives canonical quasi-isomorphisms

$$P^{\nu}_{\bullet} \otimes \overline{C}_{\bullet}(X) \to C_{\bullet}(\Sigma_{\nu}(X)). \tag{4.15}$$

We will call the subspaces $V(\nu) \subset U$ cellular. The set of all cellular subspaces is obviously cofinal in the set of all finite-dimensional G-invariant subspaces $V \subset U$.

We are now ready to define the complex P_{\bullet} in Fun $(\Lambda I_{\text{red}}^{\text{opp}}, \mathbb{Z})$. Note that for all $m, l \ge 1$ the natural C_m -action on $i_m^* J_n([l]) = J_n([ml])$ factors through a free action of the quotient group $C_m/(C_m \cap C_n)$. Therefore for any vertical morphism $v: a \to b$ in $\Lambda \mathbb{Z}$, the corresponding map

$$v: P^n_{\bullet}(b) \to P^n_{\bullet}(a)$$

is an isomorphism if deg(v) divides n, and is the natural inclusion $\mathbb{Z} = P_0^n(b) \to P_{\bullet}^n(a)$ otherwise. Moreover, for every prime p not dividing n, the restriction $i_p^*(P_{\bullet}^n)$ with respect to the functor $i_p \colon \Lambda_p \to \Lambda \cong \Lambda \mathbb{Z}_h^1 \subset \Lambda \mathbb{Z}$ is the constant functor \mathbb{Z} in degree 0, while $i_p^* P_i^n([m])$ is a free $\mathbb{Z}/p\mathbb{Z}$ -module for every $[m] \in \Lambda_p$ and all $i \ge 1$.

We now define a complex P_{\bullet} of functors in $\operatorname{Fun}(\Lambda I^{\operatorname{opp}}_{\operatorname{red}},\mathbb{Z})$ by putting

$$P_{\bullet} = \lim i_p^* P_{\bullet}^{\nu} \qquad \text{on} \quad \Lambda_p \subset \Lambda I_{\text{red}},$$

where the limit with respect to the natural inclusions $\mathbb{Z} \to P^n_{\bullet}$ is taken over all admissible sequences ν such that $\nu(pl) = 0, l \in \mathbb{N}$. Then P_{\bullet} is admissible in the sense of Proposition 3.12 and we have a canonical equivalence

$$\mathcal{DM}\Lambda(\mathbb{Z}) \cong \mathcal{D}(\Lambda\mathbb{Z}_h, P_{\bullet}, \mathbb{Z}).$$

Remark 4.6. To help the reader visualize the complex P_{\bullet} , we note that on $\Lambda_p \subset \Lambda I_{\text{red}}$ it is essentially given by

$$i_p^* \lim_{\stackrel{V}{\longrightarrow}} \overline{C}_{\bullet}(S^V),$$

where the limit is taken over all cellular subspaces in U orthogonal to $U^{C_p} \subset U$. The only difference is that we use more economical simplicial models for the spheres.

By definition, for every *n* the complex P^{ν}_{\bullet} gives an object in the category $\mathcal{D}(\Lambda \mathbb{Z}_h, P_{\bullet}, \mathbb{Z})$. The corresponding complex is $h^* P^{\nu}_{\bullet}$ (the restriction of P^{ν}_{\bullet} with respect to the embedding $h: \Lambda \mathbb{Z}_h \to \Lambda \mathbb{Z}$) and the map $\varphi: \pi^* h^* P^{\nu}_{\bullet} \to i^* h^* P^{\nu}_{\bullet}$ is induced by the action of the vertical maps on P^{nu}_{\bullet} . To be more explicit, we define a map ν_p by the formula

$$\nu_p(m) = \begin{cases} \nu(m), & m = pl, \ l \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $P_{\bullet}^{\nu} = P_{\bullet}^{\nu_p} \otimes P_{\bullet}^{\nu-\nu_p}$, and the Λ_p -component φ_p of φ is the product of the isomorphism

$$\pi_p^* h^* P_{\bullet}^{\nu_p} \cong i_p^* h^* P_{\bullet}^{\nu_p}$$

and the natural inclusion

$$\mathbb{Z} \cong \pi_p^* h^* P_{\bullet}^{\nu - \nu_p} \to i_p^* h^* P_{\bullet}^{\nu - \nu_p}.$$

If we treat the complexes P^n_{\bullet} up to quasi-isomorphism, then the following equality holds on $\Lambda \mathbb{Z}_h^m \subset \Lambda \mathbb{Z}_h$:

$$h^* P^n_{\bullet} = \begin{cases} \mathbb{Z}[2], & n = ml, \ l \ge 1, \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

We now fix a projective resolution Q_{\bullet} of the constant functor $\mathbb{Z} \in \operatorname{Fun}(\Lambda \mathbb{Z}_{h}^{\operatorname{opp}}, \mathbb{Z})$ and define a complex Q_{\bullet}^{n} by putting

$$Q^n_{\bullet} = \begin{cases} \mathbb{Z}[-2], & n = ml, \ l \ge 1, \\ \mathbb{Z} & \text{otherwise} \end{cases}$$

on $\Lambda \mathbb{Z}_h^m \subset \Lambda \mathbb{Z}_h$. Then $Q^n_{\bullet} \otimes h^* P^n_{\bullet}$ is quasi-isomorphic to the constant functor \mathbb{Z} . Since Q_{\bullet} is projective, there is an actual quasi-isomorphism

$$\varepsilon_n \colon Q_{\bullet} \to Q_{\bullet}^n \otimes h^* P_{\bullet}^n$$

Moreover, given any admissible $\nu \colon \mathbb{N} \to \mathbb{N}$ and any $m \ge 1$, we put

$$d(m,\nu)=\sum_{l\geqslant 1}\nu(ml)$$

and write Q^{ν}_{\bullet} for a complex in Fun $(\Lambda \mathbb{Z}_{h}^{\text{opp}}, \mathbb{Z})$ given by $Q_{\bullet}[-2d(m, \nu)]$ on $\Lambda \mathbb{Z}_{h}^{m} \subset \Lambda \mathbb{Z}^{h}$. Then, taking the tensor product of the maps ε_{n} , we obtain a system of quasiisomorphisms

$$\varepsilon_{\nu} \colon \mathbb{Z} \to Q^{\nu}_{\bullet} \otimes h^* P^{\nu}_{\bullet}$$
(4.16)

for all admissible ν . For every prime p we have a natural map

$$\varepsilon_{\nu-\nu_p} \colon \pi_p^* Q_{\bullet}^{\nu} \to i_p^* (Q_{\bullet}^{\nu} \otimes P_{\bullet}^{\nu-\nu_p}).$$

Composing these maps with the natural embeddings $i_p^* P_{\bullet}^{\nu - \nu_p} \to P_{\bullet}$, we equip Q_{\bullet}^{ν} with the natural structure of an object of $\mathcal{D}(\Lambda \mathbb{Z}_h, P_{\bullet}, \mathbb{Z})$.

We are now ready to define our equivariant homology functor. Let $t \in G$ -sp(U) be a prespectrum. For all admissible $\nu, \nu' \colon \mathbb{N} \to \mathbb{N}$ we have a transition map

$$\Sigma_{\nu'} t(V(\nu)) \to t(V(\nu + \nu'))$$

adjoint to the map (4.1). By (4.15), these maps induce canonical maps

$$P_{\bullet}^{\nu'} \otimes \overline{C}_{\bullet}(t(V(\nu))) \to \overline{C}(V(\nu+\nu')).$$

Let ξ^* : $\mathcal{D}(\Lambda\mathbb{Z}^{\text{opp}},\mathbb{Z}) \to \mathcal{D}(\Lambda\mathbb{Z}, \lambda^*\mathcal{T}_{\bullet},\mathbb{Z}) \cong \mathcal{DM}\Lambda(\mathbb{Z})$ be the corestriction functor with respect to the augmentation map of the A_{∞} -coalgebra $\lambda^*\mathcal{T}_{\bullet}$. Then the composites of these maps with the maps (4.16) induce transition maps

$$\xi^*\overline{C}_{\bullet}(t(V(\nu))) \to Q^{\nu}_{\bullet} \otimes \xi^*P^{\nu}_{\bullet} \otimes \xi^*\overline{C}_{\bullet}(t(V(\nu))) \to Q^{\nu}_{\bullet} \otimes \xi^*\overline{C}_{\bullet}(t(V(\nu+\nu'))).$$
(4.17)

Definition 4.7. The equivariant chain complex $C_{\bullet}(X) \in \mathcal{DMA}(\mathbb{Z})$ of a *G*-prespectrum $t \in G$ -sp(U) is given by

$$C_{\bullet}(t) = \lim_{\stackrel{\nu}{\to}} Q_{\bullet}^{\nu} \otimes \xi^* \overline{C}_{\bullet}(X(V(\nu))), \qquad (4.18)$$

where the limit is taken over all admissible maps $\nu \colon \mathbb{N} \to \mathbb{N}$ with respect to the transition maps (4.17).

Lemma 4.8. For any $t \in G$ -sp(U) with spectrification Lt, the adjunction map $t \rightarrow Lt$ induces an isomorphism

$$C_{\bullet}(t) \to C_{\bullet}(Lt).$$

Proof. Since cellular subspaces are cofinal, it suffices to take the limit in (4.2) over cellular subspaces. Then, substituting (4.2) in (4.18), we see that

$$C_{\bullet}(Lt) = \lim_{\nu \leq \nu'} Q_{\bullet}^{\nu} \otimes \xi^* \overline{C}_{\bullet}(\Omega^{V(\nu'-\nu)} t(V(\nu'))),$$

where the limit is taken over all pairs $\nu \leq \nu'$ of admissible maps $\nu, \nu' \colon \mathbb{N} \to \mathbb{N}$. Since the subset of all pairs with $\nu = \nu'$ is cofinal in this set, it suffices to take the limit over such pairs. This is exactly $C_{\bullet}(t)$. \Box

Corollary 4.9. For every $X \in G$ -Top we have

$$C_{\bullet}(\Sigma^{\infty}X) \cong \xi^*\overline{C}(X).$$

For every $X \in G$ -Sp^{naive} we have

$$C_{\bullet}(\rho_{\#}(u)X) \cong C_{\bullet}(X), \tag{4.19}$$

where $\rho_{\#}(u)$ is the change-of-universe functor associated with the embedding $\mathbb{R}^{\oplus \infty} = U^G \subset U$.

Proof. Using Lemma 4.8, we can replace Σ^{∞} and $\rho_{\#}(u)$ by $\tilde{\Sigma}^{\infty}$ and $\tilde{\rho}_{\#}(u)$ respectively. Then for $t = \tilde{\Sigma}^{\infty} X$ all the transition maps in the filtered limit of (4.18) are quasi-isomorphisms, and for $t = \tilde{\rho}_{\#}(u)X$ the only possibly non-trivial transition maps are those appearing in (4.13). \Box

4.4. Cyclotomic complexes. For every $m \ge 1$ we denote the group of *m*th roots of unity by $C_m = \mathbb{Z}/m\mathbb{Z} \subset G = U(1)$. The *m*-power map gives an isomorphism $p_m: G/C_m \to G$, and we have $p_m \circ p_n = p_{nm}, m, n \ge 1$. There are obvious canonical *G*-equivariant isomorphisms

$$u_m \colon U^{C_m} \cong U,$$

where U is the complete G-universe (4.14) and we have $u_m \circ u_n = u_{nm}$, $n, m \ge 1$.

Definition 4.10. For every $X \in G$ -Sp = G-Sp(U) and all $m \ge 1$, the extended geometric fixed-point spectrum $\widehat{\Phi}^m(X) \in G$ -Sp is given by

$$\widehat{\Phi}^m(X) = \rho_{\#}(u_m) \Phi^{C_m}(X),$$

and the extended Lewis–May fixed-point spectrum $\widehat{\Psi}^m(X) \in G$ -Sp is given by

$$\widehat{\Psi}^m(X) = \rho_{\#}(u_m) X^{C_m}.$$

For any $m \ge 1$ let $\rho_m \colon I \to I$ be multiplication by m, as in (2.20). It commutes with the \mathbb{N}^* -action and, therefore, induces an endofunctor

$$\widetilde{\rho}_m \colon \Lambda \mathbb{Z} \to \Lambda \mathbb{Z}$$

commuting with the projection $\lambda \colon \Lambda \mathbb{Z} \to \Lambda R$. By (4.11) we have

$$\widetilde{\rho}_m^* LA \cong LA^{C_m}$$

for all $A \in \Lambda^{\text{opp}}$ Sets. In particular, for any $X \in G$ -Top we have quasi-isomorphisms

$$\widetilde{\rho}_m^* \overline{C}_{\bullet}(X) \cong \overline{C}_{\bullet}(X^{C_m}). \tag{4.20}$$

By (4.13) they induce quasi-isomorphisms

$$\widetilde{\rho}_m^* C_{\bullet}(X) \cong C_{\bullet}(X^{C_m}) \tag{4.21}$$

for every naive *G*-spectrum $X \in G$ -Sp^{naive}. We want to obtain a version of this construction for genuine *G*-spectra. Since $\lambda \circ \tilde{\rho}_m = \lambda$, we tautologically have $\tilde{\rho}^* \lambda^* \mathcal{T}_{\bullet} \cong \lambda^* \mathcal{T}_{\bullet}$. This gives rise to a pullback functor

$$\widetilde{\rho}_m^* \colon \mathcal{DM}\Lambda(Z) \to \mathcal{DM}\Lambda(Z).$$

Lemma 4.11. For any $m \ge 1$ and all $X \in G$ -Sp we have a natural functorial isomorphism

$$C_{\bullet}(\widehat{\Phi}^m(X)) \cong \widetilde{\rho}_m^* C_{\bullet}(X).$$

Proof. For every map $\nu \colon \mathbb{Z} \to \mathbb{Z}$ we define $r(\nu) \colon \mathbb{Z} \to \mathbb{Z}$ by the formula $r(\nu)(ma+b) = \nu(a), \ a \in \mathbb{Z}, \ 0 \leq b < n$. If ν is admissible, then so is $r(\nu)$.

Using Lemma 4.8, we can replace $\widehat{\Phi}^m$ by the functor $\varphi^m = \widetilde{\rho}_{\#}(u_m) \circ \varphi^{C_m}$. Choose the subspaces $W(V) \subset U$ in such a way that $W(V(\nu)) = V(r(\nu))$ for all admissible $\nu : \mathbb{Z} \to \mathbb{Z}$. Then

$$C_{\bullet}(\varphi^m X) = \lim_{\stackrel{\nu}{\to}} \mathcal{Q}_{\bullet}^{\nu} \otimes \xi^* \overline{C}_{\bullet}(X(V(r\nu))^{C_m})$$
(4.22)

and, since the sequences $r(\nu)$ are cofinal in the set of all admissible sequences,

$$\widetilde{\rho}_m^* C_{\bullet}(X) = \lim_{\stackrel{\nu}{\to}} \widetilde{\rho}_m^* Q_{\bullet}^{r(\nu)} \xi^* \overline{C}_{\bullet}(X(V(r\nu))).$$
(4.23)

By definition, the pullback functor commutes with the corestriction: $\tilde{\rho}_m^* \circ \xi^* \cong \xi^* \circ \tilde{\rho}_m^*$, and we have the isomorphisms (4.20). It remains to note that, by definition,

$$\widetilde{\rho}_m^* Q_{\bullet}^{r(\nu)} \cong Q_{\bullet}^{\nu}$$

for all admissible ν . \Box

We now consider the right adjoint $\xi_* : \mathcal{DMA}(\mathbb{Z}) \to \mathcal{D}(\Lambda \mathbb{Z}^{opp}, \mathbb{Z})$ of the corestriction functor ξ^* , as in § 2.4. Then the isomorphism (4.19) induces a base-change map

$$C_{\bullet}(\rho^{\#}(u)(X)) \to \xi_* C_{\bullet}(X) \tag{4.24}$$

for every $X \in G$ -Sp. By (4.21) and (4.4) we have a natural isomorphism

$$\widetilde{\rho}_m^* C_{{\scriptscriptstyle\bullet}}(\rho^{\#}(u)(X)) \cong C_{{\scriptscriptstyle\bullet}}(\widehat{\Psi}^m(X))$$

for every genuine G-spectrum $X \in G$ -Sp and all $m \ge 1$.

Lemma 4.12. The base-change map (4.24) is an isomorphism for every $X \in G$ -Sp, whence

$$C_{\bullet}(\Psi^m(X)) \cong \widetilde{\rho}_m^* \xi_* C_{\bullet}(X)$$

for all $m \ge 1$.

Proof. It suffices to prove that the map (4.24) becomes an isomorphism after evaluation at any object $[n|m] \in \Lambda \mathbb{Z}$. By (4.4) and the definition of the equivariant homology complex $C_{\bullet}(-)$ of a naive G-spectrum, we have a natural quasi-isomorphism

$$C_{\bullet}(\rho^{\#}(u)(X))([n|m]) \cong C_{\bullet}(X^{C_m}),$$

where X^{C_m} is regarded as a non-equivariant spectrum, and $C_{\bullet}(-)$ on the right-hand side is obtained from the usual non-equivariant functor $\overline{C}_{\bullet}(-,\mathbb{Z})$ of reduced singular chain complex by passing to the limit as in Definition 4.4 (*n* is irrelevant since the functors in Definition 4.1 are locally constant). Thus we must prove that, for all $n, m \ge 1$, the natural map

$$C_{\bullet}(X^{C_m}) \to (\xi_*C_{\bullet}(X))([n|m])$$

induced by (4.24) is an isomorphism.

On the other hand, by the definition of the functor $C_{\bullet}: G\text{-}\mathrm{Sp} \to \mathcal{DMA}(\mathbb{Z})$ we have an isomorphism

$$\Sigma^V \circ C_{\bullet} \cong C_{\bullet} \circ \Sigma^V$$

for any $V = V(\nu) \subset U$, and the base-change map

$$C_{\bullet} \circ (\Sigma^V)^{-1} \to (\Sigma^V)^{-1} \circ C_{\bullet}$$

is also an isomorphism. Therefore (4.24) is an isomorphism for some $X \in G$ -Sp if and only if it is an isomorphism for $\Sigma^V X$. Since every $X \in G$ -Sp is a filtered colimit of spectra of the form $(\Sigma^V)^{-1}\Sigma^{\infty}Y$, $Y \in G$ -Top, it suffices to prove that (4.24) is an isomorphism for the suspension spectra $\Sigma^{\infty}Y$. By Corollary 4.9 we have a quasi-isomorphism

$$C_{\bullet}(\Sigma^{\infty}Y) \cong \xi^*\overline{C}_{\bullet}(Y).$$

Thus we must prove that for every $Y \in G$ -Top and all $[n|m] \in \Lambda \mathbb{Z}$, the natural map

$$C_{\bullet}((\Sigma^{\infty}Y)^{C_m}) \to (\xi_*\xi^*\overline{C}_{\bullet}(Y))([n|m])$$

induced by (4.24) is an isomorphism. The right-hand side can be computed using (2.19) and Lemma 2.14:

$$(\xi_*\xi^*\overline{C}_{\:\bullet}(Y))([n|m]) \cong \bigoplus_{l,p \geqslant 1, \, lp=m} C_{\:\bullet}\big(\mathbb{Z}/l\mathbb{Z}, \overline{C}_{\:\bullet}(Y)([nl|p])\big),$$

where, by definition, $\overline{C}_{\bullet}(Y)([nl|p])$ is equal to the reduced chain complex of the fixed point set Y^{C_p} :

$$\overline{C}_{\bullet}(Y)([nl|p]) \cong \overline{C}_{\bullet}(Y^{C_p}, \mathbb{Z}).$$

Then the desired isomorphism becomes the tom Dieck–Segal splitting (see, for example, [6], $\S1$). \Box

We recall another fundamental notion (again following [6]).

Definition 4.13. A cyclotomic structure on a genuine G-spectrum $T \in G$ -Sp is given by homotopy equivalences

$$r_m \colon \widehat{\Phi}^m T \cong T,$$

one for each integer $m \ge 1$, such that $r_1 = \text{id}$ and $r_n \circ r_m = r_{nm}$ for all integers n, m > 1.

Example 4.14. Let X be a pointed CW-complex and let $LX = \text{Maps}(S^1, X)$ be its free loop space. Then, for every finite subgroup $C \subset S^1$, the isomorphism $S^1 \cong S^1/C$ induces a homeomorphism

$$Maps(S^1, X)^C = Maps(S^1/C, X) \cong Maps(S^1, X),$$

and these homeomorphisms provide a canonical cyclotomic structure on the suspension spectrum $\Sigma^{\infty}LX$.

By Lemma 4.11, a cyclotomic structure $\{r_m\}$ on T induces a set of quasi-isomorphisms

$$r_m \colon \widetilde{\rho}_m^* C_{\:\bullet}(T) \to C_{\:\bullet}(T), \quad m \geqslant 1,$$

with $r_m \circ r_n = r_{nm}$. If we regard the equivariant chain complex $C_{\bullet}(T)$ as an object $\langle h^*C_{\bullet}(T), \varphi \rangle$ in the category $\mathcal{D}(\Lambda \mathbb{Z}_h, P_{\bullet}, \mathbb{Z})$, then the maps r_m extend $h^*C_{\bullet}(T)$ to a complex in the category $\operatorname{Fun}(\Lambda R^{\operatorname{opp}}, \mathbb{Z})$, and this extension is compatible with the map φ . Therefore $C_{\bullet}(T)$ canonically determines an object $\widetilde{C}_{\bullet}(T)$ in the derived category $\mathcal{D}(\Lambda R_h, P_{\bullet}, \mathbb{Z})$. By Proposition 3.12, this makes $C_{\bullet}(T)$ into a cyclotomic complex. Thus we can finally justify our terminology by introducing the following definition.

Definition 4.15. The cyclotomic complex $\widetilde{C}_{\bullet}(T) \in \mathcal{D}\Lambda R(\mathbb{Z})$ is called the *equiv*ariant chain complex of the cyclotomic spectrum T.

§5. Filtered Dieudonné modules

5.1. Definitions. We now want to compare cyclotomic complexes with another (and simpler) algebraic notion, which appeared earlier in another context: the notion of a filtered Dieudonné module.

Definition 5.1. Let k be a finite field of characteristic p with its Frobenius map, and let W be its ring of Witt vectors with the canonical lifting of the Frobenius map. A filtered Dieudonné module over W is a finitely generated W-module M equipped with a decreasing filtration $F^{\bullet}M$ (indexed by all integers) and a family of Frobenius-semilinear maps $\varphi_i: F^i M \to M$, one for each integer i, such that

- (i) $\varphi_i|_{F^{i+1}M} = p\varphi^{i+1};$
- (ii) the map $\sum \varphi_i \colon \bigoplus_i F^i M \to M$ is surjective.

This definition was introduced by Fontaine and Lafaille [7] as a *p*-adic analogue of the notion of a Hodge structure. Under certain assumptions, the de Rham cohomology $H_{DR}^{\bullet}(X)$ of a smooth compact algebraic variety X/W has the natural structure of a filtered Dieudonné module, where F^{\bullet} is the Hodge filtration and the maps φ^{\bullet} are induced by the Frobenius endomorphism of the special fibre $X_k = X \otimes_W k$.

The category of filtered Dieudonné modules is obviously additive, but one can say more: just as for mixed Hodge structures, a small miracle happens and the category is actually Abelian. The normalization condition (ii) of Definition 5.1 plays a crucial role here. This condition can be dropped if one is prepared to work with non-Abelian additive categories. The following notion is convenient for our purposes.

Definition 5.2. A generalized filtered Dieudonné module (gFDM for short) is an Abelian group M equipped with

(i) a decreasing filtration $F^{\bullet}M$ indexed by all integers and satisfying $M = \bigcup F^i M$, and

(ii) a family of maps $\varphi_{i,j}^p \colon F^i M \to M/p^j M$ (for all integers i, positive integers $j \ge 1$, and primes p) such that $\varphi_{i,j+1}^p = \varphi_{i,j}^p \mod p^j$ and $\varphi_{i,j}^p = p\varphi_{i+1,j}^p$ on $F^{i+1}M \subset F^i M$.

The differences between Definitions 5.2 and 5.1 are that we no longer require the normalization condition (ii), consider only prime fields rather than finite fields, and combine the structures over all primes. Note, however, that if M is finitely generated over \mathbb{Z}_p , then all other primes act as invertible maps on M, whence all the extra maps $\varphi_{\bullet,\bullet}^l$, $l \neq p$, are equal to zero. Indeed, for every prime p and every integer i we can combine all the maps $\varphi_{i,\bullet}^p$ into a single map

$$\varphi^p_i\colon F^iM\to \widehat{(M)}_p,$$

where $(M)_p$ stands for the pro-*p*-completion of the Abelian group M. If M is finitely generated over \mathbb{Z}_p , then $(\widehat{M})_p \cong M$ and $(\widehat{M})_l = 0$ for $l \neq p$.

Complexes of gFDM are defined in the obvious way. A map between such complexes is a quasi-isomorphism if it induces a quasi-isomorphism of the associated graded quotients gr^{F} . Inverting such quasi-isomorphisms, we obtain a triangulated 'derived category of gFDM'. We denote it by \mathcal{FDM} . For every gFDM M and any integer i, let M(i) be the same M with filtration F^{\bullet} twisted by i (that is, $F^{j}M(i) = F^{j-i}M$). Under this convention we introduce the following 'twisted 2-periodic' version of the category \mathcal{FDM} .

Definition 5.3. The triangulated category $\mathcal{FDM}^{\text{per}}$ is obtained by inverting quasiisomorphisms in the category of complexes of gFDMs M_{\bullet} equipped with an isomorphism $M \cong M[2](1)$.

We can now state the main result of this section.

Theorem 5.4. There is a natural equivalence

$$\mathcal{FDM}^{\mathrm{per}} \cong \mathcal{D}\Lambda R(\mathbb{Z})$$

between the twisted 2-periodic derived category of gFDMs, on the one hand, and the derived category of cyclotomic complexes of Abelian groups in the sense of Definition 3.1, on the other.

We shall prove this (in a more precise version given by Proposition 5.17) in § 5.5 after the necessary preparations. We start with some generalities on filtered objects.

5.2. Filtered objects. By a *filtered object* in an Abelian category Ab we shall understand an object $E \in Ab$ equipped with a decreasing filtration F^{\bullet} indexed by all integers. Maps and complexes of filtered objects are defined in the obvious way.

Definition 5.5. A map $f: E_{\bullet} \to E'_{\bullet}$ between complexes of filtered objects in Ab is a *filtered quasi-isomorphism* if the induced map

$$f: F^i E_{\bullet}/F^{i+1} E_{\bullet} \to F^i E'_{\bullet}/F^{i+1} E'_{\bullet}$$

is a quasi-isomorphism for every integer i.

Remark 5.6. We do not require a filtered quasi-isomorphism to induce a quasi-isomorphism, neither of the complexes E_{\bullet} , E'_{\bullet} of objects in Ab nor of the individual pieces $F^i E_{\bullet}$, $F^i E'_{\bullet}$ of the filtrations.

The filtered derived category $\mathcal{DF}(Ab)$ is obtained by inverting filtered quasiisomorphisms in the category of filtered complexes and filtered maps. The *periodic* filtered derived category $\mathcal{DF}^{per}(Ab)$ is obtained in a similar way from the category of complexes of filtered objects V_{\bullet} in Ab equipped with an isomorphism

$$u\colon V_{\bullet}\cong V_{\bullet-2}(1),$$

where V(1) stands for the twist of the filtration. Explicitly, such a complex is determined by two filtered objects V_0 , V_1 and two filtered maps

$$d_1 \colon V_1 \to V_0, \qquad d_0 \colon V_0 \to V_1(1) \cong V_{-1}$$

such that $d_1 \circ d_0 = 0 = d_0 \circ d_1$. The other terms in the complex are then given by $V_{2\bullet} = V_0$, $V_{2\bullet+1} = V_1$ with the same differentials d_0 , d_1 .

If Ab is the category of Abelian groups, then we obtain the periodic filtered derived category $\mathcal{DF}^{\text{per}}(\mathbb{Z})$ of filtered Abelian groups. For any integer n, let $\mathbb{Z}(n) \in \mathcal{DF}^{\text{per}}$ be the object V given by $V_0 = \mathbb{Z}$, $V_1 = 0$, with the filtration $F^n V_0 = V_0$ and $F^{n+1}V_0 = 0$. Then the objects $\mathbb{Z}(n)$, $n \in \mathbb{Z}$, generate the category $\mathcal{DF}^{\text{per}}(\mathbb{Z})$ in the following sense.

Lemma 5.7. Any triangulated subcategory $\mathcal{D}' \subset \mathcal{DF}^{\text{per}}(\mathbb{Z})$ closed under arbitrary sums and products and containing \mathbb{Z} with the trivial filtration $F^0\mathbb{Z} = \mathbb{Z}$, $F^1\mathbb{Z} = 0$, is equal to the whole of $\mathcal{DF}^{\text{per}}(\mathbb{Z})$.

Proof. Since \mathcal{D}' is closed under taking cones, it contains any filtered complex $\langle V_{\bullet}, F^{\bullet} \rangle$ with bounded filtration F^{\bullet} (that is, $F^i V_{\bullet} = 0$, $F^j V_{\bullet} = V_{\bullet}$ for some integers i, j). Let $\langle V_{\bullet}, F^{\bullet} \rangle$ be an arbitrary filtered complex. Then the natural maps

$$V_{\bullet} \longleftarrow \lim_{i \to i} F^{-i} V_{\bullet} \longrightarrow \lim_{i \to i} \lim_{j \to i} F^{-i} V_{\bullet} / F^{j} V_{\bullet}$$
(5.1)

induce isomorphisms on gr^F and, therefore, become isomorphisms in $\mathcal{DF}^{\operatorname{per}}(\mathbb{Z})$. Hence we can replace V_{\bullet} by the double limit on the right-hand side of (5.1). The direct limit can be computed by the telescope construction. Moreover, for every i, the inverse system $F^iV_{\bullet}/F^{\bullet}V_{\bullet}$ satisfies the Mittag–Leffler condition, so that the inverse limit can also be computed by the telescope construction. Since \mathcal{D}' is closed under sums and products, it is also closed under telescopes and, therefore, it contains V_{\bullet} . \Box

Note that by (5.1) we can represent every object in $\mathcal{DF}^{\text{per}}(\mathbb{Z})$ by a filtered complex $\langle V_{\bullet}, F^{\bullet} \rangle$ which is *admissible* in the following sense: the natural maps

$$\lim_{\stackrel{i}{\to}} F^{-i}V_{\bullet} \to V_{\bullet}, \qquad V_{\bullet} \to \lim_{\stackrel{i}{\leftarrow}} V_{\bullet}/F^{i}V_{\bullet}$$

are isomorphisms of complexes.

Working with filtered Abelian groups, it is convenient to use Rees objects. Consider the algebra $\mathbb{Z}[t]$ of polynomials in one variable t. A module M over $\mathbb{Z}[t]$ is said to be t-adically complete if the natural map

$$M \to \lim_{\stackrel{i}{\leftarrow}} M/t^i M$$

is an isomorphism (thus, in our terminology, completeness presupposes separability). We make $\mathbb{Z}[t]$ into a graded ring by assigning degree -1 to the generator t.

Lemma 5.8. The filtered derived category $\mathcal{DF}(\mathbb{Z})$ is equivalent to the full subcategory of the derived category of the Abelian category of \mathbb{Z} -graded $\mathbb{Z}[t]$ -module M_{\bullet} spanned by the t-adically complete modules.

Proof. For every filtered Abelian group $\langle M, F^{\bullet} \rangle$, the corresponding graded $\mathbb{Z}[t]$ -module \widetilde{M}_{\bullet} is called the *Rees object* of M_{\bullet} and is given by

$$\widetilde{M}_{\bullet} = \bigoplus \lim_{\stackrel{i}{\longrightarrow}} F^{\bullet} M / F^{\bullet + i} M,$$

where t is induced by the natural embeddings $F^{\bullet}M \to F^{\bullet-1}M$. We note that \widetilde{M}_{\bullet} is automatically t-adically complete, and a filtered quasi-isomorphism of complexes of Abelian groups induces a quasi-isomorphism of Rees objects.

To get the inverse correspondence, we note that every graded $\mathbb{Z}[t]$ -module M_{\bullet} has a finite resolution by modules with no t-torsion. Hence it suffices to consider the graded modules M_{\bullet} with injective maps t. Such a module M_{\bullet} is sent to

$$\widetilde{M} = \lim_{\stackrel{t}{\longrightarrow}} M_{\bullet},$$

and $F^i \widetilde{M} \subset \widetilde{M}$ is the image of the natural embedding $M_i \to \widetilde{M}$ for every integer i. \Box

The equivalence in Lemma 5.8 has an obvious periodic version. A model of the category $\mathcal{DF}^{\mathrm{per}}(\mathbb{Z})$ in terms of Rees objects is obtained by inverting quasiisomorphisms in the category of pairs $\langle \widetilde{M}_{\bullet,\bullet}, u \rangle$ formed by a complex $\widetilde{M}_{\bullet,\bullet}$ of graded *t*-adically complete $\mathbb{Z}[t]$ -modules M_{\bullet} and an isomorphism

$$\widetilde{u} \colon \widetilde{M}_{\bullet,\bullet} \cong \widetilde{M}_{\bullet+1,\bullet-2}. \tag{5.2}$$

5.3. Cyclic expansion and subdivision. Now let $Ab = \operatorname{Fun}(\Lambda^{\operatorname{opp}}, \mathbb{Z})$ be the category of cyclic Abelian groups. We put $I_0 = j_1^{\circ}\mathbb{Z}$, $I_1 = j_*\mathbb{Z}$ as in (1.23) and define $d_1 = B \colon I_1 \to I_0$, $d_0 = b_0 \circ b_1 \colon I_0 \to I_1$, where b_0 , b_1 and B are again as in (1.23). Since the sequence (1.23) is exact, we have $d_1 \circ d_0 = 0 = d_0 \circ d_1$. Moreover, if we define filtrations F^{\bullet} on I_0 and I_1 by putting $F^0I_l = I_l$, $F^1I_l = 0$, l = 0, 1, then d_0 and d_1 are filtered maps and we have a periodic filtered complex I_{\bullet} of objects in $\operatorname{Fun}(\Lambda^{\operatorname{opp}}, \mathbb{Z})$.

Definition 5.9. For any object in $\mathcal{DF}^{\text{per}}(\mathbb{Z})$ represented by a periodic complex V_{\bullet} of admissible filtered Abelian groups, we define its *cyclic expansion* $\text{Exp}(V_{\bullet})$ as the complex of cyclic Abelian groups

$$\operatorname{Exp}(V_{\bullet}) = F^0(V_{\bullet} \otimes_{\mathbb{Z}[u]} I_{\bullet}),$$

where u is the periodicity map on V_{\bullet} and I_{\bullet} , and F^0 is taken with respect to the product filtration.

In terms of the corresponding periodic complex $\widetilde{V}_{\bullet,\bullet}$ of Rees objects, the cyclic expansion is given by

$$\operatorname{Exp}(V_{\bullet})_{i} = (\widetilde{V}_{0,\bullet} \otimes I_{1})[1] \oplus (\widetilde{V}_{0,\bullet} \otimes I_{0})$$
(5.3)

with the differential $d = d_V \otimes \operatorname{id} + d_I$, where d_V is the differential on $\widetilde{V}_{0,\bullet}$ and d_I is equal to $\operatorname{id} \otimes d_1$ on the first summand and to $t\widetilde{u} \otimes d_0$ on the second, with $\widetilde{u} \colon \widetilde{V}_{\bullet,\bullet} \cong \widetilde{V}_{\bullet+1,\bullet-2}$ as in (5.2). In particular, for every object $[n] \in \Lambda$, the complex $\operatorname{Exp}(V_{\bullet})([n])$ is a sum of finitely many copies of $\widetilde{V}_{0,\bullet}$ and of its shift $\widetilde{V}_{0,\bullet}[1]$. Hence cyclic expansion commutes with arbitrary sums and products.

Lemma 5.10. Cyclic expansion induces an equivalence of categories

Exp:
$$\mathcal{DF}^{\mathrm{per}}(\mathbb{Z}) \cong \mathcal{D}_c(\Lambda^{\mathrm{opp}}, \mathbb{Z})$$

between $\mathcal{DF}^{\text{per}}(\mathbb{Z})$ and the full subcategory $\mathcal{D}_c(\Lambda^{\text{opp}},\mathbb{Z}) \subset \mathcal{D}(\Lambda^{\text{opp}},\mathbb{Z})$ spanned by the objects that are locally constant in the sense of Definition A.1.

Proof. Since the sequence (1.23) is exact, $F^1I_{\bullet} = I_{<0}$ is a resolution of the constant functor $\mathbb{Z} \in \operatorname{Fun}(\Lambda, \mathbb{Z})$. It follows by induction that the homology functors $\mathcal{H}_{\bullet}(\operatorname{Exp}(V_{\bullet})) \in \operatorname{Fun}(\Lambda^{\operatorname{opp}}, \mathbb{Z})$ of any periodic filtered complex V_{\bullet} are given by

$$\mathcal{H}_i(\operatorname{Exp}(V_{\bullet})) \cong (F^0 V_i / F^1 V_i) \otimes \mathbb{Z}.$$

In particular, a filtered quasi-isomorphism of periodic filtered complexes induces a quasi-isomorphism of their cyclic expansions and, therefore, Exp induces a triangulated functor from the category $\mathcal{DF}^{\text{per}}(\mathbb{Z})$ to the full subcategory $\mathcal{D}_c(\Lambda^{\text{opp}},\mathbb{Z}) \subset \mathcal{D}(\Lambda^{\text{opp}},\mathbb{Z})$. Moreover, for every integer n we have

$$\operatorname{Exp}(\mathbb{Z}(n)) \cong \mathbb{Z}[2n].$$

Since the fundamental group $\pi_1(|\Lambda|) \cong \pi_1(BU(1))$ is trivial, every locally constant functor $\Lambda^{\text{opp}} \to \mathbb{Z}$ -mod is constant. Therefore \mathbb{Z} generates the triangulated category $\mathcal{D}_c(\Lambda^{\text{opp}},\mathbb{Z})$ in the sense of Lemma 5.7. Since Exp commutes with arbitrary sums and products, it suffices to prove that Exp is fully faithful on the objects $\mathbb{Z}(n)$ or, in other words, that the natural map

Exp: RHom
$$\overset{\bullet}{\mathcal{D}\mathcal{F}^{\mathrm{per}}(\mathbb{Z})}(\mathbb{Z}(n),\mathbb{Z}(m)) \to \operatorname{RHom}^{\bullet}_{\mathcal{D}(\Lambda^{\mathrm{opp}},\mathbb{Z})}(\mathbb{Z}[2n],\mathbb{Z}[2m])$$

is a quasi-isomorphism for all integers n, m. This follows immediately from the isomorphism $H^{\bullet}(\Lambda^{\text{opp}}, \mathbb{Z}) \cong \mathbb{Z}[u]$. \Box

Let $n \ge 1$ be an integer, and let $\delta^n: \mathbb{Z}[t] \to \mathbb{Z}[t]$ be the map sending t to nt. Since the ideal $(nt) \subset \mathbb{Z}[t]$ lies inside $(t) \subset \mathbb{Z}[t]$, the direct image $\delta^n_* M$ of a t-adically complete $\mathbb{Z}[t]$ -module M is automatically t-adically complete.

Definition 5.11. Let $n \ge 1$ be a positive integer and let M be a filtered Abelian group with corresponding admissible Rees object M_{\bullet} as in Lemma 5.8. Then the *nth subdivision* $\text{Div}_n(M)$ is the filtered Abelian group corresponding to the graded Abelian group $\delta_*^n M_{\bullet}$.

In other words, M_{\bullet} remains the same as a graded Abelian group, but the map t is replaced by its multiple nt. We note that the underlying filtered Abelian group M may change under subdivision. For example, if $M = \mathbb{Z}(0) = \mathbb{Z}$, then

$$\operatorname{Div}_n(M) \cong \mathbb{Q}$$

with filtration $F^1 \mathbb{Q} = 0$, $F^i \mathbb{Q} = n^i \mathbb{Z} \subset \mathbb{Q}$ for $i \leq 0$.

Clearly, Div_n is an endofunctor of the category of filtered Abelian groups for every n. It descends to an endofunctor

$$\operatorname{Div}_n \colon \mathcal{DF}^{\operatorname{per}}(\mathbb{Z}) \to \mathcal{DF}^{\operatorname{per}}(\mathbb{Z})$$

of the periodic derived category $\mathcal{DF}^{\mathrm{per}}(\mathbb{Z})$.

We fix an integer $n \ge 1$ and consider the functors $i_n, \pi_n \colon \Lambda_n \to \Lambda$ in §1.1.

Lemma 5.12. For every periodic complex V. of admissible Abelian groups we have a functorial isomorphism

$$\pi_{n*}i_n^* \operatorname{Exp}(V_{\bullet}) \cong \operatorname{Exp}(\operatorname{Div}_n(V_{\bullet})).$$

Proof. We put $I'_l = \pi_{n*}i^*_n I_l$, $d'_l = \pi_{n*}i^*_n d_l$, l = 0, 1. Applying the functor $\pi_{n*}i^*_n$ to (5.3), we see that the complex $\pi_{n*}i^*_n \text{Exp}(V_{\bullet})$ is given by

$$\pi_{n*}i_n^* \operatorname{Exp}(V_{\bullet})_i = (\widetilde{V}_{0,\bullet} \otimes I_1)[1] \oplus (\widetilde{V}_{0,\bullet} \otimes I_0)$$

with the differential $d = d_V \otimes \operatorname{id} + d_I$, where d_V is the differential on $\widetilde{V}_{0,\bullet}$ and d_I is equal to $\operatorname{id} \otimes d'_1$ on the first summand and to $t\widetilde{u} \otimes d'_0$ on the second. By (1.25) we have canonical isomorphisms $I'_0 \cong I_0$, $I'_1 \cong I_1$. Under these isomorphisms, $d'_1 = d_1$ and $d'_0 = nd_0$. Therefore $t\widetilde{u} \otimes d'_0 = nt\widetilde{u} \otimes d_0$, and $\pi_{n*}i^*_n \operatorname{Exp}(V_{\bullet})$ is exactly isomorphic to the expression (5.3) for the complex $\operatorname{Exp}(\operatorname{Div}_n(V_{\bullet}))$. \Box

5.4. Stabilization. For every filtered Abelian group M with admissible Rees object M, we have a tautological map $M \to M(1)$. Applying the subdivision functor Div_n , we obtain a natural map

$$\operatorname{Div}_n(M) \to \operatorname{Div}_n(M(1)).$$
 (5.4)

Definition 5.13. In the situation of Definition 5.11, the stabilized nth subdivision $\operatorname{Stab}_n(M)$ is given by

$$\operatorname{Stab}_n(M) = \lim_{\stackrel{l}{\longrightarrow}} \operatorname{Div}_n(M(l)),$$

where the limit is taken with respect to the tautological maps (5.4).

Lemma 5.14. For every filtered Abelian group M which is admissible in the sense of § 5.2 and any prime $p \ge 2$ we have

$$\operatorname{Stab}_p(M) \cong \widehat{M}_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

with the filtration

$$F^l\widehat{M}_p = p^l\widehat{M}_p$$

where $\widehat{M}_p = \lim_{\stackrel{l}{\leftarrow}} M/p^l M$ is the pro-p-completion of the group M. Proof. At the level of Rees objects,

$$M_{\bullet} = \lim \operatorname{Div}_p(M)_{\bullet}$$

is isomorphic to M in every degree $l, M_l \cong M$, and $t \in \mathbb{Z}[t]$ acts by multiplication by p. However, this module is not t-adically complete. Thus, when we apply the equivalence in Lemma 5.8, the result is

$$\lim_{\stackrel{l}{\leftarrow}} M/t^l M_{\bullet} \cong \widehat{M}_p,$$

as required. \Box

As a corollary, we see that the periodic derived category $\mathcal{FDM}^{\text{per}}$ of generalized filtered Dieudonné modules in § 5.1 is equivalent to the category of filtered periodic complexes V_{\bullet} of Abelian groups equipped with a map

$$\varphi_p \colon V_{\bullet} \to \operatorname{Stab}_p(V_{\bullet})$$

for any prime $p \ge 2$.

Now let I_{\bullet} be the periodic complex of functors in Fun (Λ, \mathbb{Z}) as introduced in § 5.3, and let $P_{\bullet} \subset I_{\bullet}$ be its canonical truncation at 0. Explicitly we have $P_0 = \mathbb{Z}$ (the constant functor) and

$$P_{2l-1} = I^1, \qquad P_{2l} = I^0$$

for all $l \ge 1$. The complex P_{\bullet} is acyclic. Moreover, the pullback i^*P_{\bullet} with respect to the functor $i: \Lambda I_{\text{red}} \to \Lambda$ in § 3.3 is admissible in the sense of Lemma 3.10.

Let $\eta^+: \mathbb{Z} \to F^0I_{\bullet}$ and $\eta^-: \mathbb{Z} \cong P_0 \to P_{\bullet}$ be the natural embeddings. We define a map

$$\eta\colon I_{\bullet}\to P_{\bullet}\otimes (F^0I_{\bullet})$$

by putting

$$\eta = \begin{cases} \eta^- \otimes \mathrm{id} & \mathrm{on} \ I_l, \ l \leqslant 0, \\ \mathrm{id} \otimes \eta^+ & \mathrm{on} \ I_l, \ l \geqslant 0. \end{cases}$$
(5.5)

Let $F^l P_{\bullet}$, $l \ge 0$, be the stupid filtration on the complex P_{\bullet} . Then, for every $l \ge 0$, the map η induces a map

$$\eta \colon F^{-l}I_{\bullet} \to F^0I_{\bullet} \otimes F^{2l}P_{\bullet} \tag{5.6}$$

and this map is a quasi-isomorphism (both sides are quasi-isomorphic to $\mathbb{Z}[2l]$).

Lemma 5.15. For any $n \ge 1$, $l \ge 0$ and any complex $V \in \operatorname{Fun}(\Lambda_n^{\operatorname{opp}}, Z)$, the natural map

$$\eta \colon \pi_{n*}(V \otimes i_n^* F^{-l}I_{\bullet}) \to \pi_{n*}(V \otimes i_n^* F^0I_{\bullet} \otimes i_n^* F^{2l}P_{\bullet})$$
(5.7)

is a quasi-isomorphism.

Proof. Both sides of (5.7) are finite-length complexes in Fun($\Lambda_n^{\text{opp}}, \mathbb{Z}$). After evaluation at an object $[m] \in \Lambda_n$, they give complexes of free $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ -modules. Therefore we can replace π_{n*} by its derived functor $R^{\bullet}\pi_{n*}$. Then the desired assertion follows immediately since (5.6) is a quasi-isomorphism. \Box

Corollary 5.16. For every integer $n \ge 1$ and any twisted periodic complex V. of filtered Abelian groups, the map η in (5.5) induces a quasi-isomorphism

$$\operatorname{Exp}(\operatorname{Stab}_n(V_{\bullet})) \xrightarrow{\sim} \lim_{l \to \infty} \pi_{n*} i_n^* (\operatorname{Exp}(V_{\bullet}) \otimes F^l P_{\bullet}).$$

Proof. By Lemma 5.12 and Definition 5.13 we have

$$\operatorname{Exp}(\operatorname{Stab}_n(V_{\bullet})) \cong \lim_{\stackrel{l}{\longrightarrow}} \pi_{n*} i_n^* \operatorname{Exp}(V_{\bullet}(l))$$

and, by the definition of cyclic expansion,

$$\operatorname{Exp}(V_{\bullet}(l)) = F^{0}(V_{\bullet} \otimes F^{-l}I_{\bullet})$$

for all integers $l \ge 0$. \Box

5.5. Comparison. We consider the category $\mathcal{D}(\Lambda R_h, i^*P_{\bullet}, k)$ in § 3.3 with the specific choice of the complex P_{\bullet} made in § 5.4. Let $\tau \colon \Lambda R_h \cong \Lambda \times I \to \Lambda$ be the tautological projection and let V_{\bullet} be any twisted periodic complex of Abelian groups. Then the set of maps $\varphi_p \colon V_{\bullet} \to \operatorname{Stab}_p(V_{\bullet})$ for all primes p induces a map

$$\eta \circ \tau^* \operatorname{Exp}(\varphi) \colon \tau^* \operatorname{Exp}(V_{\bullet}) \to \pi_* i^* \tau^* (\operatorname{Exp}(V_{\bullet}) \otimes P_{\bullet}),$$

which gives by adjunction a map

$$\widetilde{\varphi} \colon \pi^* \tau^* \operatorname{Exp}(V_{\bullet}) \to i^* (\tau^* \operatorname{Exp}(V_{\bullet}) \otimes i^* P_{\bullet}).$$

Sending $\langle V_{\bullet}, \{\varphi_p\} \rangle$ to $\langle \tau^* \operatorname{Exp}(V_{\bullet}), \widetilde{\varphi} \rangle$, we get a comparison functor

Exp:
$$\mathcal{FDM}^{\text{per}} \to \mathcal{D}_c(\Lambda R_h, P_{\bullet}, \mathbb{Z}).$$
 (5.8)

By Proposition 3.12, the following result immediately yields Theorem 5.4.

Proposition 5.17. The functor Exp in (5.8) is an equivalence of categories.

To prove this result, we generalize it. Consider the twisted periodic derived category $\mathcal{DF}^{\text{per}}(I^{\text{opp}},\mathbb{Z})$ of functors from I^{opp} to Abelian groups. The cyclic expansion functor gives a functor

Exp:
$$\mathcal{DF}^{\mathrm{per}}(I^{\mathrm{opp}},\mathbb{Z}) \to \mathcal{D}_w(\Lambda R_h^{\mathrm{opp}},\mathbb{Z}),$$
 (5.9)

where $\mathcal{D}_w(\Lambda R_h^{\text{opp}},\mathbb{Z}) \subset \mathcal{D}(\Lambda R_h^{\text{opp}},\mathbb{Z})$ is as in Proposition 3.13. It follows immediately from Lemma 5.10 that the functor Exp is an equivalence of categories. Moreover, for every $n \ge 1$ let $\nu_n \colon I \to I$ be the functor given by multiplication by n, as in (2.20). For every twisted periodic filtered complex $M \in \mathcal{DF}^{\text{per}}(I^{\text{opp}},\mathbb{Z})$ we put

$$\widetilde{\operatorname{Stab}}_n(M) = \nu_n^* \operatorname{Stab}_n(M).$$

Let $\widetilde{\mathcal{FDM}}^{\text{per}}$ be the derived category of such complexes M equipped with a family of maps

$$\varphi_p \colon M \to \widetilde{\operatorname{Stab}}_p(M)$$

for all primes p. We denote the forgetful functor by $\tilde{h}^*: \widetilde{\mathcal{FDM}}^{\text{per}} \to \mathcal{DF}^{\text{per}}(I^{\text{opp}}, \mathbb{Z})$. It has an obvious right adjoint \tilde{h}_* given by

$$\widetilde{h}_*(M)(n) = M \oplus \prod_{p|n} \operatorname{Stab}_p(M),$$
(5.10)

where the product is taken over all prime divisors of the integer $n \in I$. Moreover, the construction of the functor Exp in (5.8) also gives a functor

$$\widetilde{\operatorname{Exp}}\colon \widetilde{\mathcal{FDM}}^{\operatorname{per}} \to \mathcal{D}_w(\Lambda R_h, i^* P_{\bullet}, \mathbb{Z})$$
(5.11)

and, by construction, we have

$$\widetilde{h}^* \circ \widetilde{\operatorname{Exp}} \cong \operatorname{Exp} \circ \widetilde{h}^*, \tag{5.12}$$

where \tilde{h}^* on the left-hand side is the restriction functor of Proposition 3.13.

Proposition 5.18. The functor (5.11) is an equivalence of categories.

Proof. As in the proof of Proposition 2.12, it follows from Proposition 3.13(ii) that the functor $\widetilde{\text{Exp}}$ has a right adjoint

$$K: \mathcal{D}_w(\Lambda R_h, i^* P_{\bullet}, \mathbb{Z}) \to \widetilde{\mathcal{FDM}}^{\mathrm{per}}.$$

Moreover, by (5.10), Corollary 5.16 and Lemma 3.14, the base-change map

$$\widetilde{\operatorname{Exp}} \circ \widetilde{h}_* \to \widetilde{h}_* \circ \operatorname{Exp}$$

induced by (5.12) is an isomorphism. Therefore, by adjunction,

$$\widetilde{h}^* \circ K \cong \operatorname{Exp}^{-1} \circ \widehat{t}^*,$$

where Exp^{-1} is the equivalence inverse to (5.9). We conclude that

$$\widetilde{h}^* \circ K \circ \widetilde{\operatorname{Exp}} \cong \operatorname{Exp}^{-1} \circ \operatorname{Exp} \circ \widetilde{h}^*, \qquad \widetilde{h}^* \circ \widetilde{\operatorname{Exp}} \circ K \cong \operatorname{Exp}^{-1} \circ \operatorname{Exp} \circ \widetilde{h}^*$$

As in the proof of Proposition 2.12, the functor \tilde{h}^* is conservative, whence Exp and Exp^{-1} are mutually inverse equivalences of categories. Therefore so are the functors Exp , K. \Box

Proof of Proposition 5.17. The tautological projection $\tau: I \to \text{pt}$ induces a functor

$$\tau^*: \mathcal{DF}^{\mathrm{per}}(\mathbb{Z}) \to \mathcal{DF}^{\mathrm{per}}(I^{\mathrm{opp}}, \mathbb{Z}).$$

This is a full embedding onto the full subcategory spanned by the locally constant functors. The adjoint functor is given by

$$\tau_* = R^{\bullet} \lim_{\stackrel{I^{\mathrm{opp}}}{\leftarrow}}.$$

By Lemma 5.14, the operation of stabilized subdivision commutes with arbitrary products, so that $\tau_* \circ \widetilde{\text{Stab}}_p \cong \text{Stab}_p \circ \tau_*$. Hence τ^* and τ_* extend to an adjoint pair of functors between $\mathcal{FDM}^{\text{per}}$ and $\widetilde{\mathcal{FDM}}^{\text{per}}$ and, therefore, $\mathcal{FDM}^{\text{per}}$ is identified with the full subcategory in $\widetilde{\mathcal{FDM}}^{\text{per}}$ spanned by all objects M with locally constant \tilde{h}^*M . By (5.12), the equivalence in Proposition 5.18 identifies this subcategory with $\mathcal{DAR}(\mathbb{Z}) \cong \mathcal{D}_c(\Lambda R_h, i^*P_{\bullet}, \mathbb{Z}) \subset \mathcal{D}_w(\Lambda R_h, i^*P_{\bullet}, \mathbb{Z})$. \Box

§6. Topological cyclic homology

We conclude the paper with a brief discussion of topological cyclic homology (again following [6]).

We fix the *G*-universe *U* as in §4.4 and recall that for every $m \ge 1$ there are canonical endofunctors $\widehat{\Psi}^m$, $\widehat{\Phi}^m$ of the category G-Sp = G-Sp(*U*). We also have natural maps can: $\widehat{\Psi}^m \to \widehat{\Phi}^m$. For every $T \in G$ -Sp and all integers r, s > 1 there is a canonical non-equivariant map

$$F_{r,s} \colon T^{C_{rs}} \to T^{C_r}.$$

On the other hand, assume that T is equipped with a cyclotomic structure. Then there is a natural map

$$R_{r,s} \colon T^{C_{rs}} \cong (\widehat{\Psi}^s(T))^{C_r} \xrightarrow{\operatorname{can}} (\widehat{\Phi}^s(T))^{C_r} \xrightarrow{r_s} T^{C_r},$$

where r_s comes from the cyclotomic structure on T. Taken together, the maps $F_{r,s}$ and $R_{r,s}$ determine a functor I(T) from the category \mathbb{I}^{opp} (see § 1.3) to the category of non-equivariant spectra: we put $I(T)(n) = T^{C_n}$ for all $n \in \mathbb{I}$, and the morphisms $F_r, R_r: s \to rs$ act by the maps $F_{r,s}, R_{r,s}$ respectively.

Definition 6.1. The topological cyclic homology TC(T) of a cyclotomic spectrum T is the non-equivariant spectrum

$$\operatorname{TC}(T) = \operatorname{holim}_{\mathbb{I}^{\operatorname{opp}}} I(T).$$

Let $M \in \mathcal{D}\Lambda R(\mathbb{Z})$ be a cyclotomic complex. We consider the functor $\widetilde{\alpha} \colon \mathbb{I} \to \widetilde{\Lambda R}$ of (1.16).

Definition 6.2. The topological cyclic homology $\mathrm{TC}_{\bullet}(M)$ of a cyclotomic complex $M \in \mathcal{D}\Lambda R(\mathbb{Z})$ is equal to $H^{\bullet}(\mathbb{I}^{\mathrm{opp}}, \tilde{\alpha}^* \xi_* M)$, where

$$\xi_* \colon \mathcal{D}\Lambda R(\mathbb{Z}) \cong \mathcal{D}_c(\widetilde{\Lambda R}, \widetilde{\lambda}^* \mathcal{T}_{\bullet}, \mathbb{Z}) \to \mathcal{D}_c(\widetilde{\Lambda R}^{\mathrm{opp}}, \mathbb{Z})$$

is the right adjoint of the corestriction functor.

In particular, take any G-spectrum T and consider its equivariant chain complex $C_{\bullet}(T) \in \mathcal{DMA}(\mathbb{Z})$ as defined in § 4.3.

Proposition 6.3. For every cyclotomic spectrum T there is a natural isomorphism

$$\operatorname{TC}_{\bullet}(C_{\bullet}(T)) \cong H_{\bullet}(\operatorname{TC}(T), \mathbb{Z}),$$

where $H_{\bullet}(-,\mathbb{Z})$ stands for the homology of a non-equivariant spectrum with coefficients in \mathbb{Z} .

Proof. By Lemma 4.12 we have

$$\widetilde{\alpha}^* \xi_* C_{\bullet}(T)(m) \cong C_{\bullet}(T^{C_m})$$

for all $m \in \mathbb{I}$. Hence $\tilde{\alpha}^* \xi_* C_{\bullet}(T) \in \mathcal{D}(\mathbb{I}^{\text{opp}}, \mathbb{Z})$ is isomorphic to $C_{\bullet}(I(T), \mathbb{Z})$, the non-equivariant chain homology complex of the system of spectra I(T) used in Definition 6.1. \Box

In view of the equivalence proved in Theorem 5.4, it would be desirable to express the topological cyclic homology functor TC. of Definition 6.2 in terms of filtered Dieudonné modules. A natural notion of homology for filtered Dieudonné modules is given by the following definition.

Definition 6.4. The syntomic cohomology of a generalized filtered Dieudonné module $M \in \mathcal{FDM}^{\text{per}}(\mathbb{Z})$ is equal to RHom[•](\mathbb{Z}, M), where $\mathbb{Z} \in \mathcal{FDM}^{\text{per}}$ is the trivial filtered Dieudonné module.

In general, syntomic cohomology differs from topological cyclic homology. However, we have the following comparison result.

Theorem 6.5. Assume that $M \in \mathcal{D}\Lambda R(\mathbb{Z}) \cong \mathcal{FDM}^{\mathrm{per}}(\mathbb{Z})$ is profinitely complete. Then we have a natural isomorphism

$$\mathrm{TC}_{\bullet}(M) \cong \mathrm{RHom}^{\bullet}(\mathbb{Z}, M).$$

Proof. In terms of cyclotomic complexes, the trivial Dieudonné module \mathbb{Z} corresponds to the corestriction $\tilde{\xi}^*\mathbb{Z}$ of the constant functor $\mathbb{Z} \in \operatorname{Fun}(\widetilde{\Lambda R}^{\operatorname{opp}},\mathbb{Z})$. By adjunction, we have

$$\operatorname{RHom}^{\bullet}(\mathbb{Z}, M) \cong H^{\bullet}(\Lambda R^{\operatorname{opp}}, \xi_* M).$$

Thus it suffices to prove that for every profinitely complete $M \in \mathcal{D}(\Lambda R^{\text{opp}}, \mathbb{Z})$, the natural map

$$H^{\bullet}(\mathbb{I}^{\mathrm{opp}}, \widetilde{\alpha}^* M) \to H^{\bullet}(\Lambda R^{\mathrm{opp}}, M)$$

is an isomorphism. Equivalently, we must prove that the map

$$H^{\bullet}([1/\mathbb{N}^*], \widetilde{\lambda}_* \widetilde{\alpha}^* M) \to H^{\bullet}(\Lambda R^{\mathrm{opp}}, \widetilde{\lambda}_* M)$$

is an isomorphism. By base change, $\lambda_* \tilde{\alpha}^* \cong \alpha^* \lambda_*$. Since right-derived Kan extensions commute with profinite completions, Proposition 1.13 enables us to replace ΛR by ΔR . But we know that $\tilde{j}^* \lambda_* M$ becomes locally constant after restricting to $\Delta \subset \Delta R$, and we claim that the map

$$H^{\bullet}([1/\mathbb{N}^*], \alpha^* M') \to H^{\bullet}(\Delta R, M')$$

is an isomorphism for every $M' \in \mathcal{D}(\Delta R, \mathbb{Z})$ with locally constant $h^*M' \in \mathcal{D}(\Delta, k)$. Indeed, by Lemma 1.6, we can compute the direct image $\delta_!$ fibrewise and, since h^*M' is locally constant, the adjunction map

$$\delta^* \delta_! M' \to M'$$

is an isomorphism. Then

$$H^{\bullet}(\Delta R, M') \cong H^{\bullet}(\Delta R, \delta^* \delta_! M') \cong H^{\bullet}([1/\mathbb{N}^*], \delta_! M').$$

Since $\delta \circ \alpha = id$, the right-hand side is precisely equal to $H^{\bullet}([1/\mathbb{N}^*], \alpha^*M')$. \Box

In conclusion, let me say that to obtain an analogous comparison isomorphism for an arbitrary cyclotomic spectrum T, one must modify the definition of topological cyclic homology TC(T) by replacing the fixed-point spectra T^{C_m} in the system I(T)by their homotopy fixed points $(T^{C_m})^{hG}$ with respect to the residual action of the group G = U(1) (the result should be related to the usual TC by an exact triangle analogous to (1.27)). I do not know whether this makes sense from a topological point of view. I am very grateful to L. Hesselholt for explaining to me why this does not matter in the profinitely complete case.

§7. Appendix

Here we collect some facts used earlier in the paper. Most of them are well known, but our terminology may be non-standard.

First, a piece of abstract nonsense. Consider a square of categories and functors

$$\begin{array}{c|c} A \xrightarrow{b} B \\ a \\ \downarrow \\ C \xrightarrow{d} D \end{array}$$

and assume that a and c admit left-adjoint functors a_1, c_1 . Then an isomorphism $d \circ a \cong c \circ b$ induces by adjunction a map

$$c_! \circ b \to a_! \circ b.$$

This map and various adjoints of it are referred to as *base-change maps* induced by the isomorphism $d \circ a \cong c \circ b$.

Given any category \mathcal{C} and objects $c, c' \in \mathcal{C}$, we write $\mathcal{C}(c, c')$ for the set of morphisms from c to c' and \mathcal{C}^{opp} for the opposite category: $\mathcal{C}^{\text{opp}}(c, c') = \mathcal{C}(c', c)$. For every small category \mathcal{C} and any category A we denote the category of functors from \mathcal{C} to A by Fun(\mathcal{C}, A). For every functor $f: \mathcal{C} \to \mathcal{C}'$, the pullback with respect to f is denoted by f^* : Fun(\mathcal{C}', A) \to Fun(\mathcal{C}, A) and the left and right Kan extensions are denoted by $f_!$ and f_* respectively (when they exist). If A is Abelian, then we denote the derived category of the category Fun(\mathcal{C}, A) by $\mathcal{D}(\mathcal{C}, A)$ and, abusing notation, write $f_!$ and f_* for the derived functors of the Kan extensions. When A = k-mod is the category of modules over a ring k, we abbreviate the symbols Fun(\mathcal{C}, k -mod), $\mathcal{D}(\mathcal{C}, k$ -mod) to Fun(\mathcal{C}, k), $\mathcal{D}(\mathcal{C}, k)$. The homology $H_{\bullet}(\mathcal{C}, k)$ and cohomology $H^{\bullet}(\mathcal{C}, k)$ of a small category \mathcal{C} are defined by taking the derived Kan extensions with respect to the projection $\mathcal{C} \to \text{pt}$. As usual, $H^{\bullet}(\mathcal{C}, k)$ is an algebra, and $H_{\bullet}(\mathcal{C}, k)$ is a module over $H^{\bullet}(\mathcal{C}, k)$. The homology can be computed using an explicit bar complex $C_{\bullet}(\mathcal{C}, k)$.

We also introduce the following slightly non-standard definition.

Definition A.1. Let \mathcal{C} be a small category. An object $E_{\bullet} \in \mathcal{D}(\mathcal{C}, k)$ is said to be *locally constant* if, for every morphism $f: a \to b$ in \mathcal{C} , the map $E_{\bullet}(f): E_{\bullet}(a) \to E_{\bullet}(a)$ is a quasi-isomorphism.

We make free use of the notions in [14]: Cartesian map, fibration, cofibration, bifibration and so on. See [2], § 1, for a brief overview with exactly the same notation as in this paper. We freely use the base-change isomorphism and projection formula ([2], Lemma 1.7). We also use the following notion from [15].

Definition A.2. A factorization system on a category C is given by two subcategories $C_v, C_h \subset C$ such that all isomorphisms in C lie both in C_v and C_h , any morphism f in C can be decomposed as $f = v \circ h$, $v \in C_v$, $h \in C_h$, and such a decomposition is unique up to a unique isomorphism.

Example A.3. If $\gamma: \mathcal{C} \to \mathcal{C}'$ is a fibration, then the fibrewise morphisms and Cartesian morphisms form a factorization system on \mathcal{C} .

Factorization systems are very useful gadgets, although they are traditionally relegated to appendices and introductions (and we follow that tradition). Definition A.2 has several corollaries and/or equivalent reformulations. For example, one can show that $C_v \cap C_h$ consists precisely of the isomorphisms in C. Moreover, morphisms in C_v have a unique lifting property with respect to morphisms in C_h , and vice versa (see [15]). We shall need a result which is not in [15]. Let C be a small category and let \overline{C} be the category of all objects in C and all isomorphisms between them. Hence we have a Cartesian square



where $v, h, \overline{v}, \overline{h}$ are embedding functors. Then the isomorphism $\overline{v}^* \circ h^* \cong \overline{h}^* \circ v^*$ induces a base-change map

$$\overline{h}_! \circ \overline{v}^* \to v^* \circ h_! \tag{A.1}$$

of functors from $\mathcal{D}(\mathcal{C}_h, k)$ to $\mathcal{D}(\mathcal{C}_v, k)$.

Lemma A.4. The base-change map (A.1) is an isomorphism.

Proof. Since the category $\mathcal{D}(\mathcal{C}_h, k)$ is generated by representable functors of the form

$$k_c(c') = k[\mathcal{C}_h(c,c')], \qquad c, c' \in \mathcal{C}_h,$$

it suffices to show that (A.1) becomes an isomorphism after application to any of the functors $k_c \in \operatorname{Fun}(\mathcal{C}_h, k)$, $c \in \mathcal{C}_h$. To do this, note that $h_!$ sends representable functors to representable functors, whence we have

$$v^* h_! k_c(c') = h_! k_c(c') = k[\mathcal{C}(c,c')]$$
 (A.2)

for every $c' \in \mathcal{C}$. But the definition of a factorization system yields a natural isomorphism

$$\mathcal{C}(c,c') = \prod_{c'' \in \mathcal{C}} (\mathcal{C}_h(c,c'') \times \mathcal{C}_v(c'',c')) / \operatorname{Aut}(c''),$$
(A.3)

and the actions of the groups $\operatorname{Aut}(c'')$ are free. Therefore (A.2) is isomorphic to

$$\overline{h}_! \bigg(\bigoplus_{c'' \in \overline{\mathcal{C}}} k[\mathcal{C}_h(c,c'')] \bigg),$$

and the expression in the brackets is equal to $\overline{v}^* k_c$. \Box

We also use the technique of A_{∞} -algebras. A brief introduction to the relevant part of it is contained in [1], § 1.5. Here we merely recall that an A_{∞} -algebra is an algebra over a certain asymmetric operad Ass_{∞} of complexes of Abelian groups. Namely, Ass_{∞} is a cofibrant resolution of the associative asymmetric operad Ass. An A_{∞} -coalgebra is an A_{∞} -algebra in the opposite category. Since operads are asymmetric, one can define algebras in an arbitrary tensor category. In particular, for any set S one can consider the category of Abelian groups graded by $S \times S$, with tensor product

$$(V \otimes V')_{s,s'} = \bigoplus_{s'' \in S} V_{s,s''} \otimes V'_{s'',s}.$$

An A_{∞} -algebra in this category is a small A_{∞} -category whose set of objects is S. More generally, given a small category C with objects C_0 and morphisms C_1 , one can consider the category of C_1 -graded Abelian groups with tensor product

$$(V \otimes V')_f = \bigoplus_{f=f' \circ f''} V_{f'} \otimes V'_{f''}.$$

A *C*-graded A_{∞} -coalgebra is an A_{∞} -coalgebra in this category.

The derived category $\mathcal{D}(\mathcal{B}_{\bullet}, \operatorname{Ab})$ of A_{∞} -functors from a small A_{∞} -category \mathcal{B}_{\bullet} to the category of complexes of objects in an Abelian category Ab is constructed by considering the DG-category of all such functors and A_{∞} -maps between them and by inverting quasi-isomorphisms. The inversion procedure presents no difficulties since the DG-category is well behaved (at least if Ab is large enough, for example, when Ab = k-mod for a ring k). Every object has an h-projective replacement and an h-injective replacement. For every A_{∞} -morphism $f: \mathcal{B}_{\bullet} \to \mathcal{B}'_{\bullet}$ we have the pullback functor $f^*: \mathcal{D}(\mathcal{B}'_{\bullet}, \operatorname{Ab}) \to \mathcal{D}(\mathcal{B}_{\bullet}, \operatorname{Ab})$, which admits the left- and right-adjoint functors

$$f_*, f_! \colon \mathcal{D}(\mathcal{B}_{\bullet}, \mathrm{Ab}) \to \mathcal{D}(\mathcal{B}'_{\bullet}, \mathrm{Ab}).$$

Given a \mathcal{C} -graded A_{∞} -coalgebra \mathcal{R}_{\bullet} , we consider the DG-category of \mathcal{C} -graded k-valued A_{∞} -comodules over \mathcal{R}_{\bullet} . One can still invert quasi-isomorphisms to obtain the triangulated derived category $\mathcal{D}(\mathcal{C}, \mathcal{R}_{\bullet}, k)$. For any A_{∞} -map $f : \mathcal{R}_{\bullet} \to \mathcal{R}'_{\bullet}$ we have the corestriction functor

$$f^*: \mathcal{D}(\mathcal{C}, \mathcal{R}_{\bullet}, k) \to \mathcal{D}(\mathcal{C}, \mathcal{R}'_{\bullet}, k).$$

For any functor $f: \mathcal{C}' \to \mathcal{C}$ we have the pullback \mathcal{C}' -graded A_{∞} -coalgebra $f^*\mathcal{R}_{\bullet}$ and the pullback functor

$$f^*: \mathcal{D}(\mathcal{C}, \mathcal{R}_{\bullet}, k) \to \mathcal{D}(\mathcal{C}', f^*\mathcal{R}_{\bullet}, k).$$

However, *h*-projective replacements do not usually exist at all, and there is no general procedure for constructing *h*-injective replacements. Therefore the existence of adjoints is non-trivial. One case when an adjoint does exist is that of the embedding functor $i: \text{pt} \to \mathcal{C}$ of the point category onto an object $c \in \mathcal{C}$. In this case, an adjoint to the pullback functor $i^*: \mathcal{D}(\mathcal{C}, \mathcal{T}, k) \to \mathcal{D}(k)$ is given by the cofree A_{∞} -comodule functor that sends $M \in \mathcal{D}(k)$ to the A_{∞} -comodule M_c with

$$M_c(c') \cong M \otimes \bigoplus_{f: c' \to c} \mathcal{T}_{\bullet}(f).$$

Another situation where things are easy is that of the trivial C-graded A_{∞} -coalgebra \mathcal{R}_{\bullet} given by $\mathcal{R}_0(f) = \mathbb{Z}$ for every $f \in \mathcal{C}_1$ and $R_i = 0$ for $i \neq 0$.

Lemma A.5. Let C be a small category and let \mathcal{R} be the trivial C-graded \mathcal{A}_{∞} coalgebra. Then every h-projective complex of functors from C^{opp} to k-mod is
h-projective as an \mathcal{A}_{∞} -comodule over \mathcal{R} , and the natural embedding

$$\mathcal{D}(\mathcal{C}^{\mathrm{opp}},k) \to \mathcal{D}(\mathcal{C},\mathcal{R},k)$$

is an equivalence of categories.

Proof. Let $\mathcal{B}_{\bullet}(c,c') = \mathbb{Z}[\mathcal{C}(c,c')]$ be the free additive category generated by \mathcal{C} . We regard it as an A_{∞} -category. Then, by definition, an A_{∞} -comodule over the trivial A_{∞} -coalgebra \mathcal{R} is the same thing as an A_{∞} -functor from $\mathcal{B}_{\bullet}^{\text{opp}}$ to complexes of k-modules. Hence it suffices to prove the assertion for A_{∞} -categories, where it is well known. \Box

Lemma A.6. Let C be a small category, \mathcal{T}_{\bullet} a C-graded A_{∞} -coalgebra, and ρ : $\mathcal{C}' \to \mathcal{C}$ a functor from a small category \mathcal{C}' such that $\rho^*\mathcal{T}_{\bullet}$ is isomorphic to the trivial \mathcal{C}' -graded A_{∞} -coalgebra. Then the pullback functor $\rho^*: \mathcal{D}(\mathcal{C}, \mathcal{T}_{\bullet}, k) \to \mathcal{D}(\mathcal{C}', k)$ admits a right-adjoint functor

$$\rho_*: \mathcal{D}(\mathcal{C}', k) \to \mathcal{D}(\mathcal{C}, \mathcal{T}_{\bullet}, k).$$

Proof. By definition, we must prove that for every object $M \in \mathcal{D}(\mathcal{C}', k)$, the functor

$$N \mapsto \operatorname{Hom}(\rho^* N, M)$$
 (A.4)

from $\mathcal{D}(\mathcal{C}, \mathcal{T}_{\bullet}, k)$ to the category of k-modules is representable. By the cobar construction, every object $M \in \mathcal{D}(\mathcal{C}', k)$ is the cone of an endomorphism of an object $M' \in \mathcal{D}(\mathcal{C}', k)$ of the form

$$M' = \prod M_i$$

where each $M_i = M'_a$ is the corepresentable functor from \mathcal{C}' to k-mod corresponding to an object $a \in \mathcal{C}'$ and a module $M' \in k$ -mod. Hence it suffices to prove that the functor (A.4) is representable when $M = M'_a$. The representing object is given by the cofree A_∞ -comodule $M'_{o(a)} \in \mathcal{D}(\mathcal{C}, \mathcal{T}_{\bullet}, k)$. \Box

Lemma A.7. Let C be a small category equipped with a factorization system as in Lemma A.4, and let \mathcal{T}_{\bullet} be a C-graded A_{∞} -coalgebra such that $h^*\mathcal{T}_{\bullet}$ is a trivial \mathcal{C}_h -graded A_{∞} -coalgebra as in Lemma A.6. Then the base-change map

$$v^* \circ h_* \to \overline{h}_* \circ \overline{v}^*$$

of functors from $\mathcal{D}(\mathcal{C}_h^{\text{opp}}, k)$ to $\mathcal{D}(\mathcal{C}_v, v^*\mathcal{T}_{\bullet}, k)$ induced by the obvious isomorphism $\overline{v}^* \circ h^* \cong \overline{h}^* \circ v^*$ is itself an isomorphism.

Proof. This is proved as in Lemma A.4, with the adjoints constructed using Lemma A.6. \Box

All the categories of A_{∞} -comodules considered in this paper ought to be symmetric tensor categories. However, to construct the tensor product, one would need to equip the A_{∞} -coalgebras with some sort of Hopf algebra structure, and this is too heavy technically. Therefore we generally avoid tensor products. We nevertheless need them in one easy case. A C-graded \mathcal{A}_{∞} -coalgebra \mathcal{R}_{\bullet} is said to be *augmented* if $\mathcal{R}_i = 0$ for i < 0 and $\mathcal{R}_0(f) = \mathbb{Z}$ for every morphism f in \mathcal{C} . Then we have an obvious augmentation map $\xi \colon \mathbb{Z}_{\mathcal{C}} \to \mathcal{R}_{\bullet}$ and the corresponding corestriction functor

$$\xi^*: \mathcal{D}(\mathcal{C}^{\mathrm{opp}}, k) \to \mathcal{D}(\mathcal{C}, \mathcal{R}_{\bullet}, k).$$

Even more can be said: for every complex M_{\bullet} in $\operatorname{Fun}(\mathcal{C}^{\operatorname{opp}}, k)$ and every A_{∞} comodule E_{\bullet} over \mathcal{R}_{\bullet} we can define the tensor product $E_{\bullet} \otimes \xi^* M_{\bullet}$ by putting

$$(E_{\bullet} \otimes \xi^* M_{\bullet})(c) = E_{\bullet}(c) \otimes M_{\bullet}(c), \qquad c \in \mathcal{C},$$

with the A_{∞} -operations being the products of the A_{∞} -operations in E_{\bullet} and the structure maps of the functor M_{\bullet} . This construction is obviously associative with respect to M_{\bullet} .

Finally, in order to construct A_{∞} -categories and A_{∞} -coalgebras, we use various categorical constructions that are associative 'up to isomorphism'. Here is the prototypical example (see [1], §1.6, for more details). Let \mathcal{C} be a small monoidal category with an associativity isomorphism satisfying the usual pentagon equation. Then \mathcal{C} is actually an algebra over the following asymmetric operad.

Definition A.8. The monoidal category operad I_n is an operad of groupoids defined by the following conditions.

- (i) On objects, I_n is the free operad generated by a single binary operation.
- (ii) There is exactly one morphism between any two objects of I_n .

The bar complex $C_{\bullet}(\mathcal{C}, k)$ is automatically an algebra over the operad $C_{\bullet}(I_{\bullet}, \mathbb{Z})$. But $C_{\bullet}(I_{\bullet}, \mathbb{Z})$ is a resolution of the associative operad Ass. The A_{∞} -operad Ass_{∞} is another resolution of Ass, and it is cofibrant. Therefore the augmentation map Ass_{∞} \rightarrow Ass factors through the map Ass $\rightarrow C_{\bullet}(I_{\bullet}, \mathbb{Z})$. Fixing such a decomposition, we make the bar complex $C_{\bullet}(\mathcal{C}, k)$ into an A_{∞} -algebra over k.

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