Motivic structures in non-commutative geometry

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1 Generalities on mixed motives.

The conjectural category $\mathcal{MM}$ of *mixed motives*, as described by Deligne, Beilinson and others, unifies and connects various cohomology theories which appear in modern algebraic geometry. Recall that one expects $\mathcal{MM}$ to be a symmetric tensor abelian category with a distiguished invertible object $\mathbb{Z}(1)$ called the *Tate motive*. One expects that for any smooth projective algebraic variety X defined over $\mathbb{Q}$, there exist a functorial *motivic cohomology complex* $H^\bullet(X) \in \mathcal{D}^b(\mathcal{MM})$ with values in the derived category $\mathcal{D}^b(\mathcal{MM})$, whose cohomology groups

$$H^i(X) \in \mathcal{MM}$$

are called *motivic cohomology groups*. If X is the projective space $\mathbb{P}^n$, $n \geq 1$, then one expects to have

$$H^{2i}(\mathbb{P}^n) \cong \mathbb{Z}(i)$$

for $0 \leq i \leq n$, and 0 otherwise. For a general X and any integer j, one defines the *absolute cohomology complex* by

$$H^\bullet_{abs}(X, \mathbb{Z}(j)) = \mathrm{RHom}^\bullet_{\mathcal{MM}}(\mathbb{Z}(-j), H^\bullet(X)),$$

with its cohomology groups $H^i_{abs}(X, \mathbb{Z}(j))$ known as *absolute cohomology groups*. It is expected that the absolute cohomology groups are related to the algebraic K-theory groups $K_\bullet(X)$ by means of a functorial *regulator map*

$$(1.1) \qquad\qquad r : K_\bullet(X) \to \bigoplus_j H^{2j-\bullet}_{abs}(X, \mathbb{Z}(j)),$$

and it is expected that the regulator map is "not far from an isomorphism" (for example, it ought to be an isomorphism modulo torsion).

The above picture, with its many refinements which we will not need, is, unfortunately, still conjectural. In practice, one has to be content with *categories of realizations*. These follow the same general pattern, but the hypothetical category $\mathcal{MM}$ is replaced with a known category Real whose definition axiomatizes the features of a particular known cohomology theory. The prototype example is that of l-adic cohomology. Recall that for any algebraic variety $X/\mathbb{Q}$, its l-adic étale cohomology groups

$$H^i_{et}(X, \mathbb{Q}_l)$$

are $\mathbb{Q}_l$-vector spaces equipped with an additional structure of an *l-adic representation* of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. These representations form a tensor symmetric abelian category $\mathrm{Rep}_l(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ with a distiguished Tate module $\mathbb{Q}_l(1)$, and one can treat l-adic cohomology as taking values in this category. One can them define a double-graded absolute cohomology theory

$$H^\bullet_{abs}(X, \mathbb{Q}_l(j)) = H^\bullet(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), H^\bullet_{et}(X, \mathbb{Q}_l(j))),$$

known as *absolute l-adic cohomology*, and construct a regulator map of the form (1.1). Conjecturally, we have an exact tensor "realization functor" $\mathcal{MM} \to \mathrm{Rep}_l(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$, l-adic cohomology is obtained by applying realization to motivic cohomology, and the étale regulator map factors through the motivic one. In practice, one can treat $\mathrm{Rep}_l(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ as a replacement for $\mathcal{MM}$, and hope that the regulator map still captures essential information about $K_\bullet(X)$.

In this paper, we will be concerned with another family of cohomology theories and realizations which appear as refinements of *de Rham cohomology*. By its very nature, de Rham cohomology of a smooth algebraic variety X has coefficients in the field or ring of definition of X. Thus it is not necessary to require that X is defined over $\mathbb{Q}$, and it is convenient to classify de Rham-type cohomology theories by their rings of definitions. There are two main examples.

(i) The ring of definition is either $\mathbb{R}$ or $\mathbb{C}$; the corresponding category of realizations is Deligne's category of mixed $\mathbb{R}$-Hodge structures, and the absolute cohomology theory is Hodge-Deligne cohomology (with a refinement by Beilinson). The regulator map is the subject of famous Beilinson Conjectures.

(ii) The ring of definition is $\mathbb{Z}_p$; the corresponding category of realizations is the category of *filtered Dieudonné modules* of Fontaine-Lafaille [FL], and the absolute cohomology theory is *syntomic cohomology* of Fontaine and Messing [FM].

The goal of this paper is to report on recent discoveries and conjectures which state, roughly speaking, that all these additional "motivic" structures on de Rham cohomology of an algebraic variety should exist in a much more general setting of periodic cyclic homology of properly understood *non-commutative* algebraic varieties. As opposed to the usual commutative setting, the "classical" case (i) is more difficult and largely conjectural; in the p-adic case (ii), most of the statements have been proved. Moreover, the p-adic story shows an unexpected relation to algebraic topology which we will also explain. Before we start, however, we should define exactly what we mean by a "non-commutative algebraic variety", and recall basic facts on cyclic homology.

2 Non-commutative setting.

We start by a brief recollection on cyclic homology; a very good overview can be found in J.-L. Loday's book [Lo], and an old overview [FT] is also quite useful. *Hochschild homology* $HH_\bullet(A/k)$ of an associative unital algebra A flat over a commutative ring k is given by

$$HH_\bullet(A) = HH_\bullet(A/k) = \mathrm{Tor}^{A^{opp} \otimes_k A}_\bullet(A, A),$$

where A^{opp} is A with multiplication written in the opposite direction. It has been discovered by Hochschild, Kostant and Rosenberg [HKR] that if A is commutative and $X = \mathrm{Spec}\, A$ is a smooth algebraic variety over k, then

$$HH_i(A) \cong H^0(X, \Omega^i(X)),$$

the space of i-forms on X over k. *Cyclic homology* $HC_\bullet(A)$ is a refinement of Hochschild homology discovered independently by A. Connes and B. Tsygan. It is functorial in A, and related to $HH_\bullet(A)$ by the *Connes' long exact sequence*

$$HH_\bullet(A) \longrightarrow HC_\bullet(A) \overset{u}{\longrightarrow} HC_{\bullet-2}(A) \longrightarrow \ ,$$

where u is a canonical *periodicity map* of degree 2. Both $HH_\bullet(A)$ and $HC_\bullet(A)$ can be represented by functorial complexes $CH_\bullet(A)$, $CC_\bullet(A)$, and the Connes' exact sequence then becomes a short exact sequence of complexes. The complex $CC_\bullet(A)$ is the total complex of a bicomplex

(2.1)

$$
\begin{array}{ccccccc}
\dots \longrightarrow & A & \overset{\mathrm{id}}{\longrightarrow} & A & \overset{0}{\longrightarrow} & A \\
& \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b} \\
\dots \longrightarrow & A \otimes A & \overset{\mathrm{id}+\tau}{\longrightarrow} & A \otimes A & \overset{\mathrm{id}-\tau}{\longrightarrow} & A \otimes A \\
& \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b} \\
\dots & \dots & & \dots & & \dots \\
& \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b} \\
\dots \longrightarrow & A^{\otimes n} & \overset{\mathrm{id}+\tau+\cdots+\tau^{n-1}}{\longrightarrow} & A^{\otimes n} & \overset{\mathrm{id}-\tau}{\longrightarrow} & A^{\otimes n} \\
& \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b}
\end{array}
$$

Here it is understood that the whole thing extends indefinitely to the left, all the even-numbered columns are the same, all odd-numbered columns are the same, and the bicomplex is invariant with respect to the horizontal shift by 2 columns which gives the periodicity map u. The map $\tau : A^{\otimes i} \to \mathcal{A}^{\otimes i}$ is the cyclic permutation of order i multiplied by $(-1)^{i+1}$, and b, b' are certain explicit differentials expressed in terms of the multiplication map $m : A^{\otimes 2} \to A$. The complex $CH_\bullet(A)$ is the rightmost column of (2.1), and also any odd-numbered column when counting from the right; the even-numbered columns are acyclic.

Periodic cyclic homology $HP_\bullet(A)$ is obtained by inverting the map u, namely, $HP_\bullet(A)$ is the homology of the complex

$$CP_\bullet(A) = \varprojlim_u CC_\bullet(A)$$

(explictily, this is the total complex of a bicomplex obtained by extending (2.1) to the right as well as to the left). *Negative cyclic homology* $HC_\bullet^-(A)$ is the homology of the complex $CC_\bullet^-(A)$ obtained as the third term in a short exact sequence

$$0 \longrightarrow CC^-(A) \longrightarrow CP_\bullet(A) \longrightarrow CC_{\bullet-2}(A) \longrightarrow 0$$

(equivalently, one extends (2.1) to the right but not to the left).

The reason cyclic homology is interesting in algebraic geometry is the following comparison theorem. In the situation of the Hochschild-Kostant-Rosenberg Theorem, let d be the dimension of $X = \operatorname{Spec} A$, and assume in addition that $d!$ is invertible in the base ring k. Then there exists a canonical isomorphism

(2.2) $$HP_\bullet(A) \cong H_{DR}^\bullet(X)((u)),$$

where the right-hand side is a shorthand for "formal Laurent power series in one variable u of degree 2 with coefficients in de Rham cohomology $H_{DR}(X)$".

By (2.2), periodic cyclic homology classes can be thought of as non-commutative generalizations of de Rham cohomology classes. Some information is lost in this generalization: because of the presense of u in the right-hand side of (2.2), what we recover from $HP_\bullet(A)$ is not the de Rham cohomology of X but rather, the de Rham cohomology of the product $X \times \mathbb{P}^\infty$ of X and the infinite projective space $\mathbb{P}^\infty$, where we moreover invert the generator $u \in H_{DR}^2(\mathbb{P}^\infty)$. Thus given a category of realizations Real and a Real-valued refinement of de Rham cohomology, the appropriate target for its non-commutative generalization is not the derived category $\mathcal{D}(\mathsf{Real})$ but the *twisted* 2-*periodic derived category* $\mathcal{D}^{per}(\mathsf{Real})$ obtained by inverting quasiisomorphisms in the category of complexes $M_\bullet$ of objects in Real equipped with an isomorphism $u : M_\bullet \cong M_\bullet(-1)[2]$, where we denote $M(n) = M \otimes \mathbb{Z}(n)$, $n \in \mathbb{Z}$.

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We note, however, that this causes no problem with the regulator map, since the summation in the right-hand side of (1.1) is the same as in the right-hand side of (2.2). Thus for a Real-valued refinement $H_{\mathsf{Real}}^\bullet(-)$ of de Rham cohomology and any smooth affine algebraic variety $X = \operatorname{Spec} A$, the regulator map (1.1) takes the form

$$K_\bullet(A) \to \operatorname{RHom}_{\mathcal{D}^{per}(\mathsf{Real})}^\bullet(k, HP_\bullet(A)) = \operatorname{RHom}_{\mathcal{D}^{per}(\mathsf{Real})}^\bullet(k, H_{\mathsf{Real}}^\bullet(X)((u))),$$

where k in the right-hand side is the unit object of Real.

Somewhat surprisingly, non-affine algebraic varieties can be included in the above picture with very little additional effort. To do it, it is convenient to use the machinery of differential graded (DG) algebras and DG categories. An excellent overview can be found in [Ke2]; for the convenience of the reader, let us summarize the relevant points.

Roughly speaking, a k-linear DG category is a category $C^\bullet$ whose Hom-sets $C^\bullet(-,-)$ are equipped with a structure of complexes of k-modules in such a way that composition maps are k-linear and compatible with the differentials (for precise definitions, see [Ke2, Section 2]). For any small k-linear DG category $C^\bullet$, one defines a triangulated *derived category of DG modules* $\mathcal{D}(C^\bullet)$ ([Ke2, Section 3]). Any k-linear DG functor $\gamma : C_1^\bullet \to C_2^\bullet$ induces a trinagulated functor $\gamma^* : \mathcal{D}(C_2^\bullet) \to \mathcal{D}(C_1^\bullet)$. The functor γ ia a *derived Morita equivalence* if the induced functor γ^* is an equivalence of triangulated categories. It turns out – this mostly due to the work of G. Tabuada and B. Toën, see [Ke2, Section 4] and references therein – that there is a closed model structure on the category of small k-linear DG categories whose weak equivalences are exactly derived Morita equivalences. Denote by $\mathsf{Morita}(k)$ the corresponding homotopy category, that is, the category of "small k-linear DG categories up to a derived Morita equivalence".

Any k-algebra A is a k-linear DG category with one object pt and $\operatorname{Hom}(\mathsf{pt},\mathsf{pt}) = A$ placed in degree 0, so that we have an embedding $\mathsf{Alg}(k) \to \mathsf{Morita}(k)$ from the category $\mathsf{Alg}(k)$ of associative k-algebras to $\mathsf{Morita}(k)$. Then, as explained in [Ke2, Section 5], Hochschild homology, cyclic homology, periodic cyclic homology and negative cyclic homology extend to functors

$$\mathsf{Morita}(k) \to \mathcal{D}(k).$$

Moreover, so does the algebraic K-theory functor $K^\bullet(-)$, and other "additive invariants" in the sense of [Ke2, Section 5].

In general, a DG category with one object pt is the same thing as an associative unital DG algebra $A^\bullet = \operatorname{Hom}^\bullet(\mathsf{pt},\mathsf{pt})$. The category of DG algebras over k has a natural closed model structure whose weak equivalences are quasiisomorphisms, and whose fibrations are surjective maps. The corresponding homotopy category $\mathsf{DG\text{-}Alg}(k)$ is the category of DG algebras "up to a quasiisomorphism". One shows that a quasiisomorphism between DG algebras is in particular a derived Morita equivalence, so that we have a natural functor

$$(2.3) \qquad\qquad\qquad\qquad \mathsf{DG\text{-}Alg}(k) \to \mathsf{Morita}(k).$$

It is not difficult to show that for every cofibrant DG algebra $A^\bullet$, the individual terms of the complex $A^\bullet$ are flat k-modules. In this case, the Hochschild, cyclic etc. homology of $A^\bullet$ are especially simple – they are given by exactly the same bicomplex (2.1) and its versions as in the case of ordinary algebras. This is manifestly invariant under quasiisomorphisms, so that the Hochschild, cyclic etc. homology obviously descend to functors from $\mathsf{DG\text{-}Alg}(k)$ to the derived category $\mathcal{D}(k)$. The DG category approach shows that there is even more invariance: even if two DG algebras $A_1^\bullet$, $A_2^\bullet$ are not quasiisomorphic but only have isomorphic images in $\mathsf{Morita}(k)$, their Hochschild, cyclic etc. homology is naturally identified. This statement is already non-trivial in the case of usual algebras, see [Lo, Section 1.2].

Definition 2.1. A DG category $T_\bullet \in \mathsf{Morita}(k)$ is *derived-affine* if it lies in the essential image of the functor (2.3).

Remark 2.2. A small k-linear DG category $C^\bullet$ with a finite number of objects is automatically derived-Morita equivalent to a DG algebra $A^\bullet$, thus affine. For example, one can take

$$A^\bullet = \bigoplus_{c,c'} C^\bullet(c,c'),$$

where the sum is taken over all pairs of objects in $C^\bullet$.

Now, it has been proved ([BV] combined with [Ke1]) that for any quasiseparated quasicompact scheme X over k, there exists a DG algebra $A^\bullet/k$ such that the derived category $\mathcal{D}(X)$ of quasicoherent sheaves on X is equivalent to the derived category $\mathcal{D}(A^\bullet)$,

$$\mathcal{D}(X) \cong \mathcal{D}(A^\bullet),$$

and such a DG algebra $A^\bullet$ is unique up to a derived Morita equivalence, so that we have a canonical functor from the category of algebraic varieties over k to the category $\mathsf{Morita}(k)$. Roughly speaking, any algebraic variety is derived Morita-equivalent to a DG algebra, or, in a succint formulation of [BV], "every algebraic variety is derived-affine".

Moreover, it turns out that the properties of X which are relevant for the present paper are reflected in the properties of a Morita-equivalent DG algebra $A^\bullet$. For example, one introduces the following (see e.g. [KKP]).

Definition 2.3. (i) A DG algebra $A^\bullet/k$ is *proper* if $A^\bullet$ is perfect as an object in the derived category $\mathcal{D}(k)$ of complexes of k-modules.

(ii) A DG algebra $A^\bullet/k$ is *smooth* if $A^\bullet$ is perfect as an object in the derived category $\mathcal{D}(A^{\bullet opp} \otimes A^\bullet)$ of $A^\bullet$-bimodules.

Then $A^\bullet$ is proper, resp. smooth if and only if X is proper, resp. smooth (in the affine case $X = \operatorname{Spec} A$, the second claim is the famous Serre regularity criterion). Moreover, the correspondence $X \mapsto A^\bullet$ is compatible with algebraic K-theory, $K_\bullet(X) \cong K_\bullet(A^\bullet)$, and if the variety X/k is smooth of dimension d, and $d!$ is invertible in k, then the Hochschild homology of such a Morita-equivalent DG algebra $A^\bullet$ is canonically isomorphic to

$$HH_i(A^\bullet) \cong \bigoplus_j H^j(X, \Omega^{i+j}(X)),$$

the so-called "Hodge cohomology" of X, while the periodic cyclic homology $HP_\bullet(A)$ is exactly as in (2.2).

Thus as far as homological invariants are concerned, one can treat DG algebras "up to a derived Morita-equivalence" as non-commutative generalizations of algebraic varieties:

- A non-commutative algebraic variety over k is a DG algebra $A^\bullet$ over k considered as an object of the Tabuada-Toën category $\mathsf{Morita}(k)$.

This is the point of view we will adopt.

3 Hodge-to-de Rham spectral sequence.

A convenient way to pack all the structures related to Hochschild homology $HH_\bullet(A^\bullet)$ of a DG algebra $A^\bullet/k$ is by considering the equivariant derived category $\mathcal{D}_{S^1}(k)$ of S^1-equivariant constructible sheaves of k-modules on the point pt. Then the claim is that the Hochschild homology complex $CH_\bullet(A^\bullet)$, while *a priori* simply a complex of k-modules, in fact underlies a canonical object $\widetilde{CH}_\bullet(A^\bullet) \in \mathcal{D}_{S^1}(k)$ (loosely speaking, "$CH_\bullet(A^\bullet)$ carries a canonical S^1-action"). The negative cyclic homology appears as S^1-equivariant cohomology

$$H_{S^1}^\bullet(\mathsf{pt}, \widetilde{CH}_\bullet(A^\bullet)),$$

the periodicity map u is the generator of $H_{S^1}^\bullet(\mathsf{pt}) \cong H^\bullet(BS^1)$, and $HP_\bullet(A^\bullet)$ is the localization $HC_\bullet^-(A^\bullet)(u^{-1})$.

Another way to pack the same data is by considering the *filtered derived category* $\mathcal{DF}(k)$ of k-modules of [BBD] – that is, the triangulated category obtained by considering complexes $V_\bullet$ of k-modules equipped with a descreasing filtration $F^\bullet$ numbered by all integers, and inverting those maps which induce quasiisomorphisms on the associated graded quotients gr^F. This has a "twisted 2-periodic" version $\mathcal{DF}^{per}(k)$, obtained from filtered complexes $V_\bullet$ equipped with an isomorphism $V_\bullet \cong V_\bullet[2](-1)$, where (-1) means renumbering the filtration: $F^i V(-1) = F^{\bullet+1}V$.

Lemma 3.1. *We have*

$$\mathcal{D}_{S^1}(k) \cong \mathcal{DF}^{per}(k).$$

Sketch of the proof. Let us just indicate the equivalence: it sends $V_\bullet \in \mathcal{D}_{S^1}(k)$ to the equivariant cohomology complex $C_{S^1}^\bullet(\mathsf{pt}, V_\bullet)(u^{-1})$, with the (generalized) filtration given by

$$F^i H_{S^1}^\bullet(\mathsf{pt}, V_\bullet)(u^{-1}) = u^i C_{S^1}^\bullet(\mathsf{pt}, V_\bullet),$$

where $u \in C_{S^1}^2(k)$ represents the generator of the equivariant cohomology ring $H_{S^1}^\bullet(\mathsf{pt}, k) \cong k[u]$. □

In the case of the Hochschild homology complex $\widetilde{CH}_\bullet(A^\bullet)$, the corresponding periodic filtered complex is $CP_\bullet(A^\bullet)$, with the filtration given by

$$F^i CP_\bullet((A^\bullet) = u^i CC_\bullet^-(A^\bullet) \subset CP_\bullet(A^\bullet).$$

One can treat $\mathcal{DF}^{per}(k)$ as a very crude "category of realization" Real in the sense of Section 1, or rather, of its periodic derived category $\mathcal{D}^{per}(\mathsf{Real})$. The expected regulator map then takes the form

$$(3.1) \qquad K_\bullet(A^\bullet) \to HC_\bullet^-(A^\bullet) = \mathrm{RHom}_{\mathcal{DF}^{per}(k)}^\bullet(k, HP_\bullet(A)).$$

Such a map does indeed exist, see [Lo, Chapter 8]. In general, it is very far from being an isomorphism. The only general result is a theorem of T. Goodwillie [Good] which shows that at least the tangent spaces to both sides are the same. Namely, given an alegbra A with an ideal $I \subset A$, one defines the relative K-theory $K_\bullet(A, I)$ spectrum as the cone of the natural map $K_\bullet(A) \to K_\bullet(A/I)$, and analogously for the cyclic homology functors. Then it has been proved in [Good] that if k is a field of characteristic 0 and $I \subset A$ is a nilpotent ideal, then the map

$$K_\bullet(A, I) \to HC^-(A, I)$$

induced by the regulator map (3.1) is a quasiisomorphism. An analogous statement also holds for DG algebras over k.

While filtered complexes are a very crude approximation to mixed motives, already on this level the smoothness and properness of a DG algebra leads to non-trivial consequences. Namely, a filtered complex gives rise to a spectral sequence. In the case of cyclic homology, it takes the form

$$(3.2) \qquad HH_\bullet(A^\bullet)((u)) \Rightarrow HP_\bullet(A^\bullet),$$

where we use the same shorthand as in (2.2). When the DG algebra $A^\bullet/k$ is Morita-equivalent to a smooth algebraic variety X/k, the filtration $F^\bullet$ on $HP_\bullet(A^\bullet) \cong H_{DR}^\bullet(X)((u))$ is just the Hodge filtration on de Rham cohomology, and (3.2) is the usual Hodge-to-de Rham spectral sequence

$$H^p(X, \Omega^q(X)) \Rightarrow H_{DR}^{p+q}(X)$$

tensored with $k((u))$. Because of this, (3.2) in general is also called "Hodge-to-de Rham spectral sequence". Then the following is a partial proof of a general conjecture of M. Kontsevich and Ya. Soibelman [KS].

Theorem 3.2 ([Ka1]). *Assume that $A^\bullet$ is a smooth and proper DG algebra over a field k of characteristic $\mathrm{char}\, k = 0$. Assume further that $A^i = 0$ for $i < 0$. Then the Hodge-to-de Rham spectral sequence (3.2) degenerates.*

The assumption $A^i = 0$, $i < 0$ is technical (note, however, that it can always be achieved for a DG algebra $A^\bullet$ corresponding to a smooth and proper algebraic variety X/k, see e.g. [O, Theorem 4]).

In the usual commutative case, the Hodge-to-de Rham degeneration statement is well-known and has two proofs. Classically, it follows from the general complex-analytic package of Hodge theory and harmonic forms. An alternative proof by Deligne and Illusie [DI] uses reduction to positive characteristic and p-adic methods. So far, it is only the second technique that has been generalized to the non-commutative case. We will now explain this.

4   Review of Filtered Dieudonné modules.

A p-adic analog of the notion of a mixed Hodge structure has been introduced in 1982 by Fontaine and Lafaille [FL]. Here is the definition.

Definition 4.1. Let k be a finite field of characteristic p, with its Frobenius map, and let W be its ring of Witt vectors, with its canonical lifting φ of the Frobenius map. A *filtered Dieudonné module* over W is a finitely generated W-module M equipped with a decreasing filtration $F^\bullet M$, indexed by all integers and such that $\cap F^i M = 0$, $\cup F^i M = M$, and a collection of Frobenius-semilinear maps $\varphi_i : F^i M \to M$, one for each integer i, such that

(i) $\varphi_i|_{F^{i+1}M} = p\varphi^{i+1}$, and

(ii) the map

$$\sum \varphi_i : \bigoplus_i F^i M \to M$$

is surjective.

We will denote by $\mathcal{FDM}(W)$ the category of filtered Dieudonné modules over W. It is an abelian category. A symmetric tensor product in $\mathcal{FDM}(W)$ is defined in the obvious way, and we have the Tate object $W(1)$ given by: $W(1) = W$ as a W-module, $F^1 W(1) = W(1)$, $F^2 W(1) = 0$, $\varphi_1 : F^1 W(1) \to W(1)$ equal to φ. We also have the derived category $\mathcal{D}(\mathcal{FDM}(W))$.

If a filtered Dieudonné module $M \in \mathcal{FDM}(W)$ is annihilated by p, then (i) of Definition 4.1 insures that the map in (ii) factors through a surjective map

$$\widetilde{\varphi} : \mathsf{gr}_F^\bullet M \to M.$$

Since both sides are k-vector spaces of the same dimension, $\widetilde{\varphi}$ must be an isomorphism. For a general filtered W-module $\langle M, F^\bullet \rangle$, one lets $\widetilde{M}$ be the cokernel of the map

$$(4.1) \qquad \bigoplus_i F^i M \xrightarrow{t - p\,\mathsf{id}} \bigoplus_i F^i M,$$

where $t : F^{\bullet+1}M \to F^\bullet M$ is the tautological embedding. Then again, (i) insures that the map $\sum_i \varphi_i$ factors through a map

$$(4.2) \qquad \widetilde{\varphi} : \widetilde{M} \to M$$

and this map must be an isomorphism if (ii) were to be satisfied. This allows to generalize the definition of a filtered Dieudonné module: instead of a finitely generated filtered W-module, one can consider a filtered W-module $\langle M, F^\bullet \rangle$ such that M is p-adically complete and complete with respect to the topology induced by $F^\bullet$ (these conditions together with the non-degeneracy conditions $\cap F^i M = 0$, $\cup F^i M = M$ insure that the map (4.1) is injective). Then a *unbounded Dieudonné module* structure on M is given by a Frobenius-semilinear isomorphism $\widetilde{\varphi}$ of the form (4.2).

I do not know whether the category of unbounded filtered Dieudonné modules is still abelian. However, complexes of unbounded filtered Dieudonné modules can be defined in the obvious way, and the correspondence $M \mapsto \widetilde{M}$ sends filtered quasiisomorphisms into quasiisomorphisms, so that we obtain a triangulated derived category $\mathcal{DFDM}(W) \supset \mathcal{D}(\mathcal{FDM}(W))$ and its twisted 2-periodic version $\mathcal{DFDM}^{per}$.

Moreover, one can drop the requirement that the map $\widetilde{\varphi}$ is an isomorphism and allow it to be an arbitrary map. Let us call the resulting objects "weak filtered Dieudonné modules". The category of weak filtered Dieudonné modules is definitely not abelian, but the above procedure stil applies: we can invert filtered quasiisomorphisms and obtain triangulated categories $\widetilde{\mathcal{DFDM}}(W)$, $\widetilde{\mathcal{DFDM}}^{per}(W)$. We then have a fully faithful inclusions $\mathcal{DFDM}(W) \subset \widetilde{\mathcal{DFDM}}(W)$, $\mathcal{DFDM}^{per}(W) \subset \widetilde{\mathcal{DFDM}}^{per}(W)$, and one can show that their essential images consist of those $M_\bullet$ in $\widetilde{\mathcal{DFDM}}(W)$, resp. $\widetilde{\mathcal{DFDM}}^{per}(W)$ for which the map $\widetilde{\varphi}$ of (4.2) is a quasiisomorphism.

Assume given a algebraic variety X smooth over W, of dimension $d < p$. Then de Rham cohomology $H^\bullet_{DR}(X/W)$ equipped with the filtration induced by the stupid filtration on the de Rham complex has the structure of a complex of generalized filtered Dieudonné modules. If X/W is proper, the groups $H^i_{DR}(X/W)$ are finitely generated, so that they are filtered Dieudonné modules in the strict sense (and the filtration is then the Hodge filtration). This Dieudonné module structure can be seen explicitly under the following strong additional assumption:

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- the Frobenius endomorphism Fr of the special fiber $X_k = X \otimes_W k$ of X/W lifts to a Frobenius-semilinear endomorphism $\widetilde{\mathsf{Fr}} : X \to X$.

Then one checks easily that for any $i \geq 0$, the natural map $\widetilde{\mathsf{Fr}}^* : \Omega^i(X/W) \to \Omega^i(X/W)$ is divisible by p^i. The Dieudonné module structure maps φ_i are induced by the corresponding maps $\frac{1}{p^i}\widetilde{\mathsf{Fr}}^*$. We note that in this special case, the map φ_i sends F^i into F^i. In the general case, the construction is due to G. Faltings [F, Theorem 4.1]; roughly speaking, it uses a comparison theorem which gives a quasiisomorphism

$$H^{\bullet}_{cris}(X_k) \cong H^{\bullet}_{DR}(X),$$

where in the left-hand side, we have the cristalline cohomology of the special fiber X_k. The Frobenius endomorphism of X_k induces an endomorphism on cristalline cohomology, and this gives the structure map φ_0. By an additional argument, one shows that $\varphi_0|F^i$ is canonically divisible by p^i, and this gives the other structure maps φ_i (in general, they do not preserve the Hodge filtration $F^{\bullet}$).

In particular, for any smooth X/W, one has the isomorphism (4.2). Its reduction mod p is an isomorphism

$$(4.3) \qquad \mathsf{gr}^{\bullet}_F H^{\bullet}_{DR}(X_k) \cong \bigoplus_i H^{\bullet-i}(X_k, \Omega^i(X_k)) \cong H^{\bullet}_{DR}(X_k)$$

between Hodge and de Rham cohomology of the special fiber X_k. If X is affine, this is nothing but the inverse to the Cartier isomorphism, discovered by P. Cartier back in the 1950-ies; as such, it depends only on the special fiber X_k and not on the lifting X/W. In the general case, it has been shown by Deligne and Illusie in [DI] that (4.3) depends on the lifting $X \otimes_W W_2(k)$ of X_k to the second Witt vectors ring $W_2(k) = W(k)/p^2$ (but not on the lifting to higher orders, nor even on the existence of such a lifting).

The absolute cohomology theory associated to the $\mathcal{FDM}$-valued refinement of de Rham cohomology is the *syntomic cohomology* of Fontaine and Messing. As it happens, the functors $\mathrm{RHom}^{\bullet}(W(-j), -)$ in the category $\mathcal{DFDM}$ are easy to compute explicitly — for any complex $M_{\bullet} \in \mathcal{DFDM}$, $\mathrm{RHom}^{\bullet}(W, -)$ is the cone of the natural map

$$F^j M_{\bullet} \xrightarrow{\ \mathsf{id}-\varphi_j\ } M_{\bullet}.$$

When applied to a smooth proper variety X/W, this gives syntomic cohomology groups $H^{\bullet}_{synt}(X, \mathbb{Z}_p(j))$. The construction can even be localized with respect to the Zariski topology on X_k, so that the syntomic cohomology is expressed as hypercohomology of X_k with coefficients in certain canonical complexes of Zariski sheaves, as in [FM].

The existence and properties of the regulator map for the syntomic cohomology have been studied by M. Gros [Gr1, Gr2]. In principle, one can construct the regulator by the standard procedure for "twisted cohomology theories" in the sense of [BO], but there is one serious problem: the filtered Dieudonné module structure on $H^{\bullet}_{DR}(X)$ only exists if $p > \dim X$. Since the standard procedure works by considering infinite projective spaces and Grassmann varieties, this condition is inevitably broken no matter what p we start with. To circumvent this, Gros had to modify (in [Gr2]) the definition of syntomic cohomology by including additional structures such as the rigid analytic space associated to X/W. The resulting picture becomes extremely complex, and at present, it is not clear whether it can be generalized to non-commutative varieties.

5   FDM in the non-commutative case.

What we do have for non-commutative varieties is the following result.

Definition 5.1. The *Hochschild cohomology* $HH^{\bullet}(A^{\bullet}/R)$ of a DG algebra $A^{\bullet}$ over a ring R is given by

$$HH^{\bullet}(A^{\bullet}/R) = \mathrm{RHom}^{\bullet}_{A^{\bullet opp}\otimes_R A}(A^{\bullet}, A^{\bullet}).$$

Theorem 5.2 ([Ka1]). *Assume given an associative DG algebra $A^{\bullet}$ over a finite field k. Assume that $A^i = 0$ for $i < 0$. Assume also that $A^{\bullet}$ is smooth, that it can be lifted to a flat DG algebra $\widetilde{A}^{\bullet}$ over $W_2(k)$, and that $HH^i(A^{\bullet}) = 0$ for $i \geq 2p - 1$. Then there exists a canonical Cartier-type isomorphism*

$$HH_{\bullet}(A^{\bullet})((u)) \cong HP_{\bullet}(A^{\bullet}).$$

Remark 5.3. If a DG algebra $A^\bullet$ is derived Morita-equivalent to a smooth algebraic variety X/k, then we have $HH^i(A^\bullet) = 0$ automatically for $i > 2 \dim X$, so that the last condition on $A^\bullet$ in Theorem 5.2 reduces to the condition $p > \dim X$ already mentioned in Section 4.

Remark 5.4. Theorem 3.2 easily follows from Theorem 5.2 by the same dimension argument as in the original proof of Deligne and Illusie in [DI]. The only non-trivial additional input is a beautiful recent theorem of B. Toën [To2] which claims that a smooth and proper DG algebra $A^\bullet$ over a field K comes from a smooth and proper DG algebra $A_R^\bullet$ over a finitely generated subring $R \subset K$, $A^\bullet \cong A_R^\bullet \otimes_R K$. This allows one to reduce problems from $\mathsf{char}\, 0$ to $\mathsf{char}\, p$.

Let us give a very rough sketch of how Theorem 5.2 is proved (for more details, see [Ka2], and the complete proof in a slightly different language is in [Ka1]). As in the commutative story, there are two cases for Theorem 5.2: the easy case when one can construct the Cartier map explicitly, and the general case. The easy case is when $A^\bullet = A$ is concentrated in degree 0, and the algebra A admits a so-called *quasi-Frobenius map*.

Lemma 5.5 ([Ka1]). *For any vector space V over the finite field k of characteristic $\mathsf{char}\, k = p > 0$, there is a canonical Frobenius-semilinear isomorphism*

$$\check{H}^\bullet(\mathbb{Z}/p\mathbb{Z}, V) \cong \check{H}^\bullet(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}),$$

where $\check{H}^\bullet(\mathbb{Z}/p\mathbb{Z}, -)$ means the Tate (co)homology of the group $\mathbb{Z}/p\mathbb{Z}$, the action of $\mathbb{Z}/p\mathbb{Z}$ on V is trivial, and the action on $V^{\otimes p}$ is by the longest cycle permutation $\sigma : V^{\otimes p} \to V^{\otimes p}$. $\square$

Definition 5.6 ([Ka1]). A *quasi-Frobenius map* for an algebra A/k is a $\mathbb{Z}/p\mathbb{Z}$-equivariant algebra map

$$\Phi : A \to A^{\otimes p}$$

which induces the standard isomorphism of Lemma 5.5 on Tate cohomology $\check{H}^\bullet(\mathbb{Z}/p\mathbb{Z}, -)$.

If the algebra A admits a quasi-Frobenius map Φ, then the construction of the Cartier isomorphism proceeds as follows. First, recall that for any algebra B equipped with an action of a group G, the *smash product algebra* $B\#G$ is the group algebra $B[G]$ but with the twisted product given by

$$(b_1 \cdot g_1)(b_2 \cdot g_2) = b_1 b_2^{g_1} \cdot g_1 g_2,$$

and one has a canonical decomposition

$$(5.1) \qquad\qquad HP_\bullet(B\#G) = \bigoplus_{\langle g \rangle} HP_\bullet(B\#G)_g$$

into components numbered by conjugacy classes of elements in G (these components are sometimes called *twisted sectors*). Next, let G be the cyclic group $\mathbb{Z}/p\mathbb{Z}$, and let $\sigma \in G$ be the generator. Then one can show that if the G-action on B is trivial, then

$$(5.2) \qquad\qquad HP_\bullet(B\#G)_\sigma \cong \widetilde{HP_\bullet(B)},$$

where $HP_\bullet(B)$ in the right-hand side is equipped with the Hodge filtration, and $\widetilde{M}$ for a filtered group M means the cokernel of the map (4.1), as in Section 4. One the other hand, if we take the p-th power $B^{\otimes p}$ with σ acting by the longest cycle permutation, then one can show that

$$(5.3) \qquad\qquad HP_\bullet(B^{\otimes p}\#G)_\sigma \cong HP_\bullet(B).$$

Both the isomorphisms (5.2) and (5.3) are completely general and valid for algebras over any ring. So is the decomposition (5.1), which is moreover functorial with respect to G-equivariant maps. We now apply this to our algebras A and $A^{\otimes p}$ over k, with the G-action as in Lemma 5.5. The quasi-Frobenius map Φ induces a map

$$\varphi : \widetilde{HP_\bullet(A)} \cong HP_\bullet(A\#G)_\sigma \to HP_\bullet(A^{\otimes p}\#G)_\sigma \cong HP_\bullet(A),$$

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and since p annihilates $HP_\bullet(A)$, we have

$$\widetilde{HP_\bullet(A)} \cong \operatorname{gr}_F^\bullet HP_\bullet(A) \cong HH_\bullet(A)((u)).$$

The map φ is the Cartier map of Theorem 5.2. One then shows that it is an isomorphism; this requires one to assume that A is smooth.

The general case of Theorem 5.2 is handled by finding a replacement for a quasi-Frobenius map; as far as the cyclic homology is concerned, the argument stays the same. One first shows that for any unital associative algebra A/k, there exists a completely canonical diagram

$$A \xleftarrow{\ \alpha\ } Q_\bullet(A) \xrightarrow{\ \Phi\ } P_\bullet(A) \xleftarrow{\ \beta\ } A^{\otimes p}$$

of DG algebras equipped with an action of $G = \mathbb{Z}/p\mathbb{Z}$ and G-equivariant maps between them. The G action on A and $Q_\bullet(A)$ is trivial. In addition, if A is smooth, the map

$$HP_\bullet(A^{\otimes p}\#G)_\sigma \to HP_\bullet(P_\bullet(A)\#G)_\sigma$$

induced by the map β is an isomorphism (although in general, this isomorphism does not preserve the Hodge filtration). Thus as before, Φ induces a canonical map

$$\overline{\varphi} : HH_\bullet(Q_\bullet(A))((u)) \cong \widetilde{HP_\bullet(Q_\bullet(A))} \to HP_\bullet(A).$$

To construct the Cartier map for the algebra A, it remains to construct a map

$$HH_\bullet(A) \to HH_\bullet(Q_\bullet(A)).$$

To do this, one applies obstruction theory and shows that the map $\alpha : Q_\bullet(A) \to A$ admits a splitting in the category $\mathsf{DG\text{-}Alg}(k)$. The homology of the DG algebra $Q_\bullet(A)$ is given by

(5.4) $$\mathcal{H}_i(Q_\bullet(A)) = A \otimes \mathsf{St}_i(k),$$

where $\mathsf{St}_\bullet(k)$ is the dual k-Steenrod algebra — that is, the dual to the algebra of k-linear cohomological operations in cohomology with coefficients in k. We have $\mathsf{St}_0(k) \cong \mathsf{St}_1(k) \cong k$, and $\mathsf{St}_i(k) = 0$ for $1 < k \leq 2p$. The map $a : Q_\bullet(A) \to A$ is an isomorphism in degree 0. The splitting is constructed degree-by-degree. In degree 1, the obstruction to splitting is exactly the same as the obstruction to lifting the algebra A/k to the ring $W_2(k)$. In any higher degree $i > 1$, the obstruction lies in the Hochschild cohomology group $HH^{2+i}(A \otimes \mathsf{St}_i(k))$, and this vanishes in the relevant range of degrees by the assumption $HH^i(A) = 0$, $i \geq 2p - 1$.

In the DG algebra case, the construction breaks down since Lemma 5.5 does not have a DG version. Thus one first has to replace a DG algebra $A^\bullet$ with a cosimplicial algebra $\mathcal{A}$ by the Dold-Kan equivalence, and then apply the above construction to $\mathcal{A}$ "pointwise". It is at this point that one has to require $A^i = 0$ for $i < 0$.

Although [Ka1] only provides a Cartier map for DG algebras defined over a finite field k, the same technology should apply to DG algebras over $W = W(k)$ with very little changes, so that for any smooth DG algebra $A^\bullet/W(k)$ with $HH^i(A^\bullet) = 0$ for $i \geq 2p - 1$, one should be able to construct a canonical isomorphism

$$\widetilde{\varphi} : \widetilde{HP_\bullet(A^\bullet)} \cong HP_\bullet(A^\bullet).$$

Equivalently, $HP_\bullet(A^\bullet)$ should carry a filtered Dieudonné module structure (in other words, underlie a canonical object of the periodic derived category $\mathcal{DFDM}^{per}(W)$). One also should be able to check that if $A^\bullet$ is Morita-equivalent to a smooth variety X/W, the comparison isomorphism (2.2) is compatible with the filtered Dieudonné module structures on both sides. However, at present, none of this has been done.

We note that the problem with the regulator map in the p-adic setting mentioned in the end of Section 4 survives in the non-commutative situation. Namely, the standard technology for constructing the regulator map (3.1) ([Lo, Section 8.4]) involves considering the group algebras $k[G]$ for $G = GL_n(A)$, for all $n \geq 1$. As n goes to infinity, the homological dimension of these group alegbras becomes arbitrarily large, and the conditions of Theorem 5.2 cannot be satified.

6 Generalities on stable homotopy.

The appearance of the Steenrod algebra in (5.4) suggests that the whole story should be related to algebraic topology. This is indeed so. To explain the relation, we need to recall some standard facts on stable homotopy theory.

6.1 Stable homotopy category and homology. Roughly speaking, the *stable homotopy category* StHom is obtained by inverting the suspension functor Σ in the category Hom of pointed CW complexes and homotopy classes of maps between them. Objects of StHom are called *spectra*. A spectrum consists of a collection of pointed CW complexes X_i, $i \geq 0$, and maps $\Sigma X_i \to X_{i+1}$ for all i (in some treatments, these data are required to satisfy additional technical conditions). For the definitions of maps between spectra and homotopies between such maps, we refer the reader to a number of standard references, for example [Ad]. Any CW complex $X \in$ Hom defines its *suspension spectrum* $\Sigma^\infty X \in$ StHom consisting of the suspensions $\Sigma^i X$. For any two CW complexes X, Y, we have

$$\mathrm{Hom}_{\mathsf{StHom}}(\Sigma^\infty X, \Sigma^\infty Y) = \varinjlim_i [\Sigma^i X, \Sigma^i Y],$$

where $[-, -]$ denotes the set of homotopy classes of maps.

Any complex of abelian groups $M_\bullet$ defines a spectrum $\mathsf{EM}(M_\bullet)$ called the *Eilenberg-Maclane spectrum of* $M_\bullet$. This is functorial in $M_\bullet$, so that for any commutative ring R, we have a functor

$$\mathsf{EM} : \mathcal{D}(R) \to \mathcal{D}(\mathrm{Ab}) \to \mathsf{StHom},$$

where $\mathcal{D}(R)$ is the derived category of the category of R-modules. This functor has a left-adjoint $H(R) :$ $\mathsf{StHom} \to \mathcal{D}(R)$, known as *homology with coefficients in* R.

The category StHom is a tensor triangulated category. Both functors EM and $H(R)$ are triangulated. Moreover, the homology functor $H(R)$ is a tensor functor – for any two spectra $X, Y \in$ StHom with smash-product $X \wedge Y$, there exists a functorial isomorphism

$$H(R)(X) \overset{\mathsf{L}}{\otimes}_R H(R)(Y) \cong H(R)(X \wedge Y).$$

The adjoint Eilenberg-Maclane functor EM is pseudotensor – we have a natural map

$$\mathsf{EM}(V_\bullet) \wedge \mathsf{EM}(W_\bullet) \to \mathsf{EM}(V_\bullet \overset{\mathsf{L}}{\otimes}_R W_\bullet)$$

for any two objects $V_\bullet, W_\bullet \in \mathcal{D}(R)$. Thus for any associative ring object $\mathcal{A}$ in StHom, its homology $H(R)(\mathcal{A})$ is a ring object in $\mathcal{D}(R)$, and conversely, for any associative ring object $A_\bullet \in \mathcal{D}(R)$, the Eilenberg-Maclane spectrum $\mathsf{EM}(A_\bullet)$ is a ring object in StHom.

In the homological setting, we know that the structure of a "ring object in $\mathcal{D}(R)$" is too weak, and the right objects to consider are DG algebras over R. To define an analogous notion for spectra is non-trivial, since the traditional topological interpretation of spectra does not behave too well as far as the products are concerned. Fortunately, new models for StHom have appeared more recently, such as for example *S-modules* of [EKMM], *orthogonal spectra* of [MM], or *symmetric spectra* of [HSS]. All these approaches give equivalent results; to be precise, let us choose for example the last one. As shown in [HSS], symmetric spectra form a symmetric monoidal category; denote it by Sym. Then in this paper, a ring spectrum will denote a monoidal object in Sym, and StAlg will denote the category of ring spectra considered up to a homotopy equivalence (formally, this is defined by putting a closed model structure on the category of ring monoidal objects in Sym whose weak equivalences are homotopy equivalences of the underlying symmetric spectra). The homology functor $H(R)$ and the Eilenberg-Maclane functor EM extend to functors

$$H(R) : \mathsf{StAlg} \to \mathsf{DG\text{-}Alg}(R), \qquad \mathsf{EM} : \mathsf{DG\text{-}Alg}(R) \to \mathsf{StAlg}\,.$$

where as in Section 2, $\mathsf{DG\text{-}Alg}(R)$ is the category of DG algebras over R considered up to a quasiisomorphism.

6.2 Equivariant categories. For any compact group G, a pointed "G-CW complex" is a pointed CW complex X equipped with a continuos action of G such that the fixed-point subset $X^g \subset X$ is a pointed subcomplex for any $g \in G$. We will denote by $\mathsf{Hom}(G)$ the category of pointed G-CW complexes and G-equivariant homotopy

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classes of G-equivariant maps between them. We note that for any closed subgroup $H \subset G$, sending X to the fixed-point subspace $X^H \subset X$ gives a well-defined functor

$$\mathsf{Hom}(G) \to \mathsf{Hom}\,.$$

This functor is representable in the following sense: for any $X \in \mathsf{Hom}(G)$, we have a homotopy equivalence

$$(6.1) \qquad\qquad X^H \cong \mathrm{Maps}_G([G/H]_+, X),$$

where $\mathrm{Maps}_G(-,-)$ means the space of G-equivariant maps with its natural topology, and $[G/H]_+$ is the pointed G-CW complex obtained by adding a (disjoint) marked point to the quotient G/H with the induced topology and G-action.

To define a stable version of the category $\mathsf{Hom}(G)$, one could again simply invert the suspension functor. However, there is a more interesting alternative: by definition, n-fold suspension Σ^n is the smash-product with an n-sphere, and in the equivariant setting, one can allow the sphere to carry a non-trivial G-action. The corresponding equivariant stable category has been constructed in [LMS]; it is known as the *genuine G-equivariant stable homotopy category* $\mathsf{StHom}(G)$. To define it, one needs to fix a real representation U of the group G which is equipped with a G-invariant inner product and contains every finite-dimensional inner-product representation countably many times; this is called a "complete G-universe". Then a genuine G-equivariant spectrum is a collection of G-CW complexes $X(V)$, one for each finite-dimensional G-invariant inner-product subspace $V \subset U$, and maps $S^W \wedge X(V) \to X(V \oplus W)$, one for each inner-product G-invariant subspace $V \oplus W \subset U$, where S^V is the one-point compactification of the underlying topological space of the representation V, with its natural G-action. As in the non-equivariant case, $\mathsf{StHom}(G)$ is a tensor triangulated category. We have a natural suspension spectrum functor $\Sigma^\infty : \mathsf{Hom}(G) \to \mathsf{StHom}(G)$, and for any two objects $X, Y \in \mathsf{Hom}(G)$, we have

$$\mathrm{Hom}_{\mathsf{StHom}(G)}(\Sigma^\infty X, \Sigma^\infty Y) = \lim_{\overrightarrow{V \subseteq U}} [S^V \wedge X, S^V \wedge Y]_G,$$

where $[-,-]_G$ is the set of G-homotopy classes of G-equivariant maps, and the limit is over all the finite-dimensional G-invariant inner-product subspaces $V \subset U$. The category $\mathsf{StHom}(G)$ does depend on U, but this is not too drastic: all complete G-universes are isomorphic, and for any isomorphism $U \cong U'$ between complete G-universes, there is a "change of universe" functor which is an equivalence between the corresponding versions of $\mathsf{StHom}(G)$.

Forgetting the G-action gives a natural forgetful functor $\mathsf{StHom}(G) \to \mathsf{StHom}$, and equipping a spectrum with a trivial G-action gives an embedding $\mathsf{StHom} \to \mathsf{StHom}(G)$. Thus for any $X \in \mathsf{StHom}$ and $Y \in \mathsf{StHom}(G)$, we have a functorial smash product $X \wedge Y \in \mathsf{StHom}(G)$. This has an adjoint: for any $X, Y \in \mathsf{StHom}(G)$, we have a natural spectrum $\mathrm{Maps}_G(X, Y) \in \mathsf{StHom}$ such that for any $Z \in \mathsf{StHom}$, there is a functorial isomorphism

$$\mathrm{Hom}_{\mathsf{StHom}(G)}(Z \wedge X, Y) \cong \mathrm{Hom}_{\mathsf{StHom}}(Z, \mathrm{Maps}_G(X, Y)).$$

For any closed subgroup $H \subset G$ and any $X \in \mathsf{StHom}(G)$, one can extend (6.1) and define the fixed point spectrum X^H by the same formula,

$$(6.2) \qquad\qquad X^H = \mathrm{Maps}_G(\Sigma^\infty [G/H]_+, X).$$

However, this does not commute with the suspension spectrum functor Σ^∞. In [LMS], a second fixed-points functor is introduced, called the *geometric fixed points functor* and denoted Φ^H. It does commute with Σ^∞, and also commutes with smash products, so that there are functorial isomorphisms

$$\Phi^H(\Sigma^\infty X) \cong \Sigma^\infty X^H, \qquad \Phi^H(X \wedge Y) \cong \Phi^H(X) \wedge \Phi^H(Y)$$

for any $X, Y \in \mathsf{StHom}(G)$. For any $X \in \mathsf{StHom}(G)$, there exists a canonical map

$$(6.3) \qquad\qquad \mathsf{can} : X^H \to \Phi^H(X),$$

functorial in X. Moreover, let $N_H \subset G$ be the normalizer of the subgroup $H \subset G$, and let $W_H = N_H/H$ be the quotient. Then Φ^H can be extended to a functor

$$\widehat{\Phi}^H : \mathsf{StHom}(G) \to \mathsf{StHom}(W_H),$$

and the same is true for the usual fixed-points functor $X \mapsto X^H$ of (6.2). The map can of (6.3) then lifts to a map of W_H-equivariant spectra. Here if $\mathsf{StHom}(G)$ is defined on a complete G-universe U, then $\mathsf{StHom}(W_H)$ should be defined on the complete W_H-universe U^H. The functor $\widehat{\Phi}^H$ has a right-adjoint which is a fully faithful embedding $\mathsf{StHom}(W_H) \to \mathsf{StHom}(G)$ (for example, if $H = G$, then this is the trivial embedding $\mathsf{StHom} \to \mathsf{StHom}(G)$).

6.3 Mackey functors. Assume from now on that the compact group G is a finite group with discrete topology. It is not difficult to extend the homology functor $H(R)$ to a functor

$$H(R) : \mathsf{StHom}(G) \to \mathcal{D}(G, R)$$

with values in the derived category of $R[G]$-modules. However, this version of equivariant homology looses a lot of information such as fixed points. A more natural target for equivariant homology is the category of the so-called *Mackey functors*. To define them, one considers an additive category $\mathcal{B}_G$ whose objects are G-orbits G/H for all subgroups $H \subset G$, and whose Hom-groups are given by

$$(6.4) \qquad \begin{aligned} \mathcal{B}^G([G/H_1], [G/H_2]) &= \mathrm{Hom}_{\mathsf{StHom}(G)}(\Sigma^\infty [G/H_1]_+, \Sigma^\infty [G/H_2]_+) = \\ &= \pi_0(\mathrm{Maps}_G(\Sigma^\infty [G/H_1]_+, \Sigma^\infty [G/H_2]_+)). \end{aligned}$$

An *R-valued G-Mackey functor* ([Dr], [Li], [tD], [M1]) is an additive functor from $\mathcal{B}_G$ to the category of R-modules. The category of such functors is an abelian category, denoted $\mathcal{M}(G, R)$.

More explicitly, for any subgroups $H_1, H_2 \subset G$, one can consider the groupoid $\mathcal{Q}([G/H_1], [G/H_2])$ of diagrams $[G/H_1] \leftarrow S \to [G/H_2]$ of finite sets equipped with a G-action, and isomorphisms between such diagrams. Then disjoint union turns these groupoids into symmetric monoidal categories, the Cartesian product turns the collection $\mathcal{Q}(-,-)$ into a 2-category with objects $[G/H]$, and it seems very likely that the mapping spectra $\mathrm{Maps}_G(\Sigma^\infty [G/H_1]_+, \Sigma^\infty [G/H_2]_+)$ are in fact obtained from the classifying spaces $|\mathcal{Q}([G/H_1], [G/H_2])|$ of symmetric monoidal groupoids $\mathcal{Q}([G/H_1], [G/H_2])$ by group completion. At present, this has not been proved ([M2]); however, the corresponding isomorphism is well-known at the level of π_0: we have

$$\pi_0(\mathrm{Maps}_G(\Sigma^\infty [G/H_1]_+, \Sigma^\infty [G/H_2]_+)) \cong \pi_0(\Omega B |\mathcal{Q}([G/H_1], [G/H_2])|),$$

so that the groups $\mathcal{B}_G(-,-)$ are given by

$$(6.5) \qquad \mathcal{B}^G([G/H_1], [G/H_2]) = \mathbb{Z}[\mathrm{Iso}(\mathcal{Q}([G/H_1], [G/H_2]))]/\{[S_1 \coprod S_2] - [S_1] - [S_2]\},$$

where Iso means the set of isomorphism classes of objects.

For any $X \in \mathsf{StHom}(G)$, individual homology groups $H_i(R)(X)$ can be equipped with a natural structure of a Mackey functor in such a way that $H_i(R)(X)([G/H]) \cong H_i(R)(X^H)$, $H \subset G$ (for more details, see [M1]). To collect these into a single homology functor $H(R)$, one has to work out a natural derived version of the abelian category $\mathcal{M}(G, R)$. This has been done recently in [Ka3]. Roughly speaking, instead of π_0 in (6.4), one should the chain homology complexes $C_\bullet(-, \mathbb{Z})$ of the corresponding spectra, and one should set

$$\mathcal{B}^G_\bullet([G/H_1], [G/H_2]) = C_\bullet(\mathrm{Maps}_G(\Sigma^\infty [G/H_1]_+, \Sigma^\infty [G/H_2]_+), \mathbb{Z}).$$

In practice, one replaces this with complexes which compute the homology of the spectra obtained by group completion from the symmetric monoidal groupoids $\mathcal{Q}([G/H_1], [G/H_2])$. This can be computed explicitly, so that the complexes $\mathcal{B}^G_\bullet(-,-)$ introduced in [Ka3, Section 3] are given by an explicit formula, and spectra are not mentioned at all. One then shows that the collection $\mathcal{B}^G_\bullet(-,-)$ is an A_∞-category in a natural way, and one defines the triangulated category $\mathcal{DM}(G, R)$ of *derived R-valued G-Mackey functors* as the derived category of A_∞-functors from $\mathcal{B}^G_\bullet$ to the category of complexes of R-modules.

In general, the category $\mathcal{DM}(G, R)$ turns out to be different from the derived category $\mathcal{D}(\mathcal{M}(G, R))$ (although both contain the abelian category $\mathcal{M}(G, R)$ as a full subcategory). On the level of slogans, one can hope that the category $\mathcal{DM}(G, R)$ is the "brave new product" of the category $\mathsf{StHom}(G)$ and the derived category $\mathcal{D}(R)$ of R-modules, taken over the non-equivariant stable homotopy category StHom, so that we have a diagram

$$\begin{array}{ccccc} \mathcal{D}(\mathcal{M}(G, R)) & \longrightarrow & \mathcal{DM}(G, R) & \longrightarrow & \mathsf{StHom}(G) \\ & & \downarrow & & \downarrow \\ & & \mathcal{D}(R) & \longrightarrow & \mathsf{StHom}, \end{array}$$

where the square is Cartesian in some "brave new" sense. On a more mundane level, it is expected that the triangulated category $\mathcal{DM}(G, R)$ reflects the structure of the category $\mathsf{StHom}(G)$ in the following way.

(i) There exists a symmetric tensor product $- \otimes -$ on the triangulated category $\mathcal{DM}(G, R)$, and for any subgroup $H \subset G$, we have natural triangulated fixed-point functors $\Phi^H, \Psi^H : \mathcal{DM}(G, R) \to \mathcal{D}(R)$.

(ii) There exists a natural triangulated equivariant homology functor

$$H_G(R) : \mathsf{StHom}(G) \to \mathcal{DM}(G, R)$$

and natural functorial isomorphisms

$$\Phi^H(H_G(R)(X)) \cong H(R)(\Phi^H(X)),$$
$$\Psi^H(H_G(R)(X)) \cong H(R)(X^H),$$
$$H_G(X \wedge Y) \cong H_G(X) \otimes H_G(Y)$$

for any $X, Y \in \mathsf{StHom}(G)$, $H \subset G$.

In fact, most of these statements has been proved in [Ka3], although only for the so-called "Spanier-Whitehead category", the full triangulated subcategory in $\mathsf{StHom}(G)$ spanned by the suspension spectra of finite CW complexes (the only thing not proved is the compatibility $\Psi^H(H_G(R)X) \cong H(R)(X^H)$ which requires one to leave the Spanier-Whitehead category). It has been also shown in [Ka3] that as in the case of spectra, the fixed point functor Φ^H extends to a functor

(6.6) $$\widehat{\Phi}^H : \mathcal{DM}(G, R) \to \mathcal{DM}(W_H, R)$$

with a fully faithful right-adjoint. These fixed-points functors allow one to give a very explicit description of the category $\mathcal{DM}(G, R)$. Namely, let $I(G)$ be the set of conjugacy classes of subgroups in G, and for any $c \in I(G)$, let

$$\mathcal{DM}_c(G, R) \subset \mathcal{DM}(G, R)$$

be the full subcategory of such $M \in \mathcal{DM}(G, R)$ that $\Phi^H(M) = 0$ unless $H \subset G$ is in the class c.

Proposition 6.1 ([Ka3]). *For any $c \in I(G)$, $\mathcal{DM}_c(G, R) \subset \mathcal{DM}(G, R)$ is an admissible triangulated subcategory, and for any subgroup $H \subset G$ is a subgroup in the class c, the functor $\widehat{\Phi}^H$ of (6.6) induces an equivalence*

$$\widehat{\varphi}^H : \mathcal{DM}_c(G, R) \cong \mathcal{D}(W_H, R).$$

Moreover, equip $I(G)$ with the partial order given by inclusion. Then it has been shown in [Ka3] that unless $c \leq c'$, $\mathcal{DM}_c(G, R)$ is left-orthogonal to $\mathcal{DM}_{c'}(G, R)$, so that $\mathcal{DM}_c(G, R)$, $c \in I(G)$ form a semiorhtogonal decomposition of the triangulated category $\mathcal{DM}(G, R)$ indexed by the partially ordered set $I(G)$ (for generalities on semiorthogonal decompositions, see [BK]). To describe the gluing data between the pieces of this semiorthogonal decomposition, one introduces the following.

Definition 6.2. Assume given a finite group G and a module V over $R[G]$. The *maximal Tate cohomology* $\check{H}^\bullet_{max}(G, V)$ is given by

$$\check{H}^\bullet_{max}(G, V) = \mathrm{RHom}^\bullet_{\mathcal{D}^b(G/R)/\mathsf{Ind}}(R, V),$$

where $\mathrm{RHom}^\bullet$ is computed in the quotient $\mathcal{D}^b(G, R)/\mathsf{Ind}$ of the bounded derived category $\mathcal{D}^b(G, R)$ by the full saturated triangulated subcategory $\mathsf{Ind} \subset \mathcal{D}^b(G, R)$ spanned by representrations $\mathsf{Ind}^H_G(W)$ induced from a representation W of a subgroup $H \subset G$, $H \neq G$.

Then for any two subgroups $H \subset H' \subset G$ with conjugacy classes $c, c' \in I$, $c \leq c'$, the gluing functor between $\mathcal{DM}_c(G, R)$ and $\mathcal{DM}_{c'}(G, R)$ is expressed in terms of maximal Tate cohomology of the group W_H and its various subgroups.

This description turns out to be very effective because maximal Tate cohomology often vanishes. For example, if the order of the group G is invertible in R, $\check{H}^\bullet_{max}(G, V) = 0$ for any $R[G]$-module V, and the category $\mathcal{DM}(G, R)$ becomes simply the direct sum of the categories $\mathcal{DM}_c(G, R) \cong \mathcal{D}(W_H, R)$ (for the abelian category $\mathcal{M}(G, R)$, a similar decomposition theorem has been proved some time ago by J. Thevenaz [Th]). On the other hand, if R is arbitrary but the group $G = \mathbb{Z}/n\mathbb{Z}$ is cyclic, then $\check{H}^\bullet_{max}(G, V) = 0$ for any V unless $n = p$ is prime, in which case $\check{H}^\bullet_{max}(G, V)$ reduces to the usual Tate cohomology $\check{H}^\bullet(G, V)$.

7   Cyclotomic traces.

Returning to the setting of Theorem 5.2, we can now explain the appearance of the Steenrod algebra in (5.4): up to a quasiisomorphism, the DG algebra $Q_\bullet(A)$ of (5.4) is in fact given by

$$Q_\bullet(A) = H(k)(\mathsf{EM}(A))^{k^*},$$

where the k^*-invariants are taken with respect to the natural action of the multiplicative group k^* of the finite field k induced by its action on k.

In particular, this shows that it is not necessary to use dimension arguments to construct a splitting $A \to Q_\bullet(A)$ of the augmentation map $Q_\bullet(A) \to A$. For example, if we are given a ring spectrum $\mathcal{A}$ with homology DG algebra $A^\bullet = H(k)(\mathcal{A})$, then a canonical map

(7.1) $A^\bullet = H(k)(\mathcal{A}) \to H(k)(\mathsf{EM}(H(k)(\mathcal{A})))$

exists simply by adjunction, and being canonical, it is in particular k^*-invariant. Thus for any DG algebra of the form $A^\bullet = H(k)(\mathcal{A})$, the same procedure as in the proof of Theorem 5.2 allows one to construct a Cartier map. However, in this case one can do much more – namely, one can compare the homological story with the theory of *cyclotomic traces* and *topological cyclic homology* known in algebraic topology. Let us briefly recall the setup (we mostly follow the very clear and concise exposition in [HM]).

7.1   Topological cyclic homology. For any unital associative algebra A over a ring k, the Hochschild homology complex $CH_\bullet(A)$ of Section 2 is in fact the standard complex of a simplicial k-module $A_\# \in \Delta^{opp}k$-mod. *Topological Hochschild homology* is a version of this construction for ring spectra. It was originally introduced by Bökstedt [B] long before the invention of symmetric spectra, and used the technology of "functors with a smash product". In the language of symmetric spectra, one starts with a unital associative ring spectrum $\mathcal{A}$, and one defines a simplicial spectrum $\mathcal{A}_\#$ by exactly the same formula as in the algebra case. The terms of $\mathcal{A}_\#$ are the iterated smash products $\mathcal{A} \wedge \cdots \wedge \mathcal{A}$, and the face and degeneracy maps are obtained from the multiplication and the unit map in $\mathcal{A}$. Then one sets

$$\mathrm{THH}(\mathcal{A}) = \mathsf{hocolim}_{\Delta^{opp}} \mathcal{A}_\#.$$

As in the algebra case, this spectrum is equipped with a canonical S^1-action, but in the topological setting this means much more: one shows that $\mathrm{THH}(\mathcal{A})$ actually underlies a canonical S^1-equivariant spectum $\mathrm{THH}(\mathcal{A}) \in \mathsf{StHom}(S^1)$.

However, this is not the end of the story. Note that the finite subgroups in S^1 are the cyclic groups $C_n = \mathbb{Z}/n\mathbb{Z} \subset S^1$ numbered by integers $n \geq 1$, and for every n, we have $S^1/C_n \cong S^1$. Fix a system of such isomorphisms which are compatible with the embeddings $C_n \subset C_{nm} \subset S^1$, $n, m \geq 1$, and fix a compatible system of isomorphisms $U^{C_n} \cong U$, where U is the complete S^1-universe used to define $\mathsf{StHom}(S^1)$. Then the following notion has been introduced in [BM].

Definition 7.1.   A *cyclotomic structure* on an S^1-equivariant spectrum T is given by a collection of S^1-equivariant homotopy equivalences

$$r_n : \widehat{\Phi}^{C_n} T \cong T,$$

one for each finite subgroup $C_n \subset S^1$, such that $r_1 = \mathsf{id}$ and $r_n \circ r_m = r_{nm}$ for any two integer $n, m > 1$.

Remark 7.2. Here it is tacitly assumed that one works with specific model of equivariant spectra, so that a spectrum means more than just an object of the triangulated category $\mathsf{StHom}(S^1)$; moreover, the functors $\widehat{\Phi}^{C_n}$ are composed with the change of universe functors so that we can treat them as endofunctors of $\mathsf{StHom}(S^1)$. Please refer to [BM] or [HM] for exact definitions.

Example 7.3. Assume given a CW complex X, and let $LX = \mathrm{Maps}(S^1, X)$ be its free loop space. Then for any finite subgroup $C \subset S^1$, the isomorphism $S^1 \cong S^1/C$ induces a homeomorphism

$$\mathrm{Maps}(S^1, X)^C = \mathrm{Maps}(S^1/C, X) \cong \mathrm{Maps}(S^1, X),$$

and these homeomorphism provide a canonical cyclotomic structure on the suspension spectrum $\Sigma^\infty LX$.

For any S^1-equivariant spectrum T and a pair of integers $r, s > 1$, one has a natural non-equivariant map

$$F_{r,s} : T^{C_{rs}} \to T^{C_r}.$$

On the other hand, assume that T is equipped with a cyclotomic structure. Then we have a natural map

$$R_{r,s} : T^{C_{rs}} \cong (T^{C_s})^{C_r} \xrightarrow{\ \mathsf{can}\ } (\widehat{\Phi}^{C_s} T)^{C_r} \xrightarrow{\ r_s\ } T^{C_r},$$

where can is the canonical map (6.3), and r_s comes from the cyclotomic structure on T. To pack together the maps $F_{r,s}$, $R_{r,s}$, it is convenient to introduce a small category $\mathbb{I}$ whose objects are all integers $n \geq 1$, and whose maps are generated by two maps $F_r, R_r : n \to m$ for each pair m, $n = rm$, $r > 1$, subject to the relations $F_r \circ F_s = F_{rs}$, $R_r \circ R_s = R_{rs}$, $F_r \circ R_s = R_s \circ F_r$. Then the maps $T_{r,s}$, $F_{r,s}$ turn the collection T^{C_n}, $n \geq 1$ into a functor $\widetilde{T}$ from $\mathbb{I}$ to the category of spectra.

Definition 7.4. The *topological cyclic homology* $\mathrm{TC}(T)$ of a cyclotomic spectrum T is given by

$$\mathrm{TC}(T) = \operatorname{holim}_{\mathbb{I}} \widetilde{T}.$$

Given a ring spectrum $\mathcal{A}$, Bökstedt and Madsen equip the S^1-equivariant spectrum $THH_\bullet(\mathcal{A})$ with a canonical cyclomomic structure. *Topological cyclic homology* $\mathrm{TC}(\mathcal{A})$ is then given by

$$\mathrm{TC}(\mathcal{A}) = \mathrm{TC}(\mathrm{THH}(\mathcal{A})).$$

Further, they construct a canonical *cyclotomic trace map*

(7.2) $$K(\mathcal{A}) \to \mathrm{TC}(\mathcal{A})$$

from the K-theory spectrum $K(\mathcal{A})$ to the topological cyclic homology spectrum.

The topological cyclic homology functor $\mathrm{TC}(\mathcal{A})$ and the cyclotomic trace were actually introduced by Bökstedt, Hsiang and Madsen in [BHM]; the more convenient formulation using cyclotomic spectra appeared slightly later in [BM]. Starting with [BHM], it has been proved in many cases that the cyclotomic trace map becomes a homotopy equivalence after taking profinite completions of both sides of (7.2). Moreover, in [Mc] MacCarthy generalized Goodwillie's Theorem and proved that after pro-p completion at any prime p, the cyclotomic trace gives an equivalence of the relative groups $\widehat{K(\mathcal{A}, I)}_p \cong \widehat{\mathrm{TC}(\mathcal{A}, I)}_p$, where $I \subset \mathcal{A}$ is a nilpotent ideal.

7.2 Cyclotomic complexes. To define a homological analog of cyclotomic spectra, one needs to replace S^1-equivariant spectra with derived Mackey functors. The machinery of [Ka3] does not apply directly to non-discrete groups, since this would require treating the groupoids $\mathcal{Q}(-,-)$ of Subsection 6.2 as topological groupoids. However, for finite subgroups $C_1, C_2 \subset S^1$, the category $\mathcal{Q}([S^1/C_1], [S^1/C_2])$ is still discrete. Thus one can define a restricted version of derived S^1-Mackey functors by discarding the only infinite closed subgroup in S^1 (which is S^1 itself). This is done in [Ka4]. The category $\mathcal{DM}\Lambda(R)$ of R-*valued cyclic Mackey functors* introduced in that paper has the following features.

  (i) For every proper finite subgroup $C = C_n \subset S^1$, $n > 1$, there is a fixed-point functor $\widehat{\Phi}_n : \mathcal{DM}\Lambda(R) \to \mathcal{DM}\Lambda(R)$ whose right-adjoint functor $\widehat{\iota}_n : \mathcal{DM}\Lambda(R) \to \mathcal{DM}\Lambda(R)$ is a full embedding. Moreover, there are canonical isomorphisms $\widehat{\Phi}_n \circ \widehat{\Phi}_m \cong \widehat{\Phi}_{mn}$.

  (ii) Let $\mathcal{D}_{S^1}(R)$ be the equivariant derived category of Section 3. Then there is a full embedding $\iota_1 : \mathcal{D}_{S^1}(R) \to \mathcal{DM}\Lambda(R)$ with a left-adjoint $\Phi_1 : \mathcal{DM}\Lambda(R) \to \mathcal{D}_{S^1}(R)$.

  (iii) The images $\mathcal{DM}\Lambda_n(R)$ of the full embeddings $\iota_n = \widehat{\iota}_N \circ \iota_1 : \mathcal{D}_{S^1}(R) \to \mathcal{DM}\Lambda(R)$, $n \geq 1$, generate the triangulated category $\mathcal{DM}\Lambda(R)$, and $\mathcal{DM}\Lambda_n(R) \subset \mathcal{DM}\Lambda(R)$ is left-orthogonal to $\mathcal{DM}\Lambda_m(R) \subset \mathcal{DM}\Lambda(R)$ unless $n = mr$ for some integer $r \geq 1$.

Thus as in the finite group case of [Ka3], the subcategories $\mathcal{DM}\Lambda_n(R) \subset \mathcal{DM}\Lambda(R)$ form a semiorthogonal decomposition of the category $\mathcal{DM}\Lambda(R)$. The gluing data between $\mathcal{DM}\Lambda_{mr}(R)$ and $\mathcal{DM}\Lambda_r(R)$ can be expressed in terms of the maximal Tate cohomology $\check{H}^\bullet_{max}(C_m, -)$ of the cyclic group $C_m = \mathbb{Z}/m\mathbb{Z}$. For any $n \geq 1$, let

$\overline{\Phi}_n : \mathcal{DM}\Lambda(R) \to \mathcal{D}(R)$ be the composition of the left-adjoint $\Phi_n = \Phi_1 \circ \widehat{\Phi}_n$ to ι_n and the forgetful functor $\mathcal{D}_{S^1}(R) \to \mathcal{D}(R)$; then the functors $\overline{\Phi}_n$ play the role of fixed points functors Φ^H. There are also functors $\Psi_n : \mathcal{DM}\Lambda(R) \to \mathcal{D}_{S^1}(R)$ analogous to the functors Ψ^H. The homology functor $H(R)$ extends to a functor

$$H_{S^1}(R) : \mathsf{StHom}(S^1) \to \mathcal{DM}\Lambda(R),$$

and we have functorial isomorphisms

$$\overline{\Phi}_n(H_{S^1}(R)(T)) \cong H(R)(\Phi^{C_n}(T)), \qquad \Psi_n(H_{S^1}(R)(T)) \cong H(R)(T^{C_n})$$

for every $n \geq 1$ and every $T \in \mathsf{StHom}(S^1)$.

Another category defined in [Ka4] is a triangulated category $\mathcal{D}\Lambda\mathrm{R}(R)$ of R-valued cyclotomic complexes. Essentially, a cyclotomic complex $M_{\bullet} \in \mathcal{D}\Lambda\mathrm{R}(R)$ is a cyclic Mackey functor $M_{\bullet}$ equipped with a system of compatible quasiisomorphisms

$$\widehat{\Phi}_n M_{\bullet} \cong M_{\bullet},$$

as in Definition 7.1 (although as in Remark 7.2, the precise definition is different for technical reasons). The homology functor $H_{S^1}(R) : \mathsf{StHom}(S^1) \to \mathcal{DM}\Lambda(R)$ extends to a functor from the category of cyclotomic spectra to the category $\mathcal{D}\Lambda\mathrm{R}(R)$. Moreover, all the constructions used in the definition of topological cyclic homology make sense for cyclotomic complexes, so that one has a natural functor

$$\mathrm{TC} : \mathcal{D}\Lambda\mathrm{R}(R) \to \mathcal{D}(R)$$

and a functorial isomorphism

(7.3) $$\mathrm{TC}(H_{S^1}(R)(T)) \cong H(R)(\mathrm{TC}(T))$$

for every cyclotomic spectrum T.

7.3   Comparison theorem. We can now formulate the comparison theorem relating Dieudonné modules and cyclotomic complexes. We introduce the following definition.

Definition 7.5. A *generalized filtered Dieudonné module* M over a commutative ring R is an R-module M equipped with a decreasing filtration $F^{\bullet}M$ and a collection of maps

$$\varphi_{i,j}^p : F^i M \to M/p^j,$$

one for every integers i, j, $j \geq 1$, and a prime p, such that

$$\varphi_{i,j+1}^p = \varphi_{i,j}^p \mod p^j, \quad \varphi_{i,j}^p|_{F^{i+1}M} = p\varphi_{i,j}^p.$$

For any integer i, we define the generalized filtered Dieudonné module $R(i)$ as R with the filtration $F^i R(i) = R$, $F^{i+1}R(i) = 0$, and $\varphi_{i,j}^p = p^i \operatorname{id}$ for any p and j. Generalized filtered Dieudonné modules in the sense of Defintion 7.5 do not form an abelian category; however, by inverting the filtered quasiisomorphisms, we can still construct the derived category $\mathcal{DFDM}_g(R)$ and its twisted 2-periodic version $\mathcal{DFDM}_g^{per}(R)$.

Definition 7.5 generalizes (4.1) in that it collects together the data for all primes p. Note, however, that one can rephrase Definition 7.5 by putting together all the maps $\varphi_{i,j}^p$, $j \geq 1$, into a single map

$$\widehat{\varphi}_i^p : F^i M \to \widehat{(M)}_p$$

into the pro-p completion $\widehat{(M)}_p$ of the module M. Then if $R = \mathbb{Z}_p$ and M is finitely generated over $\mathbb{Z}_p$, we have

$$\widehat{(M)}_p \cong M, \quad \widehat{(M)}_l = 0 \text{ for } l \neq p,$$

so that for such an M, the extra data imposed onto M in Definition 7.5 and in Definition 4.1 are the same. In general, for any prime p, we have a fully faithful embedding

$$\widetilde{\mathcal{DFDM}}(\mathbb{Z}_p) \subset \mathcal{DFDM}_g(\mathbb{Z}),$$

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where $\widetilde{\mathcal{DFDM}}(\mathbb{Z}_p)$ is as in Section 4, and similarly for the periodic categories. The essential images of these embeddings are spanned by complexes which are pro-p complete as complexes of abelian groups. Note, however, that what appears here are *weak* filtered Dieudonné modules. The requirement that the map (4.2) is a quasiisomorphism can be additionally imposed at each individual prime p; I do not know whether it is useful to impose it in the universal category $\mathcal{DFDM}_g(\mathbb{Z})$.

Here is then the main comparison theorem of [Ka4].

Theorem 7.6 ([Ka4, Section 5]). *For any commutative ring R, there is a canonical equivalence of categories*

$$\mathcal{DFDM}_g^{per}(R) \cong \mathcal{D}\Lambda\mathrm{R}(R).$$

Thus the category $\mathcal{D}\Lambda\mathrm{R}(R)$ of cyclotomic complexes over R admits an extremely simple linear-algebraic description. Roughly speaking, the reason for this is the vanishing of maximal Tate cohomology $\check{H}^\bullet(\mathbb{Z}/n\mathbb{Z}, -)$ for non-prime n mentioned at the end of Subsection 6.3. Due to this vanishing, the only non-trivial gluing between the pieces $\mathcal{DM}\Lambda_m(R)$, $\mathcal{DM}\Lambda_n(R)$ of the semiorthogonal decomposition of the category $\mathcal{DM}\Lambda(R)$ of cyclic Mackey functors occurs when $n = mp$ for a prime p (and this gluing is described by the Tate cohomology of the group $\mathbb{Z}/p\mathbb{Z}$). The gluing data provide the maps $\widehat{\varphi}_i^p$ in the equivalence of Theorem 7.6; the periodic filtered complex comes from the equivalence $\mathcal{D}_{S^1}(R) \cong \mathcal{DF}^{per}(R)$ of Lemma 3.1. These are the main ideas of the proof.

Moreover, there is a second comparison theorem which expresses topological cyclic homology in terms of generalized Dieudonné modules.

Theorem 7.7 ([Ka4, Section 6]). *Under the equivalence of Theorem 7.6, there is functorial isomorphism*

(7.4) $$\widehat{\mathrm{TC}(M_\bullet)}_f \cong \widehat{\mathrm{RHom}^\bullet(R, M_\bullet)}_f$$

for any $M_\bullet \in \mathcal{D}\Lambda\mathrm{R}(R)$, where $R = R(0) \in \mathcal{DFDM}_g^{per}(R)$ is the trivial generalized filtered Dieudonné module, and $\widehat{(-)}_f$ stands for profinite completion.

Remark 7.8. Both $\mathrm{TC}(-)$ and $\mathrm{RHom}^\bullet(R, -)$ commute with profinite completions, so that if $M_\bullet$ itself is profinitely complete, the completions in (7.4) can be dropped. In general, it is better to keep the completion; to obtain an isomorphism in the general case, one should, roughly speaking, replace T with the homotopy fixed points T^{hS^1} in the definition of topological cyclic homology $\mathrm{TC}(T)$.

Remark 7.9. It is not unreasonable to hope that Theorem 7.7 has a topological analog: one can define a triangulated category of cyclotomic spectra which is enriched over StHom, and then for any profinitely complete cyclotomic spectrum T, we have a natural homotopy equivalence

$$\mathrm{TC}(T) \cong \mathrm{Maps}(\mathbb{S}, T),$$

where $\mathrm{Maps}(-, -)$ is the mapping spectrum in the cyclotomic category, and $\mathbb{S} = \Sigma^\infty \mathsf{pt}$ is the sphere spectrum with the trivial cyclotomic structure (obtained as in Example 7.3). This would give a conceptual replacement of the somewhat *ad hoc* definition of the functor TC.

7.4   Back to ring spectra. Return now to our original situation: we have a ring spectrum $\mathcal{A} \in \mathsf{StAlg}$, and the DG algebra $A_\bullet = H(W)(\mathcal{A})$ is obtained as its homology with coefficients in the Witt vector ring $W = W(k)$ of a finite field k. Assume for simplicity that $k = \mathbb{Z}/p\mathbb{Z}$ is a prime field, so that $W = \mathbb{Z}_p$.

Then on one hand, we have the cyclotomic spectrum $\mathrm{THH}(\mathcal{A})$ of [BM], and since the homology functor $H(\mathbb{Z}_p)$ commutes with tensor products, we have a quasiisomorphism

$$H(\mathbb{Z}_p)(\mathrm{THH}(\mathcal{A})) \cong CH_\bullet(A_\bullet).$$

But the left-hand side underlies a cyclotomic complex, and by Theorem 7.6, this is equivalent to saying that it has a structure of a generalized filtered Dieudonné module. And on the other hand, $CH_\bullet(A_\bullet)$ has a Dieudonné module structure induced by the splitting map (7.1). We expect that the two structures coincide (although at present, this has not been checked).

Moreover, the functor TC commutes with $H(\mathbb{Z}_p)$ by (7.3), and Theorem 7.7 shows that we have

$$H(\mathbb{Z}_p)(\mathrm{TC}(\mathcal{A})) \cong H(\mathbb{Z}_p)(\mathrm{TC}(\mathrm{THH}(\mathcal{A}))) \cong \mathrm{TC}(CH_\bullet(A_\bullet))$$
$$\cong \mathrm{RHom}^\bullet(\mathbb{Z}_p, CH_\bullet(A)),$$

where $\mathrm{RHom}^\bullet(-,-)$ is taken in the category $\mathcal{DFDM}^{per}(\mathbb{Z}_p)$ of filtered Dieudonné modules. In other words:

- The homology functor $H(\mathbb{Z}_p)$ sends topological cyclic homology into syntomic periodic cyclic homology.

This principle can be used to study further the regulator map for syntomic homology. Namely, applying $H(\mathbb{Z}_p)$ to the cyclotomic trace map (7.2), we obtain a functorial map

$$H(\mathbb{Z}_p)(K_\bullet(\mathcal{A})) \to H(\mathbb{Z}_p)(\mathrm{TC}(\mathcal{A})),$$

and the right-hand side is the target of the desired regulator map for the DG algebra $A_\bullet$. The desired source of this map is $K_\bullet(A_\bullet) \cong K_\bullet(H(\mathbb{Z}_p)(\mathcal{A}))$. Thus the question of existence of the syntomic regulator maps reduces to a problem in algebraic K-theory: describe the relation between the homology of the K-theory of a ring spectrum, and the K-theory of its homology.

To finish the Section, let us explain how things work in a very simple particular case. Assume given a CW complex X, and let $\mathcal{A} = \Sigma^\infty \Omega X$, the suspension spectrum of the based loop space ΩX. Then since ΩX is a topological monoid, $\mathcal{A}$ is a ring spectrum. The DG algebra $A_\bullet = H(\mathbb{Z}_p)(\mathcal{A})$ is given by $A_\bullet = C_\bullet(\Omega X, \mathbb{Z}_p)$, the singular chain complex of the topological space ΩX. It is known that in this case, we have

$$CH_\bullet(A_\bullet) \cong C_\bullet(LX, \mathbb{Z}_p),$$

the singular chain complex of the free loop space LX. Analogously, we have $\mathrm{THH}(\mathcal{A}) \cong \Sigma^\infty LX$. The S^1-action on $\mathrm{THH}(\mathcal{A})$ and $CH_\bullet(A_\bullet)$ is induced by the loop rotation action on LX. The cyclotomic structure on $\mathrm{THH}(\mathcal{A})$ is that of Example 7.3. The corresponding Dieudonné module structure map φ on $CP_\bullet(A_\bullet)$ is induced by the cyclotomic structure map $LX^{\mathbb{Z}/p\mathbb{Z}} \cong LX$ of the free loop space LX. To compare this with the constructions of Section 5, specialize even further and assume that ΩX is discrete, so that $A_\bullet$ is quasiisomorphic to an algebra A concentated in degree 0. In this case $X \cong BG$ for a discrete group G, and $A = \mathbb{Z}_p[G]$ is its group algebra. Then the diagonal map $G \to G^p$ induces a map $A \to A^{\otimes p}$ which is a quasi-Frobenius map in the sense of Section 5, thus induces another Dieudonné module structure on the filtered complex $CP_\bullet(A)$. One checks easily that the two structures coincide. For a general X, the Diedonné module structure on $CP_\bullet(A_\bullet)$ can also be described explicitly in the same way as in Section 5, by using the map

$$A_\bullet \to A_\bullet^{\otimes p}$$

induced by the diagonal map $\Omega X \to (\Omega X)^p$ in place of the quasi-Frobenius map.

8 Hodge structures.

In the archimedian setting of (i) of Section 1, much less is known about periodic cyclic homology than in the non-archimedian setting of (ii). One starts with a smooth proper DG algebra $A^\bullet$ over $\mathbb{C}$ and considers its periodic cyclic homology complex $CP_\bullet(A^\bullet)$ with its Hodge filtration. In order to equip $HP_\bullet(A^\bullet)$ with an $\mathbb{R}$-Hodge structure, one needs to define a weight filtration $W_\bullet CP_\bullet(A^\bullet)$ and a complex conjugation isomorphism $\overline{} : CP_\bullet(A^\bullet) \to \overline{CP_\bullet(A^\bullet)}$. The gradings in the isomorphism (2.2) suggest that $W_\bullet$ should be simply the canonical filtration of the complex $CP_\bullet(A^\bullet)$. However, the complex conjugation is a complete mystery. There is only one approach known at present, albeit a very indirect and highly conjectural one; the goal of this section is to describe it. I have learned all this material from B. Toën and/or M. Kontsevich – it is only the mistakes here that are mine.

The so-called $\mathcal{D}^-$-stacks introduced by B. Toën and G. Vezzosi in [ToVe] generalize both Artin stacks and DG schemes and form the subject of what is now known as "derived algebraic geometry"; a very nice overview is avaiable in [To1]. Very approximately, a $\mathcal{D}^-$-stack over a ring k is a functor

$$\mathcal{M} : \Delta^{opp} \, \mathsf{Comm}(k) \to \Delta^{opp} \, \mathsf{Sets}$$

from the category of simplicial commutative algebras over k to the category of simplicial sets. This functor should satisfy some descent-type conditions, and all such functors are considered up to an appropriately defined homotopy equivalence (made sense of by the technology of closed model structures). This generalizes the Grothendieck approach to schemes which treats a scheme over k as its functor of points – a sheaf of sets on the opposite $\mathsf{Comm}(k)^{opp}$ to the category of commutative algebras over k. The category $\mathsf{Comm}(k)$ is naturally embedded in $\Delta^{opp}\,\mathsf{Comm}(k)$ as the subcategory of constant simplicial objects, and resticting a $\mathcal{D}^-$-stack $\mathcal{M}$ to $\mathsf{Comm}(k) \subset \Delta^{opp}\,\mathsf{Comm}(k)$ gives an ∞-stack in the sense of Simpson [S] (this is called the *truncation* of $\mathcal{M}$).

If k contains $\mathbb{Q}$, one may replace simplicial commutative algebras with commutative DG algebras $R_\bullet$ over k placed in non-negative homological degrees, $R_i = 0$ for $i < 0$. If we denote the category of such DG algebras by $\mathsf{DG\text{-}Comm}^-(k)$, then a $\mathcal{D}^-$-stack is a functor

$$\mathcal{M} : \mathsf{DG\text{-}Comm}^-(k) \to \Delta^{opp}\,\mathrm{Sets},$$

again satisfying some conditions, and considered up to a homotopy equivalence. The category of $\mathcal{D}^-$-stacks over k is denoted $\mathcal{D}^-\mathsf{st}(k)$. For every DG algebra $R_\bullet \in \mathsf{DG\text{-}Comm}^-(k)$, its *derived spectrum* $\mathrm{R}\,\mathrm{Spec}(R_\bullet) \in \mathcal{D}^-\mathsf{st}(k)$ sends a DG algebra $R'_\bullet \in \mathsf{DG\text{-}Comm}^-(k)$ to the simplicial set of maps from $R_\bullet$ to $R'_\bullet$, with the simplicial structure induced by the model structure on the category $\mathsf{DG\text{-}Comm}^-(k)$. We thus obtain a Yoneda-type embedding

$$\mathrm{R}\,\mathrm{Spec} : \mathsf{DG\text{-}Comm}^-(k)^{opp} \to \mathcal{D}^-\mathsf{st}(k).$$

For any DG algebra $R_\bullet \in \mathsf{DG\text{-}Comm}^-(k)$, its de Rham cohomology comple $\Omega^\bullet(R_\bullet)$ is defined in the obvious way; $\Omega^\bullet(-)$ gives a functor

$$\Omega^\bullet : \mathsf{DG\text{-}Comm}^-(k) \to \mathrm{Spaces}_\mathbb{Q}$$

from $\mathsf{DG\text{-}Comm}^-(k)$ to the category $\mathrm{Spaces}_\mathbb{Q}$ of rational homotopy types in the sense of Quillen [Q]. By the standard Kan extension machinery, $\Omega^\bullet$ extentds to a de Rham realization functor

$$\Omega^\bullet : \mathcal{D}^-\mathsf{st}(k) \to \mathrm{Spaces}_\mathbb{Q}\,.$$

Alternatively, one can take the 0-th homology algebra $H_0(R_\bullet)$ and consider its cristalline cohomology; this gives a DG algebra quasiisomorphic to $\Omega^\bullet(R_\bullet)$ (the higher homology groups behave as nilpotent extensions and do not contribute to cohomology). This shows that the de Rham realization $\Omega^\bullet(\mathcal{M})$ of a $\mathcal{D}^-$-stack $\mathcal{M} \in \mathcal{D}^-\mathsf{st}(k)$ only depends on its truncation.

Moreover, for $\mathcal{D}^-$-stacks satifying a certain finiteness condition ("locally geometric" and "locally finitely presented" in the sense of [ToVa]), instead of considering de Rham cohomology, one can take the underlying topological spaces $\mathrm{Top}(\mathcal{M}(R))$ of the simplicial complex algebraic varieties $\mathcal{M}(R)$, $R \in \mathsf{Comm}(k)$; by Kan extension, this gives a topological realization functor

$$\mathrm{Top} : \mathcal{D}^-\mathsf{st}(k) \to \mathrm{Spaces}$$

into the category of topological spaces. By the standard comparison theorems, $\mathrm{Top}(\mathcal{M})$ and $\Omega^\bullet(\mathcal{M})$ represented the same rational homotopy type.

Now, it has been proved in [ToVa] that for any associative unital DG algebra $A^\bullet$ over k, there exists a $\mathcal{D}^-$-stack $\mathcal{M}(A^\bullet)$ classifying "finite-dimensional DG modules over $A^\bullet$". By definition, for any commutative DG algebra $R_\bullet \in \mathsf{DG\text{-}Comm}^-(k)$, the simplicial set $\mathcal{M}(A^\bullet)(R_\bullet)$ is given by

- $\mathcal{M}(A^\bullet)(R_\bullet)$ is the nerve of the category $\mathsf{Perf}(A^\bullet, R_\bullet)$ of DG modules over $A^\bullet \otimes R_\bullet$ which are perfect over $R_\bullet$, and quasiisomorphisms between such DG modules.

Toën and Vaquié prove that this indeed defines a $\mathcal{D}^-$-stack. Moreover, they prove that if $A^\bullet$ satisfies certain finiteness conditions, the $\mathcal{D}^-$-stack $\mathcal{M}(A^\bullet)$ is locally geometric and locally finitely presented.

In particular, a smooth and proper DG algebra $A^\bullet \in \mathsf{DG\text{-}Alg}(k)$ satisfies the finiteness conditions needed for [ToVa], so that there exists a locally geometric and locally finitely presented $\mathcal{D}^-$-stack $\mathcal{M}(A^\bullet)$. Consider its de Rham realization $\Omega^\bullet(\mathcal{M}(A^\bullet))$. For any $R_\bullet \in \mathsf{DG\text{-}Comm}^-(k)$, the category $\mathsf{Perf}(A^\bullet, R_\bullet)$ is a symmetric monoidal category with respect to the direct sum, so that the realization $\mathrm{Top}(\mathcal{M}(A^\bullet))$ is automatically an E_∞-space.

Lemma 8.1 (Toën). *The E_∞-space $\mathrm{Top}(\mathcal{M}(A^\bullet))$ is group-like.*

Sketch of a possible proof. One has to show that $\pi_0(\text{Top}(\mathcal{M}(A^\bullet)))$ is not only a commutative monoid but also an abelian group. A point in $\text{Top}(\mathcal{M}(A^\bullet))$ is represented by a DG module $M_\bullet$ over $A^\bullet$ which is perfect over k. One observes that $M_\bullet \oplus M_\bullet[1]$ can be deformed to a acyclic DG module; thus the sum of points represented by $M_\bullet$ and $M_\bullet[1]$ lies in connected component of 0 in $\text{Top}(\mathcal{M}(A^\bullet))$. $\qquad\square$

Thus for any smooth and proper DG algebra $A^\bullet \in \mathsf{DG\text{-}Alg}(k)$, the realization $\text{Top}(\mathcal{M}(A^\bullet))$ is an infinite loop space, that is, the 0-th component of a spectrum.

Definition 8.2. The *semi-topological K-theory* $K^{st}_\bullet(A^\bullet)$ of a smooth and proper DG algebra $A^\bullet$ is given by

$$K^{st}_\bullet(A^\bullet) = \pi_\bullet(\text{Top}(\mathcal{M}(A^\bullet))),$$

the homotopy groups of the infinite loop space $\text{Top}(\mathcal{M}(A^\bullet))$.

If we are only interested in $K^{st}_\bullet(A^\bullet) \otimes k$, we may compute it using the de Rham model $\Omega^\bullet(\mathcal{M}(A^\bullet))$. Then $K^{st}_\bullet(\mathcal{M}(A^\bullet))$ is exactly the complex of primitive elements with respect to the natural cocommutative coalgebra structure on $\mathcal{M}(A^\bullet)$ induces by the direct sum map

$$\mathcal{M}(A^\bullet) \times \mathcal{M}(A^\bullet) \to \mathcal{M}(A^\bullet).$$

Since $\mathbb{Q} \subset k$, and rationally, spectra are the same as complexes of $\mathbb{Q}$-vector spaces, the groups $K^{st}_\bullet(A^\bullet) \otimes k$ are the only rational invariants one can extract from the space $\mathcal{M}(A^\bullet)$.

Assume for the moment that $A^\bullet \in \mathsf{DG\text{-}Alg}(k)$ is derived-Morita equivalent to a smooth and proper algebraic variety X/k. Then one can also consider the ∞-stack $\overline{\mathcal{M}}(X)$ of all coherent sheaves on X; for any noetherian $R \in \mathsf{Comm}(k)$, $\overline{\mathcal{M}}(X)(R)$ is by definition the nerve of the category of coherent sheaves on $M \otimes R$ and isomorphisms between them. The realization $\text{Top}(\overline{\mathcal{M}}(X))$ is again an E_∞-space, no longer group-like. By definition, we have a natural map

$$\overline{\mathcal{M}}(X) \to \mathcal{M}(A^\bullet),$$

and the induced E_∞-map of realizations.

Lemma 8.3 (Toën). *The natural E_∞-map*

$$(8.1) \qquad\qquad \text{Top}(\overline{\mathcal{M}}(X)) \to \text{Top}(\mathcal{M}(A^\bullet))$$

induces a homotopy equivalence between $\text{Top}(\mathcal{M}(A^\bullet))$ and the group completion of the E_∞-space $\text{Top}(\overline{\mathcal{M}}(X))$.

Sketch of a possible proof. Since $\text{Top}(\mathcal{M}(A^\bullet))$ is group-like by Lemma 8.1, it suffices to prove that the delooping

$$B\,\text{Top}(\overline{\mathcal{M}}(X)) \to B\,\text{Top}(\mathcal{M}(A^\bullet))$$

of the E_∞-map (8.1) is a homotopy equivalence. Delooping obviously commutes with geometric realization, so that $B\,\text{Top}(\mathcal{M}(A^\bullet))$ is the realization of the $\mathcal{D}^-$-stack $B\,\mathcal{M}(A^\bullet)$, and similarly for $B\,\text{Top}(\overline{\mathcal{M}}(X))$. Instead of taking deloopings, we can apply Waldhausen's S-construction. The resulting map

$$S\overline{\mathcal{M}}(X) \to S\,\mathcal{M}(A^\bullet)$$

is then an equivalence by Waldhausen's devissage theorem, so that it suffices to prove that the natural map

$$\text{Top}(B\,\mathcal{M}(A^\bullet)) \to \text{Top}(S\,\mathcal{M}(A^\bullet))$$

is a homotopy equivalence, and similarly for $\overline{\mathcal{M}}(X)$. For this, one argues as in Lemma 8.1: since every filtered complex can be canonically deformed to its associated graded quotient, the terms $\text{Top}(S_n\,\mathcal{M}(A^\bullet))$ of the S-construction can be retracted to n-fold products $\text{Top}(\mathcal{M}(A^\bullet) \times \cdots \times \mathcal{M}(A^\bullet))$, that is, the terms of the delooping $\text{Top}(B\,\mathcal{M}(A^\bullet))$, and similarly for $\overline{\mathcal{M}}(X)$. $\qquad\square$

Corollary 8.4. *The semitopological K-theory $K^{st}_\bullet(\mathbb{Q})$ is given by*

$$K^{st}_\bullet(k) \cong \mathbb{Z}[\beta],$$

the algebra of polynomials in one generator β of degree 2.

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Proof. By Lemma 8.3, computing $K^{st}_\bullet(k)$ reduces to studying the group completion of the realization

$$\mathrm{Top}(\overline{\mathcal{M}}(\mathsf{pt})) \cong \coprod_n \mathrm{Top}([\mathsf{pt}/GL_n]) \cong \coprod_n BU_n,$$

where $[\mathsf{pt}/GL_n]$ is the Artin stack obtained as the quotient of the point by the trivial action of the algebraic group GL_n. This group completion is well-known to be homotopy equivalent to the classifying space $\mathbb{Z} \times BU$. $\square$

Remark 8.5. At present, Lemma 8.1 and Lemma 8.3 are unpublished, as well as Corollary 8.4. The above sketches of proofs have been kindly explained to me by B. Toën. Lemma 8.1 is slightly older, and it also appears for example in [To3].

Now, since $k \supset \mathbb{Q}$ by our assumption, we have a well-defined tensor product $M_\bullet \otimes V_\bullet$ for any DG module $M_\bullet$ over $A^\bullet$ and every complex $V_\bullet$ of $\mathbb{Q}$-vector spaces. On the level of the stacks $\mathcal{M}(-)$, this tensor product turns $K^{st}_\bullet(A^\bullet)$ into a module over $K^{st}_\bullet(\mathbb{Q}) = \mathbb{Z}[\beta]$. We can now state the main conjecture.

Conjecture 8.6. *Assume that k is a ring conataining $\mathbb{Q}$, and assume that a DG algebra $A^\bullet$ DG-Alg(k) is smooth and proper. Then there exists a map*
$$c : K^{st}_\bullet(A^\bullet) \to HP_\bullet(A^\bullet)$$
such that $c(\beta(\alpha)) = u(c(\alpha))$ for any $\alpha \in K^{st}_\bullet(A^\bullet)$, where u is the periodicity map. The map c is functorial in $A^\bullet$. Moreover, the induced map

(8.2) $$K^{st}_\bullet(A^\bullet) \otimes_{\mathbb{Z}[\beta]} k[\beta, \beta^{-1}] \to HP_\bullet(A^\bullet)$$

is an isomorphism.

The reason this conjecture is relevant to the present paper is that the tensor product $K^{st}_\bullet(A^\bullet) \otimes k$ by its very definition has all the structures possessed by the de Rham cohomology of an algebraic variety. In particular, if $k = \mathbb{C}$, $K^{st}_\bullet(A^\bullet)$ has a canonical real structure.

Conjecture 8.7. *Assume that $K = \mathbb{C}$, and assume given a smooth and proper DG algebra $A^\bullet/K$ for which Conjecture 8.6 holds. Equip $CP_\bullet(A^\bullet)$ with the real structure induced from the canonical real structure on $K^{st}_\bullet(A^\bullet) \otimes K$ by the isomorphism 8.2. Then for any integer i, the periodic cyclic homology group $HP_\bullet(A^\bullet)$ this real structure and the standard Hodge filtration $F^\bullet$ is a pure $\mathbb{R}$-Hodge structure of weight i.*

The two conjectures above are a slight refinement and/or reformulation of a conjecture made by B. Toën [To3] with a reference to A. Bondal and A. Neeman, and described by L. Katzarkov, M. Kontsevich and T. Pantev in [KKP, 2.2.6].

Apart from the basic case $A^\bullet = k$ of Corollary 8.4, the only real evidence for Conjecture 8.6 comes from recent work of Fiedlander and Walker [FW], where it has been essentially proved for a DG algebra $A^\bullet$ equivalent to a smooth projective algebraic variety X/k. The definition of semi-topological K-theory used in [FW] is different from Definition 8.2, but it is very close to the homotopy groups of the group completion of the E_∞-space $\mathrm{Top}(\overline{\mathcal{M}}(X))$; Lemma 8.3 should then show that the two things are the same. Friedlander and Walker also show that their constructions are compatible with the complex conjugation, so that Conjecture 8.7 then follows by the usual Hodge theory applied to X.

In the general case, as far as I know, both Conjecture 8.6 and Conjecture 8.7 are completely open. They are now a subject of investigation by B. Toën and A. Blanc.

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