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鎌田, 正良
九州大学教養部数学教室

Tyan, Shian-Wu
Department of Mathematics, College of General Education, Kyushu University

https://doi.org/10.15017/1449045

出版情報：九州大学教養部数学雑誌. 15 (1), pp.1-11, 1985-12. College of General Education, Kyushu University
バージョン：
権利関係：

Math. Rep.
XV-1, 1985.

# On fixed point sets of differentiable involutions

Masayoshi KAMATA and Shian-Wu TYAN

(Received September 17, 1985)

## 1. Introduction

Let $(M^n, T)$ be a closed smooth manifold dimension $n$ with a differentiable involution, and let $F^m$ be the union of the $m$-dimensional components of the fixed point with the normal bundle $\nu_m$. The aim of this paper is to give some relations among the Whitney numbers of $\{[F^m, \nu_m]\}$, which are used to investigate involutions fixing a projective space and a point. Let $S^{<k>}(\xi)$ be the characteristic class of an $n$-dimensional vector bundle $\xi$ which is given by replacing the $i$-th elementary symmetric polynomial of $\alpha_i$ by the $i$-th Whitney class $W_i(\xi)$ in the symmetric polynomial

$$\sum_{i_1+i_2+\cdots+i_n=k} \alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_n^{i_n}$$

The main theorem will be:

THEOREM 1.1. *Let* $[F^m] \varepsilon H_m(F^m; Z_2)$ *be the fundamental class of* $F^m$. *Then*

(1) $\sum_{0\leq m\leq n} \langle S^{<m>}(\nu_m), [F^m]\rangle=0$

(2) $\sum_{0\leq m\leq n} \langle m S^{<m>}(\nu_m)+\sum_{1\leq j\leq\beta}\binom{\beta}{i} S^{<m-j>}(\nu_m)S_j(\nu_m)+S^{<m-\beta>}(\nu_m)S_\beta(\tau(F_m)), [F_m]\rangle=0$

*where* $S_j(\ )$ *indicates the characteristic class corresponding to the symmetric polynomial* $\sum \alpha_i^j$, and $\beta\leq n-1$.

The paper is organized as follows. In § 2 we get some relations among the characteristic classes which will be used to calculate the Wu classes of the real projective space bundle $P(\xi)$ associated with a smooth vector bundle $\xi$ in §3, and to calculate the Gysin homomorphism of the classifying map $f: P(\xi)\to RP^N$ for the canonical line bundle over $P(\xi)$ in § 4. Making use of the results obtained in § 2, the Boardman homomorphism and the Quillen theorem instead of the results of Kosniowski and Stong [8], we prove Theorem 1.1 in § 4. In § 5 we show that an involution fixing a $(2m+1)$-dimensional real projective space $RP^{2m+1}$ and a point is bordant to the $Z_2$-manifold $(RP^{2m+2}, T)$ with

$$T[x_0, x_1,\ldots, x_{2m+2}]=[-x_0, x_1,\ldots, x_{2m+2}]$$

and we investigate the dimension of a $Z_2$-manifold fixing $RP^{2m}$ and a point.

## 2. The structure of the cohomology ring of a projective space bundle

Let $\xi$ be a differentiable vector bundle of dimension $k+1$ over an $n$-dimensional smooth manifold $M^n$. Denote by $P(\xi)$ the projective space bundle. The cohomology ring $H^*(P(\xi); Z_2)$ is a $H^*(M^n; Z_2)$-module with

$$ax=\pi^*(a)\cup x,$$

where $\pi: P(\xi)\to M^n$ is the projection. Let $\eta_\xi$ be the canonical line bundle over $P(\xi)$ and let $c$ be the first Whitney class of $\eta_\xi$. The Leray-Hirsch theorem implies that $H^*(P(\xi)\ ; Z_2)$ is a free $H^*(M^n\ ; Z_2)$-module with a basis $1, c, c^2,\ldots, c^k$, and

$$c^{k+1}=W_1(\xi)c^k+W_2(\xi)c^{k-1}+\cdots+W_k(\xi)c+W_{k+1}(\xi) \tag{2.1}$$

where $W_i(\xi)$ denotes the $i$-th Whitney class of $\xi$.

Applying the splitting principle to a $(k+1)$-dimensional vector bundle $\xi$, we formally write

$$W(\xi)=\prod_{j=1}^{k+1}(1+\alpha_j)$$

Denote by $S^{<l>}(\xi)$ the characteristic class corresponding to the symmetric polynomial

$$\sum_{i_1+i_2+\cdots+i_{k+1}=l}\alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_{k+1}^{i_{k+1}}$$

Then we have the following

**Proposition** 2.2. *Let*

(2.3) $c^{k+t}=f_{t,1}c^k+f_{t,2}c^{k-1}+\dots.+f_{t,k}c+f_{t,k+1}.$

Then

(1) $f_{1,j}=W_j(\xi)$

(2) $f_{t,1}=S^{<t>}(\xi)$

(4) $f_{t,j}=S^{<t-1>}(\xi)W_j(\xi)+S^{<t-2>}(\xi)W_{j+1}(\xi)+\cdots+S^{<t+j-k-2>}(\xi)W_{k+1}(\xi)$

**Proof.** (1) is an immediate result of (2.1). It follows from (2.1) and (2.3) that

$$
(2.4)\quad \begin{cases} f_{t+1,1} = f_{t,1}W_1(\xi)+f_{t,2} \\ f_{t+1,2} = f_{t,1}W_2(\xi)+f_{t,3} \\ \quad\cdots\cdots\cdots \\ f_{t+1,k} = f_{t,1}W_k(\xi)+f_{t,k+1} \\ f_{t+1,k+1} = f_{t,1}W_{k+1}(\xi). \end{cases}
$$

Let $f_j=f_{j,1}$ for brevity. Then we have

(2.5) $f_{t+1}=f_t W_1(\xi)+f_{t-1}W_2(\xi)+\cdots+f_{t-k+1}W_k(\xi)+f_{t-k}W_{k+1}(\xi)$

We comprehend $W_j(\xi)$ to be the $j$-th elementary symmetric polynomial $\mathfrak{S}_j(\alpha)$ of $\alpha_j$, and so we have

(2.6) $f_{t+1}=f_t\mathfrak{S}_1(\alpha)+f_{t-1}\mathfrak{S}_2(\alpha)+\cdots+f_{t-k+1}\mathfrak{S}_k(\alpha)+f_{t-k}\mathfrak{S}_{k+1}(\alpha)$

Let $f_j^{(i)}=f_j^{(i-1)}+\alpha_i f_{j-1}^{(i-1)}$ where $f=f_j^{(1)}+\alpha_1 f_{j-1}$. Proceeding inductively to substitute $f_j^{(i)}$ for $f_j^{(i-1)}$ in (2.6), we finally obtain

$$f_{j+1}^{(k)}=f_j^{(k)}\alpha_{k+1}$$

and

$$f_{j+1}^{(k)}=\alpha_{k+1}^{j+1}.$$

We now suppose as the inductive hypothesis that

$$f_j^{(1)}=\sum_{i_2+\cdots+i_{k+1}=j}\alpha_2^{i_2}\cdots\alpha_{k+1}^{i_{k+1}}$$

Since $f_j^{(1)}=f_j+\alpha_1 f_{j-1}$, it follows that

$$f_j=f_j^{(1)}+\alpha_1 f_{j-1}^{(1)}+\alpha_1^2 f_{j-2}^{(1)}+\cdots+\alpha_1^{j-2}f_2^{(1)}+\alpha_1^{j-1}f_1^{(1)}+\alpha_1^j$$

and we have $f_j=S^{<j>}(\xi)$. (3) is an immediate result of (2) and (2.4).

## 3. On the Wu classes of the projective space bundle associated with a real vector bundle.

According to Adams [1], the $KO$-group $\widetilde{KO}(RP^n)$ of the $n$-dimensional projective space $RP^n$ is the cyclic group $Z/2^{f_n}$ with a generator $\eta_n - 1$, where $\eta_n$ is the canonical line bundle over $RP^n$ and $f_n$ is a number of a set $\{s \mid 0 < s \leqq n,\ s \equiv 1, 2, 4 \text{ modulo } 8\}$. Let $\xi$ be a real vector bundle of dimension $k+1$ over $RP^n$ with $\xi \sim_s u\,\eta_n$. It means that there exist trivial bundles $\theta$ and $\theta'$ such that $\xi \oplus \theta = u\,\eta_n \oplus \theta'$. Applying the splitting principle, we have

$$S^{<t>}(\xi) = \sum_{i_1+i_2+\cdots+i_{k+1}=t} \alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_{k+1}^{i_{k+1}}$$

(3.1)

$$= \sum_{j_1+j_2+\cdots+j_u=t} x^{j_1+j_2+\cdots+j_u} = \binom{-u}{t} x^t$$

where $x = W_1(\eta_n)$ is a generator of $H^1(RP^n; Z_2)$. The $i$-th Whitney class $W_i(\xi)$ of $\xi$ is $\binom{u}{i} x^i$. Using Propositon 2.2, we obtain

**Proposition 3.2.**

$$c^{k+t} = \binom{-u}{t} c^k x^t + \sum_{s=1}^{k} \binom{-u}{t-s}\binom{u}{s+1} c^{k-1} x^{t+1} + \sum_{s=1}^{k+1} \binom{-u}{t-2}\binom{u}{s+2} c^{k-2} x^{t+2}$$

$$+ \sum_{1 \leq s \leq k-j+2} \binom{-u}{t-s}\binom{u}{j+s-1} c^{k-j+1} x^{t+j-1} + \cdots + \binom{-u}{t-1}\binom{u}{k+1} x^{t+k}$$

*where $c$ is the first Whitney class of the canonical line bundle $\eta_\xi$ of the projective space bundle associated with $\xi$.*

The tangent bundle of the projective space bundle $P(\xi)$ associated with a vector bundle $\xi$ over $RP^n$ is stably equivalent to

$$\eta_\xi \otimes \pi^! \xi \oplus (n+1)\eta_n$$

where $\pi\colon P(\xi) \to RP^n$ is the projection. The Wu class $v_t(M)$ of a manifold $M$ of dimension $m$ is defined by $\langle Sq^t x, [M] \rangle = \langle x v_t(M), [M] \rangle$, where $[M]$ indicates the fundamental class of $M$. We use Proposition 3.2 to have the following (cf. [5]).

**Proposition 3.3.** *Let*

$$v_t(P(\xi)) = \lambda_0 x^t + \lambda_1 x^{t-1} c + \lambda_2 x^{t-2} c^2 + \cdots + \lambda_k x^{t-k} c^k.$$

*If* $\xi \sim_s u\eta_n$ *then*

$$\lambda_j = \sum_{a+b=t} \binom{n-t+j}{a}\binom{k-j}{b}\binom{-u}{b-j}.$$

## 4. A proof of Theorem 1.1.

Let $\xi \to F$ be a smooth real vector bundle of dimension $k+1$ over an $m$-dimensional closed smooth manifold $F$, and let $f\colon P(\xi) \to RP^N$ be the classifying map for the canonical line bundle $\eta_\xi$. We now investigate the Gysin homomorphism

$$f_!\colon H^t(P(\xi);Z_2) \overset{D}{\cong} H_{m+k-t}(P(\xi);Z_2) \overset{f_*}{\to} H_{m+k-t}(RP^N;Z_2) \overset{D^{-1}}{\cong} H^{N-m-k+t}(RP^N;Z_2)$$

where $D$ is the Poincare duality isomorphism.

PROPOSITION 4.1. *Let* $a \in H^p(F;Z_2)$ *and* $c = W_1(\eta_\xi)$. *Then*

$$f_!(ac^q) = \langle S^{<m-p>}(\xi)a, [F]\rangle \bar{x}^{N-m-k+p+q}$$

*where* $\bar{x}$ *indicates the generator of* $H^1(RP^N;Z_2)$.

PROOF. Take the dual class $(\bar{x}^{N-m-k+p+q})_* \in H_{N-k-m+p+q}(RP^N)$ which equals to $\bar{x}^{m+k-p-q} \cap [RP^N]$. Let $f_!(ac^q) = \lambda\bar{x}^{N-m-k+p+q}$. Since $f^*(\bar{x}) = c$, Proposition 2.2 implies that

$$\begin{aligned}\lambda &= \langle \bar{x}^{m+k-p-q}, f_*(ac^q \cap [P(\xi)])\rangle \\ &= \langle ac^{m+k-p}, [P(\xi)]\rangle \\ &= \langle aS^{<m-p>}(\xi)\ c^k, [P(\xi)]\rangle \\ &= \langle aS^{<m-p>}(\xi), [F]\rangle\end{aligned}$$

Q. E. D.

We now recall the Boardman map (*cf.* [6])

$$\beta\colon \mathfrak{N}^*(X) \to H^*(X;Z_2)[[t_1, t_2, \ldots.]]$$

which is a multiplicative natural transformation satisfying

(1) for the cobordism first characteristic class $W_1^N(\eta)$ of a line bundle $\eta$

$$\beta(W_1(\eta)) = W_1(\eta) + \{W_1(\eta)\}^2 t_1 + \cdots + \{W_1(\eta)\}^{i+1} t_i + \cdots$$

(2) for finite CW complex $X$, $\beta$ is injective.

We next recall the Conner-Floyd characteristic class (cf. [2], [6])

$$c_t\colon Vect(X) \to H^*(X;Z_2)[[t_1, t_2, \ldots.]]$$

which is a map assiging a real vector bundle over $X$ to a formal power series of $t_i$ with the coefficient in $H^*(X; Z_2)$ such that

(1) $c_t(f_! \xi) = f^* c_t(\xi)$

(2) $c_t(\xi \oplus \eta) = c_t(\xi) c_t(\eta)$

(3) for a line bundle $\eta$

$c_t(\eta) = 1 + W_1(\eta) t_1 + \{W_1(\eta)\}^2 t_2 + \cdots$

Denoting by $D_N$: $\mathfrak{N}^*(N) \to \mathfrak{N}_*(N)$ the Atiyah Thom Poincare duality isomorphism. Then we have the Umkehrung homomorphism $f_!$ for a map $f$: $M \to N$ between closed manifolds $M$ and $N$:

$$f_!: \mathfrak{N}^*(M) \xrightarrow{D_M} \mathfrak{N}_*(M) \xrightarrow{f_*} \mathfrak{N}_*(N) \xrightarrow{D_N^{-1}} \mathfrak{N}^*(N)$$

which satisfies $D_N f_!(1) = [M \to N] \in \mathfrak{N}_*(N)$. For the bordism group $\mathfrak{N}_n^{Z_2}$ of $Z_2$-manifolds of dimension $n$, the bordism group $\sum_{s+t=n} \mathfrak{N}_s(BO(t))$ of vector bundles and the bordism group $\mathfrak{N}_{n-1}(BO(1))$ of free $Z_2$-maifolds, there exists an exact sequence (cf. [7])

$$(4.2)\quad 0 \to \mathfrak{N}_n^{Z_2} \xrightarrow{j_*} \sum_{s+t=n} \mathfrak{N}_s(BO(t)) \xrightarrow{\partial} \mathfrak{N}_{n-1}(BO(1)) \to 0$$

where $j^*[M, T] = \sum_i [F_i, \nu_i]$, $F_i$ the fixed point component of $(M, T)$ and $\nu_i$ the normal bundle of $F_i$, and $\partial[M, \xi] = [P(\xi) \xrightarrow{f} RP^N \subset BO(1)]$, $f$ the classifying map for the canonical line bundle. We now remark that

$$(4.3)\quad \beta(f_!(1)) = f_! c_t(\nu_f),$$

where $\nu_f$ is the virtual normal bundle of $f$: $M \to N$ (cf. [9]).

A Proof Of Theorem 1.1. The exact sequence (4.2) implies that $\sum \partial[F^m, \nu_m] = 0$. Let $f_m$: $P(\nu_m) \to RP^N$ be the classifying map for the canonical line bundle $\eta_\xi$. (4.3) implies $\sum f_{m!}(c_t(\nu_{f_m})) = 0$. We now compute

$$f_{m!} c_t(\nu_{f_m}) = f_{m!}\left\{\frac{f_m^*\{c_t(\eta_N)\}^{N+1}}{c_t(\eta_{\nu_m} \otimes \pi^! \nu_m) c_t(\pi^! \tau(F^m))}\right\}$$

$$= c_t(\eta_N)^{N+1} f_{m!}\left\{\frac{1}{c_t(\eta_{\nu_m} \otimes \pi^! \eta_m) c_t(\pi^! \tau(F^m))}\right\}$$

Denote by $\mathfrak{D}$ an ideal genereated by $\{t_1, t_2, \ldots, t_{\beta-1}, t_\beta^2, t_{\beta+1}, \ldots\}$. By virtue of the splitting principle, we have

$$c_t(\eta_{\nu_m} \otimes \pi^! \nu_m) = 1 + \{(n-m) c^\beta + \sum_{1 \le j \le m} \binom{\beta}{j} S_j(\pi^! \nu_m)\ c^{\beta-j}\}\ t_\beta \mod \mathfrak{D}$$

and

$$c_t(\pi^! \tau(F)) = 1 + s_\beta(\pi^! \tau(F)) t_\beta \bmod \mathfrak{D}.$$

Since $c_t(\eta_N)$ is an invertible element, by making use of Proposition 4.1 we complete the proof.

## 5. Involutions fixing real projective spaces

Suppose that $RP^m$ is embedding in $M^n$ with the normal bundle $\nu$ which is stably equivalent to $u\eta_m$. Then

LEMMA 5.1.

(1) $S_j(\nu) = ux^j$ in $H^*(RP^m; Z_2)$

(2) $S_j(\tau(P(\nu))) = u(c+x)^j + (n+m+u)c^j + (m+1)x^j$ in $H^*(P(\nu); Z_2)$ *where $x$ is the generator of* $H^1(RP^m; Z_2)$, and $c = W_1(\eta_\nu)$.

PROOF. $\xi \sim_s \xi'$ implies that $S_j(\xi) = S_j(\xi')$ and (1) follows. $\tau(P(\nu)) \sim_s \eta \otimes \pi^! \nu \oplus (m+1)\, \pi^! \eta_m$, and

$$S_j(\tau(P(\nu))) = S_j(\eta_\nu \otimes \pi^! \nu) + (m+1)x^j.$$

Let $\nu \oplus \theta = u\, \eta_m \oplus \theta'$, with trivial bundles $\theta$ and $\theta'$. Then

$$S_j(\eta_\nu \otimes \pi^! \nu) + \dim S_j(\eta_\nu) = u\, S_j(\eta_\nu \otimes \pi^! \eta_m) + \dim \theta'\, S_j(\eta_\nu)$$

and $\quad S_j\, (\eta_\nu \otimes \pi^! \nu) = u(c+x)^j + (n+m+u)\, c^j. \qquad$ Q. E. D.

Let $(M, T)$ be a closed $Z_2$-manifold of dimension $n$ fixing the disjoint union $\sum_{1 \leq i \leq s} RP^{m_i}$ of real projective spaces, and let $\nu_i$ be the normal bundle of $RP^{m_i}$ which is stably equivalent to $u_i\, \eta_{m_i}$, where $u_i$ is a non-negative integer. Then it follows from Theorem 1.1 and Lemma 5.1 that

PROPOSITION 5.2.

(1) $\sum_{1 \leq i \leq s} \binom{-u_i}{m_i} = 0 \bmod 2$

(2) $\sum_{1 \leq i \leq s} \{m_i \binom{-u_i}{m_i} + \sum_{1 \leq j \leq \beta} \binom{\beta}{j} \binom{-u_i}{m_i - j} u_i + \binom{-u_i}{m_i - \beta} (m_i + 1)\} = 0 \bmod 2$

(3) *if* $\beta = 2^t$ *then*

$$\sum_{1 \leq i \leq s} \{m_i \binom{-u_i}{m_i} + (u_i + m_i + 1) \binom{-u_i}{m_i - \beta}\} = 0 \bmod 2$$

We then have the following

COROLLARY 5.3. *Suppose that a close $Z_2$-manifold $(M, T)$ fixing $RP^m+$ {a point} has the normal bundle $\nu$ of $RP^m$ with $\nu\sim_s u\eta_m$. Then*

(1) *if $m$ is odd, then $u$ is odd.*
(2) *if $m=2^t$, then $u$ is odd.*

PROOF. (1) is the immediate result of Proposition 5.2 (1). Applying Proposition 5.2 (3) to $\beta=2^t$, we have

$$m\binom{-u}{m}+\binom{-u}{0}(u+m+1)\equiv 0 \mod 2.$$

and $u+1\equiv 0$. Q. E. D.

We also obtain

COROLLARY 5.4. *There is no involution fixing $2k$ copies of $RP^m$ and a point such that the normal bundles of the projective spaces are stably equivalent each other.*

THEOREM 5.5. *A $Z_2$-manifold fixing a projective space $RP^{2n+1}$ of dimension $2n+1$ and a point is bordant to a $Z_2$-manifold $RP^{2n+2}$ with the action $\tilde{T}$ $[x_0, x_1, \ldots, x_{2+2}]=[-x_0, x_1, \ldots, x_{2n+2}]$.*

PROOF. Suppose that $(M, T)$ is a $Z_2$-manifold whose fixed point set is $RP^{2n+1}+$ {*a point*} and the normal bundle $\nu$ of $RP^{2n+1}$ is stably equivalent to $u\eta_{2n+1}$, where $\nu$ is of dimension $k+1$. By Corollary 5.3 (1) $u$ is odd. Conner and Floyd proved that Euler characteristic numbers modulo 2 of $M$ and the fixed point set coincide. We use this fact to prove that $\chi(M)=1$, where $\chi(\ )$ denotes the Euler characteristic modulo 2. Suppose that $k$ is odd, then the dimension of $M$ is odd and $\chi(M)=0$. This is a contradiction. Therefore $k$ is even. Generalized Whitney numbers $\langle W_\omega(N)g^*(y), [N]\rangle$ for a singular manifold $(N\rightarrow Y)$ determines the bordism class $[N\rightarrow Y]$ in $\mathfrak{N}_*(Y)$ (cf. [7]). Since $k$ is even and $u$ is odd, the first Whitney class of $P(\nu)$ is $c+x$. Let us compare the following characteristic numbers of $[P(\nu)\xrightarrow{f} RP^N]$ and $[RP^{2n+1}\overset{i}{\subset} RP^N]$, where $f$ is the classifying map for the line bundle $\eta_\nu$:

$$\langle\{W_1(P)-f^*(\bar{x})\}^{2n+k+1}, [P(\nu)]\rangle=0 \quad \text{if } k>0$$

and

$$\langle\{W_1(RP^{n+k})-i^*(\bar{x})\}^{2n+k+1}, [RP^{2n+k+1}]\rangle=1.$$

Therefore $k=0$ and $\nu$ is equivalent to $\eta_{2n+1}$. Then we have

$$j^*([M, T]) = [RP^{2n+1}, \eta_{2n+1}] + [\{a\ point\}, \theta],$$

where $\theta$ is the trivial bundle. Let $(RP^{2n+2}, \tilde{T})$ be a $Z_2$-manifold with a $Z_2$-action

$$\tilde{T}[x_0, x_1, \dots, x_{2n+2}] = [-x_0, x_1, \dots, x_{2n+2}]$$

Then

$$j^*([RP^{2n+2}, \tilde{T}] = j^*([M, T])$$

and the exact sequence (4.2) deduces that $(M, T)$ is bordant to $(RP^{2n+2}, \tilde{T})$.

Q. E. D.

THEOREM 5.6. *Suppose that an involution $(M^n, T)$ fixes $RP^{2m} + \{a\ point\}$, $m>0$. Let $\nu$ be the normal bundle of $RP^{2m}$ with $\nu \sim_s u\eta_{2m}$. Then $u$ is odd and $n \leqq 4m+1$.*

PROOF. We firstly assume that $u$ is even. Then Lemma 5.1 implies that $S_j(\tau(P(\nu)) = n c^j + x^j$. We compare generalized Whitney numbers of $[P(\nu) \xrightarrow{f} RP^N]$, $f$ the classifying map of $\eta_\nu$, and $[RP^{n-1} \overset{i}{\subset} RP^N]$. If $n>2m$, then

$$\langle c^{n-2m-1}\{S_{2m}(\tau(P(\nu))) - n f^*(\tilde{x}^{2m})\}, [P(\nu)]\rangle = 1$$

and

$$\langle x^{n-2m-1}\{S_{2m}(\tau(RP^{n-1})) - n i^*(\tilde{x}^{2m})\}, [RP^{n-1}]\rangle = 0$$

where $x = W_1(\eta_{n-1})$, $\tilde{x} = W_1(\eta_N)$ and $c = W_1(\eta_\nu)$. This is a contradiction. Then if $u$ is even, dim $M = 2m$. This means that some component of $M$ with the involution has a fixed point set consisting of a point. Since there is no involution fixing a point except the involution on a zero dimensional manifold, if $m$ is positive, then $u$ is odd. We next assume that $u$ is odd. Then Lemma 5.1 implies that

$$S_j(\tau(P(\nu)) = (c+x)^j + (n+1)c^j + x^j.$$

If $n > 4m+1$, then

$$\begin{aligned} A = c^{n-4m-2}\{&S_{4m+1}(\tau(P(\nu))) + (n+1)f^*(\tilde{x})^{4m+1} + [S_{2m+1}(\tau(P(\nu))) \\ &+ (n+1)f^*(\tilde{x})^{2m+1}][S_{2m}(\tau(P(\nu))) + (n+1)f^*(\tilde{x})^{2m}]\} \\ = c^{n-4m-2}&(c+x)^{2m+1} x^{2m} = c^{n-2m-1} x^{2m} \end{aligned}$$

and

$$B=x^{n-4m-2}\{S_{4m+1}(\tau(RP^{n-1}))+(n+1)i^*(\bar{x})^{4m+1}+[S_{2m+1}(\tau(RP^{n-1}))$$
$$+(n+1)i^*(\bar{x})^{2m+1}][S_{2m}(\tau(RP^{n-1}))+(n+1)i^*(\bar{x})^{2m}]\}=0$$

Therefore $\langle A,[P(\nu)]\rangle=1$ and $\langle B,[RP^{n-1}]\rangle=0$. This is a contradiction. Then the theorem follows.

The homogenuous space $SU(n)/SO(n)$ is diffeomorphic to a manifold $X_n=\{P \varepsilon SU(n) \mid {}^tP=P\}$. Let $Z_2$ act on $X_n$ by $P\to P^{-1}$. Denote by $F(X_n, Z_2)$ the fixed point set. Then we have

PROPOSITION 5.7. *$F(X_n, Z_2)$ is a disjoint union of the Grassmann manifolds* $\{G_{2k}(R^n)\,;\ k=0,1,2,\dots.\}$.

PROOF. Each element $P$ of $F(X_n, Z_2)$ belongs to $SO(n)$ and ${}^tP=P$. Let $F_{2k}(X_n, Z_2)$ consist of elements of $F(X_n, Z_2)$ whose trace is $n-4k$. Letting each $P$ of $F_{2k}(X_n, Z_2)$ correspond to the subspace $\{x|Px=-x\}$ in $R^n$, we see that $F_{2k}(X_n, Z_2)$ is diffeomorphic to the Grassmann manifold $G_{2k}(R^t)$.

Q. E. D.

Hence we obtain the 3-dimensional projective space $RP^3$ with the involution $[x_0, x_1, x_2, x_3]\to[-x_0, x_1, x_2, x_3]$ whose fixed point set is $RP^2+\{a\ point\}$ and the 5-dimensional manifold $X_3$ with the involution $P\to P^{-1}$ whose fixed point set is $RP^2+\{a\ point\}$.

## References

[1] F. ADAMS: *Vector fields on spheres,* Ann. Math. **75** (1962), 603-632.
[2] F. ADAMS: Stable Homotopy and Generalized Homology, Univ. of Chicago Press, Chicago (1974).
[3] J. M. BOARDMAN: *On manifolds with involution,* Bull. Amer. Math. Soc. **73** (1967), 136-138.
[4] G. E. BREDON: Introduction to Compact Transformation groups, Academic Press, (1972).
[5] E. H. BROWN and F. PETERSON: *Relations among characteristic classes* I, Topology vol. **3** (1964), 39-52.
[6] T. BRÖCKER and TAMMO TOM DIECK : *Kobordismentheorie,* Lecture Notes in Math. 178, Springer Verlag (1970).
[7] P. E. CONNER and E. E. FLOYD: Differentiable Periodic Maps, Springer Verlag (1964).
[8] C. KOSNIOWSKI and R. E. STONG: *Involutions and characteristic numbers,* Topology, vol. **17** (1978), 309-330.
[9] D. QUILLEN: *Elementary proofs of some results of cobordism theory using Steenrod operations,* Adv. in Math. **7** (1971), 29-56.

Masayoshi Kamata

Department of Mathematics, College of General Education, Kyushu University, Fukuoka 810, Japan

Shian-Wu Tyan

Szehai Junior College of Technology, Taipei, Taiwan, Republic of China