RUNE HAUGSENG

AN ALLEGEDLY SOMEWHAT FRIENDLY INTRODUCTION TO ∞-OPERADS

March 9, 2023

Contents

Chapter 1

Introduction

These notes are based on a series of 7 lectures given by the author at the University of Sevilla in October 2022. This is a preliminary version of the notes, and I would be very happy to receive comments, corrections and suggestions.[1]

Before we come to the definitions of operads and ∞-operads to the next chapter, in this introduction we will try to say a few words about what ∞-operads are supposed to do, and why the reader might be interested in learning about them. Roughly speaking, we can say that ∞-operads are a framework for working with *homotopy-coherent algebraic structures* in the setting of ∞-categories[2].

What, then, are homotopy-coherent algebraic structures? At least heuristically, these are algebraic structures where equalities are replaced by a coherent choice of (higher and higher) homotopies. Such structures were first considered in algebraic topology, with the very first probably being the A_∞-*spaces* of Stasheff [Sta63]. These are topological spaces X equipped with a multiplication $m\colon X \times X \to X$ that is only associative up to a specified homotopy in the square

$$\begin{tikzcd} X^{\times 3} \arrow[r, "m\times 1"] \arrow[d, "1\times m"] & X^{\times 2} \arrow[d, "m"] \\ X^{\times 2} \arrow[r, "m"] & X \end{tikzcd}$$

which thus gives a continuously varying family of paths $a(bc) \to (ab)c$ for $a, b, c \in X$. These homotopies must themselves be related by a homotopy of

[1]Including suggestions for further topics that ought to be covered in an introduction to ∞-operads, since I have a vague idea of expanding these notes some day…

[2]I will assume the reader already has some familiarity with ∞-categories, or is at least willing to fake it.

homotopies in the pentagon

$$\begin{tikzcd}[column sep=small] & ((ab)c)d \arrow[rr] \arrow[dl] & & (a(bc))d \arrow[dr] & \\ (ab)(cd) \arrow[drr] & & & & a((bc)d) \arrow[dll] \\ & & a(b(cd)) & & \end{tikzcd}$$

for points $a, b, c, d \in X$, and so on forever. The notion of A_∞-space was introduced because this is precisely the multiplicative structure that occurs on loop spaces. However, up to homotopy equivalence such structures can in fact always be replaced by a strictly associative multiplication (cf. Moore loops in the case of loop spaces).

Looking at iterated loop spaces leads to more complex homotopy-coherent structures, called *E_n-spaces*, first introduced by Boardman–Vogt [BV73] and May [May72]. The limiting case of *E_∞-spaces*, which is the algebraic structure on infinite loop spaces, can be thought of as additionally encoding commutativity in a fully homotopy-coherent sense. Such a structure typically *cannot* be replaced by a strict commutative multiplication on a homotopy-equivalent space[3].

Other examples of homotopy-coherent algebraic structures arise in category theory, where structures like (symmetric) monoidal (2-)categories can be viewed as "truncated" instances of homotopy-coherent structures. There are also many interesting examples of homotopy-coherent algebraic structures on chain complexes.

While they have since found many uses, operads were in fact introduced (in [BV73] and [May72]) as a way to organize the homotopy-coherence data that arises in such examples. However, such "strict" operads have certain drawbacks, analogous to those of simplicial categories as a model of ∞-categories. One advantage of instead working with ∞-operads, that is an ∞-categorical version of operads, is that they often give a less painful way to organize this kind of coherence data, where we never have to think about it explicitly, simply by exploiting that we have already set up a good theory of ∞-categories to encode homotopy-coherent diagrams.

In summary, ∞-operads are supposed to give a convenient language for working with homotopy-coherent algebraic structures.[4] We mention some general features of the theory that it might be helpful to keep in mind:

- In practice, we typically care more about the homotopy-coherent versions

[3]In fact, strictly commutative and associative multiplications can only exist on Eilenberg–MacLane spaces.

[4]In particular, if you don't happen to work with such structures, you probably don't need to care about ∞-operads!

of familiar structures like (associative or commutative) algebras and modules over them, rather than in new "exotic" algebraic structures. (There is arguably one very important exception, namely E_n-algebras. On the other hand, these can also be described as iterated associative algebras.)

- The applications of ∞-operads are more as a good language for describing homotopy-coherent algebras, rather than in the form of "big theorems" about ∞-operads themselves.
- While the algebraic structures we typically care about can also be described using other approaches (such as simplicial or topological operads) taking an ∞-categorical approach gives us many tools for constructing new symmetric monoidal ∞-categories and "doing algebra with ∞-categories" that are not easily accessible in older formalisms.

Acknowledgments

I thank Fernando Muro, Víctor Carmona Sánchez, and Ramón Flores for inviting me to give a lecture series in Sevilla, and the audience for sitting through 7 lectures on ∞-operads. These lectures were to some extent based on previous lectures on related topics, so I also thank the organizers of the YaMCATS seminar (Simona Paoli, Nicola Gambino, Steve Vickers, and Scott Balchin) for letting me talk about analytic monads in Leeds in 2018, Hongyi Chu and Bruno Stonek for inviting me to talk about ∞-operads ~~in Bonn~~ on the internet in 2020, and Thomas Nikolaus for asking me to talk about ∞-operads in Münster in 2021.

Unfortunately, mainly due to time constraints, far too much of the lectures ended up focusing on my own work. While I hope to rectify this somewhat in future versions of these notes, this at least gives me an occasion to thank my collaborators on ∞-operadic topics, who have taught me much of this material: Shaul Barkan, Hongyi Chu, David Gepner, Gijs Heuts, Joachim Kock, and Jan Steinebrunner.

Chapter 2

From operads to ∞-operads

The main goal of this chapter is to introduce Lurie's definition of ∞-operads and explain how it is a natural generalization to the ∞-categorical setting of a good (if perhaps not so well-known) approach to operads in sets. We hope to convince the reader that the definition of ∞-operads is easy to understand and that it is usable in practice (and perhaps disabuse them of the idea that ∞-operads are particularly difficult or technical).

We start by briefly reviewing operads in §2.1 before we introduce the approach to operads via their categories of operators in §2.2. Once this is done, it is easy to find an appropriate ∞-categorical analogue, which we do in §2.3.

2.1 What are operads anyway?

There are many equivalent ways to define operads (and we will see some more of these later), but one good way to think about them is that, roughly speaking, an operad is a structure like a category, but where instead of morphisms going from one object to another, morphisms in an operad have a *list* of objects as their source (but still a single object as target).

Remark 2.1.1. We will use the word *operad* to refer by default to the many-object (or "coloured") version of operads. These structures are also known as *symmetric multicategories*. The term operad was first used for the one-object case, and this usage is still common; our terminology is chosen for compatibility with that normally used in the ∞-categorical setting (where "symmetric multi-∞-category" becoems rather cumbersome). Note also that our operads here are unenriched, that is they are operads in *sets* rather than enriched in some symmetric monoidal category. This is because the notion of ∞-operads we will introduce gives precisely the ∞-categorical analogue of operads in sets.

Here is a more (but not completely) precise definition:

Definition 2.1.2. An *operad* $\mathcal{O}$ consists of a set $\operatorname{ob}\mathcal{O}$ of *objects* and for all lists of objects $Y = (y_1, \ldots, y_n)$ $(n \geq 0)$ and $x \in \operatorname{ob}\mathcal{O}$ a set $\operatorname{Hom}_{\mathcal{O}}(Y, x)$ of *multimorphisms* $Y \to x$ together with

- a composition operation that lets us compose a multimorphism $f\colon Y \to x$ with a list of multimorphisms $G = (g_i\colon Z^i \to y_i)_{i=1,\ldots,n}$ to get a multimorphism $f \circ G\colon (Z^1, \ldots, Z^n) \to x$ (where the list of lists really means their concatenation),
- an identity $\mathrm{id}_x\colon (x) \to x$ for all $x \in \operatorname{ob}\mathcal{O}$,
- a permutation symmetry whereby for each $\sigma \in \Sigma_n$ we have

  $$\operatorname{Hom}_{\mathcal{O}}((y_1, \ldots, y_n), x) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}}((y_{\sigma(1)}, \ldots, y_{\sigma(n)}), x),$$

  compatible with multiplication in the symmetric group Σ_n.

such that the composition is

- associative, i.e. $(f \circ G) \circ (H_1, \ldots, H_n) = f \circ (G \circ (H_1, \ldots, H_n))$ where $G = (g_1, \ldots, g_n)$ and $G \circ (H_1, \ldots, H_n) = (g_1 \circ H_1, \ldots, g_n \circ H_n)$,
- unital, i.e. $\mathrm{id}_x \circ f = f = f \circ \mathrm{id}_Y$ where $\mathrm{id}_Y = (\mathrm{id}_{y_1}, \ldots, \mathrm{id}_{y_n})$
- compatible with the symmetric group actions, in a sense that is "obvious" but annoying to write down[1].

Notation 2.1.3. Suppose $\mathcal{O}$ is an operad with a single object $*$. Then it is convenient to write $\mathcal{O}(n)$ for the set of n-ary multimorphisms $\operatorname{Hom}_{\mathcal{O}}((*, \ldots, *), *)$ with n $*$'s as input. The permutations give an action of Σ_n on $\mathcal{O}(n)$, and the composition operations can be written as

$$\mathcal{O}(n_1) \times \cdots \times \mathcal{O}(n_k) \times \mathcal{O}(k) \to \mathcal{O}(n_1 + \ldots + n_k).$$

Remark 2.1.4. The one-object version of operads was first introduced by May [May72] and Boardman–Vogt [BV73], while the non-symmetric (that is, without symmetric group actions) version of many-object operads (often called *multicategories*) was considered earlier by Lambek [Lam69].

Examples 2.1.5.

(i) The *commutative operad* Comm has a single object $*$ and $\mathsf{Comm}(n)$ has a single element for all $n = 0, 1, \ldots$. (Once we have defined the category of operads, Comm will be its terminal object.)

[1]Having to deal with permutation explicitly is arguably a serious disadvantage of this way of describing operads, even if we were not interested in their ∞-analogues.

(ii) The *associative operad* Assoc also has a single object $*$, and $\mathsf{Assoc}(n)$ is the set of orderings of $1, \dots, n$ with its free Σ_n-action. Composition is given by concatenation of orderings.

(iii) Let CM be the operad with two objects a, m and with

$$\mathrm{Hom}_{\mathsf{CM}}(Y, x) = \begin{cases} *, & Y = (a, \dots, a), x = a, \\ *, & Y = (a, \dots, a, m) \text{ up to permutation, } x = m, \\ \emptyset, & \text{otherwise.} \end{cases}$$

(iv) Let BM be the operad with three objects l, m, r and where for $X = (x_1, \dots, x_n)$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathsf{BM}}(X, l) &= \{\text{isomorphisms between } X \text{ and } (l, \dots, l)\}, \\ \mathrm{Hom}_{\mathsf{BM}}(X, r) &= \{\text{isomorphisms between } X \text{ and } (r, \dots, r)\}, \\ \mathrm{Hom}_{\mathsf{BM}}(X, m) &= \{\text{isomorphisms between } X \text{ and } (l, \dots, l, m, r, \dots, r)\}. \end{aligned}$$

Composition is given by concatenation of such isomorphisms.

Definition 2.1.6. Let $(\mathcal{V}, \otimes)$ be a symmetric monoidal category. We define an operad $\mathcal{V}_{\mathrm{opd}}$ with the same objects as $\mathcal{V}$ by taking[2]

$$\mathrm{Hom}_{\mathcal{V}_{\mathrm{opd}}}((v_1, \dots, v_n), w) := \mathrm{Hom}_{\mathcal{V}}(v_1 \otimes \cdots \otimes v_n, w),$$

where for $n = 0$ we interpret the empty tensor product as the unit in $\mathcal{V}$. Composition of multimorphisms is given by tensoring and composing morphisms in $\mathcal{V}$.

Definition 2.1.7. Suppose $\mathcal{O}$ and $\mathcal{P}$ are operads. A *functor* of operads $F\colon \mathcal{O} \to \mathcal{P}$ consists of an assignment $F\colon \operatorname{ob}\mathcal{O} \to \operatorname{ob}\mathcal{P}$ on objects, and maps on multimorphisms $\mathrm{Hom}_{\mathcal{O}}((y_1, \dots, y_n), x) \to \mathrm{Hom}_{\mathcal{P}}((Fy_1, \dots, Fy_n), Fx)$ compatible with composition, identities, and permutations. We write Opd for the category of operads and functors. If $\mathcal{V}$ is a symmetric monoidal category, then a functor of operads $\mathcal{O} \to \mathcal{V}_{\mathrm{opd}}$ is also called an $\mathcal{O}$-*algebra* in $\mathcal{V}$.

Examples 2.1.8. Let $\mathcal{V}$ be a symmetric monoidal category.

(i) A Comm-algebra in $\mathcal{V}$ is a commutative algebra: a functor $\mathsf{Comm} \to \mathcal{V}_{\mathrm{opd}}$ picks out a single object $A \in \mathcal{V}$ and specifies for each n a morphism $\mu_n\colon A^{\otimes n} \to A$ that is invariant under the action of Σ_n given by permuting the tensor factors, and for any partition $n = n_1 + \cdots + n_k$ we have

$$\mu_n = \mu_k \circ (\mu_{n_1} \otimes \cdots \otimes \mu_{n_k}).$$

[2] Here we're being a bit imprecise: really we need to choose a parenthesization of the tensor product and then use the associator isomorphisms when we define composition.

Here $\mu_1 = \mathrm{id}_A$ (being the image of the identity of the unique object of Comm) and $\mu_0 \colon \mathbb{1} \to A$ is a unit for the multiplication given by $\mu_2 \colon A \otimes A \to A$, which is commutative since it is invariant under reordering the two copies of A. Note also that μ_n for $n > 2$ can be expressed as an iterated composition of μ_2's.

(ii) An Assoc-algebra in $\mathcal{V}$ is an associative algebra: a functor $\mathsf{Assoc} \to \mathcal{V}_{\mathrm{opd}}$ picks out a single object $A \in \mathcal{V}$ and specifies for each n a morphism $A^{\otimes n} \to A$ for every ordering of $1, \ldots, n$, but these are all obtained from each other by reordering the factors in the tensor product. Moreover for $n > 2$ they can all be obtained by iterated composition of the binary multiplication.

(iii) A CM-algebra A in $\mathcal{V}$ consists of a commutative algebra $A(a)$ and a module $A(m)$ over $A(a)$.

(iv) A BM-algebra A in $\mathcal{V}$ consists of two associative algebras $A(l)$ and $A(r)$, and an $A(l)$-$A(r)$-bimodule $A(m)$.

Observation 2.1.9. If $\mathcal{V}$ and $\mathcal{W}$ are symmetric monoidal categories, then a functor of operads $\mathcal{V}_{\mathrm{opd}} \to \mathcal{W}_{\mathrm{opd}}$ is the same thing as a lax symmetric monoidal functor $\mathcal{V} \to \mathcal{W}$: A multimorphism $(v_1, \ldots, v_n) \to w$ factors uniquely as $(v_1, \ldots, v_n) \to v_1 \otimes \cdots \otimes v_n \to w$, so its image in $\mathcal{W}_{\mathrm{opd}}$ is determined by the functor $\mathcal{V} \to \mathcal{W}$ obtained by restricting to unary operations, together with the multimorphisms of the form $(Fv_1, \ldots, Fv_n) \to F(v_1 \otimes \cdots \otimes v_n)$ in $\mathcal{W}_{\mathrm{opd}}$. These correspond to morphisms $Fv_1 \otimes \cdots \otimes Fv_n \to F(v_1 \otimes \cdots \otimes v_n)$, which give the data of a lax symmetric monoidal structure on this functor.

Observation 2.1.10. Any category $\mathcal{C}$ can be regarded as an operad with only unary operations, that is

$$\mathrm{Hom}_{\mathcal{C}}((y_1, \ldots, y_n), x) := \begin{cases} \mathrm{Hom}_{\mathcal{C}}(y_1, x), & n = 1, \\ \emptyset, & n \neq 1. \end{cases}$$

A functor between two such operads is then the same thing as a functor between the corresponding categories, so we get a fully faithful inclusion $\mathsf{Cat} \hookrightarrow \mathsf{Opd}$. This has a right adjoint, which takes an operad $\mathcal{O}$ to the category obtained by forgetting all but the unary operations in $\mathcal{O}$.

We can characterize those operads that arise from symmetric monoidal categories:

Proposition 2.1.11. *An operad $\mathcal{O}$ arises from a symmetric monoidal category if and only if the following conditions hold:*

(1) For all lists $X = (x_1, \ldots, x_n)$ of objects in $\mathcal{O}$ there exists an object $\otimes X$ and a multimorphism $X \to \otimes X$ such that composition with this map gives an isomorphism

$$\mathrm{Hom}_{\mathcal{O}}(\otimes X, y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(X, y)$$

for all $y \in \mathbb{O}$.

(2) If the list X decomposes as a concatenation of lists $(X_1, \ldots, X_n)$, then the map $\otimes X \to \otimes(\otimes X_1, \ldots, \otimes X_n)$ that corresponds to the composite

$$X = (X_1, \ldots, X_n) \to (\otimes X_1, \ldots, \otimes X_n) \to \otimes(\otimes X_1, \ldots, \otimes X_n)$$

is an isomorphism.

See [Lei04, Section 3.3] for a proof of the non-symmetric version of this proposition.[3]

Example 2.1.12. Let $\mathscr{C}$ be a category. Define an operad $\mathscr{C}^{\amalg}$ with the same objects as $\mathscr{C}$ where

$$\mathrm{Hom}_{\mathscr{C}^{\amalg}}((c_1, \ldots, c_n), c) := \prod_{i=1}^{n} \mathrm{Hom}_{\mathscr{C}}(c_i, c),$$

with composition coming from composition in $\mathscr{C}$. This arises from a symmetric monoidal category if and only if $\mathscr{C}$ has coproducts.

2.2 Operads via categories of operators

In this section we introduce the *category of operators* of an operad, which allows us to formulate the notion of an operad entirely within the setting of ordinary categories.

Definition 2.2.1. We write $\mathbb{F}$ for the category of finite sets, and $\mathbb{F}_* := \mathbb{F}_{*/}$ for the the category of *pointed* finite sets. It is sometimes notationally convenient to use a skeleton of this category, given by the objects

$$\langle n \rangle := (\{0, 1, \ldots, n\}, 0),$$

which we will do without comment.

Observation 2.2.2. The category $\mathbb{F}_*$ can also be defined as the category whose objects are finite sets, with a morphism from S to S' given by a *span*

$$\begin{tikzcd} & T \arrow[dl, hook', "f"'] \arrow[dr, "g"] & \\ S & & S' \end{tikzcd}$$

where the backwards map f is required to be injective; composition of spans is by taking pullbacks. Here this span corresponds to the map of pointed sets $S_+ \to S'_+$ that restricts to g on $T \subseteq S$ and sends all elements not in T to the basepoint.

[3] I have not managed to find a detailed proof of the symmetric version in the literature.

Definition 2.2.3. Let $\mathcal{O}$ be an operad. Its *category of operators* $\mathcal{O}^{\otimes}$ is a category with a functor to $\mathbb{F}_*$ whose

- objects over $\langle n \rangle$ are lists $(x_1, \ldots, x_n)$ (possibly empty) of objects of $\mathcal{O}$,
- morphisms $(x_1, \ldots, x_n) \to (y_1, \ldots, y_m)$ consist of a morphism $\phi \colon \langle n \rangle \to \langle m \rangle$ in $\mathbb{F}_*$ and for $j = 1, \ldots, m$ a multimorphism $\Phi_i \colon (x_i)_{i \in \phi^{-1}(j)} \to y_j$ in $\mathcal{O}$.

Composition comes from composition in $\mathcal{O}$: given morphisms

$$(x_1, \ldots, x_n) \xrightarrow{(\phi, \Phi_i)} (y_1, \ldots, y_m) \xrightarrow{(\psi, \Psi_j)} (z_1, \ldots, z_k),$$

their composite is specified by $\psi \circ \phi$ together with the composite multimorphisms $\Psi_j \circ (\Phi_i)_{i \in \psi^{-1}(j)}$. The identity of $(x_1, \ldots, x_n)$ is $(\mathrm{id}_{\langle n \rangle}, (\mathrm{id}_{x_i})_{i=1,\ldots,n})$. Any functor of operads $F \colon \mathcal{O} \to \mathcal{P}$ induces an obvious functor $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ over $\mathbb{F}_*$, so that the category of operators gives a functor $\mathsf{Opd} \to \mathsf{Cat}_{/\mathbb{F}_*}$.

Remark 2.2.4. This definition of the category of operators goes back to the work of May and Thomason [MT78].

It turns out that we can give a purely categorical characterization of the categories that arise from this construction, and thus get a new way to think of operads. Before we can discuss this, we first need some definitions:

Definition 2.2.5. Let $p \colon \mathcal{E} \to \mathcal{B}$ be a functor. We say a morphism $\overline{f} \colon x \to y$ in $\mathcal{E}$ lying over $f \colon a \to b$ in $\mathcal{B}$ is *p-cocartesian* if for every $z \in \mathcal{E}$ over $c \in \mathcal{B}$ the commutative square

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{E}}(y, z) \arrow[r, "\overline{f}^*"] \arrow[d, "p"] & \mathrm{Hom}_{\mathcal{E}}(x, z) \arrow[d, "p"] \\ \mathrm{Hom}_{\mathcal{B}}(b, c) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathcal{B}}(a, c) \end{tikzcd}$$

is a pullback. In other words, $\overline{f}$ is cocartesian if given $g \colon b \to c$ and $\phi \colon x \to z$ with $p(\phi) = gf$, there exists a unique morphism $\overline{g} \colon y \to z$ such that $\phi = \overline{g}\overline{f}$. If S is some collection of morphisms in $\mathcal{B}$, we say that $\mathcal{E}$ *has p-cocartesian lifts of S* if for every $f \colon a \to b$ in S and x in $\mathcal{E}$ over a, there exists a p-cocartesian morphism over f with source x. We say that p is a *cocartesian fibration*[4] if $\mathcal{E}$ has p-cocartesian lifts of *all* morphisms in $\mathcal{B}$, and an *isofibration* if $\mathcal{E}$ has cocartesian lifts of *isomorphisms* in $\mathcal{B}$.

Warning 2.2.6. With this definition the notion of "having p-cocartesian lifts of S" is *not* invariant under equivalences of categories, because we are asking for a lift in $\mathcal{E}$ whose projection to $\mathcal{B}$ is *equal* (rather than isomorphic) to the morphism we start with. In fact, we can factor *any* functor as an equivalence followed by an isofibration, so up to equivalence being an isofibration is a vacuous property.

[4] or more classically a *Grothendieck opfibration*

Definition 2.2.7. We say a morphism $\phi\colon \langle n\rangle \to \langle m\rangle$ in $\mathbb{F}_*$ is *active* if $\phi^{-1}(0) = \{0\}$, i.e. nothing except the base point is sent to the base point, and *inert* if $|\phi^{-1}(i)| = 1$ for $i \neq 0$, i.e. ϕ is an isomorphism away from the base point. In the span description of $\mathbb{F}_*$, the active maps are the spans whose backwards component is an isomorphism and the inert are those whose forwards component is an isomorphism.

Lemma 2.2.8. *Every morphism in $\mathbb{F}_*$ factors uniquely up to isomorphism as a composite of an inert map followed by an active map. In other words, the inert and active maps form a factorization system on $\mathbb{F}_*$.* □

Notation 2.2.9. We write $\rho_i\colon \langle n\rangle \to \langle 1\rangle$ for the inert map given by

$$\rho_i(j) = \begin{cases} 1, & j = i, \\ 0, & j \neq i, \end{cases}$$

for $1 \leq i \leq n$.

Observation 2.2.10. Let us make some observations about how the category $\mathbb{O}^\otimes$ encodes the structure of the operad $\mathbb{O}$:

- In $\mathbb{F}_*$ there is a unique active morphism $\alpha_n\colon \langle n\rangle \to \langle 1\rangle$ for all n. Morphisms over α_n in $\mathbb{O}^\otimes$ are precisely n-ary multimorphisms in $\mathbb{O}$.
- For $\alpha\colon \langle n\rangle \to \langle m\rangle$ active in $\mathbb{F}_*$, a morphism in $\mathbb{O}^\otimes$ over α is a list of m multimorphisms in $\mathbb{O}$ with arities $\alpha^{-1}(i)$ $(i = 1, \ldots, m)$.
- The remaining data in $\mathbb{O}^\otimes$ allows us to encode the composition of multimorphisms as ordinary composition in a category.

Proposition 2.2.11. *An isofibration $p\colon \mathscr{E} \to \mathbb{F}_*$ is equivalent (over $\mathbb{F}_*$) to the category of operators of an operad if and only if the following conditions hold:*

(1) $\mathscr{E}$ has p-cocartesian lifts of the inert morphisms in $\mathbb{F}_*$.

(2) Given $X \in \mathscr{E}_{\langle n\rangle}$, cocartesian morphisms $X \to X_i$ over ρ_i, and an object $Y \in \mathscr{E}_{\langle m\rangle}$, the commutative square

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{E}}(Y,X) \arrow[r] \arrow[d] & \prod_{i=1}^n \mathrm{Hom}_{\mathscr{E}}(Y,X_i) \arrow[d] \\ \mathrm{Hom}_{\mathbb{F}_*}(\langle m\rangle,\langle n\rangle) \arrow[r] & \prod_{i=1}^n \mathrm{Hom}_{\mathbb{F}_*}(\langle m\rangle,\langle 1\rangle) \end{tikzcd}$$

is cartesian.

(3) Given objects $X_1, \ldots, X_n$ in $\mathscr{E}_{\langle 1\rangle}$, there exists an object $X \in \mathscr{E}_{\langle n\rangle}$ with cocartesian morphisms $X \to X_i$ over ρ_i.

Observation 2.2.12. In the situation above, taking cocartesian lifts of all the maps ρ_i gives a functor

$$\mathscr{E}_{\langle n\rangle} \to \prod_{i=1}^{n} \mathscr{E}_{\langle 1\rangle}.$$

It follows from the second condition above that this is fully faithful, since for $X, Y \in \mathscr{E}_{\langle n\rangle}$ with cocartesian morphisms $X \to X_i$, $Y \to Y_i$, we get an isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathscr{E}_{\langle n\rangle}}(X, Y) \cong \mathrm{Hom}_{\mathscr{E}}(X, Y)_{\mathrm{id}_{\langle n\rangle}} &\cong \prod_{i=1}^{n} \mathrm{Hom}_{\mathscr{E}}(X, Y_i)_{\rho_i} \\ &\cong \prod_{i=1}^{n} \mathrm{Hom}_{\mathscr{E}}(X_i, Y_i)_{\mathrm{id}_{\langle 1\rangle}} \cong \prod_{i=1}^{n} \mathrm{Hom}_{\mathscr{E}_{\langle 1\rangle}}(X_i, Y_i).\end{aligned}$$

The third condition above then says that this fully faithful functor is also essentially surjective on objects, i.e. an equivalence of categories.

Proof of Proposition 2.2.11. We first suppose $\mathcal{O}$ is an operad, and show that $\mathcal{O}^{\otimes} \to \mathbb{F}_*$ has the given properties. It is clear that we can lift isomorphisms from $\mathbb{F}_*$, so that this is an isofibration. To prove (1), we consider an inert map $\phi\colon \langle n\rangle \to \langle m\rangle$ in $\mathbb{F}_*$, and $X = (x_1, \ldots, x_n)$ an object of $\mathcal{O}^{\otimes}$ over $\langle n\rangle$. Set $X' = (x_{\phi^{-1}(1)}, \ldots, x_{\phi^{-1}(m)})$ and define $\overline{\phi}\colon X \to X'$ to be the map over ϕ given by $\mathrm{id}_{x_{\phi^{-1}(i)}}$ for $i = 1, \ldots, m$. We claim $\overline{\phi}$ is cocartesian. To prove this, consider a commutative triangle

$$\begin{tikzcd} \langle n\rangle \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & \langle k\rangle \\ & \langle m\rangle \arrow[ur, "\psi'"'] & \end{tikzcd}$$

and a map $\overline{\psi}\colon X \to Y$ over ψ in $\mathcal{O}^{\otimes}$. Then $\overline{\psi}$ is given by a collection of multimorphisms $\psi_j\colon (x_i)_{i\in\psi^{-1}(j)} \to y_j$. Since ϕ is inert, we can also write these as multimorphisms $(x_{\phi^{-1}(i)})_{i\in\psi'^{-1}(j)} \to y_j$, so that they also form a map $\overline{\psi}'\colon X' \to Y$ such that $\overline{\psi} = \overline{\psi}' \circ \overline{\phi}$. It is also clear that this factorization is unique, so that $\overline{\phi}$ is indeed cocartesian.

For (2), we want to see that given any $\phi\colon \langle n\rangle \to \langle m\rangle$ in $\mathbb{F}_*$ and objects $Y = (y_1, \ldots, y_n)$ over $\langle n\rangle$ and $X = (x_1, \ldots, x_m)$ over $\langle m\rangle$, then a collection of morphisms $Y \to (x_i)$ over $\rho_i\phi$ corresponds to a unique morphism $Y \to X$ over ϕ. Such a collection of morphisms is given by multimorphisms $(y_j)_{j\in(\rho_i\phi)^{-1}(1)} \to x_i$; here $(y_j)_{j\in(\rho_i\phi)^{-1}(1)} = (y_j)_{j\in\phi^{-1}(i)}$, so these multimorphisms do indeed combine to a unique morphism $Y \to X$ over ϕ.

Part (3) is now clear from our description of the inert cocartesian morphisms in $\mathcal{O}^{\otimes}$.

To prove the converse direction, suppose we have an isofibration $p\colon \mathscr{E} \to \mathbb{F}_*$ satisfying the 3 given conditions. We define an operad $\mathbb{O}$ whose objects are those of $\mathscr{E}_{\langle 1\rangle}$. Given objects $x_1, \ldots, x_n, y \in \mathscr{E}_{\langle 1\rangle}$, a multimorphism $(x_1, \ldots, x_n) \to y$ is given by an object $X \in \mathscr{E}_{\langle n\rangle}$ with cocartesian morphisms $X \to x_i$ over ρ_i (this exists by (3) and is unique up to unique isomorphism), together with a morphism $X \to y$ over α_n in $\mathscr{E}$.[5] To compose a collection of multimorphisms $(x_1^1, \ldots, x_{m_1}^1) \to y_1, \ldots, (x_1^n, \ldots, x_{m_n}^n) \to y_n$ with $(y_1, \ldots, y_n) \to z$, given by maps $X^j \to y_j$ and $Y \to z$, we choose X over $\langle M\rangle = \langle m_1 + \cdots + m_n\rangle$ with cocartesian maps $X \to x_i^j$ over $\rho_{m_1+\cdots+m_{j-1}+i}$. If π_j is the obvious inert map $\langle M\rangle \to \langle m_j\rangle$, then we have a cocartesian map from X over π_j whose target can be identified with X^j (since this is the unique object with cocartesian maps $X^j \to x_i^j$ over ρ_i). We thus have a family of maps $X \to X^j \to y_j$ which fit together into a unique map $X \to Y$ by (2). The composite multimorphism in $\mathbb{O}$ is then given by the composite $X \to Y \to z$ in $\mathscr{E}$.

We claim that we then have an equivalence $\mathscr{E} \simeq \mathbb{O}^{\otimes}$, but leave the details for any unusually diligent readers to check. A key point here is that for any objects $X \in \mathscr{E}_{\langle n\rangle}$ and $Y \in \mathscr{E}_{\langle m\rangle}$, we can describe the set $\mathrm{Hom}_{\mathscr{E}}(X, Y)_\phi$ of maps over $\phi\colon \langle n\rangle \to \langle m\rangle$ first as

$$\mathrm{Hom}_{\mathscr{E}}(X, Y)_\phi \cong \prod_{i=1}^{m} \mathrm{Hom}_{\mathscr{E}}(X, Y_i)_{\rho_i\phi}$$

by (2), where $Y \to Y_i$ is cocartesian over ρ_i, and then identify this in turn as

$$\prod_{i=1}^{m} \mathrm{Hom}_{\mathscr{E}}(\pi_{i,!}X, Y_i)_{\alpha_{n_i}}$$

where the inert-active factorization of $\rho_i\phi$ is given as

$$\langle n\rangle \xrightarrow{\pi_i} \langle n_i\rangle \xrightarrow{\alpha_{n_i}} \langle 1\rangle$$

and $X \to \pi_{i,!}X$ is cocartesian over π_i. Thus all maps in $\mathscr{E}$ are determined by those that correspond to multimorphisms in $\mathbb{O}$. $\square$

Proposition 2.2.13. *Let $\mathbb{O}$ and $\mathscr{P}$ be operads. A functor*

$$\begin{tikzcd} \mathbb{O}^{\otimes} \arrow[rr, "F"] \arrow[dr] & & \mathscr{P}^{\otimes} \arrow[dl] \\ & \mathbb{F}_* & \end{tikzcd}$$

comes from a functor of operads if and only if F preserves inert cocartesian morphisms.

[5] To form a strict 1-category rather than a (2,1)-category that's equivalent to a 1-category we should strictly speaking take isomorphism classes of this data, but we will ignore this technical point to simplify the exposition.

We omit the proof, but this amounts to the observation that F preserving inert cocartesian morphisms corresponds to $F(x_1, \dots, x_n)$ being the same as $(Fx_1, \dots, Fx_n)$. Using Proposition 2.1.11, we can also identify symmetric monoidal categories in this context:

Proposition 2.2.14. *The following are equivalent for an isofibration $p\colon \mathscr{E} \to \mathbb{F}_*$:*

(1) $\mathscr{E}$ is equivalent to the category of operators $\mathscr{V}^{\otimes}_{\mathrm{opd}}$ for a symmetric monoidal category $\mathscr{V}$.

(2) p satisfies conditions (1)–(3) in Proposition 2.2.11 and is a cocartesian fibration.

(3) p is a cocartesian fibration and the functor

$$\mathscr{E}_{\langle n\rangle} \xrightarrow{(\rho_{i,!})} \prod_{i=1}^{n} \mathscr{E}_{\langle 1\rangle},$$

induced by cocartesian morphisms over ρ_i, is an equivalence for all i.

Moreover, a functor between such cocartesian fibrations corresponds to a (strong) symmetric monoidal functor if and only if it preserves all *cocartesian morphisms.* □

(See Proposition 2.3.8 below for a proof of the equivalence of the last two conditions in the ∞-categorical context.)

Upshot 2.2.15. *We can identify* Opd *with the subcategory of* $\mathsf{Cat}_{/\mathbb{F}_*}$ *whose objects are the isofibrations satisfying conditions (1)–(3) in Proposition 2.2.11, and whose morphisms are those that preserve inert cocartesian morphisms.*

We can thus regard the conditions of Proposition 2.2.11 as a new *definition* of operads. While it looks quite different from other definitions, there are several advantages to this approach:

- The definition is concise and precise: we don't have to say that any "obvious" diagrams commute or deal with composition of Σ_n-actions (these are all hidden away in the base category $\mathbb{F}_*$).
- The definition is easy to use in practice: For example, if we want to define a symmetric monoidal category it is quite feasible to define a functor to $\mathbb{F}_*$ and check that it's a cocartesian fibration that satisfies the product condition from Proposition 2.2.14. On the other hand, in practice nobody *ever* specifies all the data that's actually required to define a symmetric monoidal category in the usual sense.

For our purposes, however, the key advantage of this approach is that it defines operads and symmetric monoidal categories in terms of ordinary categories, using only concepts that have clear generalizations to ∞-categories. We are thus led to an obvious generalization of this notion of operads to the ∞-categorical context, which we spell out in the next section.

2.3 ∞-operads and their algebras

In this section we introduce the definition of ∞-operads. This follows [Lur17, 2.1]; see also the lecture notes [Har19] by Harpaz for a good introduction to ∞-operads.

First, a few words on ∞-categories:

- I will assume the reader is already somewhat familiar with ∞-categories and how they work, or is at least willing to suspend their disbelief in the existence of such beasts.
- We will be working with ∞-categories "model-independently", that is to say we will think of an ∞-category as an object in the ∞-category of ∞-categories, rather than explicitly using some model (like quasicategories of complete Segal objects in simplicial sets).
- Our terminology should always be interpreted in an ∞-categorical context. For example, when we talk about a diagram "commuting", we mean this in the ∞-categorical sense (that is, the diagram should be a functor of ∞-categories), rather than in some stricter interpretation (which doesn't actually make sense in an ∞-category).

We also mention some basic notation:

- $\mathcal{S}$ denotes the ∞-category of spaces or ∞-groupoids, and Cat_∞ the ∞-category of (small) ∞-categories. Here $\mathcal{S}$ is a full subcategory of Cat_∞, and we write $(-)^\simeq$ for the right adjoint to the inclusion (which gives the underlying ∞-groupoid $\mathscr{C}^\simeq$ of an ∞-category $\mathscr{C}$, obtained by throwing away all non-invertible morphisms). We think of categories as a special case of ∞-categories, and so identify the $(2,1)$-category of categories with a full subcategory of Cat_∞.
- If x, y are objects of an ∞-category $\mathscr{C}$, we have a *mapping space* $\mathsf{Map}_\mathscr{C}(x,y)$; these form a functor
$$\mathsf{Map}_\mathscr{C} \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathcal{S}.$$
This transposes to the fully faithful *Yoneda embedding*
$$\mathscr{C} \hookrightarrow \mathsf{P}(\mathscr{C}),$$
where $\mathsf{P}(\mathscr{C}) := \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathcal{S})$ is the ∞-category of *presheaves* on $\mathscr{C}$.

Definition 2.3.1. Let $p \colon \mathscr{E} \to \mathscr{B}$ be a functor of ∞-categories. A morphism $\overline{f} \colon x \to y$ in $\mathscr{E}$ is *p-cocartesian* (over $f := p(\overline{f}) \colon a \to b$ in $\mathscr{B}$) if for any $z \in \mathscr{E}$, the commutative square
$$\begin{tikzcd} \mathsf{Map}_\mathscr{E}(y,z) \arrow[r, "\overline{f}^*"] \arrow[d] & \mathsf{Map}_\mathscr{E}(x,z) \arrow[d] \\ \mathsf{Map}_\mathscr{B}(b,pz) \arrow[r, "f^*"] & \mathsf{Map}_\mathscr{B}(a,pz). \end{tikzcd}$$

is a pullback square in $\mathcal{S}$. Given a set S of morphisms in $\mathcal{B}$, we say that $\mathcal{E}$ has *p-cocartesian lifts* of S if given $f: a \to b$ in $\mathcal{B}$ and $x \in \mathcal{E}$ with $px \simeq a$, there exists a p-cocartesian morphism $\overline{f}: x \to y$ such that $p(\overline{f}) \simeq f$. (To be a bit more precise, this means that in any commutative square of ∞-categories

$$\begin{tikzcd} \{0\} \arrow[r, "x"] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ {[1]} \arrow[r, "f"] \arrow[ur, dotted, "\overline{f}"] & \mathcal{B} \end{tikzcd}$$

where f lies in S, there exists a lift $\overline{f}$ that is a p-cocartesian morphism.) We say that p is a *cocartesian fibration* if $\mathcal{E}$ has p-cocartesian lifts of *all* morphisms.

We recall the *straightening theorem* for cocartesian fibrations:

Theorem 2.3.2 (Lurie, [Lur09]). *There is a natural equivalence of ∞-categories*

$$\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty) \simeq \mathsf{Cocart}(\mathcal{B}),$$

where $\mathsf{Cocart}(\mathcal{B})$ is the subcategory of $\mathsf{Cat}_{\infty/\mathcal{B}}$ whose objects are the cocartesian fibrations and whose morphisms are those that preserve cocartesian morphisms. □

We can now state Lurie's definition of an ∞-operad:

Definition 2.3.3. An ∞-*operad* is a functor of ∞-categories $p: \mathcal{O} \to \mathbb{F}_*$ such that

(1) $\mathcal{O}$ has p-cocartesian lifts of inert morphisms in $\mathbb{F}_*$. (We refer to these as *inert* morphisms in $\mathcal{O}$.)

(2) If X is an object of $\mathcal{O}_{\langle n\rangle}$ and $X \to X_i$ is an inert morphism over $\rho_i: \langle n\rangle \to \langle i\rangle$, then for any $Y \in \mathcal{O}_{\langle m\rangle}$, the commutative square

$$\begin{tikzcd} \mathsf{Map}_{\mathcal{O}}(Y, X) \arrow[r] \arrow[d] & \prod_{i=1}^{n} \mathsf{Map}_{\mathcal{O}}(Y, X_i) \arrow[d] \\ \mathrm{Hom}_{\mathbb{F}_*}(\langle m\rangle, \langle n\rangle) \arrow[r] & \mathrm{Hom}_{\mathbb{F}_*}(\langle m\rangle, \langle 1\rangle) \end{tikzcd}$$

is a pullback in $\mathcal{S}$.

(3) Given $X_1, \ldots, X_n \in \mathcal{O}_{\langle 1\rangle}$, there exists an object $X \in \mathcal{O}_{\langle n\rangle}$ with cocartesian morphisms $X \to X_i$ over ρ_i.

A *morphism of ∞-operads* is a commutative triangle

$$\begin{tikzcd} \mathcal{O} \arrow[rr, "F"] \arrow[dr] & & \mathcal{P} \arrow[dl] \\ & \mathbb{F}_* & \end{tikzcd}$$

such that F preserves inert morphisms. We also refer to such an F as an $\mathcal{O}$-*algebra* in $\mathcal{P}$. We write Opd_∞ for the subcategory of $\mathsf{Cat}_{\infty/\mathbb{F}_*}$ whose objects are ∞-operads and whose morphisms are the morphisms of ∞-operads, and $\mathsf{Alg}_{\mathcal{O}}(\mathcal{P})$ for the full subcategory of $\mathsf{Fun}_{/\mathbb{F}_*}(\mathcal{O}, \mathcal{P})$ spanned by the $\mathcal{O}$-algebras in $\mathcal{P}$.[6]

Example 2.3.4. If $\mathcal{O}$ is an operad in sets, then its category of operators $\mathcal{O}^\otimes \to \mathbb{F}_*$ is an ∞-operad. For example, if we take $\mathcal{O}$ to be the operads for (associative/commutative) algebras and modules from Examples 2.1.8, their algebras give the *correct* ∞-categorical notions of algebras and modules.

Observation 2.3.5. By the same argument as in Observation 2.2.12, for an ∞-operad $\mathcal{O}$ the functor

$$\mathcal{O}_{\langle n \rangle} \to \prod_{i=1}^{n} \mathcal{O}_{\langle 1 \rangle},$$

induced by the cocartesian morphisms over $\rho_i\colon \langle n \rangle \to \langle 1 \rangle$, is an equivalence for all n.

Definition 2.3.6. Let $\mathcal{C}$ be an ∞-category with finite products. A *commutative monoid* in $\mathcal{C}$ is a functor $M\colon \mathbb{F}_* \to \mathcal{C}$ such that the map

$$F(\langle n \rangle) \to \prod_{i=1}^{n} F(\langle 1 \rangle)$$

induced by the maps $\rho_i\colon \langle n \rangle \to \langle 1 \rangle$, is an equivalence in $\mathcal{C}$ for all n. A *symmetric monoidal ∞-category* is then just a commutative monoid in Cat_∞, or a cocartesian fibration that corresponds to such a commutative monoid.

Remark 2.3.7. In $\mathcal{S}$, this notion of a commutative monoid is the ∞-categorical analogue of Segal's *special* Γ-*spaces* [Seg74].

Just as we saw for ordinary categories, symmetric monoidal ∞-categories can be identified as ∞-operads that are cocartesian fibrations:

Proposition 2.3.8. *If $p\colon \mathcal{E} \to \mathbb{F}_*$ is a cocartesian fibration, then p is a symmetric monoidal ∞-category if and only if p is an ∞-operad.*

Proof. If p is an ∞-operad and a cocartesian fibration, we get from Observation 2.3.5 that p is a symmetric monoidal ∞-category. For the converse, we observe that conditions (1) and (3) in the definition of an ∞-operad are immediate if p is a symmetric monoidal ∞-category. To prove the last condition, we note that for $\phi\colon \langle n \rangle \to \langle m \rangle$ and objects X over $\langle n \rangle$ and Y over $\langle m \rangle$, the

[6] It is possible to enhance Opd_∞ to an $(\infty, 2)$-category where $\mathsf{Alg}_{\mathcal{O}}(\mathcal{P})$ is the ∞-category of maps from $\mathcal{O}$ to $\mathcal{P}$.

p-cocartesian morphisms in $\mathscr{C}$ provide a commutative diagram

$$\begin{tikzcd}
\mathsf{Map}_{\mathscr{C}}(X,Y)_\phi \arrow[r] & \prod_{i=1}^m \mathsf{Map}_{\mathscr{C}}(X,\rho_{i,!}Y)_{\rho_i\phi} \\
\mathsf{Map}_{\mathscr{C}_{\langle m\rangle}}(\phi_!X,Y) \arrow[u, "\sim"] \arrow[r] \arrow[dr] & \prod_{i=1}^m \mathsf{Map}_{\mathscr{C}}(\phi_!X,\rho_{i,!}Y)_{\rho_i} \arrow[u, "\sim"'] \\
& \prod_{i=1}^m \mathsf{Map}_{\mathscr{C}_{\langle 1\rangle}}(\rho_{i,!}\phi_!X,\rho_{i,!}Y), \arrow[u, "\sim"']
\end{tikzcd}$$

where the labelled morphisms are equivalences. Hence the top horizontal morphism is an equivalence if and only if the bottom diagonal morphism is one. Here the latter condition follows from p being a symmetric monoidal ∞-category, while the former for all choices of ϕ, X, and Y, gives the missing condition (2) for p to be an ∞-operad. $\square$

Notation 2.3.9. A symmetric monoidal ∞-category with underlying ∞-category $\mathscr{C}$ is often denoted $\mathscr{C}^\otimes$; then $\mathscr{C} \simeq \mathscr{C}^\otimes_{\langle 1\rangle}$. (Note that, unlike in [Lur17], we do not use the notation $\mathscr{O}^\otimes$ for a general ∞-operad.)

Definition 2.3.10. A *lax symmetric monoidal functor* between symmetric monoidal ∞-categories is a morphism of ∞-operads between them, i.e. a commutative triangle

$$\begin{tikzcd}
\mathscr{C}^\otimes \arrow[rr, "F"] \arrow[dr] & & \mathscr{D}^\otimes \arrow[dl] \\
& \mathbb{F}_* &
\end{tikzcd}$$

where F preserves inert morphisms. If F preserves *all* cocartesian morphisms, we call it a *symmetric monoidal functor*.

Example 2.3.11. If $\mathscr{O}$ is a simplicial operad (that is, an operad enriched in simplicial sets), then it has a simplicial category of operators $\mathscr{O}^\otimes \to \mathbb{F}_*$. If the simplicial sets of maps in $\mathscr{O}$ are all Kan complexes, then its homotopy-coherent nerve is an isofibration of quasicategories that represents an ∞-operad.

For later use, we also note that any ∞-operad has a canonical factorization system:

Definition 2.3.12. A *factorization system*[7] on an ∞-category $\mathscr{C}$ is a pair of wide[8] subcategories $\mathscr{C}_L, \mathscr{C}_R$ such that the composition map

$$\mathsf{Map}([1],\mathscr{C}_L) \times_{\mathscr{C}^\simeq} \mathsf{Map}([1],\mathscr{C}_R) \to \mathsf{Map}([1],\mathscr{C})$$

is an equivalence.

[7] Sometimes also called an "orthogonal factorization system".

[8] That is, containing all objects.

Definition 2.3.13. Recall that the inert and active maps form a factorization system on $\mathbb{F}_*$. If $\mathcal{O}$ is an ∞-operad, we say that a morphism in $\mathcal{O}$ is *inert* if it is cocartesian and lies over an inert morphisms in $\mathbb{F}_*$, and *active* if it just lies over an active morphism in $\mathbb{F}_*$.

Proposition 2.3.14. *If $\mathcal{O}$ is an ∞-operad, then the inert and active morphisms in $\mathcal{O}$ form a factorization system.* □

Chapter 3

Constructions and examples

In this chapter we survey a few important constructions and results on ∞-operads. We will also see how they give us new examples of ∞-operads, and in particular produce interesting symmetric monoidal ∞-categories.

3.1 Monoids and cartesian monoidal ∞-categories

This section is based on [Lur17, §2.4.1].

Definition 3.1.1. Let $\mathbb{O}$ be an ∞-operad and $\mathscr{C}$ an ∞-category with finite products. A functor $M\colon \mathbb{O} \to \mathscr{C}$ is an $\mathbb{O}$-*monoid* if for all $X \in \mathbb{O}_{\langle n \rangle}$ and inert morphisms $X \to X_i$ over $\rho_i\colon \langle n \rangle \to \langle 1 \rangle$, the induced map

$$M(X) \xrightarrow{(M(\rho_i))} \prod_{i=1}^{n} M(X_i)$$

is an equivalence in $\mathscr{C}$. We write $\mathsf{Mon}_{\mathbb{O}}(\mathscr{C})$ for the full subcategory of $\mathsf{Fun}(\mathbb{O}, \mathscr{C})$ spanned by the $\mathbb{O}$-monoids.

If $\mathscr{C}$ has finite products, we want to define a symmetric monoidal ∞-category $\mathscr{C}^{\times}$ with the tensor product given by the cartesian product, and such that $\mathbb{O}$-algebras in $\mathscr{C}^{\times}$ are naturally equivalent to $\mathbb{O}$-monoids in $\mathscr{C}$ for all ∞-operads $\mathbb{O}$.

Definition 3.1.2. Let

$$\mathsf{Ar}_{\mathrm{int}}(\mathbb{F}_*) \subseteq \mathsf{Ar}(\mathbb{F}_*) := \mathsf{Fun}([1], \mathbb{F}_*)$$

denote the full subcategory of inert morphisms.

Observation 3.1.3. The functor $\mathrm{ev}_0\colon \mathsf{Ar}_{\mathrm{int}}(\mathbb{F}_*) \to \mathbb{F}_*$ given by evaluation at $0 \in [1]$ is a cartesian fibration[1]. This follows from the inert maps being the left

[1] the dual notion of a cocartesian fibration

class in a factorization system: for an inert map $\langle n\rangle \rightarrowtail \langle m\rangle$ and an arbitrary morphism $\langle n'\rangle \to \langle n\rangle$, the cartesian morphism over this is the commutative square

$$\begin{tikzcd} \langle n'\rangle \arrow[r, tail] \arrow[d] & \langle m'\rangle \arrow[d, squiggly] \\ \langle n\rangle \arrow[r, tail] & \langle m\rangle \end{tikzcd}$$

that gives the inert-active factorization of the composite $\langle n'\rangle \to \langle m\rangle$. The functor ev_0 then corresponds under the straightening equivalence to a functor $\mathbb{F}_* \to \mathsf{Cat}$. This takes the object $\langle n\rangle$ to the category $(\mathbb{F}_*^{\mathrm{int}})_{\langle n\rangle/}$ of inert maps $\langle n\rangle \rightarrowtail \langle m\rangle$, with morphisms given by (necessarily) inert maps $\langle m\rangle \rightarrowtail \langle m'\rangle$ under $\langle n\rangle$.

Construction 3.1.4. From this cartesian fibration and the projection $\mathcal{C}\times\mathbb{F}_* \to \mathbb{F}_*$, which is a cocartesian fibration, by a general construction we can produce a cocartesian fibration $\overline{\mathcal{C}}^{\times} \to \mathbb{F}_*$ with the universal property that for $K \to \mathbb{F}_*$ we have a natural equivalence

$$\mathsf{Map}_{/\mathbb{F}_*}(K, \overline{C}^{\times}) \simeq \mathsf{Map}(K \times_{\mathbb{F}_*} \mathsf{Ar}_{\mathrm{int}}(\mathbb{F}_*), \mathcal{C}).$$

(In particular, the fibre $\overline{\mathcal{C}}^{\times}_{\langle n\rangle}$ is equivalent to $\mathsf{Fun}((\mathbb{F}_*^{\mathrm{int}})_{\langle n\rangle/}, \mathcal{C})$.) When $\mathcal{C}$ has finite products, we define $\mathcal{C}^{\times}$ to be the full subcategory of $\overline{\mathcal{C}}^{\times}$ containing the functors $F\colon (\mathbb{F}_*^{\mathrm{int}})_{\langle n\rangle/} \to \mathcal{C}$ such that for every object $\phi\colon \langle n\rangle \to \langle m\rangle$, the map

$$F(\phi) \to \prod_{i=1}^{m} F(\rho_i\phi)$$

is an equivalence.

Theorem 3.1.5. *Let $\mathcal{C}$ be an ∞-category with finite products. Then $\mathcal{C}^{\times}$ is a symmetric monoidal ∞-category. Moreover, if $\mathcal{O}$ is an ∞-operad, then the composite*

$$\mathsf{Alg}_{\mathcal{O}}(\mathcal{C}) \subseteq \mathsf{Fun}_{/\mathbb{F}_*}(\mathcal{O}, \overline{\mathcal{C}}^{\times}) \simeq \mathsf{Fun}(\mathcal{O} \times_{\mathbb{F}_*} \mathsf{Ar}_{\mathrm{int}}(\mathbb{F}_*), \mathcal{C}) \to \mathsf{Fun}(\mathcal{O}, \mathcal{C}),$$

where the last functor comes from restriction along the section $\mathbb{F}_* \to \mathsf{Ar}_{\mathrm{int}}(\mathbb{F}_*)$ given by identity morphisms, identifies $\mathsf{Alg}_{\mathcal{O}}(\mathcal{C}^{\times})$ with the full subcategory $\mathsf{Mon}_{\mathcal{O}}(\mathcal{C})$ of $\mathcal{O}$-monoids. □

If this looks way too complicated just to define the cartesian product as a symmetric monoidal structure, consider the following exercise:

Exercise 3.1.6.

(i) Define the symmetric monoidal structure on Set given by the cartesian product (in the classical sense) in complete detail, without ever saying something is "obvious". (For extra credit, formalize this on a computer.)

(ii) Do the same for the cartesian product of groupoids, as a symmetric monoidal bicategory.[2]

Variant 3.1.7. Working with Grothendieck universes as usual, we let $\widehat{\mathsf{Cat}}_\infty$ denote the (very large) ∞-category of large ∞-categories, and define $\widehat{\mathsf{Cat}}_\infty^{\text{cocomp}}$ to be the subcategory of large ∞-categories with small colimits and functors that preserve these. We then define $\widehat{\mathsf{Cat}}_\infty^{\text{cocomp},\otimes}$ to be the subcategory of $\widehat{\mathsf{Cat}}_\infty^{\times}$ whose objects are lists of cocomplete ∞-categories, and whose morphisms are componentwise given by functors

$$F\colon \mathscr{C}_1 \times \cdots \times \mathscr{C}_n \to \mathscr{D}$$

that preserve colimits in each variable (meaning that given objects $x_i \in \mathscr{C}_i$ for all $i \neq j$, the functor $F(x_1, \ldots, -, \ldots, x_n)\colon \mathscr{C}_j \to \mathscr{D}$ preserves colimits). Then it is easy to see that $\widehat{\mathsf{Cat}}_\infty^{\text{cocomp},\otimes}$ is an ∞-operad. To show that it is a symmetric monoidal ∞-category we now only need to prove that there exists a cocomplete ∞-category $\mathscr{C}_1 \otimes \cdots \otimes \mathscr{C}_n$ that corepresents such "multi-cocontinuous functors." (A similar construction could be used to obtain the tensor product of vector spaces from the cartesian product of sets.)

3.2 Day convolution

This section is based on [Lur17, §2.2.6] and [Gla16]. We start by recalling the classical case of Day convolution for ordinary categories: If $\mathscr{C}$ is a small symmetric monoidal category and $\mathscr{D}$ is a cocomplete symmetric monoidal category (where $\otimes$ preserves colimits in each variable) then the functor category $\mathsf{Fun}(\mathscr{C}, \mathscr{D})$ can be given a symmetric monoidal structure where the tensor product of F and G is the left Kan extension in the diagram

$$\begin{tikzcd}
\mathscr{C} \times \mathscr{C} \arrow[r, "F \times G"] \arrow[d, "\otimes_{\mathscr{C}}"'] & \mathscr{D} \times \mathscr{D} \arrow[r, "\otimes_{\mathscr{D}}"] & \mathscr{D} \\
\mathscr{C}. \arrow[urr, dotted, "F \otimes G"'] & &
\end{tikzcd}$$

We thus have the formula

$$(F \otimes G)(c) \cong \operatorname*{colim}_{(x,y,x\otimes_{\mathscr{C}} y \to c)} F(x) \otimes_{\mathscr{D}} F(y).$$

This is the *Day convolution* symmetric monoidal structure, first introduced in [Day70].

[2] You may stop at any time by conceding that the ∞-categorical construction is actually not so bad.

A commutative algebra in $\mathsf{Fun}(\mathscr{C},\mathscr{D})$ is the same thing as a lax symmetric monoidal functor. Heuristically, this is because a natural transformation

$$F \otimes F \to F$$

amounts to specifying, for $x, y \in \mathscr{C}$ and a map $x \otimes_{\mathscr{C}} y \to c$, a map $F(x) \otimes_{\mathscr{D}} F(y) \to F(c)$. In particular, we must provide maps $F(x) \otimes_{\mathscr{D}} F(y) \to F(x \otimes_{\mathscr{C}} y)$, and these must induce the rest by functoriality. This description of commutative algebras can be generalized to a universal property, which we may as well state in the ∞-categorical setting:

Theorem 3.2.1 (Glasman, Lurie). *Let $\mathscr{C}$ be a small symmetric monoidal ∞-category and $\mathscr{O}$ an ∞-operad. Then there exists an ∞-operad $\mathsf{Day}_{\mathscr{C}}(\mathscr{O})$ such that for any ∞-operad $\mathscr{P}$ we have a natural equivalence*

$$\mathsf{Alg}_{\mathscr{P}}(\mathsf{Day}_{\mathscr{C}}(\mathscr{O})) \simeq \mathsf{Alg}_{\mathscr{P}\times_{\mathbb{F}_*}\mathscr{C}^{\otimes}}(\mathscr{O}).$$

(In particular, we can identify $\mathsf{Day}_{\mathscr{C}}(\mathscr{O})_{\langle 1\rangle}$ with $\mathsf{Fun}(\mathscr{C},\mathscr{O}_{\langle 1\rangle})$.) Moreover, if $\mathscr{D}$ is a symmetric monoidal ∞-category such that $\mathscr{D}$ is cocomplete and $\otimes$ preserves colimits in each variable, then $\mathsf{Day}_{\mathscr{C}}(\mathscr{D}^{\otimes})$ is also a symmetric monoidal ∞-category (with the same formula for the tensor product as above).

This universal property says that $\mathsf{Day}_{\mathscr{C}}(-)$ is a right adjoint of $-\times_{\mathbb{F}_*}\mathscr{C}^{\otimes}$ (the cartesian product in Opd_{∞}). Note, however, that such a right adjoint does not exist in general, i.e. Opd_{∞} is *not* cartesian closed. We can, however, generalize the Day convolution to the case where $\mathscr{C}$ is "symmetric pro-monoidal", meaning that $\mathscr{C}^{\otimes} \to \mathbb{F}_*$ is an ∞-operad that is a flat/exponentionable/Conduché fibration[3]; see [BGS20] or [Hin20] for details.

If $\mathscr{C}$ is a small symmetric monoidal ∞-category, it can be shown that the mapping space functor has a canonical lax symmetric monoidal structure

$$\mathscr{C}^{\mathrm{op},\otimes} \times_{\mathbb{F}_*} \mathscr{C}^{\otimes} \to \mathcal{S}^{\times}.$$

If we think of $\mathsf{P}(\mathscr{C}) := \mathsf{Fun}(\mathscr{C}^{\mathrm{op}},\mathcal{S})$ as a symmetric monoidal ∞-category via the Day convolution $\mathsf{Day}_{\mathscr{C}^{\mathrm{op}}}(\mathcal{S}^{\times})$, we then get by adjunction that the Yoneda embedding is a lax symmetric monoidal functor

$$\mathscr{C}^{\otimes} \to \mathsf{P}(\mathscr{C})^{\otimes}.$$

This gives $\mathsf{P}(\mathscr{C})^{\otimes}$ a universal property for certain maps *out* of it:

Theorem 3.2.2 (Glasman [Gla16]). *Let $\mathscr{D}$ be a cocomplete symmetric monoidal ∞-category where $\otimes$ preserves colimits in each variable. Restricting along the Yoneda embedding gives an equivalence between colimit-preserving symmetric monoidal functor $\mathsf{P}(\mathscr{C}) \to \mathscr{D}$ and symmetric monoidal functors $\mathscr{C} \to \mathscr{D}$.*

[3]This means that pullback along the functor in Cat_{∞} has a right adjoint, or equivalently preserves colimits.

Remark 3.2.3. There is a particularly simple construction of the Day convolution on presheaves, due to Heine [Hei18]: We have a straightening equivalence between $\mathsf{Fun}(\mathscr{C}, \mathcal{S})$ and left fibrations over $\mathscr{C}$, which form a full subcategory $\mathsf{LFib}(\mathscr{C}) \subseteq \mathsf{Cat}_{\infty/\mathscr{C}}$. The full subcategory $\mathsf{LFib} \subseteq \mathsf{Ar}(\mathsf{Cat}_\infty)$ spanned by left fibrations is closed under cartesian products in the arrow ∞-category, and we can define $\mathsf{Day}_{\mathscr{C}}(\mathcal{S}^\times)$ as the pullback

$$\begin{tikzcd} \mathsf{Day}_{\mathscr{C}}(\mathcal{S}^\times) \arrow[r] \arrow[d] & \mathsf{LFib}^\times \arrow[d, "\mathrm{ev}_1"] \\ \mathbb{F}_* \arrow[r, "\mathscr{C}"] & \mathsf{Cat}_\infty^\times. \end{tikzcd}$$

This pullback is automatically an ∞-operad (since limits in Opd_∞ are computed in ∞-categories). To show that it is symmetric monoidal we use that a symmetric monoidal functor, such as $\mathsf{LFib}^\times \to \mathsf{Cat}_\infty^\times$, is easily seen to be a cocartesian fibration if and only if its underlying functor is one. Here $\mathrm{ev}_1 \colon \mathsf{LFib} \to \mathsf{Cat}_\infty$ is a cocartesian fibration as a consequence of left fibrations being the right part of a factorization system in Cat_∞ (with the left part being the coinitial functors), as in Observation 3.1.3. Given functors $F, G \colon \mathscr{C} \to \mathcal{S}$ corresponding to left fibrations $\mathscr{F}, \mathscr{G} \to \mathscr{C}$, this means that the Day convolution tensor $F \otimes G$ corresponds to the left fibration obtained by factoring the composite

$$\mathscr{F} \times \mathscr{G} \to \mathscr{C} \times \mathscr{C} \xrightarrow{\otimes} \mathscr{C}$$

as a coinitial functor followed by a left fibration. A similar construction using the cartesian product of cocartesian fibrations has been shown by Ramzi [Ram22] to produce the Day convolutions $\mathsf{Day}_{\mathscr{C}}(\mathsf{Cat}_\infty^\times)$.

We now want to look at how the Day convolution can be used to construct some interesting examples of symmetric monoidal ∞-categories:

Example 3.2.4 (Pushout-product)**.** The category $[1] = \{0 \to 1\}$ has a symmetric monoidal structure given by $i \otimes j = \min(i, j)$. If $\mathscr{C}$ is a symmetric monoidal ∞-category compatible with pushouts, we get from this a Day convolution on $\mathsf{Ar}(\mathscr{C}) = \mathsf{Fun}([1], \mathscr{C})$ given by

$$(x \xrightarrow{f} y) \otimes (x' \xrightarrow{g} y') \simeq x \otimes y' \amalg_{x \otimes x'} y \otimes x' \to y \otimes y'.$$

In other words, this is the *pushout-product* of arrows induced by the tensor product on $\mathscr{C}$.

We can use this to define the smash product on pointed spaces, if we first quote a result about localizations:

Theorem 3.2.5 ([Lur17, 2.2.1.9])**.** *Let $\mathscr{C}$ be a symmetric monoidal ∞-category, and suppose $\mathscr{C}_0 \subseteq \mathscr{C}$ is a full subcategory such that the inclusion $i \colon \mathscr{C}_0 \hookrightarrow \mathscr{C}$ has a left* *adjoint L. (Thus $\mathscr{C}_0$ is a localization of $\mathscr{C}$.) Assume furthermore that L-equivalences are closed under the tensor product, i.e. if $L(x \to y)$ is an equivalence then so is $L(x \otimes c \to y \otimes c)$ for all c, and define $\mathscr{C}_0^{\otimes}$ to be the full subcategory of $\mathscr{C}^{\otimes}$ on the objects whose inert cocartesian projections over $\langle 1 \rangle$ lie in $\mathscr{C}_0$. Then:*

(1) $\mathscr{C}_0^{\otimes}$ is a symmetric monoidal ∞-category[4] *and the inclusion $i^{\otimes}\colon \mathscr{C}_0^{\otimes} \hookrightarrow \mathscr{C}^{\otimes}$ is lax symmetric monoidal.*

(2) $i^{\otimes}$ has a left adjoint $L^{\otimes}$ which is a symmetric monoidal structure on L.[5]

Example 3.2.6 (Smash product on $\mathcal{S}_*$)**.** We can view $\mathcal{S}_* := \mathcal{S}_{*/}$ as a full subcategory of $\mathsf{Ar}(\mathcal{S})$, and the inclusion $i\colon \mathcal{S}_* \hookrightarrow \mathsf{Ar}(\mathcal{S})$ has a left adjoint L, given by

$$L(X \to Y) \simeq * \to Y/X := Y \amalg_X *,$$

since we then have an equivalence of mapping spaces

$$\left\{ \begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ Y/X \arrow[rr] & & Z \end{tikzcd} \right\}.$$

The localization L is compatible with the pushout-product on $\mathsf{Ar}(\mathcal{S})$: a morphism

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$

is an L-equivalence precisely when $Y/X \to Y'/X'$ is an equivalence, and we can identify $L((X \to Y) \otimes (A \to B))$ with $Y/X \wedge B/A$. Here the *smash product* $(X, x) \wedge (Y, y)$ of two pointed spaces (X, x) and (Y, y) is

$$(X \times Y)/(X \times \{y\} \amalg_{\{x\}\times\{y\}} \{x\} \times Y) \simeq L((* \to X) \otimes (* \to Y)).$$

Theorem 3.2.5 then implies that the smash product gives a symmetric monoidal structure on $\mathcal{S}_*$. (Here we did not really use anything particular to the ∞-category $\mathcal{S}$.)

Example 3.2.7 (Tensor products of commutative monoids and groups)**.** This example is based on [GGN15]. If $\mathscr{C}$ is an ∞-category with finite products, let $\mathsf{CMon}(\mathscr{C}) \subseteq \mathsf{Fun}(\mathbb{F}_*, \mathscr{C})$ denote the full subcategory of commutative monoids. We also define $\mathsf{CGrp}(\mathscr{C})$ to be the full subcategory of those commutative monoids that are *grouplike*, which can be defined as the map

$$M \times M \xrightarrow{(\mathrm{pr}_1, \mu)} M \times M$$

[4]The same definition produces an ∞-operad for an arbitrary full subcategory.

[5]In particular, the tensor product of $x, y \in \mathscr{C}_0$ is $L(x \otimes_{\mathscr{C}} y)$.

being an equivalence (where μ is the multiplication and pr_1 the projection on the first factor), or if $\mathscr{C}$ is $\mathcal{S}$ by the induced commutative monoid structure on $\pi_0 M$ being a group. Both $\mathsf{CMon}(\mathscr{C})$ and $\mathsf{CGrp}(\mathscr{C})$ are (under mild assumptions on $\mathscr{C}$) localizations of $\mathsf{Fun}(\mathbb{F}_*, \mathscr{C})$ that are compatible with Day convolution for $\wedge$ on $\mathbb{F}_*$ and the cartesian product on $\mathscr{C}$. This therefore induces tensor products of commutative monoids and grouplike commutative monoids in $\mathscr{C}$.[6]

Example 3.2.8 (Smash product of spectra)**.** A *spectrum* in the sense of stable homotopy theory can be defined as a functor $X\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$ (where $\mathcal{S}^{\mathrm{fin}}$ is the full subcategory of $\mathcal{S}$ containing the finite CW-complexes) that is

- *reduced*, that is $X(*) \xrightarrow{\sim} *$,
- *excisive*, meaning that X takes pushouts in $\mathcal{S}_*^{\mathrm{fin}}$ to pullbacks in $\mathcal{S}$.

(In particular, X must take the suspension pushout

$$\begin{tikzcd} T \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma T \end{tikzcd}$$

to a pullback square, so that $X(T) \simeq \Omega X(\Sigma T)$. As a special case, $X(S^n) \simeq \Omega X(S^{n+1})$, so that on spheres we get a spectrum in the more traditional sense; this also determines the functor, since every finite space is an iterated pushout of spheres and discs.) We can thus view the ∞-category Sp of spectra as a full subcategory of $\mathsf{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$. This is a localization, which is compatible with Day convolution for the smash product on $\mathcal{S}_*^{\mathrm{fin}}$ and the cartesian product on $\mathcal{S}$, so Sp inherits a symmetric monoidal structure from this — the *smash product* of spectra.

3.3 Algebraic patterns and non-symmetric ∞-operads

When working with homotopy-coherent algebraic structures, we can sometimes describe them in ways that are combinatorially simpler than using ∞-operads. For example, even for 1-categorical structures it can be convenient to work with *non-symmetric* (or *planar*) operads, which are defined without any symmetric group actions, when this is possible (such as for associative algebras and their (bi)modules). One can show that such non-symmetric operads are equivalent to (symmetric) operads equipped with a map to the (symmetric) operad Assoc for associative algebras. In this section, we want to discuss the ∞-categorical analogue of this comparison. To do so, it is convenient to introduce

[6] This tensor product on $\mathsf{CGrp}(\mathcal{S})$ is an ∞-categorical analogue of the tensor product of abelian groups.

some general language for describing "structures similar to ∞-operads" (especially since it will occasionally be useful to have this available later on), in the form of *algebraic patterns*. This material is based on [CH21] and [BHS22].

Definition 3.3.1. An *algebraic pattern*[7] is an ∞-category $\mathfrak{O}$ equipped with a factorization system $(\mathfrak{O}^{\mathrm{int}}, \mathfrak{O}^{\mathrm{act}})$ such that every morphism $X \to Z$ factors uniquely as $X \rightarrowtail Y \rightsquigarrow Z$ where the first morphism is *inert* and the second is *active*, as well as full subcategory $\mathfrak{O}^{\mathrm{el}} \subseteq \mathfrak{O}^{\mathrm{int}}$ consisting of *elementary* objects.

Notation 3.3.2. If $\mathfrak{O}$ is an algebraic pattern, then for $O \in \mathfrak{O}$ we write

$$\mathfrak{O}^{\mathrm{el}}_{O/} := \mathfrak{O}^{\mathrm{el}} \times_{\mathfrak{O}^{\mathrm{int}}} \mathfrak{O}^{\mathrm{int}}_{O/}$$

for the ∞-category of inert maps $O \rightarrowtail E$ with E elementary.

Definition 3.3.3. If $\mathfrak{O}$ is an algebraic pattern and $\mathscr{C}$ is an ∞-category, we say that a functor $F\colon \mathfrak{O} \to \mathscr{C}$ is a *Segal $\mathfrak{O}$-object* in $\mathscr{C}$ if for all $O \in \mathfrak{O}$, the canonical cone

$$(\mathfrak{O}^{\mathrm{el}}_{O/})^{\triangleleft} \to \mathfrak{O} \xrightarrow{F} \mathscr{C}$$

is a limit, so that we have a "Segal condition"

$$F(O) \simeq \lim_{E \in \mathfrak{O}^{\mathrm{el}}_{O/}} F(E).$$

If $\mathscr{C}$ is the ∞-category $\mathcal{S}$ of spaces we often refer to Segal $\mathfrak{O}$-objects as *Segal $\mathfrak{O}$-spaces*, while if $\mathscr{C}$ is the ∞-category Cat_∞ of ∞-categories we call them *Segal $\mathfrak{O}$-∞-categories*.

Examples 3.3.4.

(i) The pattern $\mathbb{F}_*^\flat$ consists of the category $\mathbb{F}_*$ together with its inert–active factorization system, and with $\langle 1\rangle$ as the only elementary object. Then $(\mathbb{F}_*^\flat)^{\mathrm{el}}_{\langle n\rangle/}$ is the discrete set $\{\rho_i\colon \langle n\rangle \to \langle 1\rangle\}$, and a Segal $\mathbb{F}_*^\flat$-object is precisely a commutative monoid.

(ii) As a variant of the previous example, we can consider the pattern $\mathbb{F}_*^\natural$ where we take $\mathbb{F}_*$ with the same factorization system, but now with both $\langle 1\rangle$ and $\langle 0\rangle$ as elementary objects. Then $(\mathbb{F}_*^\flat)^{\mathrm{el}}_{\langle n\rangle/}$ is the poset

$$\begin{tikzcd}
\rho_1 \arrow[dr] & \cdots \arrow[d] & \rho_n \arrow[dl] \\
& (\langle n\rangle \to \langle 0\rangle), &
\end{tikzcd}$$

[7] This name is inspired by Lurie's *categorical patterns* [Lur17], and should not be confused with the *patterns* of Getzler [Get09].

so the Segal condition for F is

$$F(\langle n\rangle) \simeq F(\langle 1\rangle) \times_{F(\langle 0\rangle)} \cdots \times_{F(\langle 0\rangle)} F(\langle 1\rangle);$$

a functor F is a Segal $\mathbb{F}_*^\natural$-object in $\mathscr{C}$ if and only if it is a commutative monoid in $\mathscr{C}_{/F(\langle 0\rangle)}$.

(iii) Let $\mathcal{O}$ be an ∞-operad. Then we can consider the pattern $\mathcal{O}^\flat$ given by the inert–active factorization system on $\mathcal{O}$, with the elementary objects being all objects over $\langle 1\rangle$. (Thus $\mathcal{O}^{\flat,\mathrm{el}} \simeq \mathcal{O}^{\simeq}_{\langle 1\rangle}$.) Here $\mathcal{O}^{\flat,\mathrm{el}}_{X/}$ is the discrete set of cocartesian morphisms $X \to X_i$ over ρ_i, and a Segal $\mathcal{O}^\flat$-object is precisely an $\mathcal{O}$-monoid.

(iv) Let Δ be the simplex category, i.e. the category of non-empty ordered sets $[n] = \{0 < 1 < \cdots < n\}$ $(n \geq 0)$. A map $\phi\colon [n] \to [m]$ is *inert* if it is a subinterval inclusion (so $\phi(i) = \phi(0) + i$) and *active* if it is boundary-preserving (so $\phi(0) = 0$ and $\phi(n) = m$). This gives an active-inert factorization system on Δ, and so an inert–active one on Δ^{op}. We define the pattern $\Delta^{\mathrm{op},\natural}$ using this factorization system, with $[0]$ and $[1]$ as the elementary objects. Then $\Delta^{\mathrm{op},\natural,\mathrm{el}}$ is $[1] \rightrightarrows [0]$, and we can depict $(\Delta^{\mathrm{op},\natural,\mathrm{el}})_{[n]/}$ as the poset

$$\begin{tikzcd}[column sep=small]
 & \{0,1\} \arrow[dl] \arrow[dr] & & \{1,2\} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \{n-1,n\} \arrow[dl] \arrow[dr] & \\
\{0\} & & \{1\} & & \{2\} \quad \cdots & & \{n-1\} & & \{n\}
\end{tikzcd}$$

(where we describe an inert map by its image). A functor $F\colon \Delta^{\mathrm{op}} \to \mathscr{C}$ is a Segal $\Delta^{\mathrm{op},\natural}$-object if

$$F([n]) \simeq F([1]) \times_{F([0])} \cdots \times_{F([0])} F([1]),$$

i.e. if it satisfies Rezk's Segal condition [Rez01] — in particular, Segal $\Delta^{\mathrm{op},\natural}$-spaces are precisely *Segal spaces*.

(v) We can also consider the pattern $\Delta^{\mathrm{op},\flat}$ where only $[1]$ is elementary. Here the Segal condition is

$$F([n]) \simeq F([1])^{\times n},$$

and Segal $\Delta^{\mathrm{op},\flat}$-objects can be identified with associative monoids. (This is the relevant pattern for non-symmetric ∞-operads.)

(vi) Let $\mathsf{Span}(\mathbb{F})$ denote the (2,1)-category whose objects are finite sets and whose morphisms from S to T are *spans* of finite sets

$$\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\
S & & T,
\end{tikzcd}$$

with composition given by taking pullbacks. We get a factorization system if we take the inert maps to be those where g is an isomorphism (i.e. the "backwards" maps) and the active ones to be those where f is an isomorphism (the "forwards" maps). We can define a pattern $\mathsf{Span}(\mathbb{F})^\flat$ where $\mathbf{1}$ (the one-point set) is the only elementary object; its Segal objects are equivalent to commutative monoids.

Other examples include the dendroidal category Ω^{op}, which can be used to describe ∞-operads (we will come back to this in the next chapter), the product $\Delta^{n,\mathrm{op}}$ which can be used to describe n-fold ∞-categories, the wreath product Θ_n^{op}, which can be used to describe (∞, n)-categories [Rez10], as well as categories of trees and graphs that are related to ∞-properads, cyclic ∞-operads, and modular ∞-operads [HRY15, HRY19, HRY20].

Remark 3.3.5. We can define a natural analogue of the category of operators for a *non-symmetric* operad as a category over Δ^{op}.[8] The categories that arise in this way can be described by a variant of the conditions from Proposition 2.2.11, which suggests a definition of non-symmetric ∞-operads as certain ∞-categories over Δ^{op}. We may as well spell out the general version of this definition over an arbitrary pattern:

Definition 3.3.6. Let $\mathfrak{O}$ be an algebraic pattern. A *weak Segal $\mathfrak{O}$-fibration* is a functor of ∞-categories $p\colon \mathscr{E} \to \mathfrak{O}$ such that:

(1) $\mathscr{E}$ has p-cocartesian lifts of inert morphisms in $\mathfrak{O}$.

(2) For $X \in \mathscr{E}_O$, if $\xi\colon (\mathfrak{O}^{\mathrm{el}}_{O/})^{\triangleleft} \to \mathscr{E}$ is a diagram of cocartesian morphisms over the objects of $\mathfrak{O}^{\mathrm{el}}_{O/}$, then for $Y \in \mathscr{E}_{O'}$, the commutative square

$$\begin{tikzcd}
\mathsf{Map}_{\mathscr{E}}(Y, X) \arrow[r] \arrow[d] & \lim_{\alpha\colon O \rightarrowtail E \in \mathfrak{O}^{\mathrm{el}}_{O/}} \mathsf{Map}_{\mathscr{E}}(Y, \xi(\alpha)) \arrow[d] \\
\mathsf{Map}_{\mathfrak{O}}(O', O) \arrow[r] & \lim_{\alpha\colon O \rightarrowtail E \in \mathfrak{O}^{\mathrm{el}}_{O/}} \mathsf{Map}_{\mathfrak{O}}(O', E)
\end{tikzcd}$$

is cartesian.

(3) The functor $\mathscr{E}_O \to \lim_{E \in \mathfrak{O}^{\mathrm{el}}_{O/}} \mathscr{E}_E$ is essentially surjective.

We will also refer to weak Segal $\mathfrak{O}$-fibrations as *$\mathfrak{O}$-∞-operads* when appropriate. (In all the examples we consider here they also coincide with the *fibrous $\mathfrak{O}$-patterns* used in [BHS22].) We will write $\mathsf{WSF}(\mathfrak{O})$ for the ∞-category of weak Segal $\mathfrak{O}$-fibrations and functors over $\mathfrak{O}$ that preserve inert cocartesian morphisms; we may also use $\mathsf{Opd}(\mathfrak{O})$ when we instead call them $\mathfrak{O}$-∞-operads.

[8] Barwick's theory of *operator categories* [Bar18] give an setting where Δ^{op} is obtained from finite ordered sets by a construction that also produces $\mathbb{F}_*$ from finite sets.

Examples 3.3.7.

(i) Weak Segal fibrations over $\mathbb{F}_*^\flat$ are precisely ∞-operads.

(ii) Weak Segal fibrations over $\mathbb{F}_*^\natural$ are *generalized ∞-operads* in the terminology of [Lur17]. They can be viewed as a generalization of ∞-operads where we don't require the fibre over $\langle 0\rangle$ to be a point, and we replace the cartesian products in the definition by fibre products over this. Let's write $\mathsf{Opd}_\infty^{\mathrm{gen}}$ for the ∞-category of these.

(iii) If $\mathcal{O}$ is an ∞-operad, then weak Segal fibrations over $\mathcal{O}^\flat$ can be identified with ∞-operads over $\mathcal{O}$.

(iv) Weak Segal fibrations over $\Delta^{\mathrm{op},\flat}$ are *non-symmetric* or *planar ∞-operads*; we'll write $\mathsf{Opd}_\infty^{\mathrm{ns}}$ for the ∞-category of these.

(v) Weak Segal fibrations over $\Delta^{\mathrm{op},\natural}$ are the non-symmetric version of generalized ∞-operads, i.e. "generalized non-symmetric ∞-operads". They can be viewed as an ∞-categorical version of the structures called *fc-multicategories* by [Lei02] and *virtual double categories* by Cruttwell and Shulman [CS10].

We point out some relations between these structures:

- Opd_∞ is a full subcategory of $\mathsf{Opd}_\infty^{\mathrm{gen}}$, which is closed under limits and filtered colimits (since these can be shown to be computed in Cat_∞ in both cases). Since these ∞-categories are presentable, it follows that there exists a localization functor $L_{\mathrm{gen}}\colon \mathsf{Opd}_\infty^{\mathrm{gen}} \to \mathsf{Opd}_\infty$, left adjoint to the inclusion.

- If $f\colon \mathcal{O} \to \mathcal{P}$ is a functor between algebraic patterns that preserves the factorization system and elementary objects, and moreover the induced functor
$$\mathcal{O}^{\mathrm{el}}_{X/} \to \mathcal{P}^{\mathrm{el}}_{f(X)/}$$
is coinitial for all X, then pullback along f gives a functor
$$f^*\colon \mathsf{WSF}(\mathcal{P}) \to \mathsf{WSF}(\mathcal{O}).$$
Under mild assumptions, this has a left adjoint. For example, we can define a functor $\mathrm{cut}\colon \Delta^{\mathrm{op}} \to \mathbb{F}_*$ that takes $[n]$ to $\langle n\rangle$ and a morphism $\phi\colon [n] \to [m]$ in Δ to the map $\mathrm{cut}(\phi)\colon \langle m\rangle \to \langle n\rangle$ given by
$$\mathrm{cut}(\phi)(i) = \begin{cases} j, & \phi(j-1) < i \leq \phi(j), \\ 0, & \text{otherwise.} \end{cases}$$
Pulling back along this gives a functor $\mathsf{Opd}_\infty \to \mathsf{Opd}_\infty^{\mathrm{ns}}$ that informally "forgets symmetric group actions". Its left adjoint is a "symmetrization" functor $\mathsf{Opd}_\infty^{\mathrm{ns}} \to \mathsf{Opd}_\infty$.

- Lurie proves a comparison result [Lur17, 2.3.3.26] that gives conditions for certain functors as above to induce equivalences. For example, [Lur17, 4.1.3.14] applies this to shows that a variant cut: $\Delta^{\mathrm{op}} \to \mathsf{Assoc}$ of the functor above gives an equivalence

$$\mathsf{Opd}_{\infty}^{\mathrm{ns}} \xrightarrow{\sim} \mathsf{Opd}_{\infty/\mathsf{Assoc}}.$$

  In [BHS22] we generalized Lurie's comparison result, which makes it easy to see that the inclusion of $\mathbb{F}_*$ as a subcategory of $\mathsf{Span}(\mathbb{F})$ induces an equivalence

$$\mathsf{Opd}_{\infty} \simeq \mathsf{Opd}(\mathsf{Span}(\mathbb{F})).$$

3.4 The Boardman–Vogt tensor product

In this section we introduce the Boardman–Vogt tensor product of ∞-operads, which also allows us to discuss the additivity theorem for the E_n-operads. This section is based on [Lur17, 2.2.5, 5.1.2] and [Bar18]; see also Harpaz's notes [Har19] for an account of (a variant of) Lurie's proof of additivity.

To motivate the Boardman–Vogt tensor product for operads in sets, we start by considering a symmetric monoidal category $\mathscr{C}$ and an operad $\mathbb{O}$ with (just for simplicity) a single object. Then the category $\mathsf{Alg}_{\mathbb{O}}(\mathscr{C})$ of $\mathbb{O}$-algebras in $\mathscr{C}$ has a symmetric monoidal structure induced by the tensor product in $\mathscr{C}$: if A and B are $\mathbb{O}$-algebras, we can make $A \otimes B$ an $\mathbb{O}$-algebra via the composite[9]

$$\mathbb{O}(n) \times_{\Sigma_n} (A \otimes B)^{\otimes n} \to \big(\mathbb{O}(n) \times_{\Sigma_n} A^{\otimes n}\big) \otimes \big(\mathbb{O}(n) \times_{\Sigma_n} (B)^{\otimes n}\big) \to A \otimes B.$$

More generally, if $\mathbb{O}$ and $\mathscr{P}$ are arbitrary operads, we can make $\mathsf{Alg}_{\mathbb{O}}(\mathscr{P})$ into an operad: a multimorphism $(A_1, \ldots, A_n) \to B$ consists of a multimorphism $(A_1 O, \ldots, A_n O) \to BO$ in $\mathscr{P}$ for every $O \in \mathbb{O}$, such that for every multimorphism $(O_1, \ldots, O_k) \to O$ in $\mathbb{O}$, the square

$$\begin{tikzcd}
(A_1(O_1), \ldots, A_1(O_k), \ldots, A_n(O_1), \ldots, A_n(O_k)) \arrow[r] \arrow[d] & (A_1(O), \ldots, A_n(O)) \arrow[d] \\
(B(O_1), \ldots, B(O_k)) \arrow[r] & B(O)
\end{tikzcd}$$

commutes.

This construction is adjoint to the *Boardman–Vogt tensor product*: functors $\mathbb{O} \to \mathsf{Alg}_{\mathscr{P}}(\mathbb{Q})$ are equivalent to functors $\mathbb{O} \otimes \mathscr{P} \to \mathbb{Q}$. Here $\mathbb{O} \otimes \mathscr{P}$ can be given a generators-and-relations description: its objects are $\mathrm{ob}\,\mathbb{O} \times \mathrm{ob}\,\mathscr{P}$, and its multimorphisms are generated by operations

$$((O, P_1), \ldots, (O, P_n)) \to (O, P)$$

[9]Note that this uses the diagonal of $\mathbb{O}(n)$, so this is not possible for more general kinds of enriched operads!

for every $O \in \mathbb{O}$ and every multimorphism $(P_1, \ldots, P_n) \to P$ in $\mathcal{P}$, and vice versa, with relations enforcing compatibility with compositions and permutations. We can also describe maps out of the Boardman–Vogt tensor as *bifunctors* of operads, which admit a simple description in terms of categories of operators. This leads to the following ∞-categorical definition:

Definition 3.4.1. Let $\mu\colon \mathbb{F}_* \times \mathbb{F}_* \to \mathbb{F}_*$ denote the *smash product* functor ($\mu(\langle n\rangle, \langle m\rangle) = \langle n\rangle \wedge \langle m\rangle \cong \langle nm\rangle$). If $\mathbb{O}, \mathcal{P}, \mathbb{Q}$ are ∞-operads, then a *bifunctor* of ∞-operads $(\mathbb{O}, \mathcal{P}) \to \mathbb{Q}$ is a commutative square

$$\begin{tikzcd} \mathbb{O} \times \mathcal{P} \arrow[r, "\phi"] \arrow[d] & \mathbb{Q} \arrow[d] \\ \mathbb{F}_* \times \mathbb{F}_* \arrow[r, "\mu"] & \mathbb{F}_* \end{tikzcd}$$

such that ϕ takes pairs of inert morphisms to inert morphisms in $\mathbb{Q}$. It can be shown that such bifunctors are corepresented by an ∞-operad $\mathbb{O} \otimes \mathcal{P}$ — the *Boardman–Vogt tensor product*.

Remark 3.4.2. We can think of $\mathbb{O} \times \mathcal{P} \to \mathbb{F}_* \times \mathbb{F}_*$ as an "external" Boardman–Vogt tensor product. As shown by Barwick [Bar18] these make sense and admit right adjoints more generally.

Remark 3.4.3. We can interpret the Boardman–Vogt tensor product in terms of the terminology of the previous section: If $\mathbb{O}$ and $\mathcal{P}$ are ∞-operads then $\mathbb{O} \times \mathcal{P}$ is *not* a weak Segal fibration for $\mathbb{F}_*^\flat \times \mathbb{F}_*^\flat$ (where the only elementary object is $(\langle 1\rangle, \langle 1\rangle)$), but it *is* one for $\mathbb{F}_*^\natural \times \mathbb{F}_*^\natural$. The smash product functor μ gives a morphism of algebraic patterns $\mathbb{F}_*^\natural \times \mathbb{F}_*^\natural \to \mathbb{F}_*^\natural$, and this induces an adjunction

$$\mu_! : \mathsf{WSF}(\mathbb{F}_*^\natural \times \mathbb{F}_*^\natural) \rightleftarrows \mathsf{Opd}_\infty^{\mathrm{gen}} : \mu^*.$$

We can then think of a bifunctor as a map

$$\begin{tikzcd} \mathbb{O} \times \mathcal{P} \arrow[rr] \arrow[dr] & & \mu^*\mathbb{Q} \arrow[dl] \\ & \mathbb{F}_*^\natural \times \mathbb{F}_*^\natural, & \end{tikzcd}$$

so $\mathbb{O} \otimes \mathcal{P}$ is the ∞-operad $L_{\mathrm{gen}}\mu_!(\mathbb{O} \times \mathcal{P})$.

Theorem 3.4.4 (Lurie)**.** *The ∞-category Opd_∞ has a symmetric monoidal structure given by the Boardman–Vogt tensor product, which preserves colimits in each variable. The unit is $\mathbb{F}_*^{\mathrm{int}}$ (which is the category of operators of the operad with one object and no non-trivial operations).*

Corollary 3.4.5. *The ∞-category Opd_∞ has internal Homs $\mathsf{ALG}_{\mathbb{O}}(\mathcal{P})$ with natural equivalences*

$$\mathsf{Alg}_{\mathbb{O} \otimes \mathcal{P}}(\mathbb{Q}) \simeq \mathsf{Alg}_{\mathbb{O}}(\mathsf{ALG}_{\mathcal{P}}(\mathbb{Q})).$$

Observation 3.4.6. We have a natural equivalence $\mathsf{Alg}_{\mathbb{F}_*^{\mathrm{int}}}(\mathcal{O}) \simeq \mathcal{O}_{\langle 1 \rangle}$, so for ∞-operads $\mathcal{P}, \mathbb{Q}$ we have

$$\mathsf{ALG}_{\mathcal{P}}(\mathbb{Q})_{\langle 1 \rangle} \simeq \mathsf{Alg}_{\mathbb{F}_* \otimes \mathcal{P}}(\mathbb{Q}) \simeq \mathsf{Alg}_{\mathcal{P}}(\mathbb{Q}),$$

so that $\mathsf{ALG}_{\mathcal{P}}(\mathbb{Q})$ is an ∞-operad with underlying ∞-category $\mathsf{Alg}_{\mathcal{P}}(\mathbb{Q})$. If $\mathbb{Q}$ is a symmetric monoidal ∞-category, it can furthermore be shown that so is $\mathsf{ALG}_{\mathcal{P}}(\mathbb{Q})$ for any ∞-operad $\mathbb{Q}$.

Definition 3.4.7. The *little discs operad E_n* is a one-object operad in Top where $E_n(k)$ is the space of embeddings of k n-dimensional discs in a bigger one; composition of these multimorphisms is given by inserting such configurations into each other. This gives rise to an ∞-operad we'll denote $\mathbb{E}_n$.

We note some important facts about these (∞-)operads:

- The space $E_n(k)$ is homotopy equivalent to the configuration space of k points in $\mathbb{R}^n$.
- The ∞-operad $\mathbb{E}_1$ is just the associative operad Assoc.
- The colimit $\mathbb{E}_\infty := \mathrm{colim}_{n \to \infty} \mathbb{E}_n$ is the commutative operad $\mathbb{F}_*$ (because the spaces in $\mathbb{E}_n$ get increasingly highly connected as n increases, so in the limit they become contractible).

Theorem 3.4.8 (Additivity)**.** *The Boardman–Vogt tensor product $\mathbb{E}_n \otimes \mathbb{E}_m$ is the ∞-operad $\mathbb{E}_{n+m}$. Equivalently, we have*

$$\mathbb{E}_1^{\otimes n} \simeq \mathbb{E}_n. \tag{3.1}$$

Remark 3.4.9. The ∞-categorical version of the additivity theorem is due to Lurie. Analogues of the result for specific models of E_n-operads were previously proven by Dunn [Dun88] and Fiedorowicz–Vogt [FV15].

The Eckmann–Hilton argument shows that a set (or more generally an object in a 1-category) with two compatible associative multiplications is just a commutative monoid. In an ∞-category, the equivalence (3.1) says that the algebras for $\mathbb{E}_n$ are objects with n compatible associative multiplications, so here we get an infinite hierarchy of algebraic structures that lie between between associative and commutative algebras. We can interpolate between these situations by considering truncated versions of ∞-categories:

Definition 3.4.10. An ∞-category $\mathcal{C}$ is an $(n, 1)$*-category* if the mapping space $\mathsf{Map}_{\mathcal{C}}(x, y)$ is an $(n-1)$*-groupoid* (meaning a space X such that all homotopy groups $\pi_i X = 0$ for $i > n-1$) for all $x, y \in \mathcal{C}$. (Thus a $(1, 1)$-category is one where the mapping spaces are sets, i.e. an ordinary category.)

Corollary 3.4.11 (Lurie). *If $\mathscr{C}$ is a symmetric monoidal $(n, 1)$-category, then*

$$\mathsf{Alg}_{\mathbb{E}_k}(\mathscr{C}) \simeq \mathsf{CAlg}(\mathscr{C})$$

whenever $k \geq n + 1$.

The key idea of the proof is that since the spaces in the ∞-operad $\mathbb{E}_k$ become increasingly connected as k increases, we eventually can't tell them apart from points when we map them into the truncated mapping spaces of $\mathscr{C}$.

Applied to n-categories, this corollary confirms a conjecture of Baez and Dolan [BD95]:

Corollary 3.4.12 (Baez–Dolan stabilization). *k-uply monoidal n-categories, by which we mean $\mathbb{E}_k$-algebras in the ∞-category Cat_n of n-categories, are equivalent to symmetric monoidal n-categories if $k \geq n + 2$.*

Remark 3.4.13. To deduce this from the previous corollary we only need to check that Cat_n is an $(n + 1, 1)$-category. This was done in [GH15], using that we can inductively define n-categories as ∞-categories enriched in $(n - 1)$-categories. The first proof of (a weaker form of) Baez–Dolan stabilization is due to Simpson [Sim98].

For example, $\mathbb{E}_k$-algebras in Cat are monoidal categories for $k = 1$, braided monoidal categories for $k = 2$, and symmetric monoidal categories for $k \geq 3$. Similarly, $\mathbb{E}_k$-algebras in 2-categories are, respectively, monoidal, braided monoidal, and sylleptic monoidal 2-categories for $k = 1, 2, 3$, and symmetric monoidal 2-categories for $k \geq 4$.[10]

We note another surprisingly useful consequence of additivity:

Corollary 3.4.14. *To define a symmetric monoidal (∞, n)-category, it suffices to define a compatible sequence of k-fold monoids (monoids for $(\Delta^{op})^{\times k}$) in (∞, n)-categories for all k.*

This is useful to construct symmetric monoidal structures on bordism (∞, n)-categories, for example. Without passing to the description of k-fold monoids as $\mathbb{E}_k$-algebras from the additivity theorem, it is not clear that this data should "converge" to a symmetric monoidal structure.

3.5 Free algebras

An important feature of algebraic structures that can be described as algebras for operads is that there is an explicit formula for free algebras: If $\mathbb{O}$ is a one-object operad in sets and $\mathscr{C}$ is a symmetric monoidal category (which has colimits

[10] At least morally speaking — it's not clear to me if these comparisons to the classical definitions of monoidal structures on 2-categories have been worked out in detail...

indexed by groupoids, and the tensor product preserves these in each variable), then the free $\mathcal{O}$-algebra on an object $X \in \mathcal{C}$ is given by

$$\coprod_{n=0}^{\infty} \mathcal{O}(n) \otimes_{\Sigma_n} X^{\otimes n}, \tag{3.2}$$

where $\mathcal{O}(n) \otimes_{\Sigma_n} X^{\otimes n}$ means the quotient of $\coprod_{\mathcal{O}(n)} X^{\otimes n}$ by the diagonal Σ_n-action. In this section we will look at an ∞-categorical version of this result, as well as a more general description of left adjoints: if $f : \mathcal{O} \to \mathcal{P}$ is a morphism of ∞-operads, then composition with f gives a functor

$$f^* : \mathsf{Alg}_{\mathcal{P}}(\mathcal{C}) \to \mathsf{Alg}_{\mathcal{O}}(\mathcal{C}),$$

and if $\mathcal{C}$ is a symmetric monoidal ∞-category (with certain well-behaved colimits) then this has a left adjoint

$$f_! : \mathsf{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathsf{Alg}_{\mathcal{P}}(\mathcal{C})$$

given by an explicit colimit formula. This section is based on [Lur17, 3.1] and [CH22].

Notation 3.5.1. If $f : \mathcal{O} \to \mathcal{P}$ is a morphism of ∞-operads and P is an object of $\mathcal{P}$, we write

$$\mathcal{O}^{\mathrm{act}}_{/P} := \mathcal{O}^{\mathrm{act}} \times_{\mathcal{P}^{\mathrm{act}}} \mathcal{P}^{\mathrm{act}}_{/P}$$

for the ∞-category of objects $O \in \mathcal{O}$ together with an active map $f(O) \rightsquigarrow P$, with active maps between these.

Definition 3.5.2. A symmetric monoidal ∞-category $\mathcal{C}$ is *compatible with K-indexed colimits* for some class K of ∞-categories if the underlying ∞-category $\mathcal{C}$ has K-indexed colimits, and for every $X \in \mathcal{C}$, the functor $X \otimes - : \mathcal{C} \to \mathcal{C}$ preserves K-indexed colimits.

Theorem 3.5.3. *Suppose $f : \mathcal{O} \to \mathcal{P}$ is a morphism of ∞-operads and $\mathcal{C}$ is a symmetric monoidal ∞-category that is compatible with $\mathcal{O}^{\mathrm{act}}_{/P}$-indexed colimits for all $P \in \mathcal{P}$. Then the functor $f^* : \mathsf{Alg}_{\mathcal{P}}(\mathcal{C}) \to \mathsf{Alg}_{\mathcal{O}}(\mathcal{C})$ has a left adjoint $f_!$ (called* operadic left Kan extension *along f) such that for $A \in \mathsf{Alg}_{\mathcal{O}}(\mathcal{P})$, the value of $f_!A$ at $P \in \mathcal{P}_{\langle 1 \rangle}$ is*

$$f_!A(P) \simeq \operatorname*{colim}_{(O, fO \xrightarrow{\alpha} P) \in \mathcal{O}^{\mathrm{act}}_{/P}} \pi(\alpha)_! A(O)$$

where π is the projection $\mathcal{P} \to \mathbb{F}_*$.

Observation 3.5.4. Let $\mathcal{O}$ be an ∞-operad. The subcategory $\mathcal{O}^{\mathrm{int}}$ of (cocartesian) inert morphisms in $\mathcal{O}$ is again an ∞-operad (with *no* non-trivial active morphisms, i.e. no non-trivial multimorphisms). Then $\mathcal{O}^{\mathrm{int}}_{\langle 1 \rangle} \simeq \mathcal{O}^{\simeq}_{\langle 1 \rangle}$, and $(\mathcal{O}^{\mathrm{int}})^{\mathrm{act}}_{/O} \simeq$

$\mathsf{Act}_{\mathcal{O}}(O)$ is the ∞-groupoid of active morphisms in $\mathcal{O}$ with target O. If $\mathcal{P}$ is another ∞-operad, we have an equivalence

$$\mathsf{Alg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{P}) \simeq \mathsf{Fun}(\mathcal{O}^{\simeq}_{\langle 1\rangle}, \mathcal{P}_{\langle 1\rangle}).$$

(This is because we can identify the left-hand side with the ∞-category of maps over $\mathbb{F}_*^{\mathrm{int}}$ that preserve all cocartesian morphisms, and the Segal condition for $\mathcal{P}$ implies that its pullback to $\mathbb{F}_*^{\mathrm{int}}$ is a right Kan extension of its restriction to $\{\langle 1\rangle\}$.)

Corollary 3.5.5. *If $\mathcal{O}$ is an ∞-operad and $\mathcal{C}$ is a symmetric monoidal ∞-category that is compatible with colimits over ∞-groupoids, then*

$$U_{\mathcal{O}} \colon \mathsf{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathsf{Alg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C}) \xrightarrow{\sim} \mathsf{Fun}(\mathcal{O}^{\simeq}_{\langle 1\rangle}, \mathcal{C})$$

has a left adjoint $\mathsf{Free}_{\mathcal{O}}$, which takes $X \colon \mathcal{O}^{\simeq}_{\langle 1\rangle} \to \mathcal{C}$ to the free $\mathcal{O}$-algebra $\mathsf{Free}_{\mathcal{O}}(X)$, *given at $O \in \mathcal{O}_{\langle 1\rangle}$ by*

$$\mathsf{Free}_{\mathcal{O}}(X)(O) \simeq \operatorname*{colim}_{(O'_1,\ldots,O'_n)\rightsquigarrow O \in \mathsf{Act}_{\mathcal{O}}(O)} X(O'_1) \otimes \cdots \otimes X(O'_n). \tag{3.3}$$

Remark 3.5.6. To recover the classical formula (3.2) from (3.3), let us (for simplicity) consider an ∞-operad $\mathcal{O}$ such that $\mathcal{O}^{\simeq}_{\langle 1\rangle} \simeq *$ (so that $\mathcal{O}$ has a single object and this has no non-trivial automorphisms); then $\mathsf{Free}_{\mathcal{O}}$ is a functor $\mathcal{C} \to \mathsf{Alg}_{\mathcal{O}}(\mathcal{C})$. Moreover, we can identify $\mathcal{O}^{\simeq}$ with the groupoid

$$\mathbb{F}_*^{\simeq} \simeq \coprod_{n=0}^{\infty} B\Sigma_n;$$

let us therefore denote the objects of $\mathcal{O}$ as $\langle n\rangle_{\mathcal{O}}$. We have a map $\mathsf{Act}_{\mathcal{O}}(\langle 1\rangle_{\mathcal{O}}) \to \mathsf{Act}_{\mathbb{F}_*}(\langle 1\rangle)$, which we can think of as a map

$$\coprod_{n=0}^{\infty} \left\{ \begin{tikzcd}[row sep=small] \langle n\rangle_{\mathcal{O}} \arrow[d, "\sim"] \arrow[dr] & \\ \langle n\rangle_{\mathcal{O}} \arrow[r] & \langle 1\rangle_{\mathcal{O}} \end{tikzcd} \right\} \to \coprod_{n=0}^{\infty} B\Sigma_n;$$

the fibre of this at a point in $B\Sigma_n$ is then the ∞-groupoid $\mathcal{O}(n)$ of n-ary operations in $\mathcal{O}$ with a Σ_n-action. We can compute a colimit over $\mathsf{Act}_{\mathcal{O}}(\langle 1\rangle_{\mathcal{O}})$ by first taking a left Kan extension along this map to $\coprod_n B\Sigma_n$ (which amounts to taking colimits over its fibres, since this is a map of ∞-groupoids) and then taking a colimit over $\coprod_n B\Sigma_n$. If we apply this to our formula (3.3) we get

$$\mathsf{Free}_{\mathcal{O}}(X) \simeq \operatorname*{colim}_{\langle n\rangle_{\mathcal{O}} \to \langle 1\rangle_{\mathcal{O}} \in \mathsf{Act}_{\mathcal{O}}(\langle 1\rangle)} X^{\otimes n} \simeq \coprod_n \operatorname{colim}_{B\Sigma_n} \operatorname{colim}_{\mathcal{O}(n)} X^{\otimes n} \simeq \coprod_n \mathcal{O}(n) \otimes_{\Sigma_n} X^{\otimes n}.$$

(More generally, if $\mathcal{O}^{\simeq}_{\langle 1\rangle}$ is connected, then the classical formula describes the left adjoint to the composite

$$\mathsf{Alg}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{U_{\mathcal{O}}} \mathsf{Fun}(\mathcal{O}^{\simeq}_{\langle 1\rangle}, \mathcal{C}) \to \mathcal{C},$$

where the second functor is given by restricting to a chosen point in $\mathcal{O}^{\simeq}_{\langle 1\rangle}$.)

We will look at a proof of the formula for free algebras in the simplest possible case, where it can be reduced to an ordinary Kan extension:

Proof of Corollary 3.5.5 for monoids. Let $\mathcal{C}$ be an ∞-category with finite products, where the cartesian product preserves colimits indexed by ∞-groupoids in each variable. If $\mathcal{O}$ is an ∞-operad, we can replace $\mathcal{O}$-algebras in $\mathcal{C}$ by $\mathcal{O}$-monoids (see Section 3.1), and view our functor $U_{\mathcal{O}}$ as the composite

$$\mathsf{Mon}_{\mathcal{O}}(\mathcal{C}) \to \mathsf{Mon}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C}) \xrightarrow{\sim} \mathsf{Fun}(\mathcal{O}^{\simeq}_{\langle 1\rangle}, \mathcal{C}),$$

where the functors are induced by restricting along the inclusions $\mathcal{O}^{\simeq}_{\langle 1\rangle} \xrightarrow{i} \mathcal{O}^{\mathrm{int}} \xrightarrow{j} \mathcal{O}$. It is easy to see that the condition for a functor $\mathcal{O}^{\mathrm{int}} \to \mathcal{C}$ to be a monoid is precisely that it is right Kan extended from $\mathcal{O}^{\simeq}_{\langle 1\rangle}$, so the restriction $\mathsf{Mon}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C}) \to \mathsf{Fun}(\mathcal{O}^{\simeq}_{\langle 1\rangle}, \mathcal{C})$ has an inverse given by right Kan extension. We claim that the functor $\mathsf{Mon}_{\mathcal{O}}(\mathcal{C}) \to \mathsf{Mon}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C})$ has a left adjoint, which is given by left Kan extension along the inclusion $j\colon \mathcal{O}^{\mathrm{int}} \to \mathcal{O}$. This left Kan extension is given by colimits along

$$(\mathcal{O}^{\mathrm{int}})_{/O} := \mathcal{O}^{\mathrm{int}} \times_{\mathcal{O}} \mathcal{O}_{/O},$$

and from the inert–active factorization system on $\mathcal{O}$ it follows that the inclusion of the ∞-groupoid $\mathsf{Act}_{\mathcal{O}}(O)$ in $(\mathcal{O}^{\mathrm{int}})_{/O}$ is cofinal; since we assume $\mathcal{C}$ has colimits indexed by ∞-groupoids, all left Kan extension along j exist, meaning that we have an adjunction

$$j_! : \mathsf{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{C}) \rightleftarrows \mathsf{Fun}(\mathcal{O}, \mathcal{C}).$$

We want to show that this restricts to an adjunction between the full subcategories of monoids. To see this, it suffices to show that if $M\colon \mathcal{O}^{\mathrm{int}} \to \mathcal{C}$ is an $\mathcal{O}^{\mathrm{int}}$-monoid, then the left Kan extension $j_!M$ is an $\mathcal{O}$-monoid, i.e. that for $O \in \mathcal{O}_{\langle n\rangle}$ the canonical map

$$(j_!M)(O) \to \prod_{i=1}^{n} (j_!M)(O_i),$$

where $O \to O_i$ is cocartesian over ρ_i, is an equivalence.

It follows from condition (2) in Definition 2.3.3 that we have an equivalence

$$\mathsf{Act}_{\mathcal{O}}(O) \xrightarrow{\sim} \prod_{i=1}^{n} \mathsf{Act}_{\mathcal{O}}(O_i),$$

which takes an active morphism $O' \rightsquigarrow O$ to the active maps $O'_i \rightsquigarrow O_i$ obtained from the inert–active factorization of the composite $O' \rightsquigarrow O \rightarrowtail O_i$. Moreover, the maps $(j_!M)(O) \to (j_!M)(O_i)$ are the maps on colimits induced by composing with these maps, so since M is an $\mathbb{O}^{\mathrm{int}}$-monoid we can identify the map we are interested in as the composite

$$\begin{aligned}
(j_!M)(O) &\simeq \operatorname*{colim}_{O' \rightsquigarrow O \in \mathsf{Act}_{\mathbb{O}}(O)} M(O') \\
&\simeq \operatorname*{colim}_{(O'_i \rightsquigarrow O_i) \in \prod_{i=1}^{n} \mathsf{Act}_{\mathbb{O}}(O_i)} \prod_{i=1}^{n} M(O'_i) \\
&\to \prod_{i=1}^{n} \operatorname*{colim}_{(O'_i \rightsquigarrow O_i) \in \mathsf{Act}_{\mathbb{O}}(O_i)} M(O'_i) \\
&\simeq \prod_{i=1}^{n} (j_!M)(O_i).
\end{aligned}$$

Here we can rewrite the colimit over $\prod_{i=1}^{n} \mathsf{Act}_{\mathbb{O}}(O_i)$ as an iterated colimit, and we then see that the penultimate map is an equivalence by our assumption that the cartesian product in $\mathscr{C}$ preserves colimits indexed by ∞-groupoids in each variable. □

For proofs of the general version of Theorem 3.5.3 and Corollary 3.5.5 we refer to [Lur17, 3.1] (which builds the left adjoint by "brute force", using an inductive simplex by simplex construction), [CH22] (which uses Day convolution to reduce to the case of monoids), and [NS22] (who prove a more general parametrized/equivariant version).

Chapter 4

Other versions of ∞-operads

In this chapter we will first look at some alternative descriptions of ∞-operads, namely as analytic monads and as dendroidal Segal spaces, and then discuss one way to define *enriched* ∞-operads.

4.1 Analytic monads

In the first part of this section we consider the monad for free $\mathcal{O}$-algebras in spaces for an ∞-operad $\mathcal{O}$, and see that in a certain sense $\mathcal{O}$ can be completely recovered from this monad, which gives a relation between ∞-operads and the class of *analytic* monads. We will then take a closer look at these monads, and in particular see how they relate one-object ∞-operads to a version of the *composition product* on symmetric sequences in $\mathcal{S}$. This section is based on [GHK22] and [Hau23].

For a fixed ∞-operad $\mathcal{O}$, let us write

$$\mathcal{O}^{\mathrm{el}} := \mathcal{O}^{\simeq}_{\langle 1 \rangle} \xrightarrow{i} \mathcal{O}^{\mathrm{int}} \xrightarrow{j} \mathcal{O}$$

for the inclusions. The forgetful functor $U_{\mathcal{O}}$ from $\mathcal{O}$-algebras in $\mathcal{S}$ can then be identified with the composite

$$\mathsf{Mon}_{\mathcal{O}}(\mathcal{S}) \xrightarrow{j^*} \mathsf{Mon}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S}) \xrightarrow{i^*} \mathsf{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{S}).$$

Here i^* is an equivalence with inverse the right Kan extension i_* along i, and as we saw in the previous section we can identify the left adjoint $\mathsf{Free}_{\mathcal{O}}$ with $j_! i_*$.

Proposition 4.1.1.

(1) The adjunction $\mathsf{Free}_{\mathcal{O}} \dashv U_{\mathcal{O}}$ is monadic.

(2) The endofunctor $T_{\mathcal{O}} := U_{\mathcal{O}} \mathsf{Free}_{\mathcal{O}}$ preserves sifted colimits and weakly contractible limits.

(3) *The unit and multiplication transformations for $T_{\mathfrak{G}}$ are cartesian transformations*[1].

Proof. To prove (1), the monadicity theorem says that it suffices to check that $U_{\mathfrak{G}}$ is conservative, i.e. detects equivalences, and preserves $U_{\mathfrak{G}}$-split simplicial colimits. For the first condition, observe that both j^* and i^* are conservative (since j is essentially surjective and i^* is an equivalence), hence so is their composite. Instead of looking at $U_{\mathfrak{G}}$-split simplicial colimits, we will prove that $U_{\mathfrak{G}}$ preserves all sifted colimits in general; since restriction along j gives a colimit-preserving functor $\mathsf{Fun}(\mathfrak{G}, \mathcal{S}) \to \mathsf{Fun}(\mathfrak{G}^{\mathrm{int}}, \mathcal{S})$, it suffices to check that monoids are closed under sifted colimits in $\mathsf{Fun}(\mathfrak{G}, \mathcal{S})$. To see this, suppose we have a diagram $F\colon \mathcal{J} \to \mathsf{Mon}_{\mathfrak{G}}(\mathcal{S})$ where $\mathcal{J}$ is sifted; for $O \in \mathfrak{G}_{\langle n \rangle}$ we have a commutative triangle

$$\begin{tikzcd} \operatorname{colim}_{x\in\mathcal{J}} F(x)(O) \arrow[r, "\sim"] \arrow[dr] & \operatorname{colim}_{x\in\mathcal{J}} \prod_{i=1}^n F(x)(O_i) \arrow[d, "\sim"] \\ & \prod_{i=1}^n \operatorname{colim}_{x\in\mathcal{J}} F(x)(O_i), \end{tikzcd}$$

where the top horizontal map is an equivalence since each $F(x)$ is a monoid and the right vertical map is an equivalence since $\mathcal{J}$ is sifted. It follows that $T_{\mathfrak{G}}$ also preserves sifted colimits, since it is the composite of $U_{\mathfrak{G}}$ and a left adjoint.

To complete the proof of (2), we need to know that colimits in $\mathcal{S}$ indexed by ∞-groupoids commute with weakly contractible colimits (i.e. those indexed by ∞-categories whose classifying spaces are contractible). This is because the straightening theorem lets us identify a functor $F\colon X \to \mathcal{S}$, where X is an ∞-groupoid, with a map of ∞-groupoids $Y \to X$, where Y is the colimit of F — thus the colimit functor $\mathsf{Fun}(X, \mathcal{S}) \to \mathcal{S}$ is equivalent to the forgetful functor $\mathcal{S}_{/X} \to \mathcal{S}$, and this preserves weakly contractible limits. Since $T_{\mathfrak{G}}$ is given by the formula

$$(T_{\mathfrak{G}}F)(O) \simeq \operatorname{colim}_{O' \rightsquigarrow O \in \mathsf{Act}_{\mathfrak{G}}(O)} F(O'),$$

where $\mathsf{Act}_{\mathfrak{G}}(O)$ is an ∞-groupoid, we see that it commutes with weakly contractible limits.

For (3), we first look at the multiplication transformation $\mu\colon T_{\mathfrak{G}}^2 F \to T_{\mathfrak{G}} F$. At $O \in \mathfrak{G}$, the value $T_{\mathfrak{G}}^2 F(O)$ is an iterated colimit, which we can rewrite as a colimit over the space

$$\mathsf{Act}^2_{\mathfrak{G}}(O) := \left\{ \begin{tikzcd} O'' \arrow[r, rightsquigarrow] & O' \arrow[d, rightsquigarrow] \\ & O \end{tikzcd} \right\}.$$

The component μ_O can then be identified with the map on colimits induced by the morphism $\mathsf{Act}^2_{\mathfrak{G}}(O) \to \mathsf{Act}_{\mathfrak{G}}(O)$ given by composition. To see that μ is

[1] Meaning that their naturality squares are all pullbacks.

cartesian, it suffices to check that the naturality square

$$\begin{tikzcd} T_{\mathbb{O}}^2 F(O) \arrow[r] \arrow[d] & T_{\mathbb{O}} F(O) \arrow[d] \\ T_{\mathbb{O}}^2 * \arrow[r] & T_{\mathbb{O}} * \end{tikzcd}$$

over the terminal object is cartesian for every O. For this it suffices to check it is an equivalence on fibres over every point $O'' \rightsquigarrow O' \rightsquigarrow O$ in $\mathsf{Act}^2_{\mathbb{O}}(O) \simeq T^2_{\mathbb{O}}*$, and using the identification between colimits and straightening, this is indeed the identity of $F(O'')$. The case of the unit transformation $\eta\colon \mathrm{id} \to T_{\mathbb{O}}$ is similar, using that its component at O can be described as the map on colimits induced by the map $* \to \mathsf{Act}_{\mathbb{O}}(O)$ picking out the identity. □

It turns out that the properties in Proposition 4.1.1 come very close to characterizing those monads that arise from ∞-operads, so we introduce some terminology for these:

Definition 4.1.2. An *analytic functor* is a functor $F\colon \mathcal{S}_{/X} \to \mathcal{S}_{/Y}$ that preserves sifted colimits and weakly contractible limits. An *analytic monad* is a monad T on $\mathcal{S}_{/X}$ such that T is an analytic functor and the unit and multiplication are cartesian natural transformations.

We have thus shown that the monad $T_{\mathbb{O}}$ is an analytic monad on $\mathcal{S}_{/\mathbb{O}^{\mathrm{el}}}$ for every ∞-operad $\mathbb{O}$. More generally, if $q\colon X \twoheadrightarrow \mathbb{O}^{\mathrm{el}}$ is a morphism of ∞-groupoids that is essentially surjective (meaning that it is surjective on π_0) then the composite $q^* \circ U_{\mathbb{O}}\colon \mathsf{Mon}_{\mathbb{O}}(\mathcal{S}) \to \mathcal{S}_{/X}$ is also the monadic right adjoint for an analytic monad (with underlying endofunctor $q^* T_{\mathbb{O}} q_!$).

Definition 4.1.3. We can define an ∞-category AnMnd whose objects are analytic monads and whose morphisms are lax morphisms of monads given by cartesian transformations. This can be defined as a subcategory of $\mathsf{Ar}(\mathsf{Cat}_\infty)^{\mathrm{op}}$, because such lax morphisms $T' \to T$ can be identified with commutative squares

$$\begin{tikzcd} \mathsf{Alg}(T) \arrow[r] \arrow[d, "U_T"] & \mathsf{Alg}(T') \arrow[d, "U_{T'}"] \\ \mathcal{S}_{/X} \arrow[r, "p^*"] & \mathcal{S}_{/X'} \end{tikzcd}$$

where p is a map of spaces, and the mate transformation is cartesian.

Definition 4.1.4. A *pinned ∞-operad* is a pair $(\mathbb{O}, q\colon X \twoheadrightarrow \mathbb{O}^{\mathrm{el}})$ where $\mathbb{O}$ is an ∞-operad and q is an essentially surjective morphism of spaces. These form an ∞-category POpd_∞.

Theorem 4.1.5 ([Hau23]). *The functor $\mathfrak{M}\colon \mathsf{POpd}_\infty \to \mathsf{AnMnd}$ that takes a pinned ∞-operad $(\mathcal{O}, q\colon X \twoheadrightarrow \mathcal{O}^{\mathrm{el}})$ to the monadic right adjoint*

$$\mathsf{Mon}_{\mathcal{O}}(\mathcal{S}) \xrightarrow{U_{\mathcal{O}}} \mathcal{S}_{/\mathcal{O}^{\mathrm{el}}} \xrightarrow{q^*} \mathcal{S}_{/X},$$

is an equivalence.

We will say a little more about how we can extract a pinned ∞-operad from an analytic monad later in this section, but first we want to look at how analytic monads can be related to the composition product of symmetric sequences. For this we first need to recall the diagrammatic description of analytic functors and cartesian transformations:

For $X, Y \in \mathcal{S}$, let $\mathsf{AnFun}(X, Y)$ denote the ∞-category of analytic functors $\mathcal{S}_{/X} \to \mathcal{S}_{/Y}$ and cartesian transformations among them. Then $\mathsf{AnEnd}(X) := \mathsf{AnFun}(X, X)$ has a monoidal structure given by composition, and we can identify the ∞-category $\mathsf{AnMnd}(X)$ of analytic monads on $\mathcal{S}_{/X}$ with that of associative algebras in $\mathsf{AnEnd}(X)$ under composition.

Theorem 4.1.6 ([GHK22]). *An analytic functor $\mathcal{S}_{/X} \to \mathcal{S}_{/Y}$ is of the form $t_! p_* s^*$ for a unique diagram of spaces*

$$X \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} Y$$

where p has finite discrete fibres. Moreover, a cartesian transformation between such functors is induced by a diagram

$$\begin{tikzcd} & E \arrow[r] \arrow[dl] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[dr] \arrow[d] & \\ X & E' \arrow[l] \arrow[r] & B' \arrow[r] & Y \end{tikzcd}$$

where the middle square is a pullback.[2]

Here we write, for a morphism of spaces $f\colon A \to B$,

- $f^*\colon \mathcal{S}_{/B} \to \mathcal{S}_{/A}$ for the functor given by pullback along f,
- $f_!\colon \mathcal{S}_{/B} \to \mathcal{S}_{/A}$ for the left adjoint to f^*, given by composition with f (or by left Kan extension along f after straightening),
- $f_*\colon \mathcal{S}_{/B} \to \mathcal{S}_{/A}$ for the right adjoint to f^* (given by right Kan extension along f after straightening),

[2]We can also define a larger ∞-category of analytic functors on varying bases in terms of such diagrams.

We can also give a diagrammatic description of composition of analytic functors, but we omit this here as it is slightly complicated.

In particular, we can describe the objects of $\mathsf{AnEnd}(*)$ as diagrams

$$* \leftarrow E \xrightarrow{p} B \rightarrow *,$$

where p has finite discrete fibres. There's a classifier for such maps p: the map

$$\coprod_n \mathbf{n}_{h\Sigma_n} \to \coprod_n B\Sigma_n$$

is terminal in $\mathsf{AnEnd}(*)$. In other words, given a map $p\colon E \to B$ with finite discrete fibres, there is a unique pullback square

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"] & \coprod_n \mathbf{n}_{h\Sigma_n} \arrow[d] \\ B \arrow[r] & \coprod_n B\Sigma_n. \end{tikzcd}$$

The map p is thus determined by a map $B \to \coprod_n B\Sigma_n$, so that we have an equivalence

$$\mathsf{AnEnd}(*) \simeq \mathcal{S}_{/\coprod_n B\Sigma_n} \simeq \prod_n \mathsf{Fun}(B\Sigma_n, \mathcal{S}).$$

An analytic endofunctor of $\mathcal{S}$ is thus determined by a *symmetric sequence*, i.e. a sequence of spaces $X(n)$ with a Σ_n-action, $n = 0, 1, \ldots$.

Moreover, under this equivalence the monoidal structure given by composition of analytic functors is described by the classical formula for the composition product of symmetric sequences, so that we have equivalences of ∞-categories between

- pointed connected ∞-operads $(\mathcal{O}, * \twoheadrightarrow \mathcal{O}^{\mathrm{el}})$,
- analytic monads on $\mathcal{S}$,
- associative algebras in symmetric sequences in $\mathcal{S}$ with respect to the composition product.

(The same idea works for any fixed space of objects, but the formulas are a bit more complicated.)

Remark 4.1.7. In sets, the relation between symmetric sequences/operads and a notion of analytic endofunctors/monads on Set is due to Joyal [Joy86].

We end by saying a little more about the proof of Theorem 4.1.5, for which we need a bit more terminology:

Definition 4.1.8. An *algebraic monad* on $\mathcal{S}_{/X}$ is one whose underlying endofunctor preserves sifted colimits.

Definition 4.1.9. A *Lawvere theory* is a pair $(\mathcal{L}, q\colon X \to \mathcal{L})$ where $\mathcal{L}$ is an ∞-category with finite products and X is a space, such that every object of $\mathcal{L}$ is a finite product of ones in the image of q. We define the ∞-category $\mathsf{Mod}(\mathcal{L})$ of *models* of $\mathcal{L}$ to be that of product-preserving functors $\mathcal{L} \to \mathcal{S}$.

If T is an algebraic monad on $\mathcal{S}_{/X} \simeq \mathsf{Fun}(X, \mathcal{S})$, then its *Lawvere theory* $\mathcal{L}(T)$ is defined by taking $\mathcal{L}(T)^{\mathrm{op}}$ to be the full subcategory of $\mathsf{Alg}(T)$ spanned by the free algebras on finite coproducts of representables, together with the map $X \to \mathcal{L}(T)$ arising from the Yoneda embedding combined with the free algebra functor.

Theorem 4.1.10 ([GGN15], [Hau23]). *If T is an algebraic monad, then $\mathcal{L}(T)$ is a Lawvere theory. Conversely, if $(\mathcal{L}, q)$ is a Lawevere theory then the restriction $\mathsf{Mod}(\mathcal{L}) \to \mathsf{Fun}(X, \mathcal{S})$ is the monadic right adjoint of an algebraic monad. Moreover, these constructions give inverse equivalences of ∞-categories between Lawvere theories and algebraic monads.*

Here the case of *one-sorted* Lawvere theories (where $q\colon * \to \mathcal{L}$) is due to Gepner–Groth–Nikolaus [GGN15], while the generalization to arbitrary spaces can be found in [Hau23].

To relate this to analytic monads, we first note that AnMnd has a terminal object Sym, which is the unique analytic monad structure on the terminal object

$$* \leftarrow \coprod_n \mathbf{n}_{h\Sigma_n} \to \coprod_n B\Sigma_n \to *.$$

Then we can identify AnMnd with a full subcategory of $\mathsf{AlgMnd}_{/\mathsf{Sym}}$. Here the Lawvere theory $\mathcal{L}(\mathsf{Sym})$ can be identified with $\mathsf{Span}(\mathbb{F})^{\natural} := (\mathsf{Span}(\mathbb{F}), \{\mathbf{1}\})$, so we can identify AnMnd with a full subcategory of Lawvere theories over $\mathsf{Span}(\mathbb{F})^{\natural}$.

Theorem 4.1.11 ([Hau23]). *The ∞-category AnMnd corresponds under taking Lawvere theories precisely to the ∞-category of weak Segal fibrations over $\mathsf{Span}(\mathbb{F})^{\natural}$.*

As we mentioned in Section 3.3, the latter ∞-category is equivalent to that of ∞-operads by pulling back along the inclusion of $\mathbb{F}_*$ in $\mathsf{Span}(\mathbb{F})$. [BHS22].

4.2 Dendroidal Segal spaces

In this section we introduce *complete dendroidal Segal spaces* as an alternative model for ∞-operads. These are presheaves on the *dendroidal category* Ω (first introduced by Moerdijk and Weiss [MW07]) that satisfy certain Segal and completeness condition. Dendroidal Segal spaces were first introduced by Cisinski and Moerdijk [CM13a], but our discussion is also based on [Koc11] and [GHK22]; see also the book [HM22] of Heuts and Moerdijk for an expository account of all things dendroidal.

As a warm-up, let's first recall that the algebraic structure of an ∞-category (i.e. a homotopy-coherently associative and unital composition) can be described by a *Segal space*, that is a simplicial space $X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ such that the restriction map

$$X([n]) \xrightarrow{\sim} X([1]) \times_{X([0])} \cdots \times_{X([0])} X([1])$$

induced by the inert inclusions $[0], [1] \hookrightarrow [n]$, is an equivalence. Thinking of $[n]$ as the category $0 \to 1 \to \cdots \to n$, we can suggestively write this map as

$$X(0 \to \cdots \to n) \xrightarrow{\sim} X(0 \to 1) \times_{X(1)} X(1 \to 2) \times_{X(2)} \cdots \times_{X(n-1)} X(n-1 \to n).$$

To think of X as an ∞-category, we view $X([0])$ as the space of objects and $X([1])$ as the space of all morphisms. Then the face maps $d_1, d_0\colon [0] \to [1]$ give maps $X([1]) \to X([0])$ that assign source and target objects to a morphism, while the degeneracy $s_0\colon [1] \to [0]$ induces a map $X([0]) \to X([1])$ that assigns an identity map to every object. The Segal condition then says that we can think of $X([n])$ as the space of composable sequences of n morphisms.

We also get a composition operation

$$X([1]) \times_{X([0])} X([1]) \xleftarrow{\sim} X([2]) \to X([1])$$

from the active face map $d_1\colon [1] \to [2]$. The commutative square

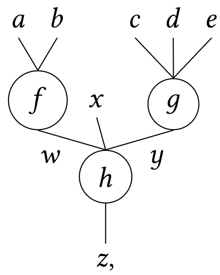
$$\begin{tikzcd} X([3]) \arrow[r] \arrow[d] & X([2]) \arrow[d] \\ X([2]) \arrow[r] & X([1]) \end{tikzcd}$$

arising from the commutative square

$$\begin{tikzcd} {[1]} \arrow[r, "d_1"] \arrow[d, "d_1"] & {[2]} \arrow[d, "d_1"] \\ {[2]} \arrow[r, "d_2"] & {[3]} \end{tikzcd}$$

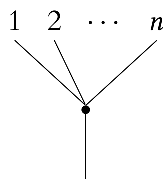
in $\mathbb{\Delta}$ can then be interpreted as saying that this composition is associative up to the specified equivalence in the square, and so forth for further coherences for associativity and units.

The idea of dendroidal Segal spaces is to describe the algebraic structure of an ∞-operad in an analogous fashion. Note that here we can think of the objects of $[n]$ as giving all the different ways we can combine morphisms in a category into something we can compose together. For operads, such "composable shapes" are trees: for example, given multimorphisms f, g, h with compatible inputs and outputs as in the tree



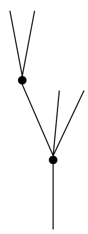
we can compose them in an operad to get a multimorphism $(a, b, x, c, d, e) \to z$. We thus want to find a category Ω whose objects are trees, so that we can describe ∞-operads algebraically as functors $F\colon \Omega^{\mathrm{op}} \to \mathcal{S}$ that satisfy a Segal condition.

Let us write C_n for the *corolla* with n inputs, that is the tree



with one vertex, and $\eta := |$ for the plain edge. If $F\colon \Omega^{\mathrm{op}} \to \mathcal{S}$ is to represent an ∞-operad, then $F(\eta)$ should be the space of objects of the ∞-operad, and $F(C_n)$ its space of n-ary multimorphisms.

To get an idea of what types of morphisms we want to have in the category Ω, consider the following tree T:



For our functor F, we want the space $F(T)$ to represent a ternary morphism together with a composable binary morphism. This corresponds to the Segal condition

$$F(T) \simeq F(C_3) \times_{F(\eta)} F(C_2).$$

This should arise from *inert* morphisms, given by the inclusions of subtrees of the shapes $C_2, C_3, \eta \hookrightarrow T$. On the other hand, we should be able to compose the data represented by T to a single multimorphism with 4 inputs, which means there should be a map

$$F(T) \to F(C_4)$$

induced by an *active* morphism $C_4 \to T$.

To proceed further, we need a definition of the trees we want as our objects. Most precise definitions of trees are quite painful to write down, but luckily there is a pretty straightforward definition of the type of tree we want here, due to Kock [Koc11]. The idea is that if E is the set of edges of a tree and V is the set of vertices[3] (or nodes), we can define V_* to be the set of pairs (v, e) where $v \in V$ and $e \in E$ is an incoming edge of the vertex v. Then we have maps

$$E \leftarrow V_* \to V \to E,$$

where the two maps on the left are the projections that take (v, e) to e and v, and the right-most map sends a vertex v to its unique outgoing edge. This diagram then completely encodes the structure of the tree, and we can give a simple criterion for such a map of sets to represent a tree:

Definition 4.2.1. A *tree* is a diagram of finite sets

$$T = \left(I \xleftarrow{s} J \xrightarrow{p} K \xrightarrow{t} I\right),$$

such that:

(1) The map t is injective. ["Every edge is the outgoing edge of at most one vertex."]

(2) The map s is injective and there is a unique element $R \in I$ (the *root* of T) that is not in the image of s. ["Every edge is incoming for at most one vertex, and the root is the unique non-incoming edge."]

(3) If we define the successor function $\sigma \colon I \to I$ by $\sigma(R) = R$ and $\sigma(i) = tp(j)$ if $i = s(j)$, then for every i we have $\sigma^k(i) = R$ for some k. ["Every edge is connected to the root by a finite sequence of vertices"]

The *leaves* of T are the elements of I that are not in the image of t.

Definition 4.2.2. The category Ω^{int} has trees (in the sense of the previous definition) as its objects, and morphisms are diagrams

$$\begin{tikzcd} I \arrow[d] & J \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & K \arrow[r] \arrow[d] & I \arrow[d] \\ I' & J' \arrow[l] \arrow[r] & K' \arrow[r] & I' \end{tikzcd} \tag{4.1}$$

where the middle square is a pullback.

[3] Our trees do not have "exterior" vertices at the root or leaves (but "nullary" vertices without incoming edges *are* allowed).

Observation 4.2.3. A morphism $T \to T'$ in Ω^{int} thus assigns edges of T to edges of T', and vertices of T to those of T', and the pullback condition says that a vertex must be sent to one with the same number of incoming edges. In fact, the conditions for T and T' to be trees force the maps on edges and vertices to be *injective*, so morphisms in Ω^{int} are precisely *subtree inclusions*.

Definition 4.2.4. For $n = 0, 1, \ldots$, the *n-corolla* C_n is the tree

$$\{0, 1, \ldots, n\} \hookleftarrow \{1, \ldots, n\} \to \{0\} \hookrightarrow \{0, 1, \ldots, n\},$$

while the plain edge η is the tree

$$* \hookleftarrow \emptyset \to \emptyset \to *.$$

We write Ω^{el} for the full subcategory of Ω^{int} spanned by these *elementary* objects.

With these definitions in hand, we can define the Segal condition we want:

Definition 4.2.5. A presheaf $F \colon \Omega^{\mathrm{int,op}} \to \mathcal{S}$ is a *Segal presheaf* (or a *Segal $\Omega^{\mathrm{int,op}}$-space* in the terminology of Section 3.3) if F is a right Kan extension of its restriction $F|_{\Omega^{\mathrm{el,op}}}$, meaning that for every object $T \in \Omega^{\mathrm{int,op}}$, we have a Segal condition

$$F(T) \simeq \lim_{E \in (\Omega^{\mathrm{el}}_{/T})^{\mathrm{op}}} F(E),$$

where $\Omega^{\mathrm{el}}_{/T} := \Omega^{\mathrm{el}} \times_{\Omega^{\mathrm{int}}} \Omega^{\mathrm{int}}_{/T}$ consists of the inclusions of all edges and corollas in T. We denote the ∞-category of Segal presheaves on Ω^{int} as $\mathsf{P}_{\mathrm{Seg}}(\Omega^{\mathrm{int}}) \subseteq \mathsf{P}(\Omega^{\mathrm{int}})$.

Next, we want to extend the category Ω^{int} to add the active maps, which should encode all the possible ways we can compose trees (as well as insert identities).

Definition 4.2.6. For T a tree, let $\mathrm{sub}(T)$ denote the set of all subtrees of T, which we can identify with the (discrete) groupoid $(\Omega^{\mathrm{int}}_{/T})^{\simeq}$. Then let $\mathrm{sub}'(T)$ be the set of subtrees of T together with a marked leaf, that is pairs $(\eta \to T', T' \to T)$ where the first map is a leaf of T'. If T is given by the diagram

$$E \leftarrow V_* \to V \to E,$$

consider the diagram

$$\overline{T} = \left(E \leftarrow \mathrm{sub}'(T) \to \mathrm{sub}(T) \to E\right),$$

where the two first maps take a marked subtree to its marked edge in T and the underlying subtree, and the last map takes a subtree to its root edge in T. We define the category Ω to have trees as its objects, with morphisms from T to T' given by diagrams of the form (4.1) from the diagram $\overline{T}$ to $\overline{T'}$.

Observation 4.2.7. A morphism from T to T' in Ω is determined by a diagram

$$\begin{tikzcd} E \arrow[d] & V_* \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[r] \arrow[d] & E \arrow[d] \\ E' & \mathrm{sub}'(T') \arrow[l] \arrow[r] & \mathrm{sub}(T') \arrow[r] & E'. \end{tikzcd}$$

Thus we assign to every vertex v of T a subtree of T' such that the incoming edges of v are in bijection with the leaves of the subtree, and the outgoing edge of v is sent to the root of the subtree. In fact, this map is determined uniquely by its underlying map $E \to E'$; we refer to [Koc11] for more details.

We get a (replete) subcategory inclusion $\Omega^{\mathrm{int}} \to \Omega$ by viewing vertices (corollas) as subtrees; morphisms in its image are *inert*. A morphism $T \to T'$ in Ω is *active* if it preserves the boundary, i.e. it takes the root and leaves of T to the root and leaves of T' (bijectively). The active and inert maps form a factorization system on Ω. Therefore, we can define an algebraic pattern structure $\Omega^{\mathrm{op},\natural}$ on the category Ω^{op} (in the sense of Section 3.3) using the corresponding inert–active factorization system, and with $\Omega^{\mathrm{el},\mathrm{op}}$ as the elementary objects.

Remark 4.2.8. Given a tree T, we can define a *free operad* on T, whose objects are the edges of T and whose n-ary multimorphisms are the subtrees of T with n leaves (giving a multimorphism from the leaves to the root); composition is given by gluing of subtrees. The category Ω can then also be identified as the full subcategory of operads (in Set) spanned by these free operads.

Definition 4.2.9. A *dendroidal Segal space* (or *Segal presheaf* on Ω, or *Segal* Ω^{op}*-space*) is a functor $F \colon \Omega^{\mathrm{op}} \to \mathcal{S}$ such that $F|_{\Omega^{\mathrm{int},\mathrm{op}}}$ is a right Kan extension of $F|_{\Omega^{\mathrm{el},\mathrm{op}}}$. We write $\mathsf{P}_{\mathrm{Seg}}(\Omega)$ for the full subcategory of $\mathsf{P}(\Omega)$ spanned by the dendroidal Segal spaces.

Dendroidal Segal spaces have the algebraic structure we want from an ∞-operad, i.e. a homotopy-coherently associative and unital composition of multimorphisms. However, to get the right ∞-category of ∞-operads we further need to invert the fully faithful and essentially surjective maps. Let us first recall how this works for Segal spaces:

Definition 4.2.10. A morphism of Segal spaces $X \to Y$ is called *fully faithful* if the square

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ X_0 \times X_0 \arrow[r] & Y_0 \times Y_0 \end{tikzcd}$$

is cartesian (where the vertical maps come from the two face maps $[0] \to [1]$ in Δ), and *essentially surjective* if the induced map $\pi_0 X_0/\!\sim_X \to \pi_0 Y_0/\!\sim_Y$, where $\sim_X$ is the equivalence relation generated by $x \sim_X x'$ if there exists an invertible morphism $x \xrightarrow{\sim} x'$ in X.

Definition 4.2.11. A Segal space X is *complete* if the square

$$\begin{tikzcd} X_0 \arrow[r, "\Delta"] \arrow[d] & X_0 \times X_0 \arrow[d, "s_0 \times s_0"] \\ X_3 \arrow[r, "(d_{\{0,2\}}, d_{\{1,3\}})"] & X_1 \times X_1 \end{tikzcd}$$

is cartesian. (The pullback here is the space of diagrams in X of the form

$$\begin{tikzcd} x' \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & x \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & x' \arrow[r] & x, \end{tikzcd}$$

that is the space of a morphism together with a left and a right inverse, so that the completeness condition says that X_0 is the space of equivalences in X.) We write $\mathsf{P}_{\mathrm{Seg}}(\Delta)$ for the full subcategory of $\mathsf{P}(\Delta)$ spanned by the Segal spaces, and $\mathsf{P}_{\mathrm{CSeg}}(\Delta)$ for the full subcategory of complete Segal spaces.

Theorem 4.2.12 (Rezk [Rez01]). *$\mathsf{P}_{\mathrm{CSeg}}(\Delta)$ is the localization of $\mathsf{P}_{\mathrm{Seg}}(\Delta)$ at the fully faithful and essentially surjective maps.*

Theorem 4.2.13 (Joyal–Tierney [JT07]). $\mathsf{P}_{\mathrm{CSeg}}(\Delta) \simeq \mathsf{Cat}_\infty$.

Now we turn to the analogous results for ∞-operads:

Observation 4.2.14. We can identify Δ with the full subcategory of Ω containing only the *linear* trees, meaning those with only unary vertices. Let $i\colon \Delta \hookrightarrow \Omega$ be the corresponding inclusion; then composition with i restricts to a functor

$$i^*\colon \mathsf{P}_{\mathrm{Seg}}(\Omega) \to \mathsf{P}_{\mathrm{Seg}}(\Delta).$$

Definition 4.2.15. We say $X \in \mathsf{P}_{\mathrm{Seg}}(\Omega)$ is *complete* if $i^*X \in \mathsf{P}_{\mathrm{Seg}}(\Delta)$ is a complete Segal space, and write $\mathsf{P}_{\mathrm{CSeg}}(\Omega)$ for the full subcategory of $\mathsf{P}_{\mathrm{Seg}}(\Omega)$ containing the complete dendroidal Segal spaces.

Definition 4.2.16. A morphism $f\colon X \to Y$ in $\mathsf{P}_{\mathrm{Seg}}(\Omega)$ is *essentially surjective* if i^*f is an essentially surjective morphism in $\mathsf{P}_{\mathrm{Seg}}(\Delta)$, and *fully faithful* if the square

$$\begin{tikzcd} X(C_n) \arrow[r] \arrow[d] & Y(C_n) \arrow[d] \\ X(\eta)^{\times(n+1)} \arrow[r] & Y(\eta)^{\times(n+1)} \end{tikzcd}$$

is a pullback for all n.

Remark 4.2.17. To explain the definition of fully faithful morphisms, note that in the square above, a point in $X(\eta)^{\times(n+1)}$ represents $n+1$ objects $x_1, \ldots, x_n, x'$ in X, and the fibre of $X(C_n)$ at this point is the space $X(x_1, \ldots, x_n; x')$ of n-ary morphisms $(x_1, \ldots, x_n) \to x'$ in X. Since a square in $\mathcal{S}$ is cartesian if and only if it is given by equivalences on fibres, the condition for f to be fully faithful is equivalent to it giving equivalences

$$X(x_1, \ldots, x_n; x') \xrightarrow{\sim} Y(f(x_1), \ldots, f(x_n); f(x'))$$

on all spaces of multimorphisms.

Theorem 4.2.18 (Cisinski–Moerdijk [CM13a]). $\mathsf{P}_{\mathrm{CSeg}}(\Omega)$ *is the localization of* $\mathsf{P}_{\mathrm{Seg}}(\Omega)$ *at the fully faithful and essentially surjective morphisms.*

Theorem 4.2.19. $\mathsf{P}_{\mathrm{CSeg}}(\Omega) \simeq \mathsf{Opd}_\infty$.

There are by now several proofs of such an equivalence between dendroidal Segal spaces and Lurie's model of ∞-operads:

- Heuts, Hinich, and Moerdijk [HHM16] proved the first comparison, but only for ∞-operads without nullary operations. (More precisely, they compare Lurie's ∞-operads to dendroidal sets as model categories, and the latter are equivalent to dendroidal Segal spaces by work of Cisinski and Moerdijk [CM13a].)
- Barwick [Bar18] proved that Lurie's ∞-operads are equivalent to complete Segal presheaves on a category of forests with levels, and Chu, Haugseng, and Heuts [CHH18] compared the latter to complete dendroidal Segal spaces.
- Hinich and Moerdijk [HM22] prove the first direct equivalence between Lurie's ∞-operads and complete denroidal Segal spaces. (They further show that they are even equivalent as symmetric monoidal ∞-categories for the Boardman–Vogt tensor product.)
- As we discussed in the previous section, it is shown in [Hau23] that Lurie's ∞-operads are equivalent to analytic monads. The main result of [GHK22] is that analytic monads have a very natural description as dendroidal Segal spaces.

Remark 4.2.20. We can think of dendroidal Segal spaces as an operadic version of Rezk's Segal spaces. Similarly, the category Ω can be used to obtain (at the model category level) dendroidal analogues of other models for ∞-categories, such as the dendroidal sets of Moerdijk and Weiss [MW07] (which are an operadic version of quasicategories), as well as dendroidal versions of Segal categories. Cisinski and Moerdijk have constructed Quillen equivalecnes that relate these dendroidal models to each other and to simplicial operads [CM13a, CM13b].

We end this section by briefly discussing the relation between dendroidal Segal spaces and analytic monads from [GHK22], since this is closely related to the way we defined the category Ω: As we mentioned in Section 4.1, analytic functors $\mathcal{S}_{/X} \to \mathcal{S}_{/Y}$ are of the form $t_!p_*s^*$ for a diagram of spaces

$$X \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} Y,$$

where p has finite discrete fibres. Above we also defined *trees* as certain diagrams (of sets) of this form. Moreover, in Theorem 4.1.6 we saw that cartesian transformations between analytic functors correspond to diagrams

$$\begin{tikzcd} X \arrow[d] & E \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r] \arrow[d] & Y \arrow[d] \\ X' & E' \arrow[l] \arrow[r] & B' \arrow[r] & Y', \end{tikzcd}$$

which was also how we defined inert morphisms between trees above.

Thus, if we define AnEnd as an ∞-category of *analytic endofunctors* $\mathcal{S}_{/X} \to \mathcal{S}_{/X}$ (where X can vary) together with cartesian transformations among these, then we can identify Ω^{int} with a full subcategory of AnEnd. Moreover, in [GHK22] we show that the restricted Yoneda embedding

$$\mathsf{AnEnd} \to \mathsf{P}(\Omega^{\mathrm{int}})$$

restricts to an equivalence

$$\mathsf{AnEnd} \simeq \mathsf{P}_{\mathrm{Seg}}(\Omega^{\mathrm{int}}).$$

Next, we can construct a forgetful functor $U\colon \mathsf{AnMnd} \to \mathsf{AnEnd}$ (taking an analytic monad to its underlying endofunctor), and this turns out to have a left adjoint $F\colon \mathsf{AnEnd} \to \mathsf{AnMnd}$ (forming free analytic monads). Furthermore, we give an explicit formula for F in terms of trees, and using this we can prove that Ω is the full subcategory of AnMnd spanned by the objects $F(T)$ where T is a tree. (In particular, the diagram we called $\overline{T}$ above is precisely $UF(T)$.)

Theorem 4.2.21 ([GHK22])**.** *The restricted Yoneda embedding along* $\Omega \hookrightarrow \mathsf{AnMnd}$ *induces an equivalence*

$$\mathsf{AnMnd} \simeq \mathsf{P}_{\mathrm{Seg}}(\Omega).$$

Combined with the equivalence from Theorem 4.1.5, this gives an equivalence $\mathsf{POpd}_\infty \simeq \mathsf{AnMnd} \simeq \mathsf{P}_{\mathrm{Seg}}(\Omega)$. It is also shown in [Hau23] that this restricts to an equivalence between Opd_∞ and $\mathsf{P}_{\mathrm{CSeg}}(\Omega)$, as we would expect.

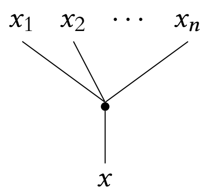
4.3 Enriched ∞-operads via Ω

So far we have only discussed ∞-operads where the multimorphisms form ∞-groupoids, which we have viewed as an ∞-categorical version of operads in sets. However, there are many examples of interesting operads that are *enriched* in other categories, such as vector spaces or chain complexes. Here we will consider one approach to define enriched ∞-operads, based on the dendroidal category Ω; this follows [CH20].

We start by using Ω to give another description of dendroidal Segal spaces, for which it will be clearer how to define an enriched version:

Definition 4.3.1. For $X \in \mathcal{S}$, define $a_X \colon \Omega^{\mathrm{op}} \to \mathcal{S}$ as the right Kan extension along $\{\eta\} \hookrightarrow \Omega^{\mathrm{op}}$ of the functor $* \to \mathcal{S}$ with value X. Then $a_X(T) \simeq X^{\times\{\text{edges in } T\}}$. We write $\Omega^{\mathrm{op}}_X \to \Omega^{\mathrm{op}}$ for the corresponding left fibration.

Observation 4.3.2. We can think of an object of Ω^{op}_X as a tree $T \in \Omega^{\mathrm{op}}$, together with a labelling of each of its edges by a point of X; we will denote such an object as $T(x_e : e \in E(T))$ where $E(T)$ is the set of edges of T. A morphism between such labelled trees is a morphism in Ω^{op} together with an identification of the corresponding labels. Moreover, we can lift the inert–active factorization system on Ω^{op} to Ω^{op}_X by taking a morphism to be inert or active if its image in Ω^{op} is so (this works because *all* morphisms are cocartesian here). We can then define an algebraic pattern structure $\Omega^{\mathrm{op},\flat}_X$ on Ω^{op}_X using this factorization system, where the elementary objects are the labelled corollas $C_n(x_1, \ldots, x_n; x)$, which we can depict as



Definition 4.3.3. Let $\mathcal{C}$ be an ∞-category with finite products. An Ω^{op}_X-*monoid* in $\mathcal{C}$ is a Segal object for the pattern $\Omega^{\mathrm{op},\flat}_X$, i.e. a functor $F \colon \Omega^{\mathrm{op}}_X \to \mathcal{C}$ such that for each object $\tilde{T} = T(x_e : e \in E(T))$ we have

$$F(\tilde{T}) \xrightarrow{\sim} \prod_{\text{corollas in } T} F(C_n(\cdots)),$$

where the labels of the corollas must match those from $\tilde{T}$. We write $\mathsf{Mon}_{\Omega^{\mathrm{op}}_X}(\mathcal{C})$ for the ∞-category of Ω^{op}_X-monoids in $\mathcal{C}$.

These monoids are contravariantly functorial in X by restriction along the functors $\Omega^{\mathrm{op}}_X \to \Omega^{\mathrm{op}}_Y$ induced by maps $f \colon X \to Y$ in $\mathcal{S}$ (which act on labelled trees by applying f to the labels). We define $\mathsf{Mon}_{\Omega^{\mathrm{op}}/\mathcal{S}}(\mathcal{C}) \to \mathcal{S}$ to be the cartesian fibration for this functor $\mathsf{Mon}_{\Omega^{\mathrm{op}}_{(-)}}(\mathcal{C}) \colon \mathcal{S}^{\mathrm{op}} \to \mathsf{Cat}_\infty$.

Proposition 4.3.4. *We have a commutative triangle*

$$\begin{tikzcd}
\mathsf{P}_{\mathrm{Seg}}(\Omega) \arrow[rr, "\sim"] \arrow[dr, "\mathrm{ev}_\eta"'] & & \mathsf{Mon}_{\Omega^{\mathrm{op}}/\mathcal{S}}(\mathcal{S}) \arrow[dl] \\
& \mathcal{S} &
\end{tikzcd}$$

where the horizontal functor is an equivalence.

In other words, a Segal presheaf $F\colon \Omega^{\mathrm{op}} \to \mathcal{S}$ is the same thing as a monoid $M\colon \Omega_X^{\mathrm{op}} \to \mathcal{S}$, where $X := F(\eta)$. The idea of the proof is that we can define the value of M at a labelled tree $\tilde{T}$ as the pullback

$$\begin{tikzcd}
M(\tilde{T}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(T) \arrow[d] \\
* \arrow[r] & \prod_{E(T)} F(\eta),
\end{tikzcd}$$

where the bottom horizontal map precisely picks out the labelling of the edges of T in $\tilde{T}$. In the other direction, we can obtain F from M by taking a left Kan extension along the projection $\Omega_X^{\mathrm{op}} \to \Omega^{\mathrm{op}}$.

More generally, if $\mathcal{C}$ is an ∞-category with finite products, Ω_X^{op}-monoids in $\mathcal{C}$ describe (the algebraic structure of) an ∞-operad enriched in $\mathcal{C}$ via the cartesian product, and with X as its space of objects: the value at $C_n(x_1, \ldots, x_n; x')$ is the object of multimorphisms from $(x_1, \ldots, x_n)$ to x', and the remaining data tells us how to compose such multimorphisms. To define enriched ∞-operads in a general symmetric monoidal ∞-category we now replace monoids by algebras (as in Section 3.1):

Definition 4.3.5. We define a functor $|{-}|\colon \Omega^{\mathrm{op}} \to \mathbb{F}_*$ on objects by $|T| := V(T) \amalg \{*\}$, where $V(T)$ is the set of vertices of the tree T. For a morphism $\alpha\colon T \to T'$ in Ω, we define $|\alpha|\colon |T'| \to |T|$ by

$$|\alpha|(v) = \begin{cases} w, & \text{if } v \text{ is a vertex of the subtree } \alpha(w), \\ *, & \text{if no such } w \text{ exists.} \end{cases}$$

This defines a functor, which is compatible with the inert–active factorization systems.

Definition 4.3.6. A Ω_X^{op}-*algebra* A in a symmetric monoidal ∞-category $\mathcal{C}$ is a commutative triangle

$$\begin{tikzcd}
\Omega_X^{\mathrm{op}} \arrow[rr, "A"] \arrow[dr] & & \mathcal{C}^{\otimes} \arrow[dl] \\
& \mathbb{F}_* &
\end{tikzcd}$$

such that A preserves inert morphisms. We write $\mathsf{Alg}_{\Omega_X^{\mathrm{op}}}(\mathscr{C})$ for the ∞-category of such algebras.

The condition that A preserves inert morphisms says informally that the value $A(\tilde{T})$ at a labelled tree $\tilde{T}$ can be viewed as the list $(A(\tilde{C}_v))_{v\in V(T)}$ where $\tilde{C}_v$ is the labelled corolla at the vertex v. If we write

$$A(x_1,\ldots,x_n;x) := A(C_n(x_1,\ldots,x_n;x))$$

then this is the object of multimorphisms from $(x_1,\ldots,x_n)$ to x, and for any labelled tree we get a composition map between these objects of multimorphisms, just as we expect for an ∞-operad in $\mathscr{C}$. We therefore also call an Ω_X^{op}-algebra in $\mathscr{C}$ a *$\mathscr{C}$-enriched ∞-preoperad* with space of objects X. (Here we use "preoperad" to underscore that we have not yet inverted the fully faithful and essentially surjective morphisms, which we need to do to obtain the correct ∞-category of enriched ∞-operads.)

Definition 4.3.7. Composition with the functors $\Omega_X^{\mathrm{op}} \to \Omega_Y^{\mathrm{op}}$ induced by maps $X \to Y$ in $\mathcal{S}$ gives a contravariant functor $\mathsf{Alg}_{\Omega_{(-)}^{\mathrm{op}}}(\mathscr{C})\colon \mathcal{S}^{\mathrm{op}} \to \widehat{\mathsf{Cat}}_\infty$, and we write $\mathsf{Alg}_{\Omega^{\mathrm{op}}/\mathcal{S}}(\mathscr{C}) \to \mathcal{S}$ for the corresponding cartesian fibration.

Observation 4.3.8. A morphism in $\mathsf{Alg}_{\Omega^{\mathrm{op}}/\mathcal{S}}(\mathscr{C})$ from A over X to B over Y consists of a morphism $f\colon X \to Y$ between spaces of objects, together with a natural transformation $\phi\colon A \to f^*B$, which in particular gives maps between objects of multimorphisms

$$A(x_1,\ldots,x_n;x) \to B(fx_1,\ldots,fx_n;fx),$$

as we would expect for a functor of enriched ∞-operads.

Definition 4.3.9. A morphism $(f,\phi)\colon A \to B$ in $\mathsf{Alg}_{\Omega^{\mathrm{op}}/\mathcal{S}}(\mathscr{C})$ is *fully faithful* if it is a cartesian morphism over f, i.e. if the components

$$A(x_1,\ldots,x_n;x) \to B(fx_1,\ldots,fx_n;fx)$$

of ϕ are all equivalences.

Any lax symmetric monoidal functor $F\colon \mathscr{C} \to \mathscr{D}$ induces a functor

$$F_*\colon \mathsf{Alg}_{\Omega^{\mathrm{op}}/\mathcal{S}}(\mathscr{C}) \to \mathsf{Alg}_{\Omega^{\mathrm{op}}/\mathcal{S}}(\mathscr{D})$$

given by composing with F. In particular, we always have a lax symmetric monoidal functor

$$u := \mathsf{Map}_{\mathscr{C}}(\mathbb{1},-)\colon \mathscr{C} \to \mathcal{S}$$

by mapping out of the monoidal unit $\mathbb{1}$, and so a functor

$$\mathsf{Alg}_{\Omega^{\mathrm{op}}/\mathcal{S}}(\mathscr{C}) \xrightarrow{u_*} \mathsf{Alg}_{\Omega^{\mathrm{op}}/\mathcal{S}}(\mathcal{S}) \simeq \mathsf{Mon}_{\Omega^{\mathrm{op}}/\mathcal{S}}(\mathcal{S}) \simeq \mathsf{P}_{\mathrm{Seg}}(\Omega)$$

taking a $\mathscr{C}$-enriched ∞-operad to its "underlying unenriched ∞-operad".

Definition 4.3.10. A morphism in $\mathsf{Alg}_{\Omega^{\mathrm{op}}/\mathcal{S}}(\mathscr{C})$ is *essentially surjective* if its image under the composite

$$\mathsf{Alg}_{\Omega^{\mathrm{op}}/\mathcal{S}}(\mathscr{C}) \to \mathsf{P}_{\mathrm{Seg}}(\Omega) \xrightarrow{i^*} \mathsf{P}_{\mathrm{Seg}}(\Delta)$$

is an essentially surjective morphism of Segal spaces, and similarly an object of $\mathsf{Alg}_{\Omega^{\mathrm{op}}/\mathcal{S}}(\mathscr{C})$ is *complete* if its image under the same functor is a complete Segal space.

Theorem 4.3.11 ([CH20])**.** *The full subcategory $\mathsf{Opd}^{\Omega}(\mathscr{C}) \subseteq \mathsf{Alg}_{\Omega^{\mathrm{op}}/\mathcal{S}}(\mathscr{C})$ comprising the complete objects is the localization at the fully faithful and essentially surjective morphisms.*

Let us also comment briefly on the relation between enriched ∞-operads and model categories of enriched operads:

Theorem 4.3.12 ([CH20])**.** *Suppose V is a symmetric monoidal model category such that the model structure can be lifted to V-operads with a fixed set of objects. Then there is an equivalence of ∞-categories*

$$\mathsf{Opd}^{\Omega}(\mathsf{V}[W^{-1}]) \simeq \mathsf{Opd}(\mathsf{V})(\mathrm{DK}^{-1}),$$

where W denotes the weak equivalences in V and DK *the* Dwyer–Kan equivalences *between V-operads, meaning those morphisms that are "weakly fully faithful and essentially surjective up to homotopy".*

The idea of the proof is to relate $\mathsf{Alg}_{\Omega_S^{\mathrm{op}}}(\mathsf{V}[W^{-1}])$ for a fixed set of objects S to algebras for the operad for S-coloured operads in $\mathsf{V}[W^{-1}]$, and then rectify these to strict algebras for the same operad in V, using comparison results of Pavlov and Scholbach [PS18].

In particular, this comparison applies to operads enriched in simplicial sets, chain complexes over a field of characteristic 0, and symmetric spectra. (In these cases the Dwyer–Kan equivalences are actually the weak equivalences in a model structure on strict operads, by work of Caviglia [Cav14] and Berger–Moerdijk [BM03]; note that the comparison between ∞-operads with simplicial operads was first proved by Cisinski and Moerdijk [CM13b].)

Bibliography

[BHS22] Shaul Barkan, Rune Haugseng, and Jan Steinebrunner, *Envelopes for algebraic patterns* (2022), available at arXiv:2208.07183.

[BD95] John C. Baez and James Dolan, *Higher-dimensional algebra and topological quantum field theory*, J. Math. Phys. **36** (1995), no. 11, 6073–6105.

[Bar18] Clark Barwick, *From operator categories to higher operads*, Geom. Topol. **22** (2018), no. 4, 1893–1959, available at arXiv:1302.5756.

[BGS20] Clark Barwick, Saul Glasman, and Jay Shah, *Spectral Mackey functors and equivariant algebraic K-theory, II*, Tunis. J. Math. **2** (2020), no. 1, 97–146, available at arXiv:1505.03098.

[BM03] Clemens Berger and Ieke Moerdijk, *Axiomatic homotopy theory for operads*, Comment. Math. Helv. **78** (2003), no. 4, 805–831.

[BV73] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin-New York, 1973.

[Cav14] Giovanni Caviglia, *A model structure for enriched coloured operads* (2014), available at arXiv:1401.6983.

[CH20] Hongyi Chu and Rune Haugseng, *Enriched ∞-operads*, Adv. Math. **361** (2020), 106913, 85, available at arXiv:1707.08049.

[CH22] ———, *Free algebras through Day convolution*, Algebr. Geom. Topol. **22** (2022), no. 7, 3401–3458, available at arXiv:2006.08269.

[CH21] ———, *Homotopy-coherent algebra via Segal conditions*, Advances in Mathematics **385** (2021), 107733, available at arXiv:1907.03977.

[CHH18] Hongyi Chu, Rune Haugseng, and Gijs Heuts, *Two models for the homotopy theory of ∞-operads*, J. Topol. **11** (2018), no. 4, 856–872, available at arXiv:1606.03826.

[CM13a] Denis-Charles Cisinski and Ieke Moerdijk, *Dendroidal Segal spaces and ∞-operads*, J. Topol. **6** (2013), no. 3, 675–704.

[CM13b] ———, *Dendroidal sets and simplicial operads*, J. Topol. **6** (2013), no. 3, 705–756, available at arXiv:1109.1004.

[CS10] G. S. H. Cruttwell and Michael A. Shulman, *A unified framework for generalized multicategories*, Theory Appl. Categ. **24** (2010), No. 21, 580–655.

[Day70] Brian Day, *On closed categories of functors*, Reports of the Midwest Category Seminar, IV, Lecture Notes in Mathematics, Vol. 137, Springer, Berlin, 1970, pp. 1–38.

[Dun88] Gerald Dunn, *Tensor product of operads and iterated loop spaces*, J. Pure Appl. Algebra **50** (1988), no. 3, 237–258.

[FV15] Z. Fiedorowicz and R. M. Vogt, *An additivity theorem for the interchange of E_n structures*, Adv. Math. **273** (2015), 421–484.

[GGN15] David Gepner, Moritz Groth, and Thomas Nikolaus, *Universality of multiplicative infinite loop space machines*, Algebr. Geom. Topol. **15** (2015), no. 6, 3107–3153.

[GH15] David Gepner and Rune Haugseng, *Enriched ∞-categories via non-symmetric ∞-operads*, Adv. Math. **279** (2015), 575–716, available at arXiv:1312.3178.

[GHK22] David Gepner, Rune Haugseng, and Joachim Kock, *∞-operads as analytic monads*, Int. Math. Res. Not. IMRN **16** (2022), 12516–12624, available at arXiv:1712.06469.

[Get09] Ezra Getzler, *Operads revisited*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, Progr. Math., vol. 269, Birkhäuser Boston, Boston, MA, 2009, pp. 675–698.

[Gla16] Saul Glasman, *Day convolution for ∞-categories*, Math. Res. Lett. **23** (2016), no. 5, 1369–1385, available at arXiv:1308.4940.

[HRY15] Philip Hackney, Marcy Robertson, and Donald Yau, *Infinity properads and infinity wheeled properads*, Lecture Notes in Mathematics, vol. 2147, Springer, Cham, 2015.

[HRY19] ———, *Higher cyclic operads*, Algebr. Geom. Topol. **19** (2019), no. 2, 863–940.

[HRY20] ———, *A graphical category for higher modular operads*, Adv. Math. **365** (2020), 107044, 61.

[Har19] Yonatan Harpaz, *Little cube algebras and factorization homology* (2019), available at https://www.math.univ-paris13.fr/ harpaz.

[Hau23] Rune Haugseng, *From analytic monads to ∞-operads through Lawvere theories* (2023), available at arXiv:2301.01199.

[Hei18] Hadrian Heine, *Restricted L_∞-algebras* (2018), available at https://repositorium.ub.uni-osnabrueck.de/handle/urn:nbn:de:gbv:700-201909201996. Ph.D. thesis, University of Osnabrück.

[HHM16] Gijs Heuts, Vladimir Hinich, and Ieke Moerdijk, *On the equivalence between Lurie's model and the dendroidal model for infinity-operads*, Adv. Math. **302** (2016), 869–1043, available at arXiv:1305.3658.

[HM22] Gijs Heuts and Ieke Moerdijk, *Simplicial and dendroidal homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 75, Springer, Cham, 2022.

[Hin20] Vladimir Hinich, *Yoneda lemma for enriched ∞-categories*, Adv. Math. **367** (2020), 107129, available at arXiv:1805.07635.

[HM22] Vladimir Hinich and Ieke Moerdijk, *On the equivalence of the Lurie's ∞-operads and dendroidal ∞-operads* (2022), available at arXiv:2206.14033.

[Joy86] André Joyal, *Foncteurs analytiques et espèces de structures*, Combinatoire énumérative (Montréal/Québec, 1985), 1986, pp. 126–159.

[JT07] André Joyal and Myles Tierney, *Quasi-categories vs Segal spaces*, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 277–326, available at arXiv:math/0607820.

[Koc11] Joachim Kock, *Polynomial functors and trees*, Int. Math. Res. Not. IMRN **3** (2011), 609–673, DOI 10.1093/imrn/rnq068.

[Lam69] Joachim Lambek, *Deductive systems and categories. II. Standard constructions and closed categories*, Category Theory, Homology Theory and their Applications, I (Battelle Institute Conference, Seattle, Wash., 1968, Vol. One), Springer, Berlin, 1969, pp. 76–122.

[Lei02] Tom Leinster, *Generalized enrichment of categories*, J. Pure Appl. Algebra **168** (2002), no. 2-3, 391–406. Category theory 1999 (Coimbra).

[Lei04] ———, *Higher operads, higher categories*, London Mathematical Society Lecture Note Series, vol. 298, Cambridge University Press, Cambridge, 2004.

[Lur09] Jacob Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. Available from http://math.ias.edu/~lurie/.

[Lur17] ———, *Higher Algebra*, 2017. Available at http://math.ias.edu/~lurie/.

[May72] J. P. May, *The geometry of iterated loop spaces*, Springer-Verlag, Berlin-New York, 1972. Lecture Notes in Mathematics, Vol. 271.

[MT78] J. P. May and R. Thomason, *The uniqueness of infinite loop space machines*, Topology **17** (1978), no. 3, 205–224.

[MW07] Ieke Moerdijk and Ittay Weiss, *Dendroidal sets*, Algebr. Geom. Topol. **7** (2007), 1441–1470.

[NS22] Denis Nardin and Jay Shah, *Parametrized and equivariant higher algebra* (2022), available at arXiv:2203.00072.

[PS18] Dmitri Pavlov and Jakob Scholbach, *Admissibility and rectification of colored symmetric operads*, J. Topol. **11** (2018), no. 3, 559–601, available at arXiv:1410.5675.

[Ram22] Maxime Ramzi, *A monoidal Grothendieck construction for ∞-categories* (2022), available at arXiv:2209.12569.

[Rez01] Charles Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007.

[Rez10] ———, *A Cartesian presentation of weak n-categories*, Geom. Topol. **14** (2010), no. 1, 521–571.

[Seg74] Graeme Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.

[Sim98] Carlos Simpson, *On the Breen-Baez-Dolan stabilization hypothesis for Tamsamani's weak n-categories* (1998), available at arXiv:math/9810058.

[Sta63] James Dillon Stasheff, *Homotopy associativity of H-spaces. I, II*, Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid. **108** (1963), 293–312.