

# THE HUREWICZ IMAGE OF $C_2$ -EQUIVARIANT $H\mathbb{F}_2$ , $H\mathbb{Z}$ , AND $H\mathbb{A}$

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ABSTRACT. We determine the  $RO(C_2)$ -graded Hurewicz image of the  $C_2$ -equivariant Eilenberg–MacLane spectra  $H\mathbb{F}_2$ ,  $H\mathbb{Z}$  and  $H\mathbb{A}$ , where  $\mathbb{F}_2$ ,  $\mathbb{Z}$  and  $\mathbb{A}$  denote the constant Mackey functor with values in  $\mathbb{F}_2$ ,  $\mathbb{Z}$ , and the Burnside Mackey functor, respectively.

Using results of the fourth author [Ma26], the answer is closely tied to Adams’ solution to the vector field problem on spheres [Ada62]: the particular element  $\frac{\theta}{\rho^{k\tau^n}}$  in the negative cone of the homotopy group of  $H\mathbb{F}_2$  is in the Hurewicz images if and only if  $S^n$  admits  $k$  linearly independent vector fields.

Moreover, using the Generalized Leibniz Rule and the Generalized Mahowald Trick by [LWX25b], we show that there are Adams differentials of arbitrary length supported by filtration 0 elements in the genuine  $C_2$ -equivariant Adams spectral sequence.

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## 1. INTRODUCTION

One of the major ideas in homotopy theory is that (generalized) homology theory is a first approximation to homotopy. For a ring spectrum  $E$ , the Hurewicz map

$$\iota_E: \pi_* S^0 \longrightarrow \pi_* E$$

packages the information in  $\pi_* S^0$  that is *detected* by  $E$ -homology, thereby yielding further information about the stable stems [Mah82, DFHH14, HHR16]. Consequently, determining the Hurewicz images of ring spectra has long been a central theme in homotopy theory. Recent progress in this direction includes computations for the topological modular forms [BHHM20, BR21, BMQ23, BBQ26], as well as for the real bordism theory and real Johnson–Wilson theories [LSWX19, HS20].

This paper concerns questions about the Hurewicz map in the  $C_2$ -equivariant setting. Naturally graded by the real representation ring  $RO(C_2)$ ,  $C_2$ -equivariant homology theories exhibit substantially richer structure than their classical counterparts; yet they remain computationally tractable. Given a  $C_2$ -Mackey functor  $\underline{M}$ , there is an associated genuine  $C_2$ -equivariant Eilenberg–MacLane spectrum  $H\underline{M}$ . If  $\underline{M} = \mathbb{F}_2$  is the constant Mackey functor,  $H\mathbb{F}_2$  is the base for the genuine  $C_2$ -equivariant Adams spectral sequence, which has been studied extensively by [BGI21, GI20, GHIR20] and has been the primary tool

in recent progress of computing the  $RO(C_2)$ -graded equivariant stable stems into a large range [GI24, BXZ24]. If  $\underline{M} = \underline{\mathbb{Z}}$  is the constant Mackey functor,  $H\underline{\mathbb{Z}}$  is a key input to the slice spectral sequence and, in particular, to the Hill–Hopkins–Ravenel approach leading to the solution of the Kervaire invariant problem [HHR16]. If  $\underline{M} = \underline{A}$ , the Burnside Mackey functor,  $H\underline{A}$  is the monoidal unit in the category of Green functors, which are equivariant analogs of rings. However, despite a wide range of applications in homotopy theory, it remains unclear which elements in the equivariant stable stems are detected by these three Eilenberg–MacLane spectra.

The goal of this paper is to explicitly compute the Hurewicz image of  $H\underline{M}$  for  $\underline{M} = \underline{\mathbb{F}}_2, \underline{\mathbb{Z}}, \underline{A}$ . In contrast to the nonequivariant case, where the homotopy groups of an Eilenberg–MacLane spectrum are concentrated in degree 0,  $\pi_*^{C_2} H\underline{M}$  are nontrivial in infinitely many bidegrees. Consequently, additional information on the equivariant stable stems is needed, which makes the problem much more subtle than the nonequivariant case.

**1.1. Main theorems.** Recall that the real representation ring  $RO(C_2)$  has a basis consisting of the trivial representation and the sign representation  $\sigma$ . We use the bi-grading  $(s, w)$  to indicate the  $RO(C_2)$ -degree  $(s - w) + w\sigma$ . We call  $s$  the *stem*,  $w$  the *weight*, and  $s - w$  the *coweight*.

Furthermore, for any integer  $n \geq 0$ , write  $n = (2a + 1)2^{c+4d}$  with  $a, c, d$  all integers and  $0 \leq c \leq 3$ . Let  $\psi(n) := 2^c + 8d$  denote the  $n^{\text{th}}$  Radon–Hurwitz number [Eck43, Ada62].

Recall from [HK01] that

$$H\underline{\mathbb{F}}_{2*,*} = \mathbb{F}_2[\rho, \tau] \oplus NC, \quad NC = \bigoplus_{r,s \geq 0} \mathbb{F}_2 \left\{ \frac{\theta}{\rho^s \tau^r} \right\}$$

where  $|\rho| = (-1, -1)$ ,  $|\tau| = (0, -1)$ , and  $|\theta| = (0, 2)$ . In particular, the first summand equals the coefficient of the  $\mathbb{R}$ -motivic Eilenberg–MacLane spectrum [Voe03]

$$H\underline{\mathbb{F}}_{2*,*}^{\mathbb{R}} = \mathbb{F}_2[\rho, \tau].$$

The algebra  $NC$  is called the *negative cone*. It is generated by elements of the form  $\frac{\theta}{\rho^s \tau^r}$  for  $s, r \geq 0$ , with relations  $\rho \cdot \theta = \tau \cdot \theta = \theta \cdot \theta = 0$ . The rest  $\mathbb{F}_2[\rho, \tau]$ -module structure of  $NC$  is hinted by the names. Fig. 1 depicts the additive structure of  $H\underline{\mathbb{F}}_{2*,*}$ , where each black dot represents  $\mathbb{F}_2$ . The coefficients of  $H\underline{\mathbb{Z}}$  and  $H\underline{A}$  are recalled in Section 4.

**Remark 1.1.** The notation  $\theta$  is also used in [LSWX19, BS20, May20]. In other references, the same element is sometimes denoted by  $\frac{\gamma}{\tau}$ ; see, for example, [GHIR20, GI24, BGI21].

We first determine the Hurewicz image of  $H\underline{\mathbb{F}}_2$ .

**Theorem 1.2 (Theorem 3.2).** The  $RO(C_2)$ -graded Hurewicz image of  $H\underline{\mathbb{F}}_2$  consists of

$$\text{Im}(\iota_{H\underline{\mathbb{F}}_2})_{s,w} = \begin{cases} \mathbb{F}_2\{\rho^{-s}\} & \text{if } s = w \leq 0, \\ \mathbb{F}_2\left\{\frac{\theta}{\rho^s \tau^{w-s-2}}\right\} & \text{if } 0 \leq s < \psi(w - s - 1) \text{ and } s - w \leq -2, \\ 0 & \text{otherwise.} \end{cases}$$

For  $s = w \leq 0$ ,  $\rho$  is the image of the Euler class of the  $C_2$ -sign representation  $\sigma$ . For  $s - w \leq -2$  (i.e. in the negative cone), a picture of the Hurewicz image of  $H\underline{\mathbb{F}}_2$  is represented in Fig. 2.

Since  $\psi(n)$  is closely related to the number of vector fields on spheres [Ada62], we have:

**Corollary 1.3.** The element  $\frac{\theta}{\rho^k \tau^n}$  is in the Hurewicz images if and only if  $S^n$  admits  $k$  many linearly independent vector fields.

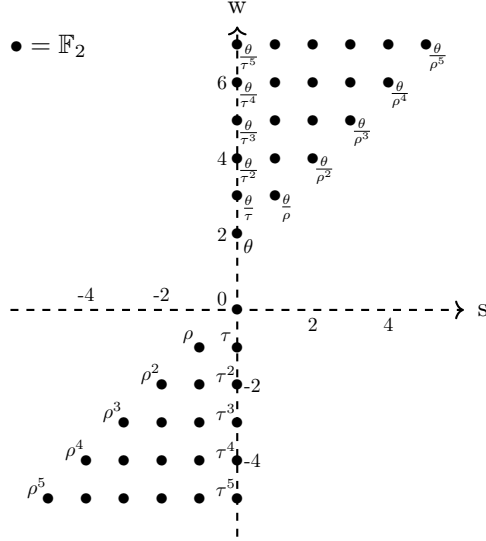
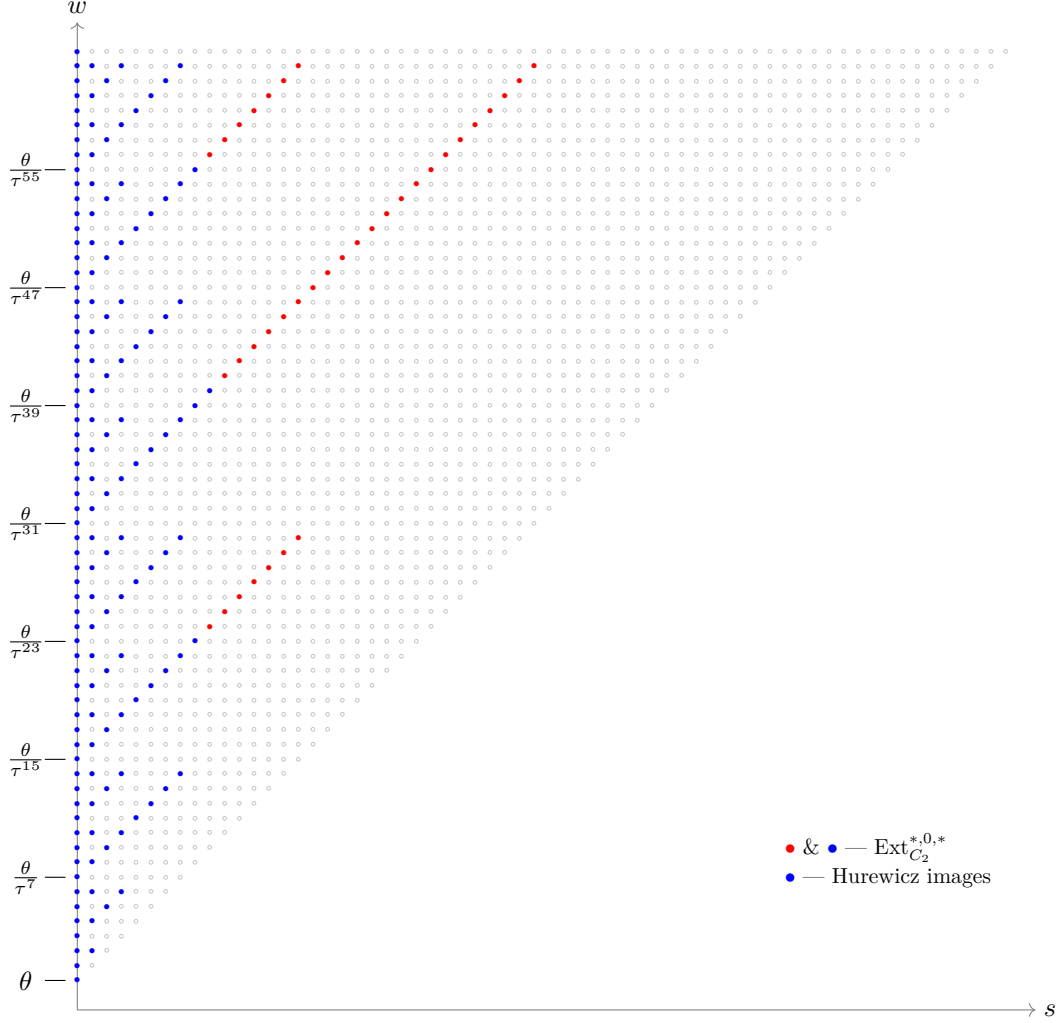


FIGURE 1.  $H\mathbb{F}_2_{s,w}$

We prove [Theorem 1.2](#) by first computing  $\text{Ext}_{C_2}^{*,0,*}$ , the 0-line of the  $E_2$ -page of the genuine  $C_2$ -equivariant Adams spectral sequence, via the  $\rho$ -Bockstein spectral sequence. The elements at degree  $(s, 0, w)$  of  $\text{Ext}_{C_2}$  with  $s - w \leq -2$  are represented by the solid (red and blue) dots in [Fig. 2](#). The Hurewicz image of  $H\mathbb{F}_2$  consists precisely of surviving permanent cycles in the 0-line. Using the correspondence between the genuine Adams spectral sequence, the Borel Adams spectral sequence, and the classical Adams spectral sequence of stunted real projective spaces [[Ma26](#)], we then determine which 0-line elements support Adams differentials. It turns out that these differentials are governed by the attaching maps of the corresponding stunted real projective spaces, and the necessary information follows as a corollary of Adams' work in his solution to the vector field problem [[Ada62](#)], which explains the appearance of the Radon–Hurwitz number  $\psi(n)$  in [Theorem 1.2](#). In [Fig. 2](#), red dots are the classes that support Adams differentials, while blue dots are the elements that are in the Hurewicz image of  $H\mathbb{F}_2$ .

Fixing  $s - w \leq -2$ , when  $s$  is the largest, the initial differentials supported by the filtration 0 elements are always  $d_2$ -differentials; as  $s$  becomes smaller, the length of the differentials will increase. An illustrative diagram for the general pattern of the  $C_2$ -equivariant genuine Adams spectral sequence at coweight  $s - w \leq -2$  and filtration 0 is shown in [Fig. 3](#). In accordance with [Fig. 2](#), the red dots indicate classes that support nontrivial Adams differentials, while the blue dots represent permanent cycles. The horizontal straight lines indicate  $\rho$ -extensions from the right to the left. In particular, for  $\frac{\theta}{\rho^{\psi(w-s-1)}\tau^{w-s-2}}$  which is the last element of coweight  $s - w$  that is not a permanent cycle by [Theorem 1.2](#), we prove that it supports an Adams differential of length depending on the 2-adic valuation of  $w - s - 1$ . In particular, there are Adams differentials of arbitrary length supported by filtration 0 elements.

**Theorem 1.4** ([Theorem 3.4](#)). Let  $v_2(\_)$  denote the 2-adic valuation. If  $v_2(w - s - 1) \geq 5$ , the element  $\frac{\theta}{\rho^{\psi(w-s-1)}\tau^{w-s-2}}$  survives to the  $E_r$ -page for  $r = v_2(w - s - 1) - 1$ , and it supports a nontrivial  $d_r$ -differential.

FIGURE 2. The Hurewicz image of  $H\mathbb{F}_2$  in the negative cone

If  $v_2(w - s - 1) = 4$ , then  $\psi(w - s - 1) = 9$  and the discussion in [Subsection 3.3](#) implies  $\frac{\theta}{\rho^s \tau^{w-s-2}}$  supports a  $d_2$ -differential for each  $9 \leq s \leq 15$ . If  $v_2(w - s - 1) \leq 3$ , then  $\psi(w - s - 1) = v_2(w - s - 1)$  and [Theorem 1.2](#) implies that every filtration 0 element of this coweight are permanent cycles.

The proof of [Theorem 1.4](#) is based on analysis of differentials in the classical Adams spectral sequence of various stunted projective spectra. One of the key steps uses the techniques of the Generalized Leibniz Rule and the Generalized Mahowald Trick, developed by Lin–Wang–Xu in their recent solutions to the last Kervaire invariant one problem [[LWX25b](#)].

Based on low-dimensional calculations, we also have a conjecture for some of the major intermediate differentials.

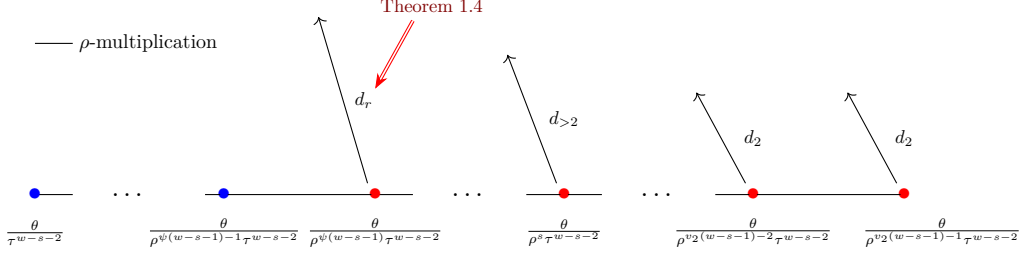


FIGURE 3.  $C_2$ -equivariant genuine Adams spectral sequence at coweight  $s - w$  and filtration 0

**Conjecture 1.5.** Let  $M$  denote the algebraic Mahowald invariant for the 2-primary Adams spectral sequence [MR93]. For  $j \gg k > 0$ ,

$$d_{k+1}\left(\frac{\theta}{\rho^{2^{j-k}+s(k)}\tau^{2^j-1}}\right) = \frac{\theta}{\tau^{2^j+2^{j-k-1}+c(k)-1}}M(h_0^k)h_{j-k},$$

where  $s(k)$  and  $c(k)$  are the stem and coweight of  $M(h_0^k)$ , with values given in the table below.

$k$	$s(k)$	$c(k)$
$4n + 1$	$8n + 1$	$4n$
$4n + 2$	$8n + 2$	$4n$
$4n + 3$	$8n + 3$	$4n$
$4n + 4$	$8n + 7$	$4n + 3$

As a direct application of [Theorem 1.2](#) and the results of [BXZ24], we also determine the Hurewicz image of  $H\mathbb{Z}$  and  $H\mathbb{A}$ .

**Theorem 1.6** ([Theorem 4.2](#)).

- (1) The  $RO(C_2)$ -graded Hurewicz image of  $H\mathbb{Z}$  consists of

$$\mathrm{Im}(\iota_{H\mathbb{Z}})_{s,w} = \begin{cases} \mathbb{Z} & \text{if } s = 0 \text{ and } w = 2k \text{ for } k \in \mathbb{Z}, \\ \mathbb{F}_2 & \text{if } s = w < 0, \\ \mathbb{F}_2 & \text{if } 0 \leq s < \psi(w - s - 1) \text{ and } s - w = 2k - 1 \text{ for } k \leq -1, \\ 0 & \text{otherwise.} \end{cases}$$

- (2) The  $RO(C_2)$ -graded Hurewicz image of  $H\mathbb{A}$  consists of

$$\mathrm{Im}(\iota_{H\mathbb{A}})_{s,w} = \begin{cases} A(C_2) & \text{if } s = w = 0, \\ \mathbb{Z} & \text{if } s = 0 \text{ and } w = -2k \text{ for } k \neq 0, \\ \mathbb{Z} & \text{if } s = w \neq 0, \\ \mathbb{F}_2 & \text{if } 0 \leq s < \psi(w - s - 1) \text{ and } s - w = 2k - 1 \text{ for } k \leq -1, \\ 0 & \text{otherwise.} \end{cases}$$

The generators of the Hurewicz images of these two spectra will be specified in [Section 4](#).

## 1.2. Notation.

- $\sigma$ : the  $C_2$ -sign representation;
- $\pi_{*,*}^{C_2}(\_)$ : the  $RO(C_2)$ -graded  $C_2$ -equivariant stable homotopy groups;
- $a$ : the Euler class of  $\sigma$  in  $\pi_{-1,-1}^{C_2}$ ;

- $\underline{\mathbb{F}}_2$ : the constant Mackey functor over  $\mathbb{F}_2$ ;
- $\underline{\mathbb{Z}}$ : the constant Mackey functor over  $\mathbb{Z}$ ;
- $\underline{A}$ : the  $C_2$ -equivariant Burnside Mackey functor;
- $H\underline{\mathbb{F}}_2$ : the genuine  $C_2$ -equivariant Eilenberg–MacLane spectrum of  $\underline{\mathbb{F}}_2$ ;
- $H\underline{\mathbb{Z}}$ : the genuine  $C_2$ -equivariant Eilenberg–MacLane spectrum of  $\underline{\mathbb{Z}}$ ;
- $H\underline{A}$ : the genuine  $C_2$ -equivariant Eilenberg–MacLane spectrum of  $\underline{A}$ ;
- $H\underline{\mathbb{F}}_2^{\mathbb{R}}$ : the  $\mathbb{R}$ -motivic Eilenberg–MacLane spectrum with  $\mathbb{F}_2$ -coefficient;
- $NC$ : the negative cone of  $H\underline{\mathbb{F}}_{2*,*}$ ;
- $\text{Ext}_{\mathbb{C}}^{*,*,*}$ : the  $E_2$ -page of the 2-primary  $\mathbb{C}$ -motivic Adams spectral sequence;
- $\text{Ext}_{\mathbb{R}}^{*,*,*}$ : the  $E_2$ -page of the 2-primary  $\mathbb{R}$ -motivic Adams spectral sequence;
- $\text{Ext}_{C_2}^{*,*,*}$ : the  $E_2$ -page of the 2-primary genuine  $C_2$ -equivariant Adams spectral sequence;
- $\text{Ext}_h^{*,*,*}$ : the  $E_2$ -page of the 2-primary Borel  $C_2$ -equivariant Adams spectral sequence;
- $\text{Ext}^{*,*}$ : the  $E_2$ -page of the classical 2-primary Adams spectral sequence;
- ${}^{\mathbb{R}}E_r^{*,*,*}$ : the  $E_r$ -page of the 2-primary  $\mathbb{R}$ -motivic Adams spectral sequence;
- $C_2 E_r^{*,*,*}$ : the  $E_r$ -page of the 2-primary genuine  $C_2$ -equivariant Adams spectral sequence;
- ${}^h E_r^{*,*,*}$ : the  $E_r$ -page of the 2-primary Borel  $C_2$ -equivariant Adams spectral sequence;
- $E_r^{*,*}$ : the  $E_r$ -page of the classical 2-primary Adams spectral sequence;
- $v_2(\_)$ : the 2-adic valuation function;
- $\psi(n)$ : the  $n^{\text{th}}$  Radon–Hurwitz number;
- $J$ : the image-of- $J$  spectrum.

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## 2. PRELIMINARIES ON $C_2$ -EQUIVARIANT HOMOTOPY

We review the basics of the  $\mathbb{R}$ -motivic, genuine  $C_2$ -equivariant and Borel  $C_2$ -equivariant Adams spectral sequences; references for this material can be found in [DI10, DI17, GHIR20, Ma26].

Let  $H\underline{\mathbb{F}}_2^{\mathbb{R}}$  and  $H\underline{\mathbb{F}}_2^h$  denote the  $\mathbb{R}$ -motivic and Borel  $C_2$ -equivariant Eilenberg–MacLane spectra with  $\mathbb{F}_2$ -coefficients, respectively; let  $\mathcal{A}^{\mathbb{R}}$ ,  $\mathcal{A}^{C_2}$  and  $\mathcal{A}^h$  denote the  $\mathbb{R}$ -motivic, genuine  $C_2$ -equivariant, and Borel  $C_2$ -equivariant Steenrod algebras, respectively. Then, the  $\mathbb{R}$ -motivic, genuine  $C_2$ -equivariant and Borel  $C_2$ -equivariant Adams spectral sequences of the spheres are

$$\begin{aligned} {}^{\mathbb{R}}E_2^{s,f,w} &= \text{Ext}_{\mathbb{R}}^{s,f,w} := \text{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s,f,w}(H\underline{\mathbb{F}}_{2*,*}^{\mathbb{R}}, H\underline{\mathbb{F}}_{2*,*}^{\mathbb{R}}) \implies (\pi_{s,w}^{\mathbb{R}})_2^\wedge \\ {}^{C_2}E_2^{s,f,w} &= \text{Ext}_{C_2}^{s,f,w} := \text{Ext}_{\mathcal{A}^{C_2}}^{s,f,w}(H\underline{\mathbb{F}}_{2*,*}, H\underline{\mathbb{F}}_{2*,*}) \implies (\pi_{s,w}^{C_2})_2^\wedge \\ {}^hE_2^{s,f,w} &= \text{Ext}_h^{s,f,w} := \text{Ext}_{\mathcal{A}^h}^{s,f,w}(H\underline{\mathbb{F}}_{2*,*}^h, H\underline{\mathbb{F}}_{2*,*}^h) \implies (\pi_{s,w}^{C_2})_2^\wedge \end{aligned}$$

where  $f$  denotes the cohomological degree, and  $(s, w)$  in the equivariant case aligns with the chosen  $RO(C_2)$ -grading, while  $(s, w)$  in the  $\mathbb{R}$ -motivic case is the pair of stem and motivic weight. Our grading convention follows that of [DI10, IWX23].

**Proposition 2.1** ([GHIR20, BXZ24]).

- (1) The splitting  $H\mathbb{F}_2_{*,*} \cong H\mathbb{F}_2^{\mathbb{R}}_{*,*} \oplus NC$  induces a splitting

$$\mathrm{Ext}_{C_2} = \mathrm{Ext}_{\mathbb{R}} \oplus \mathrm{Ext}_{NC}$$

where  $\mathrm{Ext}_{NC} = \mathrm{Ext}_{\mathcal{A}C_2}(H\mathbb{F}_2_{*,*}, NC)$ .

- (2) There is a  $\rho$ -Bockstein spectral sequence converging to  $\mathrm{Ext}_{C_2}$  such that under the splitting in part (1), the spectral sequence decomposes as

$$\begin{aligned} E_1^+ &= \mathrm{Ext}_{\mathbb{C}}[\rho] \implies \mathrm{Ext}_{\mathbb{R}} \\ E_1^- &\implies \mathrm{Ext}_{NC} \end{aligned}$$

where  $\mathrm{Ext}_{\mathbb{C}}$  is the  $E_2$ -page of the  $\mathbb{C}$ -motivic Adams spectral sequence. A Bockstein differential  $d_r$  takes a class  $x$  of degree  $(s, f, w)$  to a class  $d_r(x)$  of degree  $(s-1, f+1, w)$ .

- (3)  $E_1^-$  is generated over  $\mathbb{F}_2$  by elements of the following two types:

- Elements of the form  $\frac{\theta}{\rho^a \tau^b} x$  where  $a, b \geq 0$ , and  $x$  is an element of  $\mathrm{Ext}_{\mathbb{C}}$  that is  $\tau$ -free and not divisible by  $\tau$ . If  $x$  has degree  $(s, f, w)$  in  $\mathrm{Ext}_{\mathbb{C}}$ , then  $\frac{\theta}{\rho^a \tau^b} x$  has degree  $(s+a, f, w+a+b+2)$ .
- Elements of the form  $\frac{Q}{\rho^a \tau^b} x$  where  $0 \leq a, 0 \leq b \leq k$ , and  $x$  is an element of  $\mathrm{Ext}_{\mathbb{C}}$  that is  $\tau$ -torsion and divisible by  $\tau^k$  but not  $\tau^{k+1}$ . If  $x$  has degree  $(s, f, w)$  in  $\mathrm{Ext}_{\mathbb{C}}$ , then  $\frac{Q}{\rho^a \tau^b} x$  has degree  $(s+a+1, f-1, w+a+b+1)$ .

The relationship between the genuine and the Borel  $C_2$ -equivariant Adams spectral sequences is as follows.

**Proposition 2.2** ([Ma26, Theorem 6.2]).

$$\mathrm{Ext}_h^{s,f,w} \cong H \left( \mathrm{Ext}_{\mathbb{R}}^{s,f,w} \oplus \mathrm{Ext}_{NC}^{s,f-1,w}, \delta \right),$$

where  $\delta$  is the composition map

$$\mathrm{Ext}_{NC}^{s,f-1,w} \hookrightarrow \mathrm{Ext}_{C_2}^{s,f-1,w} \xrightarrow{d_2} \mathrm{Ext}_{C_2}^{s-1,f+1,w} \rightarrow \mathrm{Ext}_{\mathbb{R}}^{s-1,f+1,w}.$$

In particular, there is a map

$$\varphi : \mathrm{Ext}_h^{s,f,w} \rightarrow \mathrm{Ext}_{NC}^{s,f-1,w}.$$

When  $w \geq s+2$ ,  $\varphi$  is an isomorphism since both  $\mathrm{Ext}_{\mathbb{R}}^{s,f,w}$  and  $\mathrm{Ext}_{\mathbb{R}}^{s-1,f+1,w}$  are trivial. Furthermore, since  $\mathrm{Ext}_{\mathbb{R}}^{s,f-1,w}$  is trivial as well, we have

$$\mathrm{Ext}_h^{s,f,w} \cong \mathrm{Ext}_{C_2}^{s,f-1,w}.$$

Let  $j$  be the Borel completion map  $H\mathbb{F}_2 \rightarrow H\mathbb{F}_2^h$ . It induces a map

$$j_* : C_2 E_1^{s,f,w} \rightarrow {}^h E_1^{s,f,w},$$

between the  $E_1$ -page of the genuine and Borel  $C_2$ -equivariant Adams spectral sequences, which is trivial when  $w \geq s+2$ . The differentials in the genuine and Borel Adams spectral sequences have the following correspondence.

**Proposition 2.3** ([Ma26, Theorem 6.4, 6.5]).

- (1) Let  $z \in {}^{C_2}E_1^{s,f,w}$  be an element with  $j_*z = 0$ . Suppose  $z$  supports a nontrivial differential  $d_r^{C_2}([z]) = [z']$ , where  $r \geq 2$  and  $z' \in {}^{C_2}E_1^{s-1,f+r,w}$ . Suppose further that  $j_*z' = 0$ . Then there exists an element  $x \in {}^hE_1^{s,f+1,w}$  surviving to  ${}^hE_2$  with  $\varphi([x]) = [z] \in {}^{C_2}E_2$ , and an element  $x' \in {}^hE_1^{s-1,f+r,w}$  surviving to  ${}^hE_r$  with  $\varphi([x']) = [z'] \in {}^{C_2}E_2$ , such that  ${}^h d_r([x]) = [x']$ .
- (2) Let  $z \in {}^{C_2}E_1^{s,f,w}$  be an element with  $j_*z = 0$ . Suppose  $z$  survives to  ${}^{C_2}E_\infty$  and detects  $\hat{z} \in \pi_{s,w}^{C_2}$ . Then there exists an element  $x \in {}^hE_1^{s,f+1,w}$  surviving to  ${}^hE_\infty$  and detecting  $\hat{z}$ , such that  $\varphi([x]) = [z]$ .

On the other hand, the Borel Adams spectral sequence is related to the classical Adams spectral sequences for the stunted real projective spectra in the sense that the following two Mahowald squares are isomorphic, as shown in [Ma26, Section 3]:

$$\begin{array}{ccc}
& \text{Ext}_{\mathcal{A}^h} \left( H\mathbb{F}_2^{h,*} / \rho \right) [\rho] & \\
\swarrow \text{Borel Adams SS} & & \searrow \text{algebraic } \rho\text{-Bockstein SS} \\
\pi_{*,*}^{C_2} (\mathbb{S}/\rho)_2^\wedge [\rho] & & \text{Ext}_{\mathcal{A}^h} \left( H\mathbb{F}_2^{h,*} \right) \\
\swarrow \text{topological } \rho\text{-Bockstein SS} & & \searrow \text{Borel Adams SS} \\
& (\pi_{*,*}^{C_2} \mathbb{S})_2^\wedge & \\
\oplus \text{Ext}(\mathbb{F}_2) & & \\
\swarrow \text{Adams SS} & & \searrow \text{algebraic Atiyah-Hirzebruch SS} \\
\oplus \pi_* \mathbb{S}_2^\wedge & & \text{Ext} \left( \varprojlim_k H_* \mathbb{R}P_{-k}^{-w-1} \right) \\
\swarrow \text{topological Atiyah-Hirzebruch SS} & & \searrow \text{Adams SS} \\
& (\pi_* \mathbb{R}P_{-\infty}^{-w-1})_2^\wedge &
\end{array}$$

Furthermore, for the lower right spectral sequences in the above squares, we have the following description.

**Proposition 2.4** ([Ma26, Theorem 3.3, 4.1, 4.2]). The Borel Adams spectral sequence of the  $C_2$ -equivariant sphere

$$\text{Ext}_h^{s,f,w} \implies (\pi_{s,w}^{C_2} \mathbb{S})_2^\wedge$$

is either isomorphic to the classical Adams spectral sequence of the stunted real projective spectra

$$\text{Ext}^{s-w,f-1} (\Sigma \mathbb{R}P_{-\infty}^{-w-1}) \implies (\pi_{s-w} \Sigma \mathbb{R}P_{-\infty}^{-w-1})_2^\wedge$$

when  $w > 0$ , or isomorphic to

$$\text{Ext}^{s-w,f-1} (\mathbb{R}P_{-w}^\infty) \oplus \text{Ext}^{s-w,f} \implies (\pi_{s-w} \mathbb{R}P_{-w}^\infty)_2^\wedge \oplus \pi_{s-w} \mathbb{S}_2^\wedge$$

when  $w \leq 0$ .

The fiber sequence

$$\mathbb{R}P_{-\infty}^{\infty} \rightarrow \mathbb{R}P_{-w}^{\infty} \rightarrow \Sigma\mathbb{R}P_{-\infty}^{-w-1},$$

and Lin's theorem [Lin80] imply that we have the 2-adic equivalence  $\tau_{\leq -2}(\Sigma\mathbb{R}P_{-\infty}^{-w-1}) \simeq \tau_{\leq -2}(\mathbb{R}P_{-w}^{\infty})$ . In particular, when  $w \geq \max\{s+2, 0\}$ , we have

$$\mathrm{Ext}^{s-w, f-1}(\Sigma\mathbb{R}P_{-\infty}^{-w-1}) \cong \mathrm{Ext}^{s-w, f-1}(\mathbb{R}P_{-w}^{\infty}).$$

### 3. ANALYSIS OF THE GENUINE ADAMS SPECTRAL SEQUENCE AND THE HUREWICZ IMAGE OF $H\mathbb{F}_2$

**3.1. The algebraic Hurewicz image.** We first compute the 0-line of  $\mathrm{Ext}_{C_2}$ .

**Theorem 3.1.** Recall that  $v_2(\_)$  denotes the 2-adic valuation. Then

$$\mathrm{Ext}_{C_2}^{s, 0, w} = \begin{cases} \mathbb{F}_2\{\rho^{-s}\} & \text{if } s = w \leq 0, \\ \mathbb{F}_2\left\{\frac{\theta}{\rho^s \tau^{w-s-2}}\right\} & \text{if } 0 \leq s < 2^{v_2(w-s-1)} \text{ and } s-w \leq -2, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.*

- (1) If  $s-w \geq 0$ , for degree reasons,  $\mathrm{Ext}_{NC} = 0$ . In the  $\rho$ -Bockstein spectral sequence for  $\mathrm{Ext}_{\mathbb{R}}$ , by [DI17, BI22], there are differentials

$$\begin{aligned} d_1(\tau) &= \rho h_0 \\ d_{2^n}(\tau^{2^n}) &= \rho^{2^n} \tau^{2^{n-1}} h_n, \quad n \geq 1 \end{aligned}$$

which are  $\rho$ -linear. Therefore, the only elements of filtration 0 surviving to  $\mathrm{Ext}_{\mathbb{R}}$  are the  $\rho$ -powers.

- (2) If  $s-w < 0$ ,  $\mathrm{Ext}_{\mathbb{R}} = 0$ . In the  $\rho$ -Bockstein spectral sequence for  $\mathrm{Ext}_{NC}$ , the only elements of filtration 0 are of the form  $\frac{\theta}{\rho^a \tau^b}$  for some  $a, b \geq 0$  by Proposition 2.1.

By [GI24, Proposition 4.10], for each  $n \geq 0$ , there is a  $\rho$ -Bockstein differential

$$d_{2^n} \left( \frac{\theta}{\rho^{2^n} \tau^{2^{n-1}}} \right) = \frac{\theta}{\tau^{2^n + \lceil 2^{n-1} \rceil - 1}} h_n,$$

which is both  $\rho$ -divisible and  $\tau^{2^{n+1}}$ -divisible. Thus, for each  $i, k \geq 0$ , we have

$$d_{2^n} \left( \frac{\theta}{\rho^{2^n+i} \tau^{k \cdot 2^{n+1} + 2^n - 1}} \right) = \frac{\theta}{\rho^i \tau^{k \cdot 2^{n+1} + 2^n + \lceil 2^{n-1} \rceil - 1}} h_n.$$

Therefore, the only permanent cycles are  $\frac{\theta}{\rho^j \tau^{k \cdot 2^{n+1} + 2^n - 1}}$  for  $0 \leq j \leq 2^n - 1$ . The claim follows by fitting in the correct bidegree.  $\square$

**3.2. The homotopy Hurewicz image.** It then remains to determine the fate of the 0-line elements in the Adams spectral sequence.

Recall that for  $n = (2a+1)2^{c+4d}$  with  $0 \leq c \leq 3$ ,  $\psi(n) = 2^c + 8d$  is the  $n^{\mathrm{th}}$  Radon-Hurwitz number; in particular, Adams [Ada62] showed that the maximum number of continuous linearly independent vector fields on  $S^{n-1}$  is  $\psi(n) - 1$ . A by-product of Adams' solution to the vector-field problem is that the top cell of  $\mathbb{R}P_{n-\psi(n)}^{n-1}$  splits off, while it does not split in  $\mathbb{R}P_{n-\psi(n)-1}^{n-1}$  and such attaching map  $j_{\psi(n)-1}$  can be chosen as a generator of the 2-primary image-of- $J$  spectrum at stem  $\psi(n) - 1$  [BMMS86]. We denote the image-of- $J$  spectrum by  $J$ .

Note that for large  $N$ ,  $\psi(n) = \psi(2^N + n)$  and if  $n < 0$ , then  $\psi(-n) = \psi(2^N + n)$ . In particular, if  $n < 0$ , by James periodicity,

$$\mathbb{R}P_{n-\psi(-n)}^{n-1} \simeq \Sigma^{-2^N} \mathbb{R}P_{2^N+n-\psi(-n)}^{2^N+n-1} \simeq \Sigma^{-2^N} \mathbb{R}P_{2^N+n-\psi(2^N+n)}^{2^N+n-1}.$$

Thus, for  $n < 0$ , the top cell of  $\mathbb{R}P_{n-\psi(-n)}^{n-1}$  splits off, while it does not split in  $\mathbb{R}P_{n-\psi(-n)-1}^{n-1}$ .

**Theorem 3.2.** The  $RO(C_2)$ -graded Hurewicz image of  $H\mathbb{F}_2$  consists of

$$\mathrm{Im}(\iota_{H\mathbb{F}_2})_{s,w} = \begin{cases} \mathbb{F}_2\{\rho^{-s}\} & \text{if } s = w \leq 0, \\ \mathbb{F}_2\left\{\frac{\theta}{\rho^s \tau^{w-s-2}}\right\} & \text{if } 0 \leq s < \psi(w-s-1) \text{ and } s-w \leq -2, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the class  $\frac{\theta}{\rho^s \tau^{w-s-2}}$  supports a nontrivial Adams differential if and only if  $\psi(w-s-1) \leq s < 2^{v_2(w-s-1)}$ .

*Proof.*

- (1) For  $s = w$ , the powers of  $\rho$  are surviving permanent cycles for degree reasons. In particular,  $\rho$  detects the Euler class of  $\sigma$ .
- (2) By [Proposition 2.2](#) and [Proposition 2.4](#), when  $w \geq s + 2$ ,  $\mathrm{Ext}_{\mathbb{R}}^{s,f,w} = \mathrm{Ext}_{\mathbb{R}}^{s-1,f+1,w} = \mathrm{Ext}_{\mathbb{R}}^{s,f-1,w} = 0$ , and so

$$\mathrm{Ext}_{C_2}^{s,f-1,w} = \mathrm{Ext}_{NC}^{s,f-1,w} \cong \mathrm{Ext}_h^{s,f,w} \cong \mathrm{Ext}^{s-w,f-1}(\mathbb{R}P_{-w}^\infty).$$

Therefore,  $\frac{\theta}{\rho^s \tau^{w-s-2}} \in \mathrm{Ext}^{s,0,w}$  corresponds to  $1[s-w]$  in  $\mathrm{Ext}^{s-w,0}(\mathbb{R}P_{-w}^\infty)$ , since this is the only element in the bidegree of the algebraic Atiyah–Hirzebruch spectral sequence.

If  $0 \leq s < \psi(w-s-1)$ , then  $-w \geq s-w+1-\psi(w-s-1)$  so that the  $(s-w)$ -cell splits off in  $\mathbb{R}P_{-w}^{s-w}$ . Therefore,

$$\mathrm{Ext}(\mathbb{R}P_{-w}^{s-w}) \cong \mathrm{Ext}(\mathbb{R}P_{-w}^{s-w-1}) \oplus \mathrm{Ext}(S^{s-w}),$$

and  $1[s-w]$  is a permanent cycle. By naturality and degree reasons,  $1[s-w]$  is a permanent cycle in the Adams spectral sequence of  $\mathbb{R}P_{-w}^\infty$ . By [Proposition 2.3](#),  $\frac{\theta}{\rho^s \tau^{w-s-2}}$  is a permanent cycle in the genuine  $C_2$ -equivariant Adams spectral sequence.

On the other hand,  $\psi(w-s-1) \leq s < 2^{v_2(w-s-1)}$  happens only when  $v_2(w-s-1) \geq 4$ . By [\[Ada62\]](#), we have a differential in the Atiyah–Hirzebruch spectral sequence of  $\mathbb{R}P_{-w}^\infty$ ,

$$d_{\psi(w-s-1)}(1[s-w]) = j_{\psi(w-s-1)-1}[s-w-\psi(w-s-1)],$$

where  $j_{\psi(w-s-1)-1}$  is a chosen generator of  $\pi_{\psi(w-s-1)-1}J$ . Therefore, in the right part of the Mahowald square,  $1[s-w]$  must support a differential in either the algebraic Atiyah–Hirzebruch spectral sequence or the Adams spectral sequence of  $\mathbb{R}P_{-w}^\infty$ . Since  $s < 2^{v_2(w-s-1)}$ ,  $\frac{\theta}{\rho^s \tau^{w-s-2}}$  survives to  $E_2$ -page of the genuine  $C_2$ -equivariant Adams spectral sequence; so does  $1[s-w]$  to the  $E_2$ -page of the Adams spectral sequence of  $\mathbb{R}P_{-w}^\infty$ . Therefore,  $1[s-w]$  must support an Adams differential in the Adams spectral sequence of  $\mathbb{R}P_{-w}^\infty$ . By [Proposition 2.3](#) and [Proposition 2.4](#),  $\frac{\theta}{\rho^s \tau^{w-s-2}}$  supports a nontrivial differential in the genuine  $C_2$ -equivariant Adams spectral sequence. □

**Remark 3.3.** In [BCQ25, Lemma 7.4.5], Balderrama–Culver–Quigley showed that there exists a surviving permanent cycle in  $\text{Ext}_h^{s,1,w}$  of the Borel  $C_2$ -equivariant Adams spectral sequence if and only if the inclusion of the bottom cell of  $\mathbb{R}P_{w-s-1}^{w-1}$  splits. With the relationship between the genuine and Borel  $C_2$ -equivariant Adams spectral sequence revealed by Proposition 2.2–Proposition 2.4, their result would lead to an alternative proof for Theorem 3.2.

**3.3. Some concrete  $C_2$ -genuine Adams differentials.** The first nontrivial Adams differentials on the 0-line occur when  $w - s = 17$ . In this case, there are  $d_2$ -differentials

$$d_2(1[-17]) = h_1 h_3[-26]$$

in the Adams spectral sequence of  $\mathbb{R}P_{-w}^\infty$  for  $26 \leq w \leq 33$  by Lin’s data [LWX25a]. These correspond to the genuine  $C_2$ -equivariant Adams  $d_2$ -differentials on  $\frac{\theta}{\rho^j \tau^{15}}$  for  $9 \leq j \leq 15$ . Note that since  $\frac{\theta}{\rho^{15} \tau^{15}} \in \langle \rho, \frac{\theta}{\tau^{23}}, h_4 \rangle$ ,

$$\begin{aligned} d_2 \left( \frac{\theta}{\rho^{15} \tau^{15}} \right) &= d_2 \left( \langle \rho, \frac{\theta}{\tau^{23}}, h_4 \rangle \right) \\ &= \langle \rho, \frac{\theta}{\tau^{23}}, d_2(h_4) \rangle + \text{Indet} \\ &= \langle \rho, \frac{\theta}{\tau^{23}}, h_0 h_3^2 \rangle + \text{Indet} \\ &= \frac{\theta}{\tau^{22}} h_3^2 + \text{Indet}. \end{aligned}$$

The indeterminacy consists of  $\{0, \frac{\theta}{\rho^6 \tau^{19}} h_1 h_3\}$ , and the differential in the Adams spectral sequence of  $\mathbb{R}P_{-w}^\infty$  implies that  $\frac{\theta}{\rho^6 \tau^{19}} h_1 h_3$  must be a summand. Thus,

$$d_2 \left( \frac{\theta}{\rho^{15} \tau^{15}} \right) = \frac{\theta}{\tau^{22}} h_3^2 + \frac{\theta}{\rho^6 \tau^{19}} h_1 h_3.$$

Multiplying by  $\rho$ , we have

$$d_2 \left( \frac{\theta}{\rho^{9+j} \tau^{15}} \right) = \frac{\theta}{\rho^j \tau^{19}} h_1 h_3, \quad 0 \leq j \leq 5.$$

In general, fixing  $s - w$ , the differentials supported by  $\frac{\theta}{\rho^s \tau^{w-s-2}}$  for  $s = 2^{v_2(w-s-1)} - 1$  are always  $d_2$ -differentials, by a similar argument of the higher Leibniz rule as above. However, as  $s$  decreases,  $\frac{\theta}{\rho^s \tau^{w-s-2}}$  tends to support longer differentials. Although we do not know each differential supported by those 0-line elements explicitly, we are able to show the following patterns of the longest differentials as stated in Theorem 1.4.

**Theorem 3.4.** If  $v_2(w - s - 1) \geq 5$ ,  $\frac{\theta}{\rho^{\psi(w-s-1)} \tau^{w-s-2}}$  survives to the  $E_r$ -page for  $r = v_2(w - s - 1) - 1$ , and it supports a nontrivial  $d_r$ -differential.

*Proof.* To simplify notation, let  $b = s - w$ ,  $c = \psi(w - s - 1)$ .

By Proposition 2.3 and Proposition 2.4, the differential supported by  $\frac{\theta}{\rho^{\psi(w-s-1)} \tau^{w-s-2}} = \frac{\theta}{\rho^{c \tau^{-b-2}}$  in the genuine  $C_2$ -equivariant Adams spectral sequence corresponds to the differential supported by  $1[b]$  in the Adams spectral sequence of  $\mathbb{R}P_{b-c}^\infty$ . As discussed above, the top cell of  $\mathbb{R}P_{b-c+1}^b$  splits off, while it does not split off in  $\mathbb{R}P_{b-c}^b$ , and there is a subcomplex inclusion

$$\Sigma^{b-c} \text{Cof}(j_{c-1}) \hookrightarrow \mathbb{R}P_{b-c}^b$$

where  $\text{Cof}(\alpha)$  denotes the cofiber of the map  $S^{|\alpha|} \xrightarrow{\alpha} S^0$ .

If  $v_2(w-s-1) \equiv 0, 1, 2 \pmod{4}$ , in the Adams spectral sequence of  $\Sigma^{b-c} \text{Cof}(j_{c-1})$ , there is a differential

$$d_r(1[b]) = a_{c-1}[b-c],$$

where  $r = v_2(w-s-1) - 1$  and  $a_{c-1}$  is the element in  $\text{Ext}^{c-1,r}(S^0)$  that detects  $j_{c-1}$ . Pushing forward this differential to  $\mathbb{R}P_{b-c}^\infty$ , we find that  $1[b]$  must support a  $d_{r_1}$  differential for  $r_1 \geq r$ . [Lemma 3.5](#) below guarantees that  $1[b]$  must support a  $d_r$  differential.

If  $v_2(w-s-1) = 4k+3$ , consider the 3-cell complex  $T$  defined as the cofiber of the composite

$$T := \text{Cof}(S^{b-1} \xrightarrow{j_{c-1}} S^{b-c} \hookrightarrow \Sigma^{b-c} \text{Cof}(2)).$$

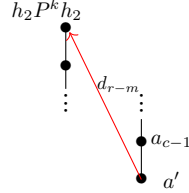
In particular, there is a natural cofiber sequence

$$S^{b-c} \xrightarrow{\partial} \Sigma^{b-c} \text{Cof}(j_{c-1}) \xrightarrow{i} T \xrightarrow{p} S^{b-c+1}. \quad (1)$$

In the Adams spectral sequence of  $\Sigma^{b-c} \text{Cof}(j_{c-1})$ , there is an Adams differential

$$d_m(1[b]) = a_{c-1}[b-c]. \quad (2)$$

Here  $a_{c-1}$  is an element in  $\text{Ext}^{c-1,m}(S^0)$  of filtration  $m$  that detects  $j_{c-1}$ . A local chart of the Adams spectral sequence of  $S^0$  around  $a_{c-1}$  is



where there is some generator  $a'$  satisfying  $a' \cdot h_0 = a_{c-1}$  and supports a differential

$$d_{r-m}(a') = h_2 P^k h_2. \quad (3)$$

The cofiber sequence in [Eq. \(1\)](#) induces a long exact sequence of the Ext-groups, and because of the 2-attaching map in  $T$ , we have

$$\begin{aligned} \partial_* : \text{Ext}(S^{b-c}) &\rightarrow \text{Ext}(\Sigma^{b-c} \text{Cof}(j_{c-1})) \\ a' &\mapsto a_{c-1}[b-c]. \end{aligned}$$

Combined with [Eq. \(3\)](#) and the Generalized Mahowald Trick [[LWX25b](#), Theorem 6.12],

$$\begin{array}{ccccccc} \Sigma^{-1}(h_2 P^k h_2[b-c+1]) & \xrightarrow{d_0^p} & h_2 P^k h_2 & & h_2 P^k h_2[b-c+1] & & \\ & & \downarrow d_{r-m} & & \uparrow d_{r-m-1}^i & & \\ & & a' & \xrightarrow{d_1^i} & a_{c-1}[b-c] & & \\ \text{Ext}(\Sigma^{-1}T) & & \text{Ext}(S^{b-c}) & & \text{Ext}(\Sigma^{b-c} \text{Cof}(j_{c-1})) & & \text{Ext}(T) \end{array}$$

there is a  $(i, E_2)$ -extension

$$d_{r-m-1}^{i, E_2}(a_{c-1}[b-c]) = h_2 P^k h_2[b-c+1].$$

By the Generalized Leibniz Rule [LWX25b, Theorem 6.1] and Eq. (2),

$$\begin{array}{ccc}
 & & h_2 P^k h_2 [b - c + 1] \\
 & \nearrow^{d_{r-m-1}^i} & \\
 a_{c-1} [b - c] & & \\
 \searrow^{d_m} & & \\
 1[b] & \xrightarrow{d_1^0} & 1[b] \\
 \text{Ext}(\Sigma^{b-c} \text{Cof}(j_{c-1})) & & \text{Ext}(T)
 \end{array}$$

we have the differential

$$d_r(1[b]) = h_2 P^k h_2 [b - c + 1] \quad (4)$$

in the Adams spectral sequence of  $T$ . Pushing forward Eq. (4) along the natural inclusions  $T \hookrightarrow \mathbb{R}P_{b-c}^b \hookrightarrow \mathbb{R}P_{b-c}^\infty$ ,  $1[b]$  must support a  $d_{r_2}$  differential for  $r_2 \geq r$ . By Lemma 3.5 proved below, there are no elements in  $\text{Ext}^{b-1, >r}(\mathbb{R}P_{b-c}^\infty)$ ; therefore,  $1[b]$  must support a  $d_r$  differential as claimed.  $\square$

We compute the relevant Ext-groups in the proof of Theorem 3.4.

**Lemma 3.5.** Using the notation in Theorem 3.4,

(1) if  $v_2(w - s - 1) \equiv 0, 3 \pmod{4}$ , then

$$\text{Ext}^{b-1, >r}(\mathbb{R}P_{b-c}^\infty) = 0;$$

(2) if  $v_2(w - s - 1) \equiv 1, 2 \pmod{4}$ , then the differential

$$d_r(1[b]) = a_{c-1} [b - c]$$

can be pushed forward to  $\mathbb{R}P_{b-c}^\infty$ .

*Proof.* Consider the algebraic Atiyah–Hirzebruch spectral sequence

$$\bigoplus_{i=b-c}^{\infty} \text{Ext}(S^i) \implies \text{Ext}(\mathbb{R}P_{b-c}^\infty).$$

- If  $v_2(w - s - 1) = 4k$ , the only possible elements of higher filtration is the order 2 element of the  $h_0$ -tower of the image-of- $J$ , coming from the  $(b - c + 1)$ -cell. Since  $b = s - w$  is odd,  $c = \psi(w - s - 1)$  is odd, we have  $b - c + 1$  is odd. Thus, the order 2 element is killed by a  $d_1$ -differential due to  $h_0$ -multiplication. Therefore,  $\text{Ext}^{b-1, >r}(\mathbb{R}P_{b-c}^\infty)$  vanishes.
- If  $v_2(w - s - 1) = 4k + 1$ ,  $j_{c-1}$  is detected by  $a_{c-1} = h_1 P^{k-1} c_0$ . We claim that  $h_1 P^{k-1} c_0 [b - c]$  survives the algebraic Atiyah–Hirzebruch spectral sequence, so that naturality guarantees that the differential is the same length as in Theorem 3.4.

For filtration reasons, the only possible sources that could hit  $h_1 P^{k-1} c_0 [b - c]$  are  $h_0^3 P^{k-2} e_0 [b - c + 1]$ ,  $P^{k-1} c_0 [b - c + 2]$ , the order 4 element of the  $h_0$ -tower of the image-of- $J$  from the  $(b - c + 3)$ -cell, and  $h_0^2 P^{k-1} h_2 [b - c + 7]$ . The first candidate can be eliminated as  $h_0^3 P^{k-2} e_0 \cdot h_0 \neq h_1 P^{k-1} c_0$ . Since  $b$  is odd,  $c$  is even,  $b - c + 3$  and  $b - c + 7$  are odd, so the order 4 element in the  $h_0$ -tower from the  $(b - c + 3)$ -cell and

$h_0^2 P^{k-1} h_2[b-c+7]$  are hit by  $d_1$ -differentials because of  $h_0$ -multiplication. Finally, since  $\psi(2^{4k+1}) = 8k+2$ , we have  $c \equiv 2 \pmod{4}$  and  $b \equiv 3 \pmod{4}$ , which implies  $b-c \equiv 1 \pmod{4}$  and  $b-c+2 \equiv 3 \pmod{4}$ . Therefore, there is no  $Sq^2$ -action from  $(b-c)$ -cell to  $(b-c+2)$ -cell, and the last candidate is ruled out.

- If  $v_2(w-s-1) = 4k+2$ ,  $j_{c-1}$  is detected by  $a_{c-1} = P^k h_2$ . We claim that  $P^k h_2[b-c]$  survives the algebraic Atiyah–Hirzebruch spectral sequence, so that naturality guarantees that the differential is the same length as in [Theorem 3.4](#).

For filtration reasons, the only possible sources that could hit  $P^k h_2[b-c]$  are  $h_1 P^{k-1} c_0[b-c+3]$  and the order 2 element in the  $h_0$ -tower of the image-of- $J$  from the  $(b-c+5)$ -cell. Since  $\psi(2^{4k+2}) = 8k+4$ , we have  $c \equiv 0 \pmod{4}$  and  $b \equiv 3 \pmod{4}$ , which implies  $b-c+3 \equiv 2 \pmod{4}$ . Therefore, there is a  $Sq^2$  action from  $(b-c+3)$ -cell to  $(b-c+5)$ -cell, which implies the differential

$$d_2(P^{k-1} c_0[b-c+5]) = h_1 P^{k-1} c_0[b-c+3],$$

so  $h_1 P^{k-1} c_0[b-c+3]$  can be eliminated. Computation of the lambda algebra shows that the order 2 element in the  $h_0$ -tower from the  $(b-c+5)$ -cell is a permanent cycle [\[CLM88\]](#). As a result,  $P^k h_2[b-c]$  is a surviving cycle.

- If  $v_2(w-s-1) = 4k+3$ , the only possible elements of higher filtration is  $h_0^2 P^k h_2[b-c+4]$ . Since  $b$  is odd,  $c$  is even, we have  $b-c+4$  is odd, so  $h_0^2 P^k h_2[b-c+4]$  is hit by  $h_0 P^k h_2[b-c+5]$ . Therefore,  $\text{Ext}^{b-1, >r}(\mathbb{R}P_{b-c}^\infty)$  vanishes. □

#### 4. THE HUREWICZ IMAGE OF $H\mathbb{Z}$ AND $HA$

Recall from [\[Dug05, Lew88\]](#) that we have

**Proposition 4.1.**

- (1) The  $RO(C_2)$ -graded abelian group structure of  $H\mathbb{Z}_{*,*}$  is

$$H\mathbb{Z}_{s,w} = \begin{cases} \mathbb{Z} & \text{if } s = 0 \text{ and } w \text{ is even,} \\ \mathbb{F}_2 & \text{if } s < 0 \text{ and } s - w = 2k \text{ for } k \geq 0, \\ \mathbb{F}_2 & \text{if } s \geq 0 \text{ and } s - w = 2k - 1 \text{ for } k \leq -1, \\ 0 & \text{otherwise.} \end{cases}$$

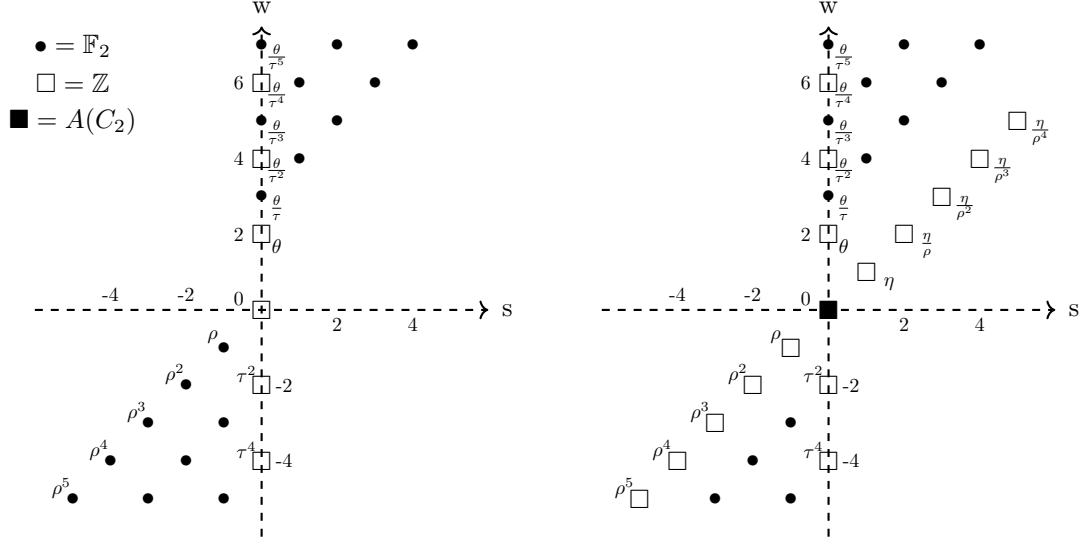
- (2) The  $RO(C_2)$ -graded abelian group structure of  $HA_{*,*}$  is

$$HA_{s,w} = \begin{cases} A(C_2) & \text{if } s = w = 0, \\ \mathbb{Z} & \text{if } s = 0 \text{ and } w = 2k \text{ for } k \neq 0, \\ \mathbb{Z} & \text{if } s = w \neq 0, \\ \mathbb{F}_2 & \text{if } s < 0 \text{ and } s - w = 2k \text{ for } k \geq 1, \\ \mathbb{F}_2 & \text{if } s \geq 0 \text{ and } s - w = 2k - 1 \text{ for } k \leq -1, \\ 0 & \text{otherwise.} \end{cases}$$

The additive structures of  $H\mathbb{Z}_{*,*}$  and  $HA_{*,*}$  are depicted in [Fig. 4](#), where each black dot represents  $\mathbb{F}_2$ , each hollow square represents  $\mathbb{Z}$ , and the black square represents  $A(C_2)$ . The references for multiplicative structures are [\[Dug05, Theorem 2.8\]](#) and [\[Sik22, Theorem 7.7\]](#).

Since the Hurewicz map  $\iota_{H\mathbb{F}_2} : S^0 \rightarrow H\mathbb{F}_2$  factors through

$$S^0 \longrightarrow HA \longrightarrow H\mathbb{Z} \longrightarrow H\mathbb{F}_2,$$


 FIGURE 4.  $H\underline{Z}_{s,w}$  and  $H\underline{A}_{s,w}$ 

by abuse of notation, we denote the generators of  $H\underline{A}_{*,*}$  and  $H\underline{Z}_{*,*}$  by the same names as their images in  $H\mathbb{F}_2$ , and the  $\mathbb{F}_2[\rho]$ -module structure is inherited similarly. For  $H\underline{A}$ ,  $H\underline{A}_{0,0} = A(C_2) = \mathbb{Z}\{1, [C_2]\}$ . There is an additional generator  $\eta \in H\underline{A}_{1,1}$ , which is the Hurewicz image of the non-nilpotent element  $\eta_{C_2} \in \pi_{1,1}^{C_2}$  [BXZ24], since both satisfy the multiplicative relation  $\eta \cdot \rho = 2 - [C_2]$ . The notation  $\frac{\eta}{\rho^j}$  suggests that multiplication by  $\rho$  induces isomorphisms

$$\cdot \rho : \pi_{n+1, n+1} H\underline{A} \rightarrow \pi_{n, n} H\underline{A}, \quad n \geq 1.$$

**Theorem 4.2.**

- (1) The  $RO(C_2)$ -graded Hurewicz image of  $H\underline{Z}$  consists of

$$\mathrm{Im}(\iota_{H\underline{Z}})_{s,w} = \begin{cases} \mathbb{Z}\{1\} & \text{if } s = w = 0, \\ \mathbb{Z}\{\frac{\theta}{\tau^{2k-2}}\} & \text{if } s = 0 \text{ and } w = 2k \text{ for } k > 0, \\ \mathbb{Z}\{2\tau^{2k}\} & \text{if } s = 0 \text{ and } w = -2k \text{ for } k > 0, \\ \mathbb{F}_2\{\rho^{-s}\} & \text{if } s = w < 0, \\ \mathbb{F}_2\{\frac{\theta}{\rho^s \tau^{w-s-2}}\} & \text{if } 0 \leq s < \psi(w-s-1) \text{ and } s-w = 2k-1 \text{ for } k \leq -1, \\ 0 & \text{otherwise.} \end{cases}$$

(2) The  $RO(C_2)$ -graded Hurewicz image of  $H\underline{A}$  consists of

$$\mathrm{Im}(\iota_{H\underline{A}})_{s,w} = \begin{cases} A(C_2) & \text{if } s = w = 0, \\ \mathbb{Z}\{\frac{\theta}{\tau^{2k-2}}\} & \text{if } s = 0 \text{ and } w = 2k \text{ for } k > 0, \\ \mathbb{Z}\{2\tau^{2k}\} & \text{if } s = 0 \text{ and } w = -2k \text{ for } k > 0, \\ \mathbb{Z}\{\rho^{-s}\} & \text{if } s = w < 0, \\ \mathbb{Z}\{2^{s-n(s)-1}\frac{\eta}{\rho^{s-1}}\} & \text{if } s = w > 0, \\ \mathbb{F}_2\{\frac{\theta}{\rho^s\tau^{w-s-2}}\} & \text{if } 0 \leq s < \psi(w-s-1) \text{ and } s-w = 2k-1 \text{ for } k \leq -1, \\ 0 & \text{otherwise.} \end{cases}$$

where

$$n(s) = \begin{cases} 4t-1 & s = 8t, \\ 4t & s = 8t + j \text{ for } j = 1, 2, 3, 4, \\ 4t+1 & s = 8t + 5, \\ 4t+2 & s = 8t + 6, \\ 4t+3 & s = 8t + 7. \end{cases}$$

*Proof.*

(1) Since the Hurewicz map  $\iota_{H\underline{\mathbb{F}}_2}$  factors through  $\iota_{H\underline{\mathbb{Z}}}$ , [Theorem 3.2](#) forces the result as claimed, with exceptional cases only in degrees  $(s, w)$  for  $s = 0$  and  $w$  even.

If  $w > 0$ , the generators  $\omega_{-k} \in \pi_{0,2k}^{C_2}$  that detects  $\frac{\theta}{\tau^{2k-2}}$  are torsion-free [[BXZ24](#)], and since its image in  $H\underline{\mathbb{F}}_{2*,*}$  is nontrivial, it must have nontrivial image in  $H\underline{\mathbb{Z}}_{*,*}$ .

If  $w < 0$ ,  $\tau^{2k} \in H\underline{\mathbb{Z}}_{0,-2k}$  satisfies the multiplicative relation

$$\tau^{2k} \cdot \frac{\theta}{\tau^{2k-2}} = 2.$$

On the other hand, according to [[BXZ24](#)], there are a torsion-free generators  $\omega_k \in \pi_{0,-2k}^{C_2}$  satisfying

$$\omega_k \cdot \omega_{-k} = 2[C_2].$$

Since  $[C_2]$  has image 2 in  $H\underline{\mathbb{Z}}_{*,*}$ , the image of  $\omega_k$  in  $H\underline{\mathbb{Z}}_{*,*}$  must be  $2\tau^{2k}$  in  $H\underline{\mathbb{Z}}_{*,*}$ .

(2) Since the Hurewicz map  $\iota_{H\underline{\mathbb{Z}}}$  factors through  $\iota_{H\underline{A}}$ , part (1) implies the result as claimed, except the cases for  $s = w > 0$ .

Let  $a \in \pi_{-1,-1}^{C_2}$  denote the Euler class of  $\sigma$ , whose  $H\underline{\mathbb{F}}_2$ -Hurewicz image is  $\rho$ .

According to [[BXZ24](#)], the torsion-free generator of  $\pi_{s,s}^{C_2}$  is  $\frac{\eta_{C_2}^s}{2^{n(s)}}$  subject to the relation

$$\frac{\eta_{C_2}^N}{a} = \frac{\eta_{C_2}^{N+1}}{2}$$

for large enough  $N$ . In particular,

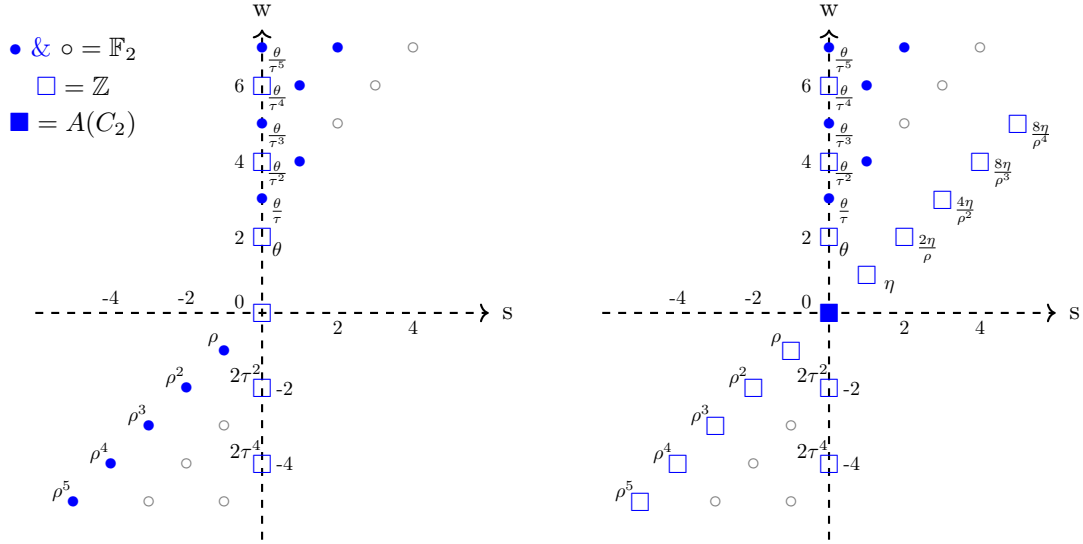
$$\frac{\eta_{C_2}^s}{2^{n(s)}} \cdot a^{s-1} = 2^{s-n(s)-1} \cdot \eta_{C_2}.$$

On the other hand, from the definition,

$$\frac{\eta}{\rho^{s-1}} \cdot \rho^{s-1} = \eta$$

in  $H\underline{A}_{*,*}$ . Therefore, the generators  $\frac{\eta_{C_2}^s}{2^{n(s)}}$  all map injectively into  $H\underline{A}_{s,s}$ :

$$\iota_{H\underline{A}} \left( \frac{\eta_{C_2}^s}{2^{n(s)}} \right) = 2^{s-n(s)-1} \frac{\eta}{\rho^{s-1}}.$$

FIGURE 5. The Hurewicz image of  $H\mathbb{Z}_{s,w}$  and  $H\mathbb{A}_{s,w}$ 

□

Fig. 5 represents the Hurewicz image of  $H\mathbb{Z}$  and  $H\mathbb{A}$ . The generators with names that are in the Hurewicz image are indicated in blue.

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