THE $RO(\mathcal{K})$-GRADED HOMOTOPY OF KLEIN-FOUR NORMED MACKEY FUNCTORS

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Abstract. We compute the $RO(\mathcal{K})$-graded coefficients of the equivariant Eilenberg–Mac Lane spectrum associated to various Hill–Hopkins–Ravenel norms of the constant-$\mathbb{F}_2$ Mackey functor, where $\mathcal{K}$ is the Klein-four group. Further, we analyze the multiplicative structure of these $\mathrm{RO}(\mathcal{K})$-graded Tambara functors.

Contents

1. Introduction

In equivariant stable homotopy theory, ordinary cohomology is represented by an equivariant Eilenberg–Mac Lane G-spectrum $H\underline{M}$, where $\underline{M}$ is a Mackey functor for the group G. One may expect that the coefficients of an equivariant Eilenberg–Mac Lane spectrum are easy to understand, but this is more complicated than in the non-equivariant setting. The homotopy of a G-spectrum E can be considered as $\mathrm{RO}(G)$-graded, where $\mathrm{RO}(G)$ is the real representation ring of G. In the case that $E = H\underline{M}$ for some G-Mackey functor $\underline{M}$, the $\mathrm{RO}(G)$-graded coefficients of E correspond to the Bredon homology of virtual real representation spheres of G with coefficients in $\underline{M}$.

For $G = C_2$, the $RO(G)$-graded coefficients of any equivariant Eilenberg–Mac Lane spectrum are quite well-understood [Si]. This is far from true for $G = \mathcal{K} = C_2 \times C_2$, the Klein-four group. One computational difficulty that arises in this context is that $\mathrm{RO}(\mathcal{K})$ is a free abelian group of rank four. Despite this, some computations of the $\mathrm{RO}(\mathcal{K})$-graded coefficients of $H\underline{M}$ have been done [HoKr, GY, EB, Sl, K]. In this paper, we make further contributions to these computations. We compute a portion of the $\mathrm{RO}(\mathcal{K})$-graded homotopy Mackey functors of $\underline{M} = N_H^{\mathcal{K}}\underline{\mathbb{F}_2}$, where H is a proper subgroup of $\mathcal{K}$ and $\underline{\mathbb{F}_2}$ the constant H-Mackey functor at $\mathbb{F}_2$. Here, $N_H^{\mathcal{K}}(-) : \mathcal{M}\mathrm{ack}_H \to \mathcal{M}\mathrm{ack}_{\mathcal{K}}$ is the Mackey functor norm. We also introduce a method for calculating the Mackey functors $N_H^G(\underline{\mathbb{F}_2})$ for any group G and subgroup H, based on Tambara ideals of the Burnside Tambara functor.

Hill, Hopkins, and Ravenel introduced the norm functor $N_H^G(-) : \mathcal{S}\mathrm{p}^H \to \mathcal{S}\mathrm{p}^G$ in [HHR] and have studied various norms $N_{C_2}^G \mathrm{BP}_{\mathbb{R}}$, where $\mathrm{BP}_{\mathbb{R}}$ is the Real Brown-Peterson C_2-spectrum. Despite the success of this analysis, many computations concerning equivariant spectra related to this norm construction remain mysterious. Along these lines, [MSZ, Theorem 4.4] gives partial information about the homotopy of the geometric fixed points $\Phi^{C_2} N_{C_2}^{C_4} \mathrm{BP}_{\mathbb{R}} \simeq N_e^{C_2} H\mathbb{F}_2$. The first few $\mathbb{Z}$-graded homotopy Mackey functors are listed in Table 1.1 using notation as indicated in Table 7.1, and the $RO(C_2)$-graded homotopy groups $\pi_{x+y\sigma} N_e^{C_2} H\mathbb{F}_2$ are displayed for $x \leq 6$ and $x + y \leq 6$ in Figure 7.3. See the beginning of Section 7 for a discussion of Figure 7.3. A complete calculation of the homotopy of $N_{C_2}^{C_4} \mathrm{BP}_{\mathbb{R}}$ and $N_e^{C_2} H\mathbb{F}_2$ is out of reach. Part of the reason for this is that the underlying homotopy groups of $N_e^{C_2} H\mathbb{F}_2$, together with the C_2-action, form the dual Steenrod algebra with the action of the antipode. There is no known formula for the fixed points of this action. In contrast, the $\mathrm{RO}(C_2)$-graded homotopy groups of both $H_{C_2} N_e^{C_2} \mathbb{F}_2$, the 0^{th}-Postnikov truncation of $N_e^{C_2} H\mathbb{F}_2$, and $H_{C_2}\underline{\mathbb{F}_2}$ are completely understood [D, Si]. The $RO(C_2)$-graded homotopy groups of the Eilenberg–Mac Lane spectra $H_{C_2} N_e^{C_2} \mathbb{F}_2$ and $H_{C_2}\underline{\mathbb{F}_2}$ are displayed in Figure 7.2 and Figure 7.1, respectively.

Motivation for the study of $N_{C_2}^{C_4} \mathrm{BP}_{\mathbb{R}}$ comes from chromatic homotopy theory [HSWX]. The quaternion group Q_8 also plays an important role in chromatic homotopy theory, which suggests the study of $N_{C_2}^{Q_8} \mathrm{BP}_{\mathbb{R}}$, where C_2 is the center of Q_8. This is expected to be difficult; however, given that the quotient Q_8/C_2 is isomorphic to $\mathcal{K}$, one may wish to compute $\Phi^{C_2} N_{C_2}^{Q_8} \mathrm{BP}_{\mathbb{R}} \simeq N_e^{\mathcal{K}} H\mathbb{F}_2$. Again, this is out of reach, though its Postnikov truncation $H_{\mathcal{K}} N_e^{\mathcal{K}} \mathbb{F}_2$ can be completely computed. The $RO(\mathcal{K})$-graded homotopy groups of $H\underline{\mathbb{F}_2}$ were previously computed in [HoKr, EB], while some of the homotopy Mackey functors were determined in [GY]. These are depicted in a range in Figure 7.4. In this article, we compute a portion of the $\mathrm{RO}(\mathcal{K})$-graded homotopy Mackey functors of $H_{\mathcal{K}} N_e^{\mathcal{K}} \mathbb{F}_2$. It is common to use the symbol $\bigstar$ to denote a grading over $RO(G)$. We will use the symbol $\blacklozenge$ to denote a grading over the $\mathrm{Aut}(\mathcal{K})$-fixed subgroup $\mathbb{Z}\{1, \overline{\rho}\} \subset RO(\mathcal{K})$, where $\overline{\rho}$ is the reduced regular representation.

Theorem A. *The homotopy Mackey functors* $\underline{\pi}_{\blacklozenge} H_{\mathcal{K}} N_e^{\mathcal{K}} \mathbb{F}_2$ *are as described in Section 4 and displayed in Figures 7.6 and 7.7, where* $\blacklozenge \in \mathbb{Z}\{1, \overline{\rho}\} \subset RO(\mathcal{K})$.

We also consider the intermediate norm $N_C^{\mathcal{K}}\underline{\mathbb{F}_2}$, where $C \leq K$ is an order 2 subgroup. The corresponding Eilenberg–Mac Lane spectrum $H_{\mathcal{K}} N_C^{\mathcal{K}} \underline{\mathbb{F}_2}$ is the Postnikov truncation of the normed spectrum $N_C^{\mathcal{K}} H_C \underline{\mathbb{F}_2}$. The latter is a useful intermediary between the mysterious $N_e^{\mathcal{K}} H\mathbb{F}_2$ and the well-understood $H_{\mathcal{K}}\underline{\mathbb{F}_2}$. For definiteness, we specialize to the case that $C = D$ is the diagonal subgroup of $\mathcal{K}$, though the other choices can be obtained by using the $\mathrm{Aut}(\mathcal{K})$-action.

Theorem B. *The homotopy Mackey functors* $\underline{\pi}_{\blacklozenge} H_{\mathcal{K}} N_D^{\mathcal{K}} \underline{\mathbb{F}_2}$ *are as described in Section 5 and displayed in Figures 7.8 and 7.9, where* $\blacklozenge \in \mathbb{Z}\{1, \overline{\rho}\} \subset RO(\mathcal{K})$.

TABLE 1.1. The homotopy Mackey functors $\underline{\pi}_n N_e^{C_2} H\mathbb{F}_2$, $n \leq 6$. See Table 7.1 for the Mackey functor Lewis diagrams.

n	0	1	2	3	4	5	6
$\underline{\pi}_n N_e^{C_2} H\mathbb{F}_2$	$N_e^{C_2}\mathbb{F}_2$	$\underline{g} \oplus \underline{f}$	$N_e^{C_2}\mathbb{F}_2$	$\underline{g} \oplus \uparrow_e^{C_2} \mathbb{F}_2$	$\uparrow_e^{C_2} \mathbb{F}_2$	$\uparrow_e^{C_2} \mathbb{F}_2$	$N_e^{C_2}\mathbb{F}_2 \oplus \uparrow_e^{C_2} \mathbb{F}_2$

1.1. **Conventions.** We write e for a trivial group and $C_2 := \langle \gamma \mid \gamma^2 = 1 \rangle$ for a finite group of order two. Our main group of interest is the Klein four-group $\mathcal{K} := C_2 \times C_2$; its nontrivial subgroups are $L := C_2 \times e$, $D := \langle (\gamma, \gamma) \rangle$, and $R := e \times C_2$.

We write both $\mathbb{Z}/2$ and $\mathbb{F}_2$ for the ring of order 2, guided by aesthetics.

We use different fonts to differentiate between non-equivariant, C_2-equivariant, and $\mathcal{K}$-equivariant homotopy theory. We write H for non-equivariant Eilenberg–Mac Lane spectra, $\mathbf{H}$ for C_2-equivariant Eilenberg–Mac Lane spectra, and $\boldsymbol{\mathcal{H}}$ for $\mathcal{K}$-equivariant Eilenberg–Mac Lane spectra. Similarly, we will often abbreviate the Mackey functors $N_e^{C_2}\mathbb{F}_2, N_e^{\mathcal{K}}\mathbb{F}_2$, and $N_D^{\mathcal{K}}\underline{\mathbb{F}_2}$ by $\mathbf{N}, \boldsymbol{\mathcal{N}}$, and $\boldsymbol{\mathcal{N}}_D$, respectively.

Equivariant spectra will always be considered as indexed over a complete universe, so that their homotopy is valued in Mackey functors. Our calculations require many C_2- or $\mathcal{K}$-Mackey functors; notation and definition for these can be found in Table 7.1 and Table 7.2.

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2. $\mathcal{K}$-NORMS VIA TAMBARA IDEALS

The Hill–Hopkins–Ravenel norm $N_H^G \colon \mathcal{S}\mathrm{p}^H \to \mathcal{S}\mathrm{p}^G$ is captured on the level of Mackey functors by a functor $N_H^G \colon \mathcal{M}\mathrm{ack}(H) \to \mathcal{M}\mathrm{ack}(G)$. As a non-additive functor between additive categories, the norm is often difficult to compute. We introduce a technique for computing $N_H^G(\underline{\mathbb{F}_2})$ based on Tambara ideals, and use it to compute $N_e^{\mathcal{K}}(\mathbb{F}_2)$ and $N_C^{\mathcal{K}}(\underline{\mathbb{F}_2})$, where C is any of the order two subgroups of $\mathcal{K}$.

2.1. **Norms of Mackey and Tambara Functors.** Recall that the category of G-Mackey functors is the category of additive functors $\mathcal{A}_G \to \mathcal{A}\mathrm{b}$, where $\mathcal{A}_G$ is the Burnside category for G.

Definition 2.1 ([Ho, Definition 2.3.2])**.** The functor $N_H^G \colon \mathcal{M}\mathrm{ack}(H) \to \mathcal{M}\mathrm{ack}(G)$ is given by left Kan extension along coinduction $\mathcal{F}\mathrm{in}_H(G, -) \colon \mathcal{A}_H \to \mathcal{A}_G$.

Tambara functors are the G-commutative monoids in the category of G-Mackey functors. Alternatively, the category of G-Tambara functors is the category of product preserving functors $\underline{T} \colon \mathcal{P}^G \to \mathcal{S}\mathrm{et}$ such that each $\underline{T}(U)$ is a commutative ring, where $\mathcal{P}^G$ is the category of polynomials of finite G-sets [T, Section 8]. Concretely, a Tambara functor is a Mackey functor valued in commutative rings whose restrictions are ring homomorphisms satisfying Frobenius reciprocity (a Green functor) equipped with norm maps (of multiplicative monoids) satisfying Tambara reciprocity.

The norm n_H^G from H-Tambara functors to G-Tambara functors is slightly different from the norm of Mackey functors.

Definition 2.2 ([BH, Definition 6.8, Proposition 6.9])**.** The functor $n_H^G \colon \mathcal{T}\mathrm{amb}(H) \to \mathcal{T}\mathrm{amb}(G)$ is given by left Kan extension along the inclusion $\mathcal{P}^H \to \mathcal{P}^G$; it is left adjoint to the restriction functor $\mathrm{res}_H^G \colon \mathcal{T}\mathrm{amb}(G) \to \mathcal{T}\mathrm{amb}(H)$.

We recall a theorem relating the two functors:

Theorem 2.3 ([Ho, Theorem 2.3.3])**.** *The following square commutes, where vertical arrows are forgetful functors:*

$$\begin{tikzcd}
\mathcal{T}\mathrm{amb}(H) \arrow[r, "n_H^G"] \arrow[d] & \mathcal{T}\mathrm{amb}(G) \arrow[d] \\
\mathcal{M}\mathrm{ack}(H) \arrow[r, "N_H^G"] & \mathcal{M}\mathrm{ack}(G)
\end{tikzcd}$$

2.2. **Tambara Ideals and Norms.** Like commutative rings, Tambara functors have a robust theory of ideals. We use non-unital Tambara functors to define Tambara ideals.

Let $\mathcal{E}\mathrm{pi}^G \subseteq \mathcal{F}\mathrm{in}^G$ be the category of finite G-sets and surjections. Note that $\mathcal{E}\mathrm{pi}^G$ is a pullback stable subcategory of finite G-sets, i.e. pullbacks in $\mathcal{F}\mathrm{in}^G$ of morphisms in $\mathcal{E}\mathrm{pi}^G$ are again in $\mathcal{E}\mathrm{pi}^G$. Let $\mathcal{P}^G_{\mathcal{E}\mathrm{pi}}$ be the category of polynomials with exponents in $\mathcal{E}\mathrm{pi}^G$ [BH, Definition 2.7].

Definition 2.4 ([BH, Definition 4.15])**.** A *non-unital Tambara functor* is a product preserving functor $\underline{T}\colon \mathcal{P}^G_{\mathcal{E}\mathrm{pi}} \to \mathcal{S}\mathrm{et}$ such that each $\underline{T}(X)$ is an abelian group.

Concretely, a non-unital Tambara functor is a Tambara functor valued in non-unital rings, that is, a Mackey functor valued in non-unital commutative rings whose restrictions are ring homomorphisms satisfying Frobenius reciprocity (a non-unital Green functor) equipped with norm maps (of non-unital multiplicative monoids) satisfying Tambara reciprocity.

The definition below is equivalent to the original definition given by Nakaoka [N, Definition 2.1].

Definition 2.5 ([H1, sentence before definition 5.1])**.** A *Tambara ideal* $\underline{I}$ of a Tambara functor $\underline{T}$ is a sub-non-unital Tambara functor of $\underline{T}$ with a morphism of non-unital Tambara functors $\underline{T} \boxtimes \underline{I} \to \underline{I}$.

Practically speaking, a Tambara ideal $\underline{I}$ of a Tambara functor $\underline{T}$ is a collection of ideals $\underline{I}(G/H) \subseteq \underline{T}(G/H)$ closed under restriction, transfer, and norm.

Example 2.6. Recall that the Burnside ring for C_2 is

$$A(C_2) \cong {}^{\mathbb{Z}[t]}\!/\!_{(t^2 - 2t)},$$

where $t = [{}^{C_2}\!/\!_{e}]$. Let $\underline{I}$ be the Tambara ideal generated by $2 \in \underline{A}(C_2/C_2)$ inside the C_2-Burnside Tambara functor $\underline{A}$. This is the smallest Tambara ideal of $\underline{A}$ containing $2 \in \underline{A}(C_2/C_2)$. This ideal must contain

$$2 = \mathrm{res}(2) \in \underline{A}(C_2/e)$$

at the underlying level and

$$2t = \mathrm{tr}\,\mathrm{res}(2), \quad 2 + t = \mathrm{nm}\,\mathrm{res}(2) \in \underline{A}(C_2/C_2)$$

at the fixed level. Altogether, a minimal generating set for $\underline{I}(C_2/C_2)$ is $(2, t)$ and a minimal generating set for $\underline{I}(C_2/e)$ is (2). The quotient Tambara functor $\underline{A}/\underline{I}$ is therefore isomorphic to $\underline{\mathbb{F}_2}$.

$$\begin{tikzcd}
(2,t) \arrow[d, "\mathrm{res}"] & \mathbb{Z}[t]/(t^2-2t) \arrow[d, "t\mapsto 2"] & \mathbb{F}_2 \arrow[d, "1"] \\
(2) \arrow[u, bend left, "\mathrm{nm}"] \arrow[u, bend right, "\mathrm{tr}"'] & \mathbb{Z} \arrow[u, bend left, "a\mapsto a+\frac{a^2-a}{2}t"] \arrow[u, bend right, "a\mapsto at"'] & \mathbb{F}_2 \arrow[u, bend left, "a\mapsto a^2"] \arrow[u, bend right, "2"'] \\
\underline{I} \arrow[r, hook] & \underline{A} \arrow[r, two heads] & \underline{\mathbb{F}_2}
\end{tikzcd}$$

As in the preceeding example, it is always the case that $\underline{\mathbb{F}_2}$ is the quotient of $\underline{A}$ by the Tambara ideal generated by $2 \in \underline{A}(G/G)$, for any finite group G. Writing $\underline{\mathbb{F}_2}$ as the quotient of $\underline{A}$ by this Tambara ideal is a productive strategy to compute its norms.

Proposition 2.7. *The norm $n_H^G(\underline{\mathbb{F}_2})$ is the quotient of the G-Tambara functor $\underline{A}$ by the Tambara ideal generated by $2 \in \underline{A}(G/H)$.*

Proof. The H-Tambara ideal $\underline{I} \subseteq \underline{A}$ generated by $2 \in \underline{A}(H/H)$ is the image of the H-Tambara functor homomorphism $\underline{A}[x_{H/H}] \to \underline{A}$ determined by $x \mapsto 2$, where $\underline{A}[x_{H/H}]$ is the free H-Tambara functor generated at the top level [BH, Definition 5.4].

We may therefore write the H-Tambara functor $\underline{\mathbb{F}_2}$ as a reflexive coequalizer in the category $\mathcal{T}\mathrm{amb}(H)$:

$$\begin{tikzcd}
\underline{A}[x_{H/H}] \arrow[r, shift left=2, "x\mapsto 2"] \arrow[r, shift right=2, "x\mapsto 0"'] & \underline{A} \arrow[l] \arrow[r, two heads] & \underline{\mathbb{F}_2}
\end{tikzcd}$$

Since $n_H^G \colon \mathcal{T}\mathrm{amb}(H) \to \mathcal{T}\mathrm{amb}(G)$ is a left adjoint, it preserves coequalizers. The Tambara norm n_H^G sends the H-Burnside functor to the G-Burnside functor. By [HMQ, Proposition 4.2], we know that $n_H^G(\underline{A}[x_{H/H}])$ is isomorphic to $\underline{A}[x_{G/H}]$. Thus, we have a reflexive coequalizer in $\mathcal{T}\mathrm{amb}(G)$:

$$\begin{tikzcd}
\underline{A}[x_{G/H}] \arrow[r, shift left=2, "x\mapsto 2"] \arrow[r, shift right=2, "x\mapsto 0"'] & \underline{A} \arrow[l] \arrow[r, two heads] & n_H^G(\underline{\mathbb{F}_2})
\end{tikzcd}$$

This expresses $n_H^G(\underline{\mathbb{F}_2})$ as the quotient of the G-Tambara functor $\underline{A}$ by the ideal generated by $2 \in \underline{A}(G/H)$. ☐

Example 2.8. $n_e^{C_2}(\mathbb{F}_2) \cong \underline{A}/\underline{J}$, where $\underline{J}$ is the Tambara ideal of $\underline{A}$ generated by $2 \in \underline{A}(C_2/e)$. At the fixed level, minimal generators for this ideal are $(2t, 2+t)$.

$$\begin{tikzcd}
(2t,2+t) \arrow[d, "\mathrm{res}"] & \mathbb{Z}[t]/(t^2-2t) \arrow[d, "t\mapsto 2"] & \mathbb{Z}/4 \arrow[d, "1"] \\
(2) \arrow[u, bend left, "\mathrm{nm}"] \arrow[u, bend right, "\mathrm{tr}"'] & \mathbb{Z} \arrow[u, bend left, "a\mapsto a+\frac{a^2-a}{2}t"] \arrow[u, bend right, "a\mapsto at"'] & \mathbb{Z}/2 \arrow[u, bend left, "a\mapsto a^2"] \arrow[u, bend right, "2"'] \\
\underline{J} \arrow[r, hook] & \underline{A} \arrow[r, two heads] & n_e^{C_2}(\mathbb{F}_2)
\end{tikzcd}$$

Remark 2.9. If $\underline{I}$ is a Tambara ideal of a Tambara functor $\underline{T}$, we might expect a statement like the following to be true: $N_H^G(\underline{I})$ is a Tambara ideal of $n_H^G(\underline{T})$, and the norm of the quotient $n_H^G(\underline{T}/\underline{I})$ is the quotient of the norm $n_H^G(\underline{T})/N_H^G(\underline{I})$. There are several obstacles to making this statement precise. Although $N_H^G \colon \mathcal{M}\mathrm{ack}(H) \to \mathcal{M}\mathrm{ack}(G)$ is a left adjoint, it is not an exact functor (nor even additive). Furthermore, $\underline{T}/\underline{I}$ is not a colimit in H-Tambara functors, and there's no reason it should be preserved by n_H^G.

2.3. **The Norm $N_e^{\mathcal{K}}(\mathbb{F}_2)$.** Recall that the Burnside ring for $\mathcal{K}$ is

$$A(\mathcal{K}) \cong \mathbb{Z}[t_L, t_D, t_R] \big/ t_Lt_D = t_Lt_R = t_Dt_R,\, t_\bullet^2 = 2t_\bullet,$$

where $t_L = [\mathcal{K}/L]$, $t_D = [\mathcal{K}/D]$, and $t_R = [\mathcal{K}/R]$. The relation $t_\bullet^2 = 2t_\bullet$ holds for all $\bullet \in \{L, D, R\}$. Note that the class $[\mathcal{K}/e]$ is unnecessary as a generator, since $[\mathcal{K}/e] = t_Lt_D = t_Lt_R = t_Dt_R$. The Burnside Tambara functor $\underline{A}$ for the Klein four-group is displayed in Figure 2.1. In the Tambara functor $\underline{A}$, the norm nm_e^L is given by

$$\mathrm{nm}_e^L(a) = a + \left(\tfrac{a^2-a}{2}\right) s_L,$$

and the norm $\mathrm{nm}_L^{\mathcal{K}}$ is given by

$$\mathrm{nm}_L^{\mathcal{K}}(a\, s_L + b) \;=\; \left(a^2 - a + ba\right) t_Dt_R \;+\; \left(\tfrac{b^2-b}{2}\right) t_L \;+\; a\, t_D \;+\; a\, t_R + b,$$

and similarly for norms to or from D and R.

Figure 2.1. The Burnside $\mathcal{K}$-Tambara functor $\underline{A}$.

$$\begin{tikzcd}[column sep=small]
 & & \mathbb{Z}[t_L,t_D,t_R] \big/ (t_Lt_D = t_Lt_R = t_Dt_R, t_\bullet^2 = 2t_\bullet) \arrow[dll, "\substack{s_L \mapsfrom t_D, t_R \\ 2 \mapsfrom t_L}"] \arrow[d, "\substack{t_L, t_R \mapsto s_D \\ t_D \mapsto 2}"] \arrow[drr, "\substack{t_L, t_D \mapsto s_R \\ t_R \mapsto 2}"] & & \\
\mathbb{Z}[s_L] \big/ s_L^2 - 2s_L \arrow[urr, bend left, "\mathrm{nm}_L^K"] \arrow[urr, bend right, "\substack{1 \mapsto t_L \\ s_L \mapsto t_Dt_R}"] \arrow[drr, "s_D \mapsto 2"] & & \mathbb{Z}[s_D] \big/ s_D^2 - 2s_D \arrow[u, bend left, "\mathrm{nm}_D^K"] \arrow[u, bend right, "\substack{1 \mapsto t_D \\ s_D \mapsto t_Lt_R}"'] \arrow[d, "s_D \mapsto 2"] & & \mathbb{Z}[s_R] \big/ s_R^2 - 2s_R \arrow[ull, bend right, "\mathrm{nm}_R^K"'] \arrow[ull, bend left, "\substack{t_R \mapsfrom 1 \\ t_Lt_D \mapsfrom s_R}"'] \arrow[dll, "2 \mapsfrom s_D"'] \\
 & & \mathbb{Z} \arrow[ull, bend left, "\mathrm{nm}_e^L"] \arrow[ull, bend right, "\cdot s_D"'] \arrow[u, bend left, "\mathrm{nm}_e^D"] \arrow[u, bend right, "\cdot s_D"'] \arrow[urr, bend right, "\mathrm{nm}_e^R"'] \arrow[urr, bend left, "\cdot s_D"'] & &
\end{tikzcd}$$

Proposition 2.10. *The value of the $\mathcal{K}$-Tambara functor $n_e^{\mathcal{K}}(\mathbb{F}_2)$ at the trivial orbit $\mathcal{K}/\mathcal{K}$ is the ring*

$$n_e^{\mathcal{K}}(\mathbb{F}_2)(\mathcal{K}/\mathcal{K}) \cong \mathbb{Z}/8[b_L, b_R] \big/ (2b_Lb_R, 2b_R, b_L^2, b_R^2, b_Lb_R),$$

where b_L and b_R are the images of $t_L + 2$ and $t_R + 2$, respectively, under the surjection $\underline{A} \twoheadrightarrow n_e^{\mathcal{K}}(\mathbb{Z}/2)$. This surjection sends t_D to $b_L + b_R + 2$.

Proof. We apply Proposition 2.7: $n_e^{\mathcal{K}}(\mathbb{F}_2)$ is the quotient of $\underline{A}$ by the ideal generated by $2 \in \underline{A}(\mathcal{K}/e)$. Thus, $n_e^{\mathcal{K}}(\mathbb{F}_2)(\mathcal{K}/\mathcal{K})$ is the quotient of $\underline{A}(\mathcal{K}/\mathcal{K})$ by all norms and transfers of 2. In other words, we are asking for the quotient of $\mathbb{Z}[t_L, t_D, t_R]$ by the ideal generated by the relations in the Burnside ring $A(\mathcal{K})$:

$$\begin{gathered} t_Lt_D - t_Lt_R \\ t_Lt_R - t_Dt_R \\ t_L^2 - 2t_L \\ t_D^2 - 2t_D \\ t_R^2 - 2t_R \end{gathered}$$

and the relations imposed by the ideal of $\underline{A}$ generated by $2 \in \underline{A}(\mathcal{K}/e)$:

$$\begin{aligned} \operatorname{tr}_e^{\mathcal{K}}(2) &= 2t_Lt_D \\ \operatorname{nm}_e^{\mathcal{K}}(2) &= 2t_Lt_D + t_L + t_D + t_R + 2. \\ \operatorname{tr}_L^{\mathcal{K}}(\operatorname{nm}_e^L(2)) &= 2t_L + t_Dt_R \\ \operatorname{nm}_L^{\mathcal{K}}(\operatorname{tr}_e^L(2)) &= 2t_D + 2t_R + 2t_Dt_R \\ \operatorname{tr}_D^{\mathcal{K}}(\operatorname{nm}_e^D(2)) &= 2t_D + t_Lt_R \\ \operatorname{nm}_D^{\mathcal{K}}(\operatorname{tr}_e^D(2)) &= 2t_L + 2t_R + 2t_Lt_R \\ \operatorname{tr}_R^{\mathcal{K}}(\operatorname{nm}_e^R(2)) &= 2t_R + t_Lt_D \\ \operatorname{nm}_R^{\mathcal{K}}(\operatorname{tr}_e^R(2)) &= 2t_L + 2t_D + 2t_Lt_D. \end{aligned} \tag{2.11}$$

By SAGE, a Gröbner basis for this ideal is:

$$t_L^2 + 4, \quad t_Lt_R + 4, \quad t_R^2 + 4, \quad t_L + t_D + t_R + 2, \quad 2t_L + 4, \quad 2t_R + 4, \quad 8.$$

Let I be the ideal generated by the above. Write $\bar{t}_L$, $\bar{t}_D$, and $\bar{t}_R$ for the images of t_L, t_D, and t_R in the quotient $\mathbb{Z}[t_L, t_D, t_R]/I$. Since $8 \in I$, the $\mathbb{Z}$ becomes a $\mathbb{Z}/8$. We may eliminate any one of $\bar{t}_L, \bar{t}_D$, or $\bar{t}_R$ using the relation $t_L + t_D + t_R + 2$. Thus, we have

$$\mathbb{Z}[t_L, t_D, t_R]\big/_I \cong \mathbb{Z}/8[\bar{t}_L, \bar{t}_R]\big/_{(2\bar{t}_L + 4, 2\bar{t}_R + 4, \bar{t}_L^2 + 4, \bar{t}_R^2 + 4, \bar{t}_L\bar{t}_R + 4)}.$$

Setting $b_L = \bar{t}_L + 2$ and $b_R = \bar{t}_R + 2$, we have

$$\mathbb{Z}[t_L, t_D, t_R]\big/_I \cong \mathbb{Z}/8[b_L, b_R]\big/_{(2b_L, 2b_R, b_L^2, b_R^2, b_Lb_R)}.$$

□

Together with the fact that

$$\operatorname{res}_H^{\mathcal{K}} n_e^{\mathcal{K}}(\mathbb{F}_2) \cong \operatorname{res}_H^{\mathcal{K}} n_H^{\mathcal{K}} n_e^H(\mathbb{F}_2) \cong \left(n_e^H(\mathbb{F}_2)\right)^{\boxtimes 2} \cong n_e^H(\mathbb{F}_2^{\otimes 2}) \cong n_e^H(\mathbb{F}_2) \tag{2.12}$$

for any nontrivial proper subgroup H of $\mathcal{K}$, this proposition allows us to determine the $\mathcal{K}$-Tambara functor $n_e^{\mathcal{K}}(\mathbb{F}_2)$, pictured in Figure 2.2. The norms can be determined from Tambara reciprocity:

Proposition 2.13 (Tambara Reciprocity [HM, Theorem 2.5])**.** *Let H be any of the order 2 subgroups of $\mathcal{K}$. In any $\mathcal{K}$-Tambara functor, we have:*

$$\operatorname{nm}_H^{\mathcal{K}}(a + b) = \operatorname{nm}_H^{\mathcal{K}}(a) + \operatorname{nm}_H^{\mathcal{K}}(b) + \operatorname{tr}_H^{\mathcal{K}}(a\,(\gamma H \cdot b)),$$

where γH is the non-identity coset of H inside $\mathcal{K}$.

FIGURE 2.2. The Tambara functor $n_e^{\mathcal{K}}(\mathbb{F}_2)$.

$$\begin{tikzcd}
 & {}^{\mathbb{Z}}/_{8}[b_L, b_R]\big/\left(2b_L, 2b_R, b_L^2, b_R^2, b_L b_R\right) \arrow[dl, "0 \leftarrow\!\shortmid b_L{,} b_R"'] \arrow[d, "b_L{,} b_R \mapsto 0"] \arrow[dr, "b_L{,} b_R \mapsto 0"] & \\
\mathbb{Z}/4 \arrow[ur, bend left, "\mathrm{nm}_L^K"] \arrow[ur, bend right, "1 \mapsto b_L - 2"'] \arrow[dr] & \mathbb{Z}/4 \arrow[u, bend left, "\mathrm{nm}_D^K"] \arrow[u, bend right, "1 \mapsto b_L + b_R - 2"'] \arrow[d] & \mathbb{Z}/4 \arrow[ul, bend left, "\mathrm{nm}_R^K"] \arrow[ul, bend right, "b_R - 2 \leftarrow\!\shortmid 1"'] \arrow[dl] \\
 & \mathbb{Z}/2 \arrow[ul, bend left, "1"] \arrow[ul, bend right, "2"'] \arrow[u, bend left, "1"] \arrow[u, bend right, "2"'] \arrow[ur, bend left, "1"] \arrow[ur, bend right, "2"'] &
\end{tikzcd}$$

Note that because $n_e^{\mathcal{K}}(\mathbb{F}_2)$ is a quotient of the Burnside $\mathcal{K}$-Tambara functor, the Weyl actions are trivial. Thus, in the Tambara reciprocity formula for $n_e^{\mathcal{K}}(\mathbb{F}_2)$, $a\,(\gamma H \cdot b)$ becomes simply ab.

In particular, $\mathrm{nm}_L^{\mathcal{K}}(0) = 0$, $\mathrm{nm}_L^{\mathcal{K}}(1) = 1$, $\mathrm{nm}_L^{\mathcal{K}}(2) = b_L$ and $\mathrm{nm}_L^{\mathcal{K}}(3) = b_L - 3$. Similar considerations yield formulas for $\mathrm{nm}_D^{\mathcal{K}}$ and $\mathrm{nm}_R^{\mathcal{K}}$.

Example 2.14. Recall [BGHL, Section 5.2] that the geometric fixed points $\Phi^H \underline{M}$ of a $\mathcal{K}$-Mackey functor is obtained by first quotienting at every level by all transfers up from subgroups not containing H and then forgetting all levels $\mathcal{K}/J$, where J does not contain H. This interacts with norms according to the formula [BGHL, Theorem 5.15]

$$\Phi^H N_J^{\mathcal{K}} \underline{M} \cong N_e^{K/H} \operatorname{res}_e^J \underline{M}.$$

In particular, we find that

$$\Phi^L N_e^{\mathcal{K}}(\mathbb{F}_2) \cong \Phi^L N_L^{\mathcal{K}} N_e^L(\mathbb{F}_2) \cong N_e^{\mathcal{K}/L} \operatorname{res}_e^L N_e^L(\mathbb{F}_2) \cong N_e^{\mathcal{K}/L}(\mathbb{F}_2),$$

which is the Green functor

$$\begin{tikzcd}
\mathbb{Z}/4 \arrow[d] \\
\mathbb{Z}/2. \arrow[u, bend right, "2"']
\end{tikzcd}$$

Indeed, the elements $b_L + b_R - 2$ and $b_R - 2$ in $N_e^{\mathcal{K}}(\mathbb{F}_2)$ are in the image of the transfers from D and R, respectively. Setting those elements equal to zero produces $\mathbb{Z}/4$ as a quotient ring:

$$\frac{\mathbb{Z}/8 \oplus \mathbb{Z}/2\{b_L, b_R\}}{-2 + b_R, -2 + b_L + b_R} \cong \mathbb{Z}/4.$$

2.4. **The Norm $N_{C_2}^{\mathcal{K}}(\underline{\mathbb{F}_2})$.** We can use Proposition 2.7 to compute the norms $n_H^{\mathcal{K}}(\underline{\mathbb{F}_2})$ for $H \in \{L, D, R\}$. We will focus on the case $H = D$; the other cases are similar.

Proposition 2.15. *The value of the $\mathcal{K}$-Tambara functor $n_D^{\mathcal{K}}(\underline{\mathbb{F}_2})$ at the trivial orbit $\mathcal{K}/\mathcal{K}$ is the ring*

$$n_D^{\mathcal{K}}(\underline{\mathbb{F}_2})(\mathcal{K}/\mathcal{K}) \cong {}^{\mathbb{Z}/4[c]}\big/_{(2c, c^2)},$$

where c is the image of $t_R + 2$ under the surjection $\underline{A} \twoheadrightarrow n_D^{\mathcal{K}}(\underline{\mathbb{F}_2})$.

Proof. Let $\underline{I}$ be the Tambara ideal of $\underline{A}$ generated by $2 \in \underline{A}(\mathcal{K}/D)$. By Proposition 2.7, $n_D^{\mathcal{K}}(\underline{\mathbb{F}_2})$ is the quotient of $\underline{A}$ by $\underline{I}$. Since $\operatorname{res}_e^D(2) = 2$, this ideal contains the ideal of $\underline{A}$ generated by $2 \in \underline{A}(\mathcal{K}/e)$. Hence, $n_D^{\mathcal{K}}(\underline{\mathbb{F}_2})$ is a further quotient of $n_e^{\mathcal{K}}(\mathbb{F}_2)$ by the Tambara ideal $\underline{J}$ generated by $2 \in n_e^{\mathcal{K}}(\mathbb{F}_2)(\mathcal{K}/D)$.

At the top level $n_e^{\mathcal{K}}(\mathbb{F}_2)(\mathcal{K}/\mathcal{K})$, the ideal $\underline{J}(\mathcal{K}/\mathcal{K})$ is generated by $\operatorname{nm}_D^{\mathcal{K}}(2)$ and $\operatorname{tr}_D^{\mathcal{K}}(2)$. The $\mathcal{K}/L$ and $\mathcal{K}/R$ levels do not contribute any generators because $\underline{J}(\mathcal{K}/L) = \underline{J}(\mathcal{K}/R) = 0$. By Tambara reciprocity,

$$\begin{aligned}
\operatorname{nm}_D^{\mathcal{K}}(2) = \operatorname{nm}_D^{\mathcal{K}}(1+1) &= \operatorname{nm}_D^{\mathcal{K}}(1) + \operatorname{nm}_D^{\mathcal{K}}(1) + \operatorname{tr}_D^{\mathcal{K}}(1) \\
&= 1 + 1 + (b_L + b_R - 2) \\
&= b_L + b_R
\end{aligned}$$

$$\begin{aligned}
\operatorname{tr}_D^{\mathcal{K}}(2) &= 2(b_L + b_R - 2) \\
&= 2b_L + 2b_R - 4 \\
&= -4
\end{aligned}$$

So $n_D^{\mathcal{K}}(\underline{\mathbb{F}_2})(\mathcal{K}/\mathcal{K})$ is the quotient of $n_e^{\mathcal{K}}(\mathbb{F}_2)(\mathcal{K}/\mathcal{K})$ by the ideal $(b_L + b_R, -4)$. From Proposition 2.10,

$$n_e^{\mathcal{K}}(\mathbb{F}_2)(\mathcal{K}/\mathcal{K}) \cong {}^{\mathbb{Z}/8[b_L, b_R]}\big/_{(2b_L, 2b_R, b_L^2, b_R^2, b_L b_R)}.$$

The relation $b_L + b_R$ allows us to identify the two generators, and the -4 allows us to replace the $\mathbb{Z}/8$ by a $\mathbb{Z}/4$. Writing c for the image of b_R in the quotient, we have

$$n_D^{\mathcal{K}}(\underline{\mathbb{F}_2})(\mathcal{K}/\mathcal{K}) \cong {}^{\mathbb{Z}/4[c]}\big/_{(2c, c^2)}. \qquad \square$$

Together with the facts that $\operatorname{res}_D^{\mathcal{K}} n_D^{\mathcal{K}}(\underline{\mathbb{F}_2}) = \underline{\mathbb{F}_2}$ as in (2.12) and $\operatorname{res}_H^{\mathcal{K}} n_D^{\mathcal{K}}(\underline{\mathbb{F}_2}) \cong n_e^{C_2}(\mathbb{Z}/2)$ via the double coset formula for $H \in \{L, R\}$, this proposition allows us to determine the $\mathcal{K}$-Tambara functor $n_D^{\mathcal{K}}(\underline{\mathbb{F}_2})$, pictured in Figure 2.3.

The norm $\operatorname{nm}_D^{\mathcal{K}}$ is determined by $\operatorname{nm}_D^{\mathcal{K}}(0) = 0$ and $\operatorname{nm}_D^{\mathcal{K}}(1) = 1$. The norms $\operatorname{nm}_H^{\mathcal{K}}$ for $H \in \{L, R\}$ are determined by $\operatorname{nm}_H^{\mathcal{K}}(0) = 0$, $\operatorname{nm}_H^{\mathcal{K}}(1) = 1$, and Tambara reciprocity (Proposition 2.13). In particular, $\operatorname{nm}_H^{\mathcal{K}}(2) = c$ and $\operatorname{nm}_H^{\mathcal{K}}(3) = 1 + c$ for $H \in \{L, R\}$.

The following result about $n_D^{\mathcal{K}}\underline{\mathbb{F}_2}$-modules will be of use later in the analysis of exact sequences of Mackey functors.

Lemma 2.16. *Suppose that $\underline{M}$ is a module over $n_D^{\mathcal{K}}\underline{\mathbb{F}_2}$ such that $\underline{M}(\mathcal{K}/D) = 0$. Then $\underline{M}(\mathcal{K}/\mathcal{K})$ is 2-torsion.*

Proof. Let $x \in \underline{M}(\mathcal{K}/\mathcal{K})$. Since $\underline{M}(\mathcal{K}/D)$ vanishes, we certainly have that $\operatorname{tr}_D^{\mathcal{K}} \operatorname{res}_D^{\mathcal{K}} x$ must be zero. On the other hand, Frobenius reciprocity gives

$$\operatorname{tr}_D^{\mathcal{K}} \operatorname{res}_D^{\mathcal{K}}(x) = \operatorname{tr}_D^{\mathcal{K}} \operatorname{res}_D^{\mathcal{K}}(1 \cdot x) = \operatorname{tr}_D^{\mathcal{K}} \operatorname{res}_D^{\mathcal{K}}(1) \cdot x = 2 \cdot x. \qquad \square$$

Figure 2.3. The $\mathcal{K}$-Tambara functor $n_D^{\mathcal{K}}(\underline{\mathbb{F}_2})$.

$$\begin{tikzcd}
& {}^{\mathbb{Z}}/_{4}[c]\big/_{(2c,c^2)} \arrow[dl, "0 \mapsfrom c"'] \arrow[d, "0 \mapsfrom c"] \arrow[dr, "c \mapsto 0"] & \\
{}^{\mathbb{Z}}/_{4} \arrow[ur, bend left=40, "\mathrm{nm}_L^{\mathcal{K}}"] \arrow[ur, bend right=10, "2+c"'] \arrow[dr] & {}^{\mathbb{Z}}/_{2} \arrow[u, bend left=20, "\mathrm{nm}_D^{\mathcal{K}}"] \arrow[u, bend right=20, "2"'] \arrow[d, "1"] & {}^{\mathbb{Z}}/_{4} \arrow[ul, bend left=10, "\mathrm{nm}_R^{\mathcal{K}}"] \arrow[ul, bend right=40, "2+c"'] \arrow[dl] \\
& {}^{\mathbb{Z}}/_{2} \arrow[ul, bend left=40, "1"] \arrow[ul, bend right=10, "2"'] \arrow[u, bend left=20, "1"] \arrow[u, bend right=20, "0"'] \arrow[ur, bend left=10, "1"] \arrow[ur, bend right=40, "2"'] &
\end{tikzcd}$$

3. Homotopical Background

In this section, we collect some facts that will be useful in our computations below.

3.1. Eilenberg–Mac Lane $\mathcal{K}$-Spectra.

We review here some facts about equivariant Eilenberg–Mac Lane spectra for the group $\mathcal{K}$. Similar statements hold more generally for any finite group.

The Eilenberg–Mac Lane spectrum of a $\mathcal{K}$-Mackey $\underline{M}$ functor is the $\mathcal{K}$-spectrum $\mathfrak{H}\underline{M}$ whose homotopy Mackey functors are determined by

$$\underline{\pi}_n \mathfrak{H}\underline{M} = \begin{cases} \underline{M} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

Although the only nonzero integer-graded homotopy group of $\mathfrak{H}\underline{M}$ is $\underline{\pi}_0$, suspensions of $\mathfrak{H}\underline{M}$ by representation spheres may have more complicated homotopy. See, for example, Figure 7.1 in the C_2-equivariant case. An important property of the functor $\mathfrak{H}\colon \mathcal{M}\mathrm{ack}(\mathcal{K}) \to \mathcal{S}\mathrm{p}^{\mathcal{K}}$ is that it sends short exact sequences of $\mathcal{K}$-Mackey functors to cofiber sequences of $\mathcal{K}$-spectra.

The Eilenberg–Mac Lane spectrum functor also commutes with several change of group functors. For H a subgroup of $\mathcal{K}$, let $\uparrow_H^{\mathcal{K}}$ and $\downarrow_H^{\mathcal{K}}$ denote the induction and restriction functors, respectively, either between $\mathcal{K}$- and H-Mackey functors or between $\mathcal{K}$- and H-spectra.

Proposition 3.1. *Let $\underline{M}$ be a $\mathcal{K}$-Mackey functor and $\underline{N}$ an H-Mackey functor. Then*

(a) $\downarrow_H^{\mathcal{K}} \mathfrak{H}(\underline{M}) \simeq \mathbf{H}\left(\downarrow_H^{\mathcal{K}} \underline{M}\right)$

(b) $\uparrow_H^{\mathcal{K}} \mathbf{H}(\underline{N}) \simeq \mathfrak{H}\left(\uparrow_H^{\mathcal{K}} \underline{N}\right)$.

However, taking the Eilenberg–Mac Lane spectrum does not commute with norms.

3.2. $\mathrm{RO}(\mathcal{K})$-Graded Suspensions.

The abelian group $\mathrm{RO}(\mathcal{K})$ is a free abelian group of rank 4 generated by the trivial representation 1 and three one-dimensional representations σ_L, σ_D, and σ_R. For a subgroup $H \in \{L, D, R\}$ of $\mathcal{K}$, the representation σ_H corresponds to the sign

TABLE 3.1. The inflation functor $\phi_H^*\colon \mathcal{M}\mathrm{ack}(C_2) \to \mathcal{M}\mathrm{ack}(\mathcal{K})$, where we identify $C_2 \cong \mathcal{K}/H$. We also write $\phi_{LDR}^*\underline{M} := \phi_L^*\underline{M} \oplus \phi_D^*\underline{M} \oplus \phi_R^*\underline{M}$.

$\underline{M}$	$\phi_L^*\underline{M}$	$\phi_D^*\underline{M}$	$\phi_R^*\underline{M}$
$\begin{tikzcd} \underline{M}(C_2/C_2) \arrow[d, "\mathrm{res}"'] \\ \underline{M}(C_2/e) \arrow[u, bend right, "\mathrm{tr}"'] \end{tikzcd}$	$\begin{tikzcd} & \underline{M}(C_2/C_2) \arrow[dl, "\mathrm{res}"'] & \\ \underline{M}(C_2/e) \arrow[ur, bend right, "\mathrm{tr}"'] & 0 & 0 \\ & 0 & \end{tikzcd}$	$\begin{tikzcd} & \underline{M}(C_2/C_2) \arrow[d, "\mathrm{res}"'] & \\ 0 & \underline{M}(C_2/e) \arrow[u, bend right, "\mathrm{tr}"'] & 0 \\ & 0 & \end{tikzcd}$	$\begin{tikzcd} \underline{M}(C_2/C_2) \arrow[dr, "\mathrm{res}"] & & \\ 0 & 0 & \underline{M}(C_2/e) \arrow[ul, bend right, "\mathrm{tr}"'] \\ & 0 & \end{tikzcd}$

representation σ of $\mathcal{K}/H \cong C_2$, where $\mathcal{K}$ acts via the quotient homomorphism $\mathcal{K} \to \mathcal{K}/H$. We also write $\rho = 1 + \sigma_L + \sigma_D + \sigma_R$ for the regular representation of $\mathcal{K}$ and $\overline{\rho} = \rho - 1$ for the reduced regular representation.

Because we are interested in the RO($\mathcal{K}$)-graded homotopy of $\mathcal{K}$-spectra, we will frequently suspend by (virtual) representation spheres S^V for $V \in \mathrm{RO}(\mathcal{K})$. We are primarily interested in $\overline{\rho}$-suspensions of $\mathcal{K}$-spectra, but we will also find occasion to use σ_H-suspensions. The suspension Σ^{σ_H} fits into a cofiber sequence

$$\mathcal{K}/H_+ \wedge X \to X \to \Sigma^{\sigma_H} X.$$

The fiber has an alternative description, by the shearing isomorphism:

Proposition 3.2 (Shearing)**.** *Let X be a $\mathcal{K}$-spectrum and let H be a subgroup of $\mathcal{K}$. Then*

$$\mathcal{K}/H_+ \wedge X \simeq \uparrow_H^{\mathcal{K}} \downarrow_H^{\mathcal{K}} X,$$

where $\downarrow_H^{\mathcal{K}}$ denotes the restriction from $\mathcal{K}$-spectra to H-spectra, and $\uparrow_H^{\mathcal{K}}$ the induction.

In particular, when X is an Eilenberg–Mac Lane spectrum, we have

$$\mathcal{K}/H_+ \wedge \mathfrak{H}\underline{M} \simeq \uparrow_H^{\mathcal{K}} \downarrow_H^{\mathcal{K}} \mathfrak{H}\underline{M} \simeq \mathfrak{H}\left(\uparrow_H^{\mathcal{K}} \downarrow_H^{\mathcal{K}} \underline{M}\right).$$

A useful consequence of this is the following:

Corollary 3.3. *Let X be a $\mathcal{K}$-spectrum and let $H \in \{L, D, R\}$. If the restriction of X to H is contractible, then*

$$\Sigma^{\sigma_H} X \simeq X.$$

3.3. **Inflation Functors.** Our main computations are $\mathcal{K}$-equivariant. However, we often make comparisons to C_2-equivariant computations, as C_2 appears both as a subgroup and a quotient of $\mathcal{K}$. For reference, we display in Figure 7.1 and Figure 7.2 the $RO(C_2)$-graded homotopy Mackey functors of $\mathbf{H}\underline{\mathbb{F}_2}$ and $\mathbf{H}N_e^{C_2}\mathbb{F}_2$, respectively. The Mackey functors appearing in those charts are as shown in Table 7.1.

We follow [H2, Section 4] in writing $\phi_H^*\underline{M}$ for the inflation along the quotient $\mathcal{K} \to \mathcal{K}/H$ of the $\mathcal{K}/H$-Mackey functor $\underline{M}$, for any subgroup $H \leq \mathcal{K}$. Since the groups $\mathcal{K}/H$ are canonically isomorphic for all H in $\{L, D, R\}$, we write $\phi_{LDR}^*\underline{M}$ as shorthand for the sum $\phi_L^*\underline{M} \oplus \phi_D^*\underline{M} \oplus \phi_R^*\underline{M}$ for any C_2-Mackey functor $\underline{M}$. See Table 3.1. As in [GY], we write $\underline{g}$ for the fully inflated Mackey functor $\phi_{\mathcal{K}}^*\mathbb{F}$ (see Table 7.2).

The inflation functor ϕ_H^* in fact extends to a "geometric inflation" functor on spectra, and $\phi_H^*\mathbf{H}\underline{M} \simeq \mathfrak{H}\phi_H^*\underline{M}$. In fact, by considering the Postnikov tower for any $\mathcal{K}/H$-spectrum X, one gets more generally an isomorphism of $\mathcal{K}$-Mackey functors

$$\underline{\pi}_n\phi_H^* X \cong \phi_H^*\underline{\pi}_n X. \tag{3.4}$$

We will use this frequently. More importantly, for any $\mathcal{K}$-representation V, we have [H2, Corollary 4.6]

$$\Sigma^V\mathfrak{H}\phi_H^*\underline{M} \simeq \phi_H^*\left(\Sigma^{V^H} H\underline{M}\right) \tag{3.5}$$

where V^H is considered as a representation of $\mathcal{K}/H$.

This, combined with Figure 7.1, gives the following useful equivalences:

Proposition 3.6. *We have equivalences*

- *(a)* $\Sigma^{\overline{\rho}}\mathfrak{H}\underline{g} \simeq \mathfrak{H}\underline{g}$
- *(b)* $\Sigma^{\overline{\rho}}\mathfrak{H}\phi_H^*\underline{\mathbb{F}_2}^* \simeq \Sigma^1\mathfrak{H}\phi_H^*\underline{f}$ *for* $H \in \{L, D, R\}$
- *(c)* $\Sigma^{\overline{\rho}}\mathfrak{H}\phi_H^*\underline{f} \simeq \Sigma^1\mathfrak{H}\phi_H^*\underline{\mathbb{F}_2}$ *for* $H \in \{L, D, R\}$.

where $\underline{f}$ *is as displayed in Table 7.1.*

4. THE HOMOTOPY OF $\mathfrak{H}N_e^{\mathcal{K}}\mathbb{F}_2$

In this section, we compute the homotopy Mackey functors $\underline{\pi}_{\blacklozenge}\mathfrak{H}\mathfrak{N}$, where $\mathfrak{N}$ will often be used as an abbreviation for $N_e^{\mathcal{K}}\mathbb{F}_2$. The results are displayed in Figure 7.6.

Remark 4.1. For $k \geq 0$, the homology groups $\underline{H}_n(S^{k\overline{\rho}};\mathfrak{N}) \cong \underline{\pi}_{n-k\overline{\rho}}\mathfrak{H}\mathfrak{N}$ are concentrated in degrees $n \geq 0$. We will refer to this portion of $\underline{\pi}_{\blacklozenge}\mathfrak{H}\mathfrak{N}$ as the **positive cone**. It appears in the fourth quadrant of Figure 7.6. Similarly, the cohomology groups $\underline{H}^n(S^{k\overline{\rho}};\mathfrak{N}) \cong \pi_{-n+k\overline{\rho}}\mathfrak{H}\mathfrak{N}$ are concentrated in degrees $n \geq 0$. We will refer to this portion as the **negative cone**. It appears in the second quadrant of Figure 7.6.

We proceed with a computation of the positive cone for $\mathfrak{H}\mathfrak{N}$ followed by that of the negative cone.

4.1. The Positive Cone of $\mathfrak{H}N_e^{\mathcal{K}}\mathbb{F}_2$.

Here we compute the homotopy Mackey functors of the $\mathcal{K}$-spectra $\Sigma^{k\overline{\rho}}\mathfrak{H}\mathfrak{N}$ for $k \geq 0$.

We start with the case $k = 1$. Our analysis will rely on cofiber sequences of equivariant Eilenberg–Mac Lane spectra arising from short exact sequences of Mackey functors. In particular, the following Mackey functor $\underline{E}$ will be of use.

Definition 4.2. Let $\underline{E}$ be the cokernel of $\underline{\mathbb{F}_2}^* \hookrightarrow N_e^{\mathcal{K}}\mathbb{F}_2$. This Mackey functor is displayed to the right.

$$\begin{tikzcd}
 & & \mathbb{Z}/4 \oplus \mathbb{Z}/2\{b_L, b_R\} \arrow[dll, shift right, "{\begin{pmatrix}1 & 0 & 0\end{pmatrix}}"'] \arrow[d, shift right, "{\begin{pmatrix}1 & 0 & 0\end{pmatrix}}"'] \arrow[drr, shift left, "{\begin{pmatrix}1 & 0 & 0\end{pmatrix}}"'] & & \\
\mathbb{Z}/2 \arrow[urr, shift right, "{\begin{pmatrix}2 \\ 1 \\ 0\end{pmatrix}}"'] & & \mathbb{Z}/2 \arrow[u, shift right, "{\begin{pmatrix}2 \\ 1 \\ 1\end{pmatrix}}"'] & & \mathbb{Z}/2 \arrow[ull, shift left, "{\begin{pmatrix}2 \\ 0 \\ 1\end{pmatrix}}"'] \\
 & & & & \\
 & & 0 & &
\end{tikzcd}$$

The Mackey functor $\underline{E}$ also fits into a short exact sequence

$$\phi_{LDR}^*(\underline{\mathbb{F}_2}^*) \hookrightarrow \underline{E} \twoheadrightarrow \underline{g}. \tag{4.3}$$

Lemma 4.4. *The nonzero homotopy Mackey functors of* $\Sigma^{\overline{\rho}}\mathcal{H}\underline{E}$ *are*

$$\underline{\pi}_n\Sigma^{\overline{\rho}}\mathcal{H}\underline{E} = \begin{cases} \underline{g} & n=0 \\ \phi^*_{LDR}\underline{f} & n=1 \end{cases}$$

Proof. Applying the functors $\Sigma^{\overline{\rho}}$ and $\mathcal{H}$ to the short exact sequence (4.3) yields a cofiber sequence:

$$\Sigma^{\overline{\rho}}\mathcal{H}\phi^*_{LDR}\underline{\mathbb{F}_2}^* \to \Sigma^{\overline{\rho}}\mathcal{H}\underline{E} \to \Sigma^{\overline{\rho}}\mathcal{H}\underline{g}.$$

But Proposition 3.6 provides equivalences $\Sigma^{\overline{\rho}}\mathcal{H}\phi^*_{LDR}\underline{\mathbb{F}_2}^* \simeq \Sigma^1\mathcal{H}\phi^*_{LDR}\underline{f}$ and $\Sigma^{\overline{\rho}}\mathcal{H}\underline{g} \simeq \mathcal{H}\underline{g}$, so the homotopy of $\Sigma^{\overline{\rho}}\mathcal{H}\underline{E}$ can be read off from the associated long exact sequence. □

Proposition 4.5. *The nonzero homotopy Mackey functors of* $\Sigma^{\overline{\rho}}\mathcal{H}N_e^{\mathcal{K}}\mathbb{F}_2$ *are*

$$\underline{\pi}_n\Sigma^{\overline{\rho}}\mathcal{H}N_e^{\mathcal{K}}\mathbb{F}_2 \cong \begin{cases} \underline{\mathbb{F}_2} & n=3 \\ \phi^*_{LDR}\underline{f} & n=1 \\ \underline{g} & n=0. \end{cases}$$

Proof. By applying the functors $\Sigma^{\overline{\rho}}$ and $\mathcal{H}$ to the defining short exact sequence for $\underline{E}$, we have a cofiber sequence

$$\Sigma^{\overline{\rho}}\mathcal{H}\underline{\mathbb{F}_2}^* \to \Sigma^{\overline{\rho}}\mathcal{H}\mathcal{N} \to \Sigma^{\overline{\rho}}\mathcal{H}\underline{E}.$$

We understand the homotopy of the fiber: $\Sigma^{\overline{\rho}}\mathcal{H}\underline{\mathbb{F}_2}^*$ is $\Sigma^3\mathcal{H}\underline{\mathbb{F}_2}$ by [GY, Proposition 4.2]. We also understand the homotopy of the cofiber by Lemma 4.4. The desired homotopy may then be read off from the associated long exact sequence. □

The computation of the homotopy of $\Sigma^{\overline{\rho}}\mathcal{H}\mathcal{N}$ gives a fiber sequence

$$\Sigma^3\mathcal{H}\underline{\mathbb{F}_2} \to \Sigma^{\overline{\rho}}\mathcal{H}\mathcal{N} \to \Sigma^{\overline{\rho}}\mathcal{H}\underline{E}. \tag{4.6}$$

It turns out that the $(k-1)\overline{\rho}$-suspension of the map $\Sigma^3\mathcal{H}\underline{\mathbb{F}_2} \to \Sigma^{\overline{\rho}}\mathcal{H}\mathcal{N}$ detects much of the homotopy of $\Sigma^{k\overline{\rho}}\mathcal{H}\mathcal{N}$, as follows from analysis of the suspensions of the cofiber.

Proposition 4.7. *For* $k \geq 2$, *the nonzero homotopy Mackey functors of* $\Sigma^{k\overline{\rho}}\mathcal{H}\underline{E}$ *are*

$$\underline{\pi}_n\Sigma^{k\overline{\rho}}\mathcal{H}\underline{E} \cong \begin{cases} \phi^*_{LDR}\underline{\mathbb{F}_2} & n=k \\ \underline{g}^3 & n\in[2,k-1] \\ \underline{g} & n=0. \end{cases}$$

Proof. As in Lemma 4.4, we have a fiber sequence

$$\Sigma^1\mathcal{H}\phi^*_{LDR}\underline{f} \to \Sigma^{\overline{\rho}}\mathcal{H}\underline{E} \to \mathcal{H}\underline{g}.$$

Suspending this $(k-1)\overline{\rho}$ times yields another fiber sequence

$$\Sigma^{1+(k-1)\overline{\rho}}\mathcal{H}\phi^*_{LDR}\underline{f} \to \Sigma^{k\overline{\rho}}\mathcal{H}\underline{E} \to \Sigma^{(k-1)\overline{\rho}}\mathcal{H}\underline{g}.$$

Applying the equivalences

$$\Sigma^{\overline{\rho}}\mathcal{H}\underline{g} \simeq \mathcal{H}\underline{g} \qquad \text{and} \qquad \Sigma^{\overline{\rho}}\mathcal{H}\phi^*_{LDR}\underline{f} \simeq \Sigma^1\mathcal{H}\phi^*_{LDR}\underline{\mathbb{F}_2}$$

from Proposition 3.6, we see that this fiber sequence is equivalent to

$$\Sigma^{2+(k-2)\overline{\rho}}\mathcal{H}\phi^*_{LDR}\underline{\mathbb{F}_2} \to \Sigma^{k\overline{\rho}}\mathcal{H}\underline{E} \to \mathcal{H}\underline{g}. \tag{4.8}$$

Then by (3.5), the left term becomes

$$\Sigma^{2+(k-2)\overline{\rho}}\mathcal{H}\phi^*_{LDR}\underline{\mathbb{F}_2} \simeq \bigvee_{H\in\{L,D,R\}} \phi^*_H\left(\Sigma^{(2+(k-2)\overline{\rho})}\mathbf{H}\underline{\mathbb{F}_2}\right)$$

We may compute the homotopy of the above using (3.4) and the computation of the C_2-Mackey functors $\underline{\pi}_*^{C_2}\mathbf{H}\underline{\mathbb{F}_2}$, as in Figure 7.1. The result then follows from the long exact sequence associated to (4.8). ☐

In the case $k = 2$, the cofiber sequence (4.6) and Proposition 4.7 give the following computation of $\Sigma^{2\overline{\rho}}\mathcal{H}\mathcal{N}$.

Corollary 4.9. *The nontrivial homotopy Mackey functors of* $\Sigma^{2\overline{\rho}}\mathcal{H}N_e^{\mathcal{K}}\mathbb{F}_2$ *are*

$$\underline{\pi}_n\Sigma^{2\overline{\rho}}\mathcal{H}N_e^{K}\mathbb{F}_2 \cong \begin{cases} \underline{\pi}_{n-3}\Sigma^{\overline{\rho}}\mathcal{H}\underline{\mathbb{F}}_2 & n \geq 3 \\ \phi^*_{LDR}\underline{\mathbb{F}}_2 & n = 2 \\ \underline{g} & n = 0. \end{cases}$$

More generally, combining (4.6) with Proposition 4.7 gives the following.

Corollary 4.10. *For $k \geq 2$, we have isomorphisms*

$$\underline{\pi}_n\Sigma^{k\overline{\rho}}\mathcal{H}N_e^{\mathcal{K}}\mathbb{F}_2 \cong \underline{\pi}_n\Sigma^{3+(k-1)\overline{\rho}}\mathcal{H}\underline{\mathbb{F}_2} = \underline{\pi}_{n-3}\Sigma^{(k-1)\overline{\rho}}\mathcal{H}\underline{\mathbb{F}_2}$$

for $n \geq k+1$.

However, starting with $k = 3$, the homotopy of the left and right terms in the sequence

$$\Sigma^{3+(k-1)\overline{\rho}}\mathcal{H}\underline{\mathbb{F}_2} \to \Sigma^{k\overline{\rho}}\mathcal{H}\mathcal{N} \to \Sigma^{k\overline{\rho}}\mathcal{H}\underline{E} \tag{4.11}$$

begin to overlap. In order to find the lower homotopy Mackey functors of $\Sigma^{k\overline{\rho}}\mathcal{H}\mathcal{N}$, it is convenient to consider a different sequence. The kernel of the surjection $\mathcal{N} \twoheadrightarrow \underline{\mathbb{F}}_2$ is $\underline{B}(2,0) \oplus \underline{g}^2$, and we now describe its homotopy.

Proposition 4.12. *The nonzero homotopy Mackey functors of* $\Sigma^{\overline{\rho}}\mathcal{H}(\underline{B}(2,0) \oplus \underline{g}^2)$ *are*

$$\underline{\pi}_n\Sigma^{\overline{\rho}}\mathcal{H}(\underline{B}(2,0) \oplus \underline{g}^2) \cong \begin{cases} \underline{mg} & n = 1 \\ \underline{g}^3 & n = 0. \end{cases}$$

For $k \geq 2$, the nonzero homotopy Mackey functors of $\Sigma^{k\overline{\rho}}\mathcal{H}(\underline{B}(2,0) \oplus \underline{g}^2)$ *are*

$$\underline{\pi}_n\Sigma^{k\overline{\rho}}\mathcal{H}(\underline{B}(2,0) \oplus \underline{g}^2) \cong \begin{cases} \phi^*_{LDR}\underline{\mathbb{F}}_2 & n = k \\ \underline{g}^3 & n \in [2, k-1] \\ \underline{g}^2 & n = 1 \\ \underline{g}^3 & n = 0. \end{cases}$$

Proof. Since $\Sigma^{k\overline{\rho}}\mathcal{H}\underline{g}$ is equivalent to $\mathcal{H}\underline{g}$, the claim amounts to the computation of $\Sigma^{k\overline{\rho}}\mathcal{H}\underline{B}(2,0)$.

The Mackey functor $\underline{B}(2,0)$ is the cokernel of the inclusion $\underline{\mathbb{Z}}^* \hookrightarrow \underline{\mathbb{Z}}$, so the homotopy of $\Sigma^{\overline{\rho}}\mathcal{H}\underline{B}(2,0)$ can be calculated from the cofiber sequence

$$\Sigma^{\overline{\rho}}\mathcal{H}\underline{\mathbb{Z}}^* \longrightarrow \Sigma^{\overline{\rho}}\mathcal{H}\underline{\mathbb{Z}} \longrightarrow \Sigma^{\overline{\rho}}\mathcal{H}\underline{B}(2,0)$$

and the equivalence $\Sigma^{\overline{\rho}}\mathcal{H}\underline{\mathbb{Z}}^* \simeq \Sigma^3\mathcal{H}\underline{\mathbb{Z}}$ [Sl, Proposition 4.2], together with the computation of $\underline{\pi}_n\Sigma^{\overline{\rho}}\mathcal{H}\underline{\mathbb{Z}}$ [Sl, Proposition 9.1]. Note that [Sl] reports the homotopy of the $\rho = (1+\overline{\rho})$-suspensions of $\mathcal{H}\underline{\mathbb{Z}}$, whereas we are interested in the $\overline{\rho}$-suspensions. The long exact sequence associated to the cofiber sequence above immediately yields $\underline{\pi}_0\Sigma^{\overline{\rho}}\mathcal{H}\underline{B}(2,0)$ and $\underline{\pi}_1\Sigma^{\overline{\rho}}\mathcal{H}\underline{B}(2,0)$.

It also shows that $\underline{\pi}_n\Sigma^{\bar{\rho}}\mathcal{H}\underline{B}(2,0)$ vanishes in degrees $n<0$, $n=2$, and $n>4$. In degrees 3 and 4, the long exact sequence is

$$\begin{tikzcd}
 & \Sigma^3\mathcal{H}\underline{\mathbb{Z}} & \Sigma^{\bar{\rho}}\mathcal{H}\underline{\mathbb{Z}} & \Sigma^{\bar{\rho}}\mathcal{H}\underline{B}(2,0) \\
n=4 & & 0 \arrow[r] & \underline{\pi}_4\Sigma^{\bar{\rho}}\mathcal{H}\underline{B}(2,0) \arrow[dll, rounded corners] \\
n=3 & \underline{\mathbb{Z}} \arrow[r] & \underline{\mathbb{Z}} \arrow[r] & \underline{\pi}_3\Sigma^{\bar{\rho}}\mathcal{H}\underline{B}(2,0) \arrow[dll, rounded corners] \\
n=2 & 0 & &
\end{tikzcd}$$

Since $\underline{B}(2,0)$ vanishes at the underlying level, the underlying level of the map $\underline{\mathbb{Z}}\to\underline{\mathbb{Z}}$ must be an isomorphism. Because these are constant Mackey functors, this determines the homomorphism $\underline{\mathbb{Z}}\to\underline{\mathbb{Z}}$ entirely; it must be an isomorphism. Hence, $\underline{\pi}_3\Sigma^{\bar{\rho}}\mathcal{H}\underline{B}(2,0)$ and $\underline{\pi}_4\Sigma^{\bar{\rho}}\mathcal{H}\underline{B}(2,0)$ are zero.

We next turn to the calculation of the $k\bar{\rho}$-suspensions of $\mathcal{H}\underline{B}(2,0)$. Given the calculation of $\underline{\pi}_n\Sigma^{\bar{\rho}}\mathcal{H}\underline{B}(2,0)$ in the previous paragraph, there is a Postnikov sequence

$$\Sigma^1\mathcal{H}\underline{mg}\to\Sigma^{\bar{\rho}}\mathcal{H}\underline{B}(2,0)\to\mathcal{H}\underline{g}.$$

To use this sequence to compute the homotopy of $\Sigma^{k\bar{\rho}}\underline{B}(2,0)$, we must first compute the homotopy of $\Sigma^{1+(k-1)\bar{\rho}}\mathcal{H}\underline{mg}$. As in [GY, Proposition 7.4], there is a short exact sequence of Mackey functors $\phi^*_{LDR}\underline{f}\hookrightarrow\underline{mg}\to\underline{g}^2$, which gives a fiber sequence

$$\Sigma^1\mathcal{H}\phi^*_{LDR}\underline{\mathbb{F}_2}=\Sigma^{\bar{\rho}}\mathcal{H}\phi^*_{LDR}\underline{f}\to\Sigma^{\bar{\rho}}\mathcal{H}\underline{mg}\to\Sigma^{\bar{\rho}}\mathcal{H}\underline{g}^2=\mathcal{H}\underline{g}^2.$$

Suspending again by $\Sigma^{(k-2)\bar{\rho}}$ gives a fiber sequence

$$\Sigma^{1+(k-2)\bar{\rho}}\mathcal{H}\phi^*_{LDR}\underline{\mathbb{F}_2}\to\Sigma^{(k-1)\bar{\rho}}\mathcal{H}\underline{mg}\to\mathcal{H}\underline{g}^2.$$

Using (3.5), we may calculate the homotopy of $\Sigma^{1+(k-2)\bar{\rho}}\mathcal{H}\phi^*_{LDR}\underline{\mathbb{F}_2}$ as in the proof of Proposition 4.7 (see also Figure 7.1). Unwinding the cofiber sequences yields the homotopy of $\Sigma^{\bar{\rho}}\mathcal{H}B(2,0)$. ☐

We now use the previous results to determine the Mackey functors $\underline{\pi}_n\Sigma^{k\bar{\rho}}\mathcal{H}\mathcal{N}$ for $k\geq 3$.

Theorem 4.13. *For $k\geq 3$, the nonzero homotopy Mackey functors of $\Sigma^{k\bar{\rho}}\mathcal{H}N_e^{\mathcal{K}}\mathbb{F}_2$ are*

$$\underline{\pi}_n\Sigma^{k\bar{\rho}}\mathcal{H}N_e^{\mathcal{K}}\mathbb{F}_2\cong\begin{cases}\underline{\pi}_n\Sigma^{k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2} & n\geq k+2\\ \underline{g}^{2k-3} & n=k+1\\ \phi^*_{LDR}\underline{\mathbb{F}_2}\oplus\underline{g}^{2k-5} & n=k\\ \underline{g}^{2n-2} & n\in[3,k-1]\\ \underline{g}^3 & n=2\\ \underline{g} & n=0.\end{cases}$$

Proof. The homotopy Mackey functors in dimensions 0, 1, and 2 are given by Proposition 4.7. Those in dimensions at least $k+1$ are given by Corollary 4.10. The answer is stated differently here to emphasize the relation to $\Sigma^{k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}$. These spectra are related via the cofiber sequence

$$\Sigma^{k\bar{\rho}}\mathcal{H}\underline{B}(2,0)\oplus\underline{g}^2\to\Sigma^{k\bar{\rho}}\mathcal{H}\mathcal{N}\to\Sigma^{k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}. \tag{4.14}$$

By Proposition 4.12, the map $\Sigma^{k\bar{\rho}}\mathcal{H}\mathcal{N}\to\Sigma^{k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}$ induces an isomorphism of homotopy Mackey functors in degrees at least $k+2$. In fact, we claim that it is an *injection* in all degrees. Consider, for instance the long exact sequence associated to (4.14) for $k = 3$, where the only Mackey functor remaining to be determined is $\underline{\pi}_3\Sigma^{3\bar{\rho}}\mathcal{H}\mathcal{N}$.

$$\begin{tikzcd}
 & & \underline{\pi}_n\Sigma^{3\bar{\rho}}\mathcal{H}\underline{B}(2,0)\oplus\underline{g}^2 & \underline{\pi}_n\Sigma^{3\bar{\rho}}\mathcal{H}\mathcal{N} & \underline{\pi}_n\Sigma^{3\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2} & \\
n=4 & {} \arrow[r, dashed] & 0 \arrow[r, "0"] & \underline{g}^3 \arrow[r, hook] & \phi^*_{LDR}\underline{\mathbb{F}_2}\oplus\underline{g}^3 \arrow[dll, rounded corners] & \\
n=3 & & \phi^*_{LDR}\underline{\mathbb{F}_2} \arrow[r, "0"] & ? \arrow[r, hook] & \phi^*_{LDR}\underline{\mathbb{F}_2}\oplus\underline{g}^4 \arrow[dll, rounded corners] & \\
n=2 & & \underline{g}^3 \arrow[r, "0"] & \underline{g}^3 \arrow[r, hook] & \underline{g}^5 \arrow[dll, rounded corners] & \\
n=1 & & \underline{g}^2 \arrow[r, "0"] & 0 \arrow[r, "0"] & \underline{g}^3 \arrow[r, dashed] & {}
\end{tikzcd}$$

The connecting homomorphism $\underline{\pi}_4\Sigma^{3\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2} \to \underline{\pi}_3\mathcal{H}\underline{B}(2,0)\oplus\underline{g}^2$ is an isomorphism upon restricting to any C_2, which forces it to be a surjection of Mackey functors. Thus $\underline{\pi}_3\Sigma^{3\bar{\rho}}\mathcal{H}\mathcal{N}$ is a sub-Mackey functor of $\underline{\pi}_3\Sigma^{3\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2} \cong \phi^*_{LDR}\underline{\mathbb{F}_2}\oplus\underline{g}^4$. On the other hand, the sequence (4.11) forces a short exact sequence

$$\underline{g} \hookrightarrow \underline{\pi}_3\Sigma^{3\bar{\rho}}\mathcal{H}\mathcal{N} \twoheadrightarrow \phi^*_{LDR}\underline{\mathbb{F}_2}$$

This combines to force an isomorphism $\underline{\pi}_3\Sigma^{3\bar{\rho}}\mathcal{H}\mathcal{N} \cong \phi^*_{LDR}\underline{\mathbb{F}_2}\oplus\underline{g}$. A similar argument works for higher values of k. □

Remark 4.15. We argued that the map $\Sigma^{k\bar{\rho}}\mathcal{H}\mathcal{N} \to \Sigma^{k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}$ induces an isomoprhism on Mackey functors in degrees at least $k+2$ by showing that the fiber in (4.14) has no homotopy above degree k, in Proposition 4.12. In other words, we used that the homology of $S^{k\bar{\rho}}$ with coefficients in $\underline{B}(2,0)\oplus\underline{g}^2$ vanishes above degree k. An alternative argument for this is that $S^{k\bar{\rho}}$ has a $\mathcal{K}$-CW structure in which all cells in degrees $k+1$ or higher are $\mathcal{K}$-free. In particular, it is the $\mathcal{K}$-CW structure associated to the k-fold smash product $\left(S^{\bar{\rho}}\right)^{\wedge k} \simeq S^{k\bar{\rho}}$. We need only observe that $\mathcal{K}/H \times \mathcal{K}/J \cong \mathcal{K}/e$ for any two distinct order two subgroups H and J. Since $\underline{B}(2,0)\oplus\underline{g}^2$ vanishes at the underlying level, the homology of $S^{k\bar{\rho}}$ with these coefficents will vanish in degrees at least $k+1$.

The blue shading in Figures 7.4 and 7.6 highlights the regions in which the isomorphism $\underline{\pi}_n\Sigma^{k\bar{\rho}}\mathcal{H}\mathcal{N} \cong \underline{\pi}_n\Sigma^{k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}$ was shown to hold in the positive cones.

4.2. The Negative Cone of $\mathcal{H}N_e^{\mathcal{K}}\mathbb{F}_2$.

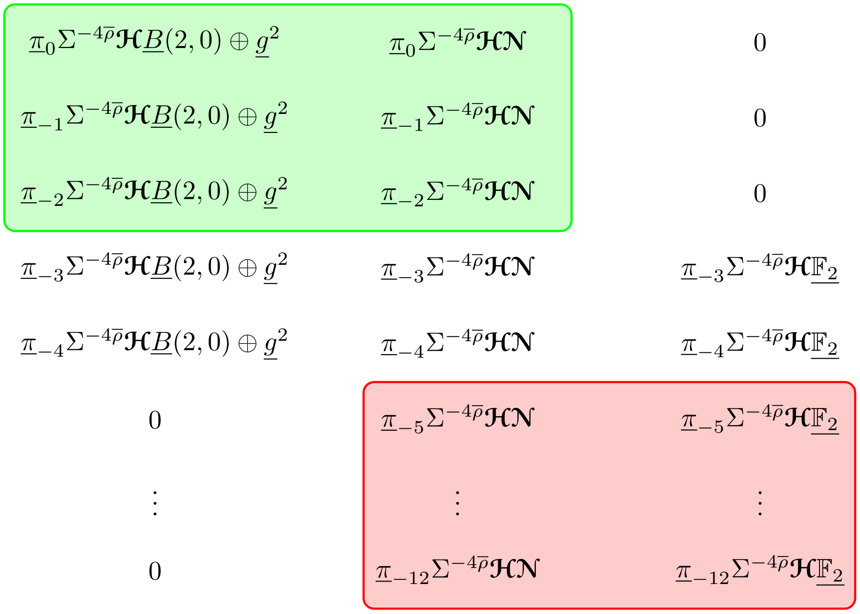
We now turn to the case of $\Sigma^{k\bar{\rho}}\mathcal{H}N_e^{\mathcal{K}}\mathbb{F}_2$ for k negative. The strategy is largely the same as in Section 4.1: most of the answer follows easily from a cofiber sequence, while some extension problems are resolved by considering a separate cofiber sequence.

We will again use the Mackey functor $\underline{B}(2,0)$. An argument as in the proof of Proposition 4.12 gives the following computation.

Proposition 4.16. *The nonzero homotopy Mackey functors of* $\Sigma^{-\bar{\rho}}\mathcal{H}\underline{B}(2,0)\oplus\underline{g}^2$ *are*

$$\underline{\pi}_n\Sigma^{-\bar{\rho}}\mathcal{H}\underline{B}(2,0)\oplus\underline{g}^2 \cong \begin{cases} \underline{g}^3 & n=0 \\ \underline{mg}^* & n=-1. \end{cases}$$

FIGURE 4.1. The fiber sequence $\underline{\pi}_*\Sigma^{-4\bar{\rho}}\mathcal{H}\underline{B}(2,0) \oplus \underline{g}^2 \to \underline{\pi}_*\Sigma^{-4\bar{\rho}}\mathcal{H}\mathcal{N} \to \underline{\pi}_*\Sigma^{-4\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}$. In the green region, the homotopy of $\Sigma^{-4\bar{\rho}}\mathcal{H}\mathcal{N}$ matches the homotopy of the fiber, and in the red region, the homotopy is the same as the homotopy of $\mathcal{H}\underline{\mathbb{F}_2}$.



$$\begin{array}{ccc}
\underline{\pi}_0\Sigma^{-4\bar{\rho}}\mathcal{H}\underline{B}(2,0)\oplus\underline{g}^2 & \underline{\pi}_0\Sigma^{-4\bar{\rho}}\mathcal{H}\mathcal{N} & 0 \\
\underline{\pi}_{-1}\Sigma^{-4\bar{\rho}}\mathcal{H}\underline{B}(2,0)\oplus\underline{g}^2 & \underline{\pi}_{-1}\Sigma^{-4\bar{\rho}}\mathcal{H}\mathcal{N} & 0 \\
\underline{\pi}_{-2}\Sigma^{-4\bar{\rho}}\mathcal{H}\underline{B}(2,0)\oplus\underline{g}^2 & \underline{\pi}_{-2}\Sigma^{-4\bar{\rho}}\mathcal{H}\mathcal{N} & 0 \\
\underline{\pi}_{-3}\Sigma^{-4\bar{\rho}}\mathcal{H}\underline{B}(2,0)\oplus\underline{g}^2 & \underline{\pi}_{-3}\Sigma^{-4\bar{\rho}}\mathcal{H}\mathcal{N} & \underline{\pi}_{-3}\Sigma^{-4\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2} \\
\underline{\pi}_{-4}\Sigma^{-4\bar{\rho}}\mathcal{H}\underline{B}(2,0)\oplus\underline{g}^2 & \underline{\pi}_{-4}\Sigma^{-4\bar{\rho}}\mathcal{H}\mathcal{N} & \underline{\pi}_{-4}\Sigma^{-4\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2} \\
0 & \underline{\pi}_{-5}\Sigma^{-4\bar{\rho}}\mathcal{H}\mathcal{N} & \underline{\pi}_{-5}\Sigma^{-4\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2} \\
\vdots & \vdots & \vdots \\
0 & \underline{\pi}_{-12}\Sigma^{-4\bar{\rho}}\mathcal{H}\mathcal{N} & \underline{\pi}_{-12}\Sigma^{-4\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}
\end{array}$$

For $k \geq 2$, the nonzero homotopy Mackey functors of $\Sigma^{-k\bar{\rho}}\mathcal{H}\underline{B}(2,0) \oplus \underline{g}^2$ are

$$\underline{\pi}_n\Sigma^{k\bar{\rho}}\mathcal{H}\underline{B}(2,0) \oplus \underline{g}^2 \cong \begin{cases} \underline{g}^3 & n = 0 \\ \underline{g}^2 & n = -1 \\ \underline{g}^3 & n \in [-k+1, -2] \\ \phi^*_{LDR}\underline{\mathbb{F}}_2^* & n = -k. \end{cases}$$

Remark 4.17. Rather than arguing as in Proposition 4.12, an alternative method to obtain Proposition 4.16 is to use Brown-Comenetz duality, as in [Sl, Section 8], since the Mackey functors $\underline{B}(2,0)$ and $\underline{g}$ are self-dual.

Returning to $\Sigma^{-k\bar{\rho}}\mathcal{H}\mathcal{N}$, the lower homotopy groups are captured by suspensions of $\mathcal{H}\underline{\mathbb{F}_2}$, as we now describe (see also Figure 4.1). In Figures 7.4 and 7.6, we use red shading to indicate the region of the negative cone where $\pi_{n+k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}$ and $\pi_{n+k\bar{\rho}}\mathcal{H}\mathcal{N}$ agree.

Proposition 4.18. *The map $\Sigma^{-k\bar{\rho}}\mathcal{H}N_e^{\mathcal{K}}\mathbb{F}_2 \to \Sigma^{-k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}$ is an isomorphism on $\underline{\pi}_n$ for $n < -k$, while the map $\Sigma^{-k\bar{\rho}}\mathcal{H}\underline{B}(2,0) \oplus \underline{g}^2 \to \Sigma^{-k\bar{\rho}}\mathcal{H}N_e^{\mathcal{K}}\mathbb{F}_2$ is an isomorphism on $\underline{\pi}_n$ for $n \in \{0, -1, -2\}$.*

Proof. This follows from the cofiber sequence

$$\Sigma^{-k\bar{\rho}}\mathcal{H}\underline{B}(2,0) \oplus \underline{g}^2 \to \Sigma^{-k\bar{\rho}}\mathcal{H}\mathcal{N} \to \Sigma^{-k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2},$$

together with the fact that $\Sigma^{-k\bar{\rho}}\mathcal{H}\underline{B}(2,0) \oplus \underline{g}^2$ is $-k$-connective and $\Sigma^{-k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2} \simeq \Sigma^{-3-(k-1)\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}^*$ has no homotopy above dimension -3. □

Note that Proposition 4.18 captures all of the homotopy groups in the case of $-k$ equal to either -1 or -2. The remaining cases are described in the next result.

Theorem 4.19. *For $-k \leq -3$, the nonzero homotopy Mackey functors of $\Sigma^{-k\bar{\rho}}\mathcal{H}N_e^{\mathcal{K}}\mathbb{F}_2$ are*

$$\underline{\pi}_n\Sigma^{-k\bar{\rho}}\mathcal{H}N_e^{\mathcal{K}}\mathbb{F}_2 \cong \begin{cases} \underline{g}^3 & n = 0 \\ \underline{g}^2 & n = -1 \\ \underline{g}^3 & n = -2 \\ \underline{g}^{-2n-2} & n \in [-k+1, -3] \\ \phi^*_{LDR}\underline{\mathbb{F}_2}^* \oplus \underline{g}^{2k-5} & n = -k \\ \underline{\pi}_n\Sigma^{-k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}}_2 & n \leq -k-1. \end{cases}$$

Proof. By Proposition 4.18, it remains to capture the homotopy in degrees $n \in [-k, -3]$. The Mackey functors listed here are the only possibilities that are simultaneously compatible with the fiber sequence

$$\Sigma^{-k\bar{\rho}}\mathcal{H}\underline{B}(2,0) \oplus \underline{g}^2 \to \Sigma^{-k\bar{\rho}}\mathcal{H}\mathcal{N} \to \Sigma^{-k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2},$$

as well as the fiber sequence

$$\Sigma^{-k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}^* \to \Sigma^{-k\bar{\rho}}\mathcal{H}\mathcal{N} \to \Sigma^{-k\bar{\rho}}\mathcal{H}\underline{E}.$$

For example, in the case $k = 3$, the first fiber sequence provides the short exact sequence

$$\phi^*_{LDR}\underline{\mathbb{F}_2}^* \hookrightarrow \underline{\pi}_{-3}\Sigma^{-3\bar{\rho}}\mathcal{H}\mathcal{N} \twoheadrightarrow \underline{g},$$

while the second provides the short exact sequence

$$\underline{g}^3 \hookrightarrow \phi^*_{LDR}\underline{\mathbb{F}_2}^* \oplus \underline{g}^4 \twoheadrightarrow \underline{\pi}_{-3}\Sigma^{-3\bar{\rho}}\mathcal{H}\mathcal{N}.$$

It follows that $\underline{\pi}_{-3}\Sigma^{-3\bar{\rho}}\mathcal{H}\mathcal{N}$ must be $\phi^*_{LDR}\underline{\mathbb{F}_2}^* \oplus \underline{g}$. □

Remark 4.20. Dual to the proof of Theorem 4.13, here the map $\underline{\pi}_n\Sigma^{-k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}^* \to \underline{\pi}_n\Sigma^{-k\bar{\rho}}\mathcal{H}N_e^{\mathcal{K}}\mathbb{F}_2$ is *surjective* for $k < 0$.

The homotopy $\underline{\pi}_{n+k\bar{\rho}}\mathcal{H}\mathcal{N} = \underline{\pi}_{n+k\bar{\rho}}\mathcal{H}N_e^{\mathcal{K}}\mathbb{F}_2$ is displayed in Figure 7.6.

5. THE HOMOTOPY OF $\mathcal{H}N_D^{\mathcal{K}}\mathbb{F}_2$

We compute the homotopy Mackey functors $\underline{\pi}^{\mathcal{K}}_{n+k\bar{\rho}}(\mathcal{H}\mathcal{N}_D)$, where $\mathcal{N}_D = N_D^{\mathcal{K}}\underline{\mathbb{F}_2}$. The Tambara functor structure of $\mathcal{N}_D$ is displayed in Figure 2.3. As in the previous section, we first compute the positive cone.

5.1. The Positive Cone of $\mathcal{H}N_D^{\mathcal{K}}\underline{\mathbb{F}_2}$.

We compute the homotopy Mackey functors of the $\mathcal{K}$-spectra $\Sigma^{k\bar{\rho}}\mathcal{H}\mathcal{N}_D$. As usual we begin with a short exact sequence of Mackey functors.

Let us write $\overline{\mathcal{N}_D}$ for the sub-Mackey functor of $\mathcal{N}_D$ generated at the proper subgroups, as displayed in Figure 5.1. We then have a short exact sequence

$$\overline{\mathcal{N}_D} \hookrightarrow \mathcal{N}_D \twoheadrightarrow \underline{g}. \tag{5.1}$$

Furthermore, we have a short exact sequence of Mackey functors

$$\underline{v}_D^* \hookrightarrow \overline{\mathcal{N}_D} \twoheadrightarrow \varphi_D^*\mathbb{F}_2^* \oplus \underline{n}_D^*. \tag{5.2}$$

Proposition 5.3. *There is an equivalence $\Sigma^{\bar{\rho}}\mathcal{H}\underline{n}_D^* \simeq \Sigma^1\mathcal{H}\underline{n}_D$.*

FIGURE 5.1. The $\mathcal{K}$-Mackey functor $\overline{\mathbf{N}_D} = \overline{N_D^{\mathcal{K}}\underline{\mathbb{F}_2}}$, which is the kernel of the augmentation $N_D^{\mathcal{K}}(\underline{\mathbb{F}_2}) \to \underline{g}$.

$$\begin{tikzcd}
 & \mathbb{Z}/2 \oplus \mathbb{Z}/2\{c\} \arrow[dl, "(2\ 0)"'] \arrow[d, bend right, "0"'] \arrow[dr, "(2\ 0)"] & \\
\mathbb{Z}/4 \arrow[ur, bend right, "\binom{1}{1}"'] \arrow[dr, "1"'] & \mathbb{Z}/2 \arrow[u, bend right, "\binom{1}{0}"'] \arrow[d, bend right, "1"'] & \mathbb{Z}/4 \arrow[ul, bend right, "\binom{1}{1}"'] \arrow[dl, "1"] \\
 & \mathbb{Z}/2 \arrow[ul, bend right, "2"'] \arrow[u, bend right, "0"'] \arrow[ur, bend right, "2"'] &
\end{tikzcd}$$

Proof. First, note that since $\underline{n}^*_D$ restricts to zero on D, it follows that the natural map $\mathcal{H}\underline{n}^*_D \to \Sigma^{\sigma_D}\mathcal{H}\underline{n}^*_D$ is an equivalence by Corollary 3.3. The cofiber sequence

$$\begin{tikzcd}
\mathcal{K}/L_+ \wedge \mathcal{H}\underline{n}^*_D \arrow[r] & \mathcal{H}\underline{n}^*_D \arrow[r] & \Sigma^{\sigma_L}\mathcal{H}\underline{n}^*_D \\
\mathcal{H}\left(\uparrow_L^{\mathcal{K}}\underline{f}\right) \arrow[u, phantom, "\wr"] & &
\end{tikzcd}$$

allows us compute the homotopy Mackey functors of $\Sigma^{\sigma_L}\mathcal{H}\underline{n}^*_D \simeq \Sigma^{\sigma_L+\sigma_D}\mathcal{H}\underline{n}^*_D$; the nonzero homotopy Mackey functors are $\phi^*_L\underline{f}$ in degree one and $\phi^*_R\underline{f}$ in degree zero. This gives a Postnikov fiber sequence

$$\Sigma^1\mathcal{H}\phi^*_L\underline{f} \to \Sigma^{\sigma_L+\sigma_D}\mathcal{H}\underline{n}^*_D \to \mathcal{H}\phi^*_R\underline{f}.$$

Suspending by σ_R then gives a fiber sequence

$$\begin{tikzcd}
\Sigma^1\mathcal{H}\phi^*_L\underline{f} \arrow[r] & \Sigma^{\overline{\rho}}\mathcal{H}\underline{n}^*_D \arrow[r] & \Sigma^{\sigma_R}\mathcal{H}\phi^*_R\underline{f} \\
 & & \Sigma^1\mathcal{H}\phi^*_R\underline{\mathbb{F}_2}, \arrow[u, phantom, "\wr"]
\end{tikzcd}$$

where we have again used the fact that if the restriction of X to R is contractible, then $\Sigma^{\sigma_R}X \simeq X$ (Corollary 3.3). It follows that $\Sigma^{\overline{\rho}}\mathcal{H}\underline{n}^*_D$ has homotopy concentrated in degree 1, with Mackey functor an extension of $\phi^*_R\underline{\mathbb{F}_2}$ by $\phi^*_L\underline{f}$. The only potential ambiguity in the extension is the restriction from K to L. But as the result must be symmetric in L and R, we conclude the restriction to L must be nontrivial since the restriction to R is so. Hence, this extension must be $\underline{n}^*_D$. □

Proposition 5.4. *For $k \geq 1$, the nonzero homotopy Mackey functors of $\Sigma^{k\overline{\rho}}\mathcal{H}\underline{n}_D$ are*

$$\underline{\pi}_n\Sigma^{k\overline{\rho}}\mathcal{H}\underline{n}_D \cong \begin{cases} \phi^*_{LR}\underline{\mathbb{F}_2} & n = k \\ \underline{g}^2 & n \in [1, k-1] \\ \underline{g} & n = 0. \end{cases}$$

Proof. We have a short exact sequence of Mackey functors $\phi_{LR}^*\underline{f} \hookrightarrow \underline{n}_D \twoheadrightarrow \underline{g}$. Suspending by $\overline{\rho}$ and use of Proposition 3.6 give a cofiber sequence

$$\Sigma^1\mathcal{H}\phi_{LR}^*\underline{\mathbb{F}_2} \to \Sigma^{\overline{\rho}}\mathcal{H}\underline{n} \to \mathcal{H}\underline{g}.$$

The homotopy of the higher $\overline{\rho}$-suspensions follow by use of (3.5) from the relevant C_2-equivariant computations, as in Figure 7.1. ☐

Proposition 5.5. *The nonzero homotopy Mackey functors of* $\Sigma^{\overline{\rho}}\mathcal{H}\underline{v}_D^*$ *are*

$$\underline{\pi}_n\Sigma^{\overline{\rho}}\mathcal{H}\underline{v}_D^* \cong \begin{cases} \underline{\mathbb{F}_2} & n=3 \\ \phi_D^*\underline{f} & n=2. \end{cases}$$

For $k \geq 2$*, the nonzero homotopy Mackey functors of* $\Sigma^{k\overline{\rho}}\mathcal{H}\underline{v}_D^*$ *are*

$$\underline{\pi}_n\Sigma^{k\overline{\rho}}\mathcal{H}\underline{v}_D^* \cong \begin{cases} \underline{\pi}_n\Sigma^{k\overline{\rho}}\mathcal{H}\underline{\mathbb{F}}_2 & n \geq k+2 \\ \phi_D^*\underline{\mathbb{F}_2} \oplus \underline{g}^{2k-3} & n = k+1 \\ \underline{g}^{2n-4} & n \in [3,k]. \end{cases}$$

Proof. We have a short exact sequence $\phi_D^*\underline{\mathbb{F}_2}^* \hookrightarrow \underline{\mathbb{F}_2}^* \twoheadrightarrow \underline{v}_D^*$, which gives rise to a cofiber sequence

$$\begin{tikzcd}
\Sigma^{\overline{\rho}}\mathcal{H}\phi_D^*\underline{\mathbb{F}_2}^* \arrow[r] \arrow[d, phantom, "\wr"] & \Sigma^{\overline{\rho}}\mathcal{H}\underline{\mathbb{F}_2}^* \arrow[r] \arrow[d, phantom, "\wr"] & \Sigma^{\overline{\rho}}\mathcal{H}\underline{v}_D^*. \\
\Sigma^1\mathcal{H}\phi_D^*\underline{f} & \Sigma^3\mathcal{H}\underline{\mathbb{F}_2} &
\end{tikzcd}$$

We may rotate this and suspend further to get a cofiber sequence

$$\Sigma^{3+(k-1)\overline{\rho}}\mathcal{H}\underline{\mathbb{F}_2} \to \Sigma^{k\overline{\rho}}\mathcal{H}\underline{v}_D^* \to \Sigma^{2+(k-2)\overline{\rho}}\mathcal{H}\phi_D^*\underline{\mathbb{F}_2}.$$

The homotopy Mackey functors of suspensions of $\mathcal{H}\underline{\mathbb{F}_2}$ can be read off from Figure 7.4, and the homotopy Mackey functors of $\Sigma^{2+(k-2)\overline{\rho}}\mathcal{H}\phi_D^*\underline{\mathbb{F}_2}$ follow from (3.5) and Figure 7.1.

On the other hand, we also have a short exact sequence $\phi_{LR}^*\underline{f} \hookrightarrow \underline{v}_D^* \twoheadrightarrow \underline{f}$ which gives a cofiber sequence

$$\begin{tikzcd}
\Sigma^{k\overline{\rho}}\mathcal{H}\phi_{LR}^*\underline{f} \arrow[r] \arrow[d, phantom, "\wr"] & \Sigma^{k\overline{\rho}}\mathcal{H}\underline{v}_D^* \arrow[r] & \Sigma^{k\overline{\rho}}\mathcal{H}\underline{f}. \\
\Sigma^{1+(k-1)\overline{\rho}}\mathcal{H}\phi_{LR}^*\underline{\mathbb{F}_2} & &
\end{tikzcd}$$

The homotopy Mackey functors of the right term of this last cofiber sequence are computed in [GY, Corollary 7.5], and the homotopy Mackey functors of $\Sigma^{1+(k-1)\overline{\rho}}\mathcal{H}\phi_{LR}^*\underline{\mathbb{F}_2}$ follow from (3.5) and Figure 7.1. The stated homotopy Mackey functors of $\Sigma^{k\overline{\rho}}\mathcal{H}\underline{v}_D^*$ are the only ones compatible with both long exact sequences and Lemma 2.16. ☐

Proposition 5.6. *The nonzero homotopy Mackey functors of* $\Sigma^{\overline{\rho}}\mathcal{H}N_D^{\mathcal{K}}\underline{\mathbb{F}_2}$ *are*

$$\underline{\pi}_n\Sigma^{\overline{\rho}}\mathcal{H}N_D^{\mathcal{K}}\underline{\mathbb{F}_2} \cong \begin{cases} \underline{\mathbb{F}_2} & n=3 \\ \phi_D^*\underline{f} & n=2 \\ \phi_D^*\underline{f} \oplus \underline{n}_D & n=1 \\ \underline{g} & n=0. \end{cases}$$

Proof. Suspending (5.1) gives a cofiber sequence

$$\begin{tikzcd}
\Sigma^{\overline{\rho}}\mathcal{H}\overline{\mathbf{N}_D} \arrow[r] & \Sigma^{\overline{\rho}}\mathcal{H}\mathbf{N}_D \arrow[r] & \Sigma^{\overline{\rho}}\mathcal{H}\underline{g} \arrow[d, phantom, "\wr"] \\
& & \mathcal{H}\underline{g}.
\end{tikzcd}$$

It therefore suffices to compute the homotopy Mackey functors of $\Sigma^{\overline{\rho}}\mathcal{H}\overline{\mathbf{N}_D}$, for which we use the following suspension of (5.2):

$$\Sigma^{\overline{\rho}}\mathcal{H}\underline{v}_D^* \to \Sigma^{\overline{\rho}}\mathcal{H}\overline{\mathbf{N}_D} \to \Sigma^{\overline{\rho}}\mathcal{H}\big(\phi_D^*\underline{\mathbb{F}_2}^* \oplus \underline{n}_D^*\big).$$

The result now follows by combining Propositions 3.6, 5.3 and 5.5. ☐

The homotopy of the higher suspensions is as follows.

Theorem 5.7. *For $k \geq 2$, the nonzero homotopy Mackey functors of $\Sigma^{k\overline{\rho}}\mathcal{H}N_D^{\mathcal{K}}\underline{\mathbb{F}_2}$ are*

$$\underline{\pi}_n\Sigma^{k\overline{\rho}}\mathcal{H}N_D^{\mathcal{K}}\underline{\mathbb{F}_2} \cong \begin{cases} \underline{\pi}_n\Sigma^{k\overline{\rho}}\mathcal{H}\underline{\mathbb{F}_2} & n \geq k+2 \\ \phi_D^*\underline{\mathbb{F}_2} \oplus \underline{g}^{2k-3} & n = k+1 \\ \phi_{LDR}^*\underline{\mathbb{F}_2} \oplus \underline{g}^{2k-4} & n = k \\ \underline{g}^{2n-1} & n \in [2, k-1] \\ \underline{g} & n \in [0,1]. \end{cases}$$

Proof. The argument is the same as for Proposition 5.6, using the cofiber sequence

$$\Sigma^{k\overline{\rho}}\mathcal{H}\underline{v}_D^* \to \Sigma^{k\overline{\rho}}\mathcal{H}\overline{\mathbf{N}_D} \to \Sigma^{k\overline{\rho}}\mathcal{H}\big(\phi_D^*\underline{\mathbb{F}_2}^* \oplus \underline{n}_D^*\big).$$

for $k \geq 2$. The left term has homotopy in degrees at least 3, while the right term has homotopy in degrees at most k. Therefore the homotopy Mackey functors in degrees greater than k follow from Proposition 5.5. And the homotopy below degree 3 follows from Proposition 5.4.

For the Mackey functors in degrees k and lower, we employ a similar argument to that used in the proof of Theorem 4.13. Consider, for instance, the case $k = 4$, where the Mackey functors $\underline{\pi}_3$ and $\underline{\pi}_4$ remain to be determined. In the relevant degrees, the long exact sequence arising from the above cofiber sequence takes the form

$$\begin{tikzcd}
 & & \underline{\pi}_n\Sigma^{4\overline{\rho}}\mathcal{H}\underline{v}_D^* & \underline{\pi}_n\Sigma^{4\overline{\rho}}\mathcal{H}\overline{\mathbf{N}_D} & \underline{\pi}_n\Sigma^{4\overline{\rho}}\mathcal{H}(\phi_D^*\underline{\mathbb{F}_2}^* \oplus \underline{n}_D^*) \\
n = 5 & {} \arrow[r, dashed] & \phi_D^*\underline{\mathbb{F}_2} \oplus \underline{g}^5 \arrow[r, "\cong"] & \phi_D^*\underline{\mathbb{F}_2} \oplus \underline{g}^5 \arrow[r, "0"] & 0 \arrow[dll, rounded corners] \\
n = 4 & & \underline{g}^4 \arrow[r, hook] & ? \arrow[r] & \phi_D^*\underline{\mathbb{F}_2} \oplus \phi_{LR}^*\underline{\mathbb{F}_2} \arrow[dll, rounded corners] \\
n = 3 & & \underline{g}^2 \arrow[r] & ?? \arrow[r, two heads] & \underline{g} \oplus \underline{g}^2 \arrow[dll, rounded corners] \\
n = 2 & & 0 \arrow[r, "0"] & \underline{g}^3 \arrow[r, "\cong"] & \underline{g} \oplus \underline{g}^2 \arrow[r, dashed] & {}
\end{tikzcd}$$

On the other hand, the long exact sequence induced by (the rotation of) the cofiber sequence for $\underline{g}\oplus\underline{n}_D \hookrightarrow \boldsymbol{\mathcal{N}}_D \twoheadrightarrow \underline{\mathbb{F}_2}$ takes the form

$$\begin{tikzcd}
 & & \underline{\pi}_n\Sigma^{4\overline{\rho}}\mathcal{H}\boldsymbol{\mathcal{N}}_D & \underline{\pi}_n\Sigma^{4\overline{\rho}}\mathcal{H}\underline{\mathbb{F}_2} & \underline{\pi}_n\Sigma^{1+4\overline{\rho}}\mathcal{H}(\underline{g}\oplus\underline{n}_D) & \\
n=5 & {} \arrow[r, dashed] & \phi_D^*\underline{\mathbb{F}_2}\oplus\underline{g}^5 \arrow[r, hook] & \phi_{LDR}^*\underline{\mathbb{F}_2}\oplus\underline{g}^5 \arrow[r, two heads] & \phi_{LR}^*\underline{\mathbb{F}_2} \arrow[dll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
n=4 & & ? \arrow[r, hook] & \phi_{LDR}^*\underline{\mathbb{F}_2}\oplus\underline{g}^6 \arrow[r] & \underline{g}^2 \arrow[dll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
n=3 & & ?? \arrow[r] & \underline{g}^7 \arrow[r, two heads] & \underline{g}^2 \arrow[dll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
n=2 & & \underline{g}^3 \arrow[r, hook] & \underline{g}^5 \arrow[r, two heads] & \underline{g}^2 \arrow[r, dashed] & {}
\end{tikzcd}$$

The claimed values for $\underline{\pi}_4$ and $\underline{\pi}_3$ are the only possibilities compatible with both long exact sequences. Note that although one sequence computes the homotopy of $\boldsymbol{\mathcal{N}}_D$ and the other computes the homotopy of $\overline{\boldsymbol{\mathcal{N}}_D}$, these agree in positive degree as in the proof of Proposition 5.6. The argument works just as well for larger values of k. ☐

We display these results in the fourth quadrant of Figure 7.8. The region of the fourth quadrant of Figure 7.8 where the homotopy agrees with $\mathcal{H}\underline{\mathbb{F}_2}$ is highlighted in blue.

5.2. The Negative Cone of $\mathcal{H}N_D^{\mathcal{K}}\underline{\mathbb{F}_2}$.

We now compute the negative cone with coefficients in $\boldsymbol{\mathcal{N}}_D = N_D^{\mathcal{K}}\underline{\mathbb{F}_2}$, which we display in the second quadrant of Figure 7.8.

Proposition 5.8. *The nonzero homotopy Mackey functors of $\Sigma^{-\overline{\rho}}\mathcal{H}N_D^{\mathcal{K}}\underline{\mathbb{F}_2}$ are*

$$\underline{\pi}_{-n}\Sigma^{-\overline{\rho}}\mathcal{H}N_D^{\mathcal{K}}\underline{\mathbb{F}_2} \cong \begin{cases} \underline{\mathbb{F}_2}^* & n=3 \\ \underline{n}_D^* & n=1 \\ \underline{g} & n=0. \end{cases}$$

Proof. The short exact sequence of Mackey functors $\underline{g}\oplus\underline{n}_D \hookrightarrow \boldsymbol{\mathcal{N}}_D \twoheadrightarrow \underline{\mathbb{F}_2}$ gives a cofiber sequence

$$\begin{tikzcd}
\Sigma^{-\overline{\rho}}\mathcal{H}(\underline{g}\oplus\underline{n}_D) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \Sigma^{-\overline{\rho}}\mathcal{H}\boldsymbol{\mathcal{N}}_D \arrow[r] & \Sigma^{-\overline{\rho}}\mathcal{H}\underline{\mathbb{F}_2} \arrow[d, phantom, sloped, "\cong"] \\
\mathcal{H}\underline{g}\vee\Sigma^{-1}\mathcal{H}\underline{n}_D^* & & \Sigma^{-3}\mathcal{H}\underline{\mathbb{F}_2}^*
\end{tikzcd}$$

according to Proposition 5.3. The result follows from the associated long exact sequence. ☐

Theorem 5.9. *For $k\geq 2$, the nonzero homotopy Mackey functors of $\Sigma^{-k\overline{\rho}}\mathcal{H}N_D^{\mathcal{K}}\underline{\mathbb{F}_2}$ are*

$$\underline{\pi}_{-n}\Sigma^{-k\overline{\rho}}\mathcal{H}N_D^{\mathcal{K}}\underline{\mathbb{F}_2} \cong \begin{cases} \underline{\pi}_{-n}\Sigma^{-k\overline{\rho}}\mathcal{H}\underline{\mathbb{F}_2} & n\geq k+1 \\ \phi_{LR}^*\underline{\mathbb{F}_2}^*\oplus\underline{g}^{2k-5} & n=k \\ \underline{g}^{2n-3} & n\in[3,k-1] \\ \underline{g}^2 & n=2 \\ \underline{g} & n=0,1. \end{cases}$$

Proof. As in the proof of Proposition 5.8, we have a cofiber sequence

$$\begin{tikzcd}
\Sigma^{-k\bar{\rho}}\mathcal{H}\big(\underline{g}\oplus\underline{n}_D\big) \arrow[r] \arrow[d, phantom, "\wr"] & \Sigma^{-k\bar{\rho}}\mathcal{H}\mathcal{N}_D \arrow[r] & \Sigma^{-k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2} \arrow[d, phantom, "\wr"] \\
\mathcal{H}\underline{g}\vee\Sigma^{-1-(k-1)\bar{\rho}}\mathcal{H}\underline{n}_D^* & & \Sigma^{-3-(k-1)\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}^*
\end{tikzcd}$$

The homotopy Mackey functors of $\Sigma^{-k\bar{\rho}}\mathcal{H}\underline{n}_D^*$ are the duals of those given in Proposition 5.4 by Brown-Comenetz duality. Thus $\Sigma^{-k\bar{\rho}}\mathcal{H}(\underline{g}\oplus\underline{n}_D)$ is $(-k)$-connective, so that the homotopy Mackey functors of $\Sigma^{-k\bar{\rho}}\mathcal{H}\mathcal{N}_D$ agree with those of $\Sigma^{-k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}$ below degree $-k$. Similarly, $\Sigma^{-k\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}$ is (-3)-coconnective, so that the homotopy Mackey functors of $\Sigma^{-k\bar{\rho}}\mathcal{H}\mathcal{N}_D$ follow in degree -2 or higher. The intermediate homotopy Mackey functors are the only ones compatible with Lemma 2.16 and the long exact sequences arising from the above cofiber sequence as well as the cofiber sequence $\Sigma^{-k\bar{\rho}}\mathcal{H}\big(\underline{g}\oplus\phi_D^*\underline{\mathbb{F}_2}^*\big)\to\Sigma^{-k\bar{\rho}}\mathcal{H}\mathcal{N}\to\Sigma^{-k\bar{\rho}}\mathcal{H}\mathcal{N}_D$. ☐

We display these results in the second quadrant of Figure 7.8. The region of the second quadrant of Figure 7.8 where the homotopy agrees with $\mathcal{H}\underline{\mathbb{F}_2}$ is highlighted in red.

6. Multiplicative Structure

We briefly describe some of the multiplicative structure in the bigraded Green functors $\underline{\pi}_{\blacklozenge}\mathcal{H}\mathcal{N}$ and $\underline{\pi}_{\blacklozenge}\mathcal{H}\mathcal{N}_D$.

The $RO(\mathcal{K})$-graded Green ring $\pi_\bigstar^{\mathcal{K}}\mathcal{H}\underline{\mathbb{F}_2}$ was described in [EB] and [HS]. The portion graded by honest, as opposed to virtual, representations [1] is described in [EB, Theorem 4.14] as

$$\pi^{\mathcal{K}}_{\bigstar\geq 0}\mathcal{H}\underline{\mathbb{F}_2}\cong\frac{\mathbb{F}_2[a_L,a_D,a_R,t_L,t_D,t_R]}{a_Lt_Dt_R+t_La_Dt_R+t_Lt_Da_R}$$

where a_H is the Euler class for σ_H, in degree $-\sigma_H$, and t_H is the orientation class, in degree $1-\sigma_H$. We have chosen to focus on the $\mathrm{Aut}(\mathcal{K})$-invariant subring $\pi^{\mathcal{K}}_{\blacklozenge\geq 0}\mathcal{H}\underline{\mathbb{F}_2}$ and similarly for our other Eilenberg–Mac Lane spectra. It follows that this bigraded ring can be described as

$$\pi^{\mathcal{K}}_{\blacklozenge\geq 0}\mathcal{H}\underline{\mathbb{F}_2}\cong\mathbb{F}_2[a,\mathbf{x}_L,\mathbf{y}_D,\mathbf{z}_R,\mathbf{v}_L,\mathbf{w}_R,\mathbf{u}]/I,$$

where the generators are

$$\begin{aligned}\mathbf{x}_L&=t_La_Da_R, & \mathbf{y}_D&=a_Lt_Da_R, & \mathbf{z}_R&=a_La_Dt_R,\\ \mathbf{v}_L&=a_Lt_Dt_R, & \mathbf{w}_R&=t_Lt_Da_R, & \mathbf{u}&=t_Lt_Dt_R,\end{aligned}$$

and I is the ideal generated by

$$\begin{aligned}&\mathbf{x}_L\mathbf{y}_D+a\mathbf{w}_R, && \mathbf{y}_D\mathbf{z}_R+a\mathbf{v}_L, && \mathbf{x}_L\mathbf{z}_R+a\mathbf{v}_L+a\mathbf{w}_R,\\ &\mathbf{x}_L\mathbf{v}_L+a\mathbf{u}, && \mathbf{w}_R\mathbf{z}_R+a\mathbf{u}, && \mathbf{v}_L\mathbf{y}_D+\mathbf{y}_D\mathbf{w}_R+a\mathbf{u},\\ &{\mathbf{v}_L}^2+\mathbf{y}_D\mathbf{u}+\mathbf{z}_R\mathbf{u}, && \mathbf{v}_L\mathbf{w}_R+\mathbf{y}_D\mathbf{u}, && {\mathbf{w}_R}^2+\mathbf{x}_L\mathbf{u}+\mathbf{y}_D\mathbf{u}.\end{aligned}$$

The negative cone is more poorly behaved. However, it contains a class Θ in degree $-3+\bar{\rho}$ that is infinitely divisible by each of the mutliplicative generators of the positive cone.

In Figure 7.5, vertical (purple) lines indicate multiplications by $a=a_La_Da_R$. We use rainbows to indicate multiplication by $\mathbf{x}_L$, $\mathbf{y}_D$, and $\mathbf{z}_R$, though we omit the subscripts from the generator names (as indicated in the Key), in order to avoid clutter in the figures. Thus the rainbow connecting the pentagon in (1,-1) to the pentagon in (2,-2) and the vertical line from (2,-1) to (2,-2) indicate that basis elements in (2,-2) are ${\mathbf{x}_L}^2$, ${\mathbf{y}_D}^2$, ${\mathbf{z}_R}^2$, $a\mathbf{v}_L$, and $a\mathbf{w}_R$. On the other hand, the rainbow from (2,-1) to (3,-2) indicates that a basis in (3,-2) is given by $\mathbf{x}_L\mathbf{w}_R$,

---

[1] This portion is called the "positive cone" in [EB], which is slightly different from our usage of the term in this article.

$\mathbf{v}_L\mathbf{z}_R$, $\mathbf{y}_D\mathbf{v}_L$, and $a\mathbf{u}$. Note that $\mathbf{y}_D\mathbf{w}_R$ is equal to $\mathbf{y}_D\mathbf{v}_L + a\mathbf{u}$, and that $\mathbf{x}_L\mathbf{v}_L$ and $\mathbf{z}_R\mathbf{w}_R$ are both equal to $a\mathbf{u}$, though this is not indicated in the figure.

Thus, our use of multiplication lines is to indicate choices of basis elements, rather than to display all possible multiplications. For example, the element a in (0,-1) also supports a rainbow, thought we have not included it in the figure.

As indicated in Figure 7.7, the multiplicative structure of $\underline{\pi}_{\blacklozenge}^{\mathcal{K}}\mathcal{HN}$ is largely the same as that of $\underline{\pi}_{\blacklozenge}^{\mathcal{K}}\mathcal{H}\underline{\mathbb{F}_2}$. Important differences include:

1. the absence of $\mathbf{v}_L$ and $\mathbf{w}_R$ and their corresponding rainbow
2. the absence of $\mathbf{x}_L$, $\mathbf{y}_D$, and $\mathbf{z}_R$, though their restrictions appear
3. the elements 4, b_L, and b_R in $\pi_0^{\mathcal{K}}$ are infinitely a-divisible
4. the rainbows indicate multiplication by the generators of $\pi_{2-2\overline{\rho}}\mathcal{HN}$. These generators correspond to the elements ${\mathbf{x}_L}^2$, ${\mathbf{y}_D}^2$, and ${\mathbf{z}_R}^2$ in $\pi_{2-2\overline{\rho}}\mathcal{H}\underline{\mathbb{F}_2}$, up to a-multiples.

The multiplicative structure of $\underline{\pi}_{\blacklozenge}^{\mathcal{K}}\mathcal{HN}_D$ indicated in Figure 7.9 is intermediate between that of $\underline{\pi}_{\blacklozenge}^{\mathcal{K}}\mathcal{HN}$ and $\underline{\pi}_{\blacklozenge}^{\mathcal{K}}\mathcal{H}$. For instance, the element $\mathbf{x}_L + \mathbf{z}_R$ is present in $\pi_{1-\overline{\rho}}\mathcal{HN}_D$ but not in $\pi_{1-\overline{\rho}}\mathcal{HN}$. Here, the yellow arcs in the diagonal $x + y = 1$ denote multiplication by the corresponding generator in $\pi_{2-2\overline{\rho}}\mathcal{HN}_D$; as discussed above, this is ${\mathbf{y}_D}^2$, modulo a. However, we again warn the reader that many multiplications are not indicated in Figure 7.9. For example, in $\pi_{\blacklozenge}^{\mathcal{K}}\mathcal{H}\underline{\mathbb{F}_2}$, we have the relation ${\mathbf{x}_L}^2 \cdot \mathbf{y}_D\mathbf{v}_L = a^2\mathbf{w}_R\mathbf{u}$, which gives rise to an analogous formula in $\pi_{\blacklozenge}^{\mathcal{K}}\mathcal{HN}_D$.

Remark 6.1. We have here discussed the multiplicative structure on $\underline{\pi}_{\blacklozenge}\mathcal{HN}$ and $\underline{\pi}_{\blacklozenge}\mathcal{HN}_D$. In other words, we have described the graded Green functors. These have more structure: they are graded Tambara functors. However, here there is not much additional data carried in the norms. In the case of $\underline{\pi}_{\blacklozenge}\mathcal{HN}$, we have a norm

$$\mathrm{nm}_e^{\mathcal{K}} \colon \pi_n H\mathbb{F}_2 \to \pi_{n+n\overline{\rho}}\mathcal{HN}.$$

This can only be nonzero when n is equal to zero, in which case this norm has been described in Section 2.3. Similarly, for the intermediate norm

$$\mathrm{nm}_L^{\mathcal{K}} \colon \pi_{n+k\sigma}\mathbf{H}\underline{\mathbb{F}_2} \to \pi_{n+n\sigma_L+k\sigma_D+k\sigma_R}\mathcal{HN}$$

to land in the $\blacklozenge$-grading, one must have $n = k$. Then the source group is only nonzero when n is equal to zero, in which case this norm has been described in Section 2.3.

7. Tables and Charts

Here, we display charts of homotopy Mackey functors for C_2-equivariant and $\mathcal{K}$-equivariant Eilenberg–Mac Lane spectra, including our main computations from Sections 4 and 5.

We first display C_2-equivariant charts. Table 7.1 is useful for reading Figure 7.1, Figure 7.2, and Figure 7.3. Figure 7.3 was obtained from the work of [MSZ]. In more detail, the homotopy Mackey functors $\underline{\pi}_n N_e^{C_2} H\mathbb{F}_2$ are described in [MSZ, Theorem 4.4] for $n \leq 8$. The data presented in Figure 7.3 was then deduced from the computation of [MSZ] by the facts that $N_e^{C_2} H\mathbb{F}_2$ is connective and that its geometric fixed points are $\Phi^{C_2} N_e^{C_2} H\mathbb{F}_2 \simeq H\mathbb{F}_2$. The main mechanism used in this process is the long exact sequence

$$\cdots \to {\uparrow}_e^{C_2} \pi_{\dim V}\left(H\mathbb{F}_2 \wedge H\mathbb{F}_2\right) \to \underline{\pi}_V N_e^{C_2} H\mathbb{F}_2 \xrightarrow{a} \underline{\pi}_{V-\sigma} N_e^{C_2} H\mathbb{F}_2 \to \cdots$$

The shaded region of Figure 7.3 has not been computed.

Table 7.2 is useful for reading the $\mathcal{K}$-equivariant charts Figures 7.4 to 7.9. As discussed in Sections 4 and 5, the shaded regions in Figures 7.6 and 7.8 indicate where those charts agree with the previously known Figure 7.4. The charts Figures 7.5, 7.7 and 7.9 indicate multiplicative structure. This is described in Section 6.

Figure 7.1. $\underline{\pi}^{C_2}_{x+y\bar{\rho}}\mathbf{H}\underline{\mathbb{F}_2}$

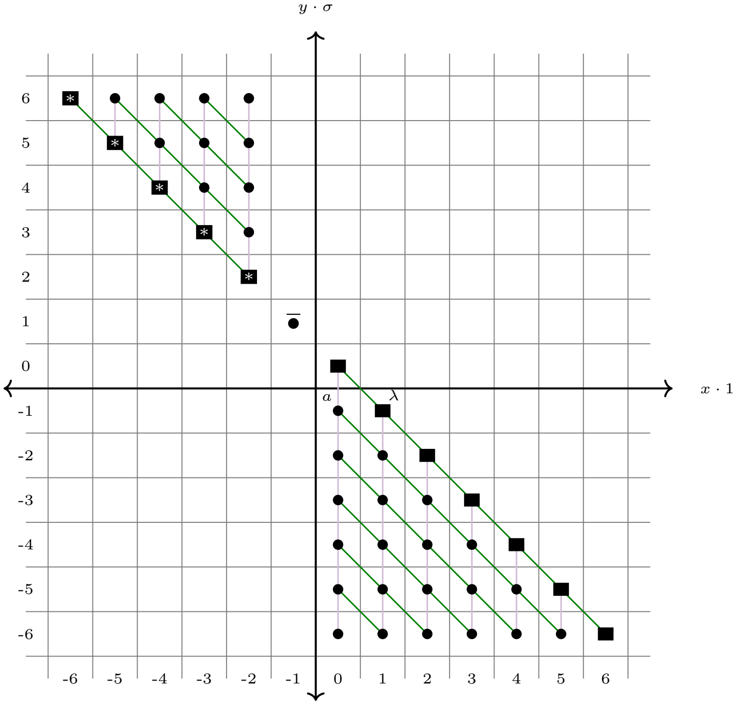


Figure 7.2. $\underline{\pi}^{C_2}_{x+y\bar{\rho}}\mathbf{H}N_e^{C_2}\mathbb{F}_2$

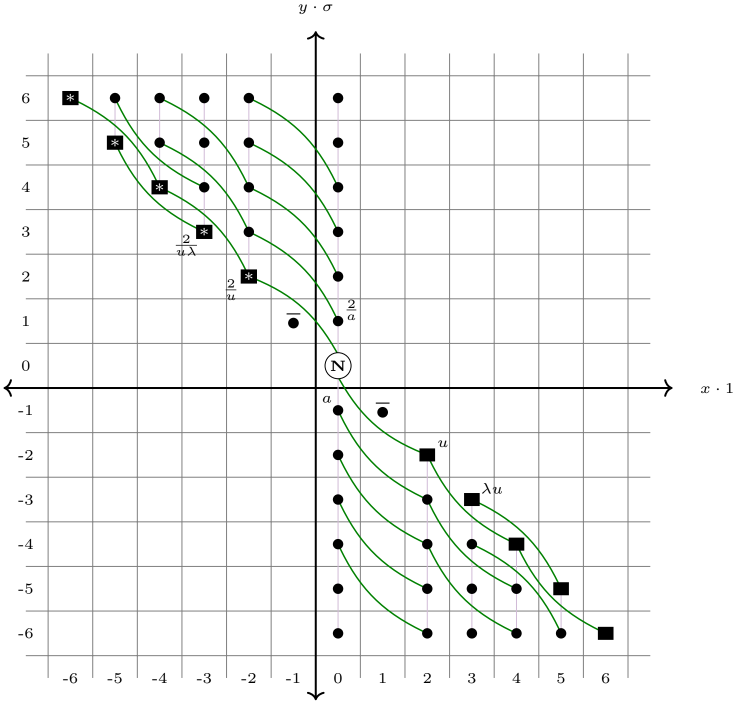


Table 7.1. Some C_2-Mackey functors

● $= \underline{g} = \phi^*_{C_2}\mathbb{F}_2$	$\bar{\bullet} = \underline{f}$	■ $= \underline{\mathbb{F}_2}$	⧆ $= \underline{\mathbb{F}_2}^*$	Ⓝ $= N_e^{C_2}\mathbb{F}_2$	$\hat{\bullet} = \uparrow_e^{C_2}\mathbb{F}_2$
$\mathbb{F}_2$ 0	0 $\mathbb{F}_2$	$\begin{tikzcd} \mathbb{F}_2 \arrow[d, "1"'] \\ \mathbb{F}_2 \end{tikzcd}$	$\begin{tikzcd} \mathbb{F}_2 \\ \mathbb{F}_2 \arrow[u, "1"'] \end{tikzcd}$	$\begin{tikzcd} \mathbb{Z}/4 \arrow[d, bend right, "1"'] \\ \mathbb{Z}/2 \arrow[u, bend right, "2"'] \end{tikzcd}$	$\begin{tikzcd} \mathbb{F}_2 \arrow[d, bend right, "\Delta"'] \\ \mathbb{F}_2\{C_2/e\} \arrow[u, bend right, "\nabla"'] \end{tikzcd}$

Figure 7.3. $\underline{\pi}^{C_2}_{x+y\bar{\rho}} N_e^{C_2} H\mathbb{F}_2$

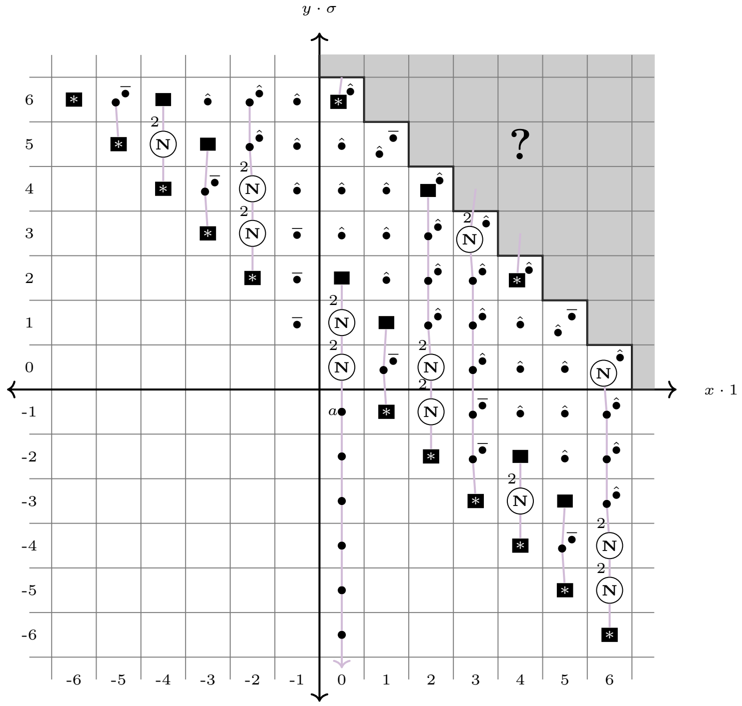


Table 7.2. Some K_4-Mackey functors

■ $= \underline{\mathbb{F}_2}$	⊛ $= \underline{\mathbb{F}_2}^*$	○ $= \underline{B}(2,0)$
$$\begin{tikzcd} & \mathbb{F}_2 \arrow[dl, "1"'] \arrow[d, "1"'] \arrow[dr, "1"] & \\ \mathbb{F}_2 \arrow[dr, "1"'] & \mathbb{F}_2 \arrow[d, "1"'] & \mathbb{F}_2 \arrow[dl, "1"] \\ & \mathbb{F}_2 & \end{tikzcd}$$	$$\begin{tikzcd} & \mathbb{F}_2 & \\ \mathbb{F}_2 \arrow[ur, "1"] & \mathbb{F}_2 \arrow[u, "1"'] & \mathbb{F}_2 \arrow[ul, "1"'] \\ & \mathbb{F}_2 \arrow[ul, "1"] \arrow[u, "1"'] \arrow[ur, "1"'] & \end{tikzcd}$$	$$\begin{tikzcd} & \mathbb{Z}/4 \arrow[dl, bend right, "1"'] \arrow[d, bend right, "1"'] \arrow[dr, bend left, "1"'] & \\ \mathbb{Z}/2 \arrow[ur, bend right, "2"'] & \mathbb{Z}/2 \arrow[u, bend right, "2"'] & \mathbb{Z}/2 \arrow[ul, bend right, "2"'] \\ & 0 & \end{tikzcd}$$
⬟ $= \phi^*_{LDR}(\underline{\mathbb{F}_2})$	⬟ $= \phi^*_{LDR}(\underline{\mathbb{F}_2}^*)$	▬ $= \phi^*_{LDR}(\underline{f})$
$$\begin{tikzcd} & \mathbb{F}_2^3 \arrow[dl, "p_1"'] \arrow[d, "p_2"'] \arrow[dr, "p_3"] & \\ \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 \\ & 0 & \end{tikzcd}$$	$$\begin{tikzcd} & \mathbb{F}_2^3 & \\ \mathbb{F}_2 \arrow[ur, "\iota_1"] & \mathbb{F}_2 \arrow[u, "\iota_2"'] & \mathbb{F}_2 \arrow[ul, "\iota_3"'] \\ & 0 & \end{tikzcd}$$	$$\begin{tikzcd} & 0 & \\ \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 \\ & 0 & \end{tikzcd}$$
⏢ $= \underline{mg}$	⏢ $= \underline{mg}^*$	• $= \underline{g}$
$$\begin{tikzcd} & \mathbb{F}_2^2 \arrow[dl, "p_1"'] \arrow[d, "\nabla"'] \arrow[dr, "p_2"] & \\ \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 \\ & 0 & \end{tikzcd}$$	$$\begin{tikzcd} & \mathbb{F}_2^2 & \\ \mathbb{F}_2 \arrow[ur, "\iota_1"] & \mathbb{F}_2 \arrow[u, "\Delta"'] & \mathbb{F}_2 \arrow[ul, "\iota_2"'] \\ & 0 & \end{tikzcd}$$	$$\begin{tikzcd} & \mathbb{F}_2 & \\ 0 & 0 & 0 \\ & 0 & \end{tikzcd}$$
◠ $= \underline{n}_D$	◠ $= \underline{n}_D^*$	$\underline{v}_D^*$
$$\begin{tikzcd} & \mathbb{F}_2 \arrow[dl, "1"'] \arrow[dr, "1"] & \\ \mathbb{F}_2 & 0 & \mathbb{F}_2 \\ & 0 & \end{tikzcd}$$	$$\begin{tikzcd} & \mathbb{F}_2 & \\ \mathbb{F}_2 \arrow[ur, "1"] & 0 & \mathbb{F}_2 \arrow[ul, "1"'] \\ & 0 & \end{tikzcd}$$	$$\begin{tikzcd} & 0 & \\ \mathbb{F}_2 & 0 & \mathbb{F}_2 \\ & \mathbb{F}_2 \arrow[ul, "1"] \arrow[ur, "1"'] & \end{tikzcd}$$

FIGURE 7.4. $\underline{\pi}^{\mathcal{K}}_{x+y\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}$

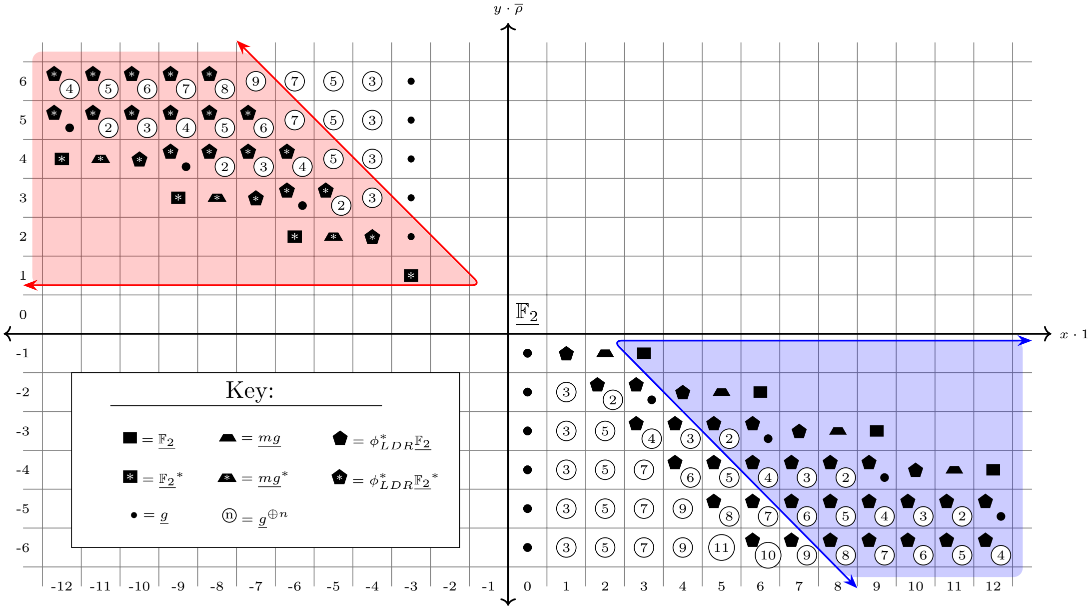


FIGURE 7.5. $\underline{\pi}^{\mathcal{K}}_{x+y\bar{\rho}}\mathcal{H}\underline{\mathbb{F}_2}$, with multiplicative structure emphasized

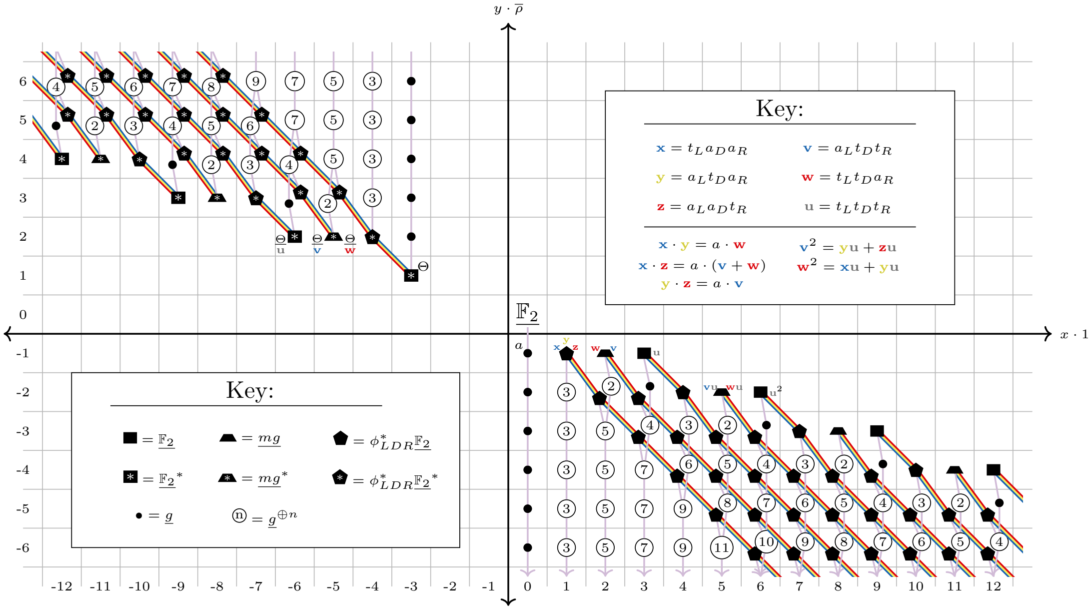


FIGURE 7.6. $\underline{\pi}^{\mathcal{K}}_{x+y\bar{\rho}}\mathcal{H}N_e^{\mathcal{K}}\mathbb{F}_2$

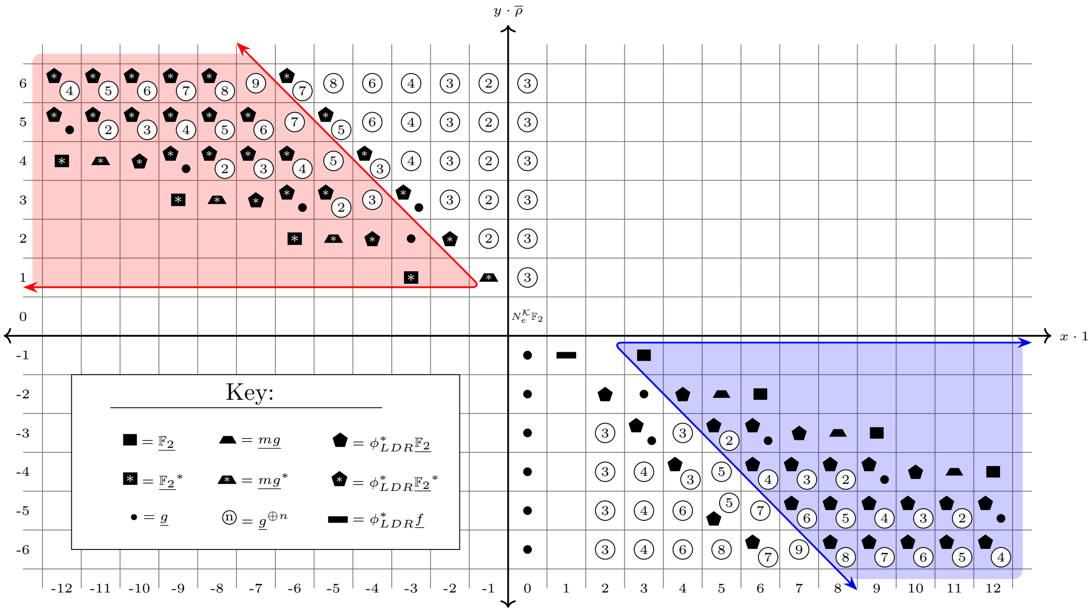


FIGURE 7.7. $\underline{\pi}^{\mathcal{K}}_{x+y\bar{\rho}}\mathcal{H}N_e^{\mathcal{K}}\mathbb{F}_2$, with multiplicative structure emphasized

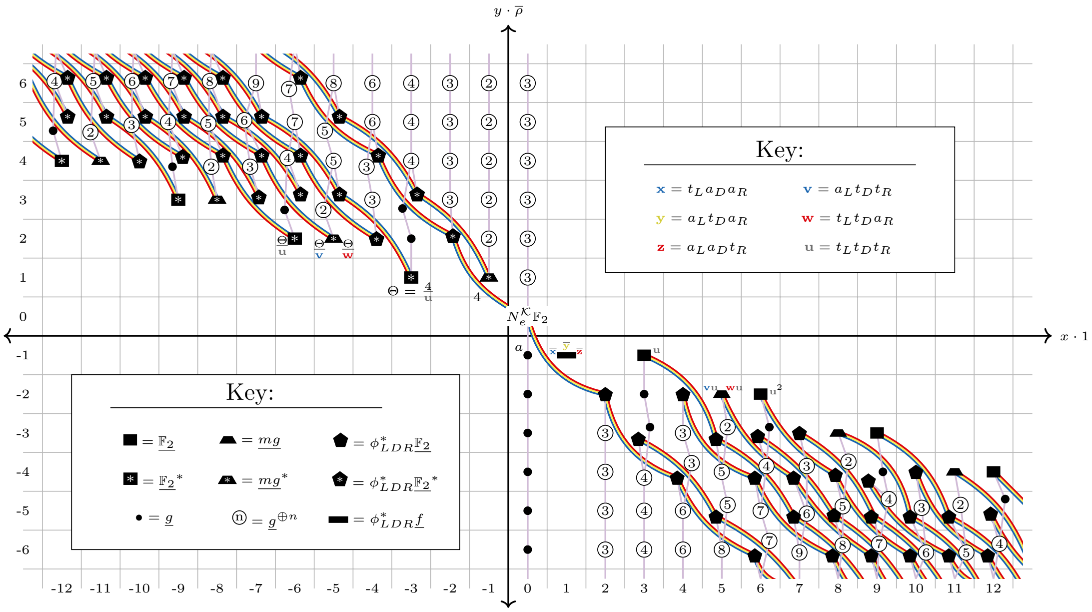


FIGURE 7.8. $\underline{\pi}^{\mathcal{K}}_{x+y\overline{\rho}}\mathfrak{H}N^{\mathcal{K}}_D\underline{\mathbb{F}_2}$

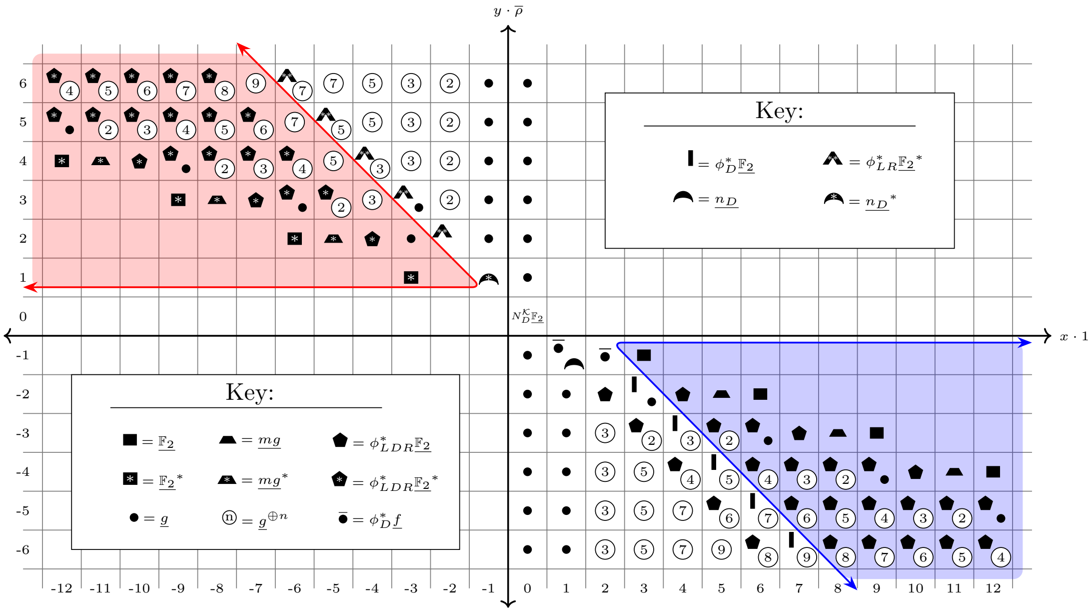
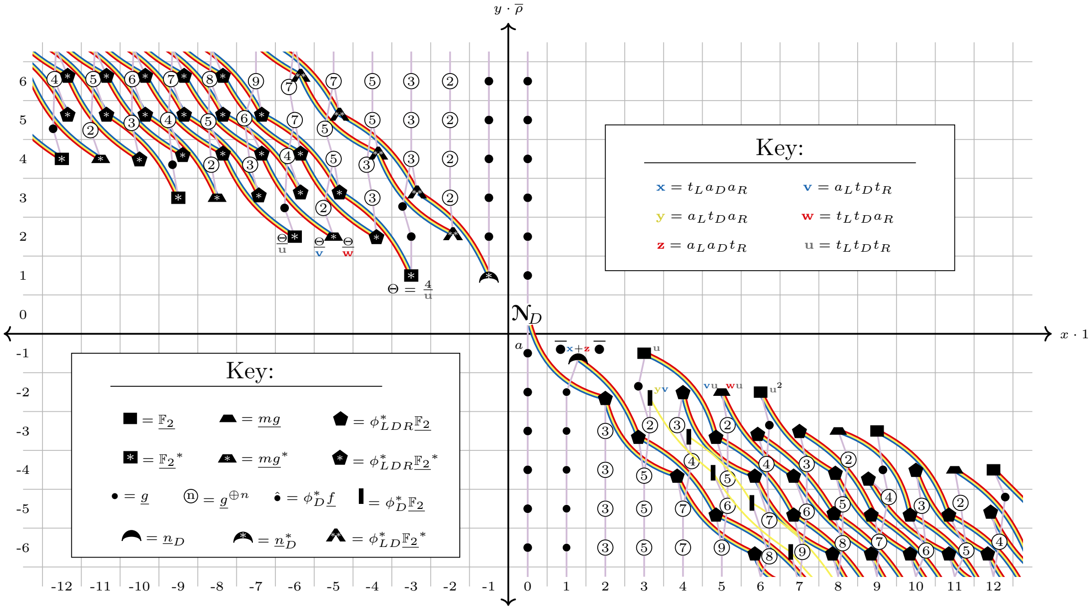


FIGURE 7.9. $\underline{\pi}^{\mathcal{K}}_{x+y\overline{\rho}}\mathfrak{H}N^{\mathcal{K}}_D\mathbb{F}_2$, with multiplicative structure emphasized



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