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Graphs arising from the dual Steenrod algebra

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Abstract

We extend Wood's graph theoretic interpretation of certain quotients of the mod 2 dual Steenrod algebra to quotients of the mod p dual Steenrod algebra where p is an odd prime and to quotients of the C_2-equivariant dual Steenrod algebra. We establish connectedness criteria for graphs associated to monomials in these algebra quotients and investigate questions about trees and Hamilton cycles in these settings. We also give graph theoretic interpretations of algebraic structures such as the coproduct and antipode arising from the Hopf algebra structure on the mod p dual Steenrod algebra and the Hopf algebroid structure of the C_2-equivariant dual Steenrod algebra.

Keywords Steenrod algebra · Graphs · Equivariant

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1 Introduction

The action of the mod p Steenrod algebra A and its dual $A^\vee$ on mod p cohomology and homology, respectively, plays a foundational role in homotopy theory computations. In particular, it serves as input to the Adams spectral sequence [1], which is one of the most powerful tools for computing stable homotopy groups. A primary example of the Adams spectral sequence's use includes Hill-Hopkins-Ravenel's solution of the Kervaire invariant one problem and Lin-Wang-Xu's solution of the last Kervaire invariant one problem [3, 8]. These solutions used the Adams spectral sequence, and thereby information coming from the action of the dual Steenrod algebra on homology, to answer the question of when a framed $4k+2$ dimensional manifold can be surgically converted into a sphere.

Because the dual Steenrod algebra plays a central role in stable homotopy calculations, it is worthwhile to study it from a variety of points of view (see, for example [14–17]). The purpose of this paper is to study the dual Steenrod algebra $A^\vee$ from a graph theory perspective. In particular, we extend a graphical construction due to Wood [18, §8] and studied by Yearwood [19] and Larson [6] at the prime 2 to odd primes p and the C_2-equivariant setting. For the sake of exposition, we discuss the odd primary and C_2-equivariant results separately.

1.1 Odd primary results

Let p be an odd prime and recall the mod p dual Steenrod algebra $A^\vee$ is a graded Hopf algebra with

$$A^\vee \cong \mathbb{F}_p[\xi_1, \xi_2, \xi_3, \cdots] \otimes E[\tau_0, \tau_1, \cdots],$$

where $|\xi_i| = 2(p^i - 1)$ and $|\tau_i| = 2p^i - 1$. The coproduct on $A^\vee$ is given by

$$\psi(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i$$

$$\psi(\tau_n) = \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \tau_i$$

while the unit map $\eta : \mathbb{F}_p \to A^\vee$ and counit map $\epsilon : A^\vee \to \mathbb{F}_p$ are isomorphisms in degree 0. Finally, the antipode $c : A^\vee \to A^\vee$ is defined recursively by

$$c(\xi_0) = 1 \qquad\qquad c(\tau_0) = \tau_0$$

$$\sum_{i=0}^{n} \xi_{n-i}^{p^i} c(\xi_i) = 0 \qquad\qquad \tau_n + \sum_{i=0}^{n-1} \xi_{n-i}^{p^i} c(\tau_i) = 0.$$

Given an integer $n \geq 0$, the quotient $A^\vee(n)$ is defined by

$$A^\vee(n) = A^\vee \Big/ (\xi_1^{p^n}, \xi_2^{p^{n-1}}, \cdots, \xi_n^p, \xi_{n+1}, \xi_{n+2}, \cdots, \tau_{n+1}, \tau_{n+2}, \cdots).$$

In other words,

$$A^\vee(n) \cong \mathbb{F}_p[\xi_1, \xi_2, \cdots, \xi_n, \tau_0, \tau_1, \cdots, \tau_n] \Big/ (\xi_1^{p^n}, \xi_2^{p^{n-1}}, \cdots, \xi_n^p, \tau_0^2, \tau_1^2, \cdots, \tau_n^2).$$

These $A^\vee(n)$ are Hopf algebra quotients so the Hopf algebra structure on the dual Steenrod algebra $A^\vee$ descends to the quotient $A^\vee(n)$ [2, 12]. Dualizing these quotient Hopf algebras yields finite subalgebras $A(n)$ of the Steenrod algebra A which are generated by reduced p-th power operations P^i, where $i \leq p^{n-1}$, and the Bockstein operation β.

Both the sub-Hopf algebras $A(n)$ and their duals $A^\vee(n)$ facilitate computation of homotopy groups via the Adams spectral sequence. Many Adams spectral sequence computations begin with Ext over the Steenrod algebra A. This initial computation can often be done over the simpler finite sub-Hopf Algebra $A(n)$ instead via a change-of-rings isomorphism [11, A1.3.12].

One of the main purposes of this paper is to give a graph theoretic interpretation of the quotients $A^\vee(n)$ when p is an odd prime. Similarly to Wood's construction at the prime 2 [18, §8], this is accomplished by associating to every monomial $x \in A^\vee(n)$ a graph G_x on the vertex set $\{1, 2, 2p, 2p^2, \cdots, 2p^n\}$. We then study the properties of the ensuing graphs.

We defer an explicit description of the construction of these graphs to Section 3 and focus on the statements of the main results here. Taking addition to be represented as disjoint union of graphs, it is sufficient to restrict our study to monic monomials $x \in A^\vee(n)$.

Suppose x is a monic monomial in $A^\vee(n)$. Then

$$x = \tau_0^{\epsilon_0} \tau_1^{\epsilon_1} \cdots \tau_n^{\epsilon_n} \xi_1^{r_1} \xi_2^{r_2} \cdots \xi_n^{r_n}$$

where $\epsilon_i \in \{0, 1\}$ and $0 \leq r_i \leq p^{n+1-i} - 1$. We can rewrite x using the p-adic expansion of r_i, that is

$$r_i = \sum_{m=0}^{n-i} c_{i,n-i-m} \, p^{n-i-m}.$$

Specifically, we can write

$$x = \tau_0^{\epsilon_0}\tau_1^{\epsilon_1}\cdots\tau_n^{\epsilon_n}\prod_{i=1}^{n}\prod_{m=0}^{n-i}\xi_i^{c_{i,n-i-m}p^{n-i-m}}. \tag{1.1}$$

Rewriting the monomial x in this way facilitates extracting a connectedness criterion for the graph G_x from the data of the monomial x. We call x as written in (1.1) a monic monomial in standard form (Definition 3.1). In order to state this connectedness criterion, we first need to introduce some notation.

Let t be an integer with $1 \leq t \leq n+1$ and define T_t (Definition 2.4) to be the set of all $(t+1)$-tuples $(b_0, b_1, , \cdots b_{t-1}, b_t)$ where $b_i \in \{0, 1, \cdots, n\}$ with no two entries being the same. Further let $T_t' \subset T_t$ (Definition 2.5) be the subset of T_t where $b_0 = r$ and $b_t = s$.

Theorem (Theorem 3.11) *The graph G_x is connected if and only if*

(1) the integers

$$C(r,s) := \sum_{t=1}^{n}\sum_{T_t'}\prod_{k=1}^{t} c_{|b_k-b_{k-1}|,\max(b_{k-1},b_k)-1}$$

are positive for all integers r and s with $0 \leq r < s \leq n$,

(2) and the sum

$$\sum_{k=0}^{n}\epsilon_k > 0.$$

In [19], Yearwood observed that the graphs G_x can also naturally be viewed as directed graphs with edges oriented in the direction of the larger vertex. We will denote the directed version of the graph G_x by G_x^{dir}. With the directed version of Wood's construction in mind, Larson studied unilaterality, the appropriate connectedness property for directed graphs, in the $p = 2$ setting. In the odd primary setting we identify exactly when the directed graph G_x^{dir} is unilateral. In order to state our result, we define T_t'' (Definition 2.9) to be the set of all $(t+1)$-tuples $(b_0, b_1, \cdots, b_t)$ with $b_i \in \{0, 1, \cdots, n+1\}$ and $r = b_0 < b_1 < \cdots < b_t = s$.

Theorem (Theorem 3.12) *The directed graph G_x^{dir} is unilateral if and only if*

(1) the integers

$$U(r,s) := \sum_{t=1}^{n}\sum_{T_t''}\prod_{k=1}^{t} c_{b_{k-1},b_k}$$

are positive for all integers r, s with $0 \leq r < s \leq n+1$,

(2) and furthermore x is divisible by τ_0.

With these connectedness results in hand, we can also identify the graphs G_x that are trees.

Theorem (Theorem 3.13) *Suppose x is a monic monomial in $A^\vee(n)$ such that G_x is connected. Then G_x is a tree if and only if*

(1) the sum

$$\sum_{i=1}^{n} \alpha_p(r_i) = n,$$

where $\alpha_p(r_i)$ denotes the number of nonzero digits in the base-p expansion of r_i,
(2) and exactly one τ_k divides the monic monomial x, for some $0 \leq k \leq n$.

Among the questions posed by Wood in [18, §8] is whether there is an algebraic criterion that describes when the graph G_x has a Hamilton cycle. Larson answered this question affirmatively in the $p = 2$ setting ([6, Theorem 1.4]). We give an analogue of this result in the odd primary setting.

Theorem (Theorem 3.16) *Suppose $x = \tau_0^{\epsilon_0} \cdots \tau_n^{\epsilon_n} \xi_1^{r_1} \cdots \xi_n^{r_n} \in A^\vee(n)$ where $\epsilon_i \in \{0, 1\}$ and $0 \leq r_i \leq p^{n+1-i} - 1$. The corresponding graph G_x has a Hamilton cycle if*

(1) $n > 0$,
(2) for each j with $0 \leq j \leq n$,

$$f_{\text{in}}(2p^j) + f_{\text{out}}(2p^j) \geq \frac{n}{2},$$

(3) and

$$\sum_{k=0}^{n} \epsilon_k \geq \frac{n}{2}.$$

Finally, we generalize the results of [19] and [6] giving a graphical interpretation of some of the Hopf algebra structure on $A^\vee(n)$. Specifically we give a graphical interpretation of the coproduct and antipode on elements the $\xi_i^{p^j}$ and $\tau_i \in A^\vee(n)$ when p is an odd prime.

Theorem (Theorem 3.17)

I. The coproduct $\psi(\xi_i^{p^j}) \in A^\vee(n)$ is the sum of tensors of all pairs of edges that make length 2 directed paths from the vertex $2p^j$ to the vertex $2p^{i+j}$ in the complete graph on the vertices $\{1, 2, 2p, 2p^2, \cdots, 2p^n\}$ considered as a directed graph.

II. Similarly, coproduct $\psi(\tau_i) \in A^\vee(n)$ is the sum of tensors of all pairs of edges that make length 2 directed paths from the vertex 1 to the vertex $2p^i$.

Theorem (Theorem 3.20)

I. The antipode $c(\xi_i^{p^j}) \in A^\vee(n)$ is the signed sum of all directed paths from the vertex $2p^j$ to the vertex $2p^{i+j}$ in the complete graph on the vertices $\{1, 2, 2p, 2p^2, \cdots, 2p^n\}$ considered as a directed graph. The positive terms in the sum correspond exactly to paths consisting of an even number of edges from the vertex 2 to the vertex $2p^{i+j}$.

II. *Similarly, the antipode* $c(\tau_i) \in A^\vee(n)$ *is the signed sum of all directed paths from the vertex* 1 *to the vertex* $2p^i$. *The positive terms correspond exactly to paths consisting of an even number of edges from the vertex* 1 *to the vertex* $2p^i$.

As a corollary, we obtain another characterization of the unilaterality of the directed graphs G_x^{dir}.

Corollary (Corollary 3.23) *Suppose* x *is a monic monomial in* $A^\vee(n)$. *Then the directed graph* G_x^{dir} *is unilateral if and only if*

(1) for each $\xi_i^{p^j} \in A^\vee(n)$, *at least one summand of* $c(\xi_i^{p^i})$ *is a factor of* x
(2) and τ_0 *is a factor of* x.

1.2 C_2-equivariant Results

While homotopy theory studies the properties of topological spaces that are preserved under continuous deformations (i.e. homotopies), similar questions can be asked about spaces with rotation and reflection symmetries. This leads to the field of equivariant homotopy theory where a group G acts on the topological spaces under consideration. In this setting, there are also familiar invariants such as homology and homotopy (see [9] for a comprehensive introduction). However, their definition is now more complicated. Specifically, instead of being integer graded, these equivariant theories are graded on the additive group underlying the real representation ring $RO(G)$. Intuitively, this comes from the fact that we are now interested in probing our equivariant spaces not only with the ordinary nonequivariant spheres S^n but also with equivariant representation spheres S^V, which are formed by taking the one-point compactification of a G-representation V and assigning the point at infinity the trivial action.

Similarly to the nonequivariant setting, equivariant Steenrod algebras act on equivariant (co)homology theories and this serves as input to equivariant Adams spectral sequences. In general, equivariant dual Steenrod algebras are more complicated than their classical nonequivariant counterparts. For instance, the C_p-equivariant dual Steenrod algebra is not free over its coefficients [5, 13]. Further complicating matters, we do not currently have, nor expect to have in the near future, complete computations of the G-equivariant dual Steenrod algebra when G an arbitrary finite or compact Lie group. Hence in this paper, we focus on the simplest nontrivial case where the group of equivariance is $G = C_2$, the cyclic group of order 2. In this case, the C_2-equivariant dual Steenrod algebra actually has a form remarkably similar to the nonequivariant mod p dual Steenrod algebra.

In order to introduce the C_2-equivariant dual Steenrod algebra, we observe a few representation theoretic preliminaries. The group C_2 has two irreducible real representations, the one dimensional trivial representation and the one dimensional sign representation σ. Throughout, we will let ρ denote the C_2-regular representation $1+\sigma$. This leads to a bigraded theory which we can view as indexed by $(i-j)+j\sigma$. Here i is the topological dimension while j is the weight or twisted dimension. And indeed, the C_2-equivariant dual Steenrod algebra is bigraded. Moreover, it also has the structure of a Hopf algebroid rather than a Hopf algebra as in the nonequivariant mod p setting.

To describe the C_2-equivariant dual Steenrod algebra as a Hopf algebroid, we must first introduce the bigraded coefficient ring

$$\mathbb{M}_2 \cong \mathbb{F}_2[a, u] \oplus \frac{\mathbb{F}_2[a, u]}{(a^\infty, u^\infty)}\{\theta\},$$

where $|a| = (-1, -1)$, $|u| = (0, -1)$. A more detailed description of $\mathbb{M}_2$ can be found in Section 4.1. The Hopf algebroid $(\mathbb{M}_2, A_{C_2}^\vee)$ has underlying algebra

$$A_{C_2}^\vee \cong \mathbb{M}_2[\xi_1, \xi_2, \cdots, \tau_0, \tau_1, \cdots]/(\tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1}) \tag{1.2}$$

where $|\xi_i| = \rho(2^i - 1)$ and $|\tau_i| = 2^i\rho - \sigma$. The coproduct is given by

$$\psi(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \xi_i$$

$$\psi(\tau_n) = \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \tau_i$$

(see [4, Theorem 6.41] or [7, Theorem 2.14] for the full Hopf algebroid structure).

Given an integer $n \geq 0$, define the quotient $A_{C_2}^\vee(n)$ by

$$A_{C_2}^\vee(n) = \frac{A_{C_2}^\vee}{(\xi_1^{2^n}, \xi_2^{2^{n-1}}, \cdots, \xi_n^2, \xi_{n+1}, \xi_{n+2}, \cdots, \tau_{n+1}, \tau_{n+2}, \cdots)}.$$

In other words,

$$A_{C_2}^\vee(n) \cong \mathbb{M}_2[\xi_1, \xi_2, ..., \xi_n, \tau_0, \tau_1, ..., \tau_n] \Big/ (\xi_i^{2^{n-i+1}}, \ \tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1}).$$

One of the primary purposes of this paper is to give a graph theoretic interpretation of the quotients $A_{C_2}^\vee(n)$. Similarly to Wood's construction at the prime 2 [18, §8], this is accomplished by associating to every monomial $x \in A_{C_2}^\vee(n)$ a graph G_x on the vertex set $\{\sigma, \rho, 2\rho, 2^2\rho, \cdots, 2^n\rho\}$. We then study the properties of the ensuing graphs. We defer a description of the construction of these graphs to Section 4 and focus on the statements of the main results here.

Given a monomial $x \in A_{C_2}^\vee(n)$ we can use the relation $\tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1}$ to write x as a finite sum consisting of terms of the form

$$w\tau_0^{\epsilon_0}\tau_1^{\epsilon_1} + \cdots + \tau_n^{\epsilon_n}\xi_1^{r_1}\xi_2^{r_2}\cdots\xi_n^{r_n} \tag{1.3}$$

where $w \in \mathbb{M}_2$, $\epsilon_i \in \{0, 1\}$ and $0 \leq r_i \leq 2^{n+1-i} - 1$. Because addition is represented as disjoint union in our graphical interpretation, it is sufficient to focus on monic monomials of the form (1.3). Similarly to Wood's construction in the nonequivariant setting at the prime 2, we will also utilize the 2-adic expansion of r_i,

$$r_i = \sum_{m=0}^{n-i} c_{i,n-i-m} \cdot 2^{n-i-m}.$$

Specifically, given a monic monomial of the form (1.3) we can write

$$x = \tau_0^{\epsilon_0}\tau_1^{\epsilon_1}\cdots\tau_n^{\epsilon_n}\left(\prod_{i=1}^{n}\prod_{m=0}^{n-i}\xi_i^{c_{i,n-i-m}\cdot 2^{n-i-m}}\right). \tag{1.4}$$

We call x written as in (1.4) a monic monomial in standard form (Definition 4.2). Rewriting x in this way again facilitates the extraction of a connectedness criterion for the graph G_x from the data of the monomial x. In fact, this connectedness criterion ends up having the same statement as in the nonequivariant p an odd prime case (3.11).

Theorem (Theorem 4.6) *Suppose $x \in A_{C_2}^{\vee}(n)$ is a monic monomial in standard form. Then the graph G_x is connected if and only if*

(1) the integers

$$C(r,s) := \sum_{t=1}^{n}\sum_{T_t'}\prod_{k=1}^{t} c_{|b_k-b_{k-1}|,\max(b_{k-1},b_k)-1}$$

are positive for all integers r and s with $0 \le r < s \le n$ and
(2) the sum

$$\sum_{k=0}^{n}\epsilon_k > 0.$$

Similarly to the nonequivariant setting, the graph G_x can also naturally be viewed as a directed graph with edges oriented in the direction of the vertex of larger representation dimension. We will again denote the directed version of the graph G_x by G_x^{dir}. Much like the nonequivariant p an odd prime case, we have a similar criterion for unilaterality.

Theorem (Theorem 4.7) *Suppose $x \in A_{C_2}^{\vee}(n)$ is a monic monomial in standard form. Then the directed graph G_x^{dir} is unilateral if and only if*

(1) the integers

$$U(r,s) := \sum_{t=1}^{n}\sum_{T_t''}\prod_{k=1}^{t} c_{b_{k-1},b_k}$$

are positive for all integers r, s with $0 \le r < s \le n+1$,
(2) and x is divisible by τ_0.

Similarly to the nonequivariant p an odd prime case, we also give an algebraic criterion that describes when the graph G_x has a Hamilton cycle.

Theorem (Theorem 4.11) *Suppose* $x = \tau_0^{\epsilon_0} \cdots \tau_n^{\epsilon_n} \xi_1^{r_1} \cdots \xi_n^{r_n}$ *is a monic monomial in standard form in* $A_{C_2}^{\vee}(n)$*. The corresponding graph* G_x *has a Hamilton cycle if*

(1) $n > 0$,
(2) for each j *with* $0 \leq j \leq n$,

$$\deg_{\text{in}}(2^j\rho) + \deg_{\text{out}}(2^j\rho) \geq \frac{n}{2},$$

(3) and

$$\sum_{k=0}^{n} \epsilon_k \geq \frac{n}{2}.$$

Finally, we give a graphical interpretation of some of the Hopf algebroid structure on $A_{C_2}^{\vee}(n)$. Specifically we give a graphical interpretation of the coproduct and antipode on the elements $\xi_i^{p^j}$ and $\tau_i \in A_{C_2}^{\vee}(n)$.

Theorem (Theorem 4.12)

I. *The coproduct* $\psi(\xi_i^{p^j}) \in A_{C_2}^{\vee}(n)$ *is the sum of tensors of all pairs of edges that make length 2 directed paths from the vertex* $2^j\rho$ *to the vertex* $2^{i+j}\rho$ *in the complete graph on the vertices* $\{\sigma, \rho, 2\rho, 2^2\rho, \cdots 2^n\rho\}$ *considered as a directed graph.*
II. *Similarly, the coproduct* $\psi(\tau_i) \in A_{C_2}^{\vee}(n)$ *is the sum of tensors of all pairs of edges that make length 2 directed paths from the vertex* σ *to the vertex* $2^i\rho$.

Theorem (Theorem 4.13)

I. *The antipode* $c(\xi_i^{p^j}) \in A_{C_2}^{\vee}(n)$ *is the sum of all directed paths from the vertex* $2^j\rho$ *to the vertex* $2^{i+j}\rho$ *in the complete graph on the vertices* $\{\sigma, \rho, 2\rho, 2^2\rho, \cdots 2^n\rho\}$ *considered as a directed graph.*
II. *Similarly, the antipode* $c(\tau_i) \in A_{C_2}^{\vee}(n)$ *is the sum of all directed paths from the vertex* σ *to the vertex* $2^i\rho$.

As a corollary, we also obtain another characterization of the unilaterality of the directed graphs G_x^{dir}.

Corollary (Theorem 4.14) *Suppose* x *is a monic monomial in standard form in* $A_{C_2}^{\vee}(n)$*. Then the directed graph* G_x^{dir} *is unilateral if and only if*

(1) for each $\xi_i^{p^j} \in A_{C_2}^{\vee}(n)$*, at least one summand of* $c(\xi_i^{p^i})$ *is a factor of* x
(2) and τ_0 *is a factor of* x.

1.3 Open Questions

These results suggest some natural questions for further study, some of which partially overlap with questions posed by Larson in [6, §5.4].

(1) In [18, §8], Wood points out that the mod 2 Steenrod algebra A (as opposed to its dual) and some of its subalgebras $A(n)$ can be interpreted graph theoretically. What would the results of [6, 19] or this paper look like in that setting?
(2) Given the Hopf algebra structure on $A^\vee(n)$, can one give a graph theoretic interpretation of the coproduct ψ and antipode c on an arbitrary monomial $x \in A^\vee(n)$? Similarly, given the Hopf algebroid structure on $A^\vee_{C_2}(n)$, can one give a graph theoretic interpretation of the coproduct ψ and antipode c on an arbitrary monomial $x \in A^\vee(n)$?
(3) In [18, §5] Wood describes two procedures one can perform in the mod 2 Steenrod algebra A, called stripping and strapping, that together allow one to derive all of the Adem relations from the single relation $\mathrm{Sq}_1\,\mathrm{Sq}_1 = 0$. A step in the process of recovering the Adem relations involves assigning to each monomial $\xi_i^{2^j} \in A^\vee$ a "stripping operator" which is analogous to how Wood's construction assigns an edge of a graph to each $\xi_i^{2^j}$. Can this analogy be leveraged to obtain further graph theoretic results in both the p-primary and C_2-equivariant settings?
(4) Can we extend our graphical constructions to represent comodules over the quotients $A^\vee(n)$ or $A^\vee_{C_2}(n)$?

1.4 Outline of the paper

In Section 2 we recall Wood's construction of the graphs corresponding to monomials $x \in A^\vee(n)$ when $p = 2$ [18, §8]. We also discuss graph theoretic results in this setting due to Larson and Yearwood [6, 19]. In Section 3 we describe a construction yielding graphs corresponding to monomials $x \in A^\vee(n)$ when p is an odd prime. This section also contains the proofs of our odd primary results. In Section 4 we extend our odd primary construction and results to the C_2-equivariant setting.

1.5 Definitions, conventions, and notation

We recall some foundational definitions and set notation.

(1) A ***graph*** $G = (V_G, E_G)$ is a structure that contains a set of objects V_G, which are called ***vertices***, and relations on the vertices coming from E_G, which are called ***edges***. The subscript G on V_G and E_G will often be suppressed when the graph is clear from context.
(2) Given an edge $e \in E_G$ relating two vertices v_0 and v_1, we say v_0 and v_1 are the ***ends*** of e.
(3) For $e \in E$ connecting vertices $u, v \in V$, we will write $e\colon u \to v$ and for n copies of e, we will write $e^n : u \xrightarrow{n} v$.
(4) The ***degree*** of a vertex v is given by the number of edges that touch that vertex, we will denote this by $\deg(v)$.
(5) A ***walk*** is a sequence of edges and vertices in a graph. When the starting and ending vertices are the same, we say the walk is a ***closed walk***.
(6) A ***path*** is a sequence of edges and vertices where no vertex is visited more than once, except possibly the start and end vertices.

(7) Note that a path is a specific type of walk, so when the ending and starting vertices are the same, the path is considered a ***closed path***.
(8) A ***cycle***, also known as a closed path, is a path in a graph that begins and ends at the same vertex.
(9) A ***connected*** graph is one for which there is a path between any given pair of vertices.
(10) A ***tree*** is a graph that is connected and acyclic.
(11) A ***complete*** graph is a graph where any given pair of vertices is connected by an edge.
(12) A ***Hamiltonian path*** is a path that visits every vertex exactly once.
(13) A ***Hamiltonian cycle*** is a cycle that visits every vertex exactly once.
(14) A ***directed edge*** is an edge where the ends are distinguished - one is the head and one is the tail. In particular, a directed edge is specified as an ordered pair of vertices (tail, head).
(15) A ***directed graph*** is a graph where every edge is directed.
(16) A directed graph D is ***unilateral*** if for every pair of distinct vertices $v_i,\ v_j \in D$, there is a directed path starting at v_i and ending at v_j or vice versa.
(17) Given a graph G with an ordered vertex set $V_G = \{v_1, v_2, \cdots, v_n\}$, the associated ***adjacency matrix*** A_G is an $n \times n$ matrix with entry $a_{i,j}$ given by the number of edges from v_i to v_j.

2 Wood's graphical construction at the prime 2

In this section, we recall Wood's graphical construction at the prime 2 and describe results due to Yearwood [19] and Larson [6]. These results inspire our work at odd primes and in the C_2-equivariant setting so it is helpful recollect them here.

2.1 The mod 2 dual Steenrod algebra

The mod 2 dual Steenrod algebra is

$$A^\vee \cong \mathbb{F}_2[\xi_1, \xi_2, \cdots],$$

where $|\xi_n| = 2^n - 1$. The coproduct is given by

$$\psi(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \xi_i$$

and the antipode $c : A^\vee \to A^\vee$ is given recursively by

$$c(\xi_0) = 1$$

$$\sum_{k=0}^{i} \xi_{i-k}^{2^k} c(\xi_k) = 0.$$

Milnor solved this recursion in [10, Lemma 10] obtaining

$$c(\xi_i) = \sum_{\pi} \prod_{k=1}^{\ell(\pi)} \xi_{\pi(k)}^{2^{\sigma(k)}}$$

where the sum is over all ordered partitions π of i, $\ell(\pi)$ is the length of π, $\pi(k)$ is the kth term of π, and $\sigma(k)$ is the sum of the first $k-1$ terms of π.

The Hopf algebra quotient $A^\vee(n)$ is given by

$$A^\vee(n) \cong A^\vee \Big/ (\xi_1^{2^{n+1}}, \xi_2^{2^n}, \xi_3^{2^{n-1}}, \cdots, \xi_{n+1}^2, \xi_{n+2}, \xi_{n+3}, \cdots).$$

In other words,

$$A^\vee(n) \cong \mathbb{F}_2[\xi_1, \xi_2, \cdots, \xi_{n+1}] \Big/ (\xi_1^{2^{n+1}}, \xi_2^{2^n}, \xi_3^{2^{n-1}}, \cdots, \xi_{n+1}^2).$$

2.2 Wood's graphical construction ($p = 2$)

Suppose x is a monomial in $A^\vee(n)$. Then

$$x = \xi_1^{r_1} \xi_2^{r_2} \cdots \xi_n^{r_n} \xi_{n+1}^{r_{n+1}},$$

where $r_i \in \{0, 1, \cdots, 2^{n+1-i} - 1\}$ and where each r_i has the 2-adic expansion

$$r_i = \sum_{m=0}^{n+1-i} c_{i,n+1-i-m} \cdot 2^{n+1-i-m}.$$

We can rewrite the monomial x as

$$x = \prod_{i=1}^{n+1} \prod_{m=0}^{n+1-i} \xi_i^{c_{i,n+1-i-m} \cdot 2^{n+1-i-m}}.$$

This rewritten form of x is useful for constructing the graphical interpretation of monomials in $A^\vee(n)$ so we make the following definition.

Definition 2.1 Given a monomial $x \in A^\vee(n)$, the **standard form** of x is

$$x = \prod_{i=1}^{n+1} \prod_{m=0}^{n+1-i} \xi_i^{c_{i,n+1-i-m} \cdot 2^{n+1-i-m}},$$

where the exponents on the ξ_i are written in terms of their 2-adic expansion.

Given a monomial $x \in A^{\vee}(n)$, Wood defines the graph $G_x = (V_x, E_x)$ by giving the ordered vertex set $V_x = \{2^0, 2^1, 2^2, \cdots, 2^{n+1}\}$ and the edge set

$$E_x = \bigcup \left\{ \xi_i^{c_{i,j} \cdot 2^j} : 2^j \xrightarrow{c_{i,j}} 2^{i+j} \right\}$$

[18, §8]. Notice that the degree $|\xi_i^{2^j}|$ of the edge $\xi_i^{2^j}$, that is, the grading of $\xi_i^{2^j}$ when viewed as an element of $A^{\vee}(n)$, is the absolute value of the difference of the ends of $\xi_i^{2^j}$.

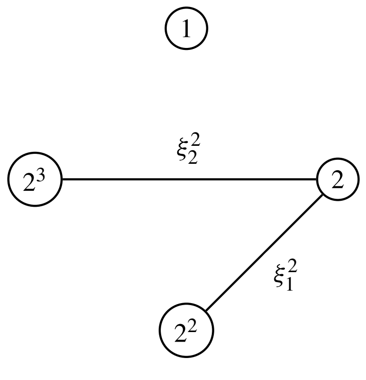
Considering entries in the upper triangle of the adjacency matrix A_x associated to the graph G_x, we observe that

$$a_{l,k} = c_{k-l,l-1} \qquad (\text{when } l < k), \tag{2.1}$$

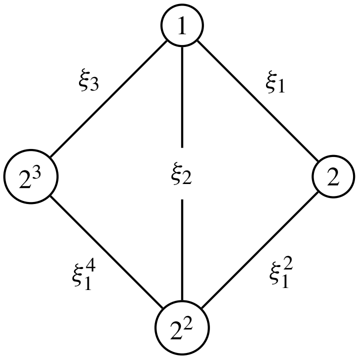
that is,

$$A_x = \begin{bmatrix} 0 & c_{1,0} & c_{2,0} & & c_{n,0} & c_{n+1,0} \\ c_{1,0} & 0 & c_{1,1} & & c_{n-1,1} & c_{n,1} \\ c_{2,0} & c_{1,1} & 0 & & c_{n-2,2} & c_{n-1,2} \\ & & & \ddots & & \\ c_{n,0} & c_{n-1,1} & c_{n-2,2} & & 0 & c_{1,n} \\ c_{n+1,0} & c_{n,1} & c_{n-1,2} & & c_{1,n} & 0 \end{bmatrix}. \tag{2.2}$$

Example 2.2 The monomial $\xi_1^2 \xi_2^2 \in A^{\vee}(2)$ has associated graph $G_{\xi_1^2 \xi_2^2}$:



Example 2.3 The monomial $\xi_1^7 \xi_2 \xi_3 \in A^{\vee}(2)$ has associated graph $G_{\xi_1^7 \xi_2 \xi_3}$:



2.3 Connectedness criteria

In [6], Larson gives connectedness criteria for the graph G_x given the only the data of the monomial x. We recall some of Larson's results here for comparison with our odd primary and C_2-equivariant results.

Throughout this section we will assume x is a given monomial in $A^{\vee}(n)$. Thus we can write

$$x = \xi_1^{r_1}\xi_2^{r_2}\cdots\xi_n^{r_n}\xi_{n+1}^{r_{n+1}}$$

for some $r_i \in \{0, 1, \cdots, 2^{n+1-i}-1\}$. We will also make use of x written in standard form, that is writing

$$x = \prod_{i=1}^{n+1}\prod_{m=0}^{n+1-i} \xi_i^{c_{i,n+1-i-m}\cdot 2^{n+1-i-m}}.$$

Recall that a graph is *connected* if for any two distinct vertices there exists a path between them. Given two vertices of a graph, we can count the number of paths between of a given length. To count the number of acyclic paths between two vertices of a given length, say of t edges, we define the following set.

Definition 2.4 Suppose t is an integer with $1 \le t \le n+1$. Define T_t to be the set of all $(t+1)$-tuples $(b_0, b_1, , \cdots b_{t-1}, b_t)$ where $b_i \in \{0, 1, \cdots, n\}$ and no two entries are the same.

We can interpret tuples $(b_0, b_1, \cdots b_{t-1}, b_t) \in T_t$ as corresponding to acyclic paths of length t from the vertex 2^{b_0} to the vertex 2^{b_t}. Specifically, the tuple $(b_0, b_1, \cdots b_{t-1}, b_t)$ represents a path through the vertices $2^{b_0}, 2^{b_1}, \cdots, 2^{b_{t-1}}, 2^{b_t}$. Note that b_{k-1} is not necessarily less than b_k so the path need not pass through the vertices of G_x in ascending order.

Thus, if we are interested in indicating whether a particular acyclic path exists in our graph, we can begin by checking whether there is an edge between the vertices $2^{b_{k-1}}$ and 2^{b_k}. This is indicated by the value of

$$a_{\max(b_{k-1},b_k),\min(b_{k-1},b_k)} = \begin{cases} 1 & \text{if there is an edge between } 2^{b_{k-1}} \text{ and } 2^{b_k} \\ 0 & \text{otherwise} \end{cases},$$

in the adjacency matrix A_x. Using the observation that $a_{l,k} = c_{k-l,l-1}$ when $l < k$ (2.1), we can rewrite $a_{\max(b_{k-1},b_k),\min(b_{k-1},b_k)}$ in terms of the coefficients $c_{i,j}$ appearing in the standard form of x. Specifically,

$$a_{\max(b_{k-1},b_k),\min(b_{k-1},b_k)} = c_{|b_k-b_{k-1}|,\max(b_{k-1},b_k)-1}$$

Thus the path corresponding to $(b_0, b_1, \cdots, b_{t-1}, b_t)$ exists in our graph if and only if

$$\prod_{k=1}^{t} c_{|b_k-b_{k-1}|,\max(b_{k-1},b_k)-1} = 1.$$

Therefore, to count the total number of acyclic paths between two given vertices, say the vertex 2^r and the vertex 2^s in our graph G_x, we make the following definitions.

Definition 2.5 Given integers $r, s \in \{0, 1, \cdots, n\}$, define T_t' to be the set of all $(t+1)$-tuples $(r, b_1, \cdots, b_{t-1}, s)$ where $b_i \in \{0, 1, \cdots, n+1\}$.

In particular, T_t' is the subset of T_t where the first element in each $(t+1)$-tuple is set equal to r and the last element in each $(t+1)$-tuple is set equal to s.

Definition 2.6 Given integers $r, s \in \{0, 1, \cdots, n\}$, let

$$C(r, s) := \sum_{t=1}^{n+1} \sum_{T_t'} \prod_{k=1}^{t} c_{|b_k-b_{k-1}|,\max(b_{k-1},b_k)-1}.$$

Here, the outer sum is taken over all possible lengths of paths between the vertices. Since there are $n+2$ vertices, the longest path that is not a cycle has length $n+1$. The inner sum is taken over T_t', that is, over all the acyclic paths of length t between the vertices 2^r and 2^s.

Using this notation, Larson proves the following connectedness criterion.

Theorem 2.7 ([6, Theorem 1.1]) *The graph G_x is connected if and only if the integers*

$$C(r, s) := \sum_{t=1}^{n+1} \sum_{T_t'} \prod_{k=1}^{t} c_{|b_k-b_{k-1}|,\max(b_{k-1},b_k)-1}$$

are positive for all integers r and s with $0 \le r < s \le n+1$.

2.4 Trees and Hamilton cycles

Letting $\alpha(m)$ denote the number of nonzero digits in the base 2 expansion of an integer m, Larson also proves:

Theorem 2.8 ([6, Theorem 1.2]) *The graph G_x is a tree if and only if G_x is connected and*

$$\sum_{i=1}^{n+1} \alpha(r_i) = n + 1.$$

In [19], Yearwood observed that the graphs G_x can naturally be viewed as directed graphs with edges oriented in the direction of the larger vertex (i.e., $2^\alpha \to 2^\beta$ if $0 \le \alpha < \beta \le n+1$). We will denote the directed version of the graph G_x by G_x^{dir}. With the directed version of Wood's construction in mind, Larson studied the unilaterality, the appropriate connectedness property for directed graphs (see Section 1.5). To count the total number of acyclic directed paths between two given vertices, say the vertex 2^r and the vertex 2^s, we make the following definitions.

Definition 2.9 Given integers $r, s \in \{0, 1, \cdots, n\}$, let T_t'' be the set of all $(t+1)$-tuples $(b_0, b_1, \cdots, b_t)$ where $b_i \in \{0, 1, \cdots, n+1\}$ and $r = b_0 < b_1 < \cdots < b_t = s$.

In particular, T_t'' is the subset of T_t' where the elements in each $(t+1)$-tuple are strictly increasing.

Definition 2.10 Given integers $r, s \in \{0, 1, \cdots, n\}$, let

$$U(r, s) := \sum_{t=1}^{n+1} \sum_{T_t''} \prod_{k=1}^{t} c_{b_{k-1}, b_k}.$$

Then $U(r, s)$ is the directed graph analogue of $C(r, s)$ and thus we can think of the outer and inner sums similarly.

Using this notation, Larson proves the following characterization of unilaterality.

Theorem 2.11 *([6, Theorem 1.3]) The directed graph G_x^{dir} is unilateral if and only if $U(r, s)$ is positive for all integers r, s with $0 \le r < s \le n+1$.*

Larson also studied gave an algebraic criteron describing when a Hamilton cycle must occur in the graph G_x.

Theorem 2.12 ([6, Theorem 1.4]) *The graph G_x has a Hamilton cycle if $n > 0$ and for every vertex 2^i of G_x,*

$$\#\{1 \le k \le j : c_{j,j-k} = 1\} + \#\{1 \le k \le n+1-j : a_{j+k,j} = 1\} \ge \frac{n}{2}.$$

Moreover, the directed graph G_x^{dir} has a directed Hamilton path if and only if x is divisible by $\xi_1^{2^{n+1}-1}$.

2.5 The Hopf algebra structure

Following Lemmas 3.1.7 and 3.1.8 of [19], Larson also gave a graph theoretic interpretation of the coproduct and antipode on monomials $\xi_i^{2^j} \in A^\vee(n)$.

Theorem 2.13 *([6, Theorem 1.5]) The coproduct $\psi(\xi_i^{2^j}) \in A^\vee(n)$ is the sum of tensors of all pairs of edges that make length 2 directed paths from the vertex 2^j to the vertex 2^{i+j} in the complete graph on the vertices $\{2^0, 2^1, 2^2, \cdots, 2^{n+1}\}$ considered* *as a directed graph. Moreover, the antipode* $c(\xi_i^{2^j}) \in A^\vee(n)$ *is the sum of all directed paths from the vertex* 2^j *to the vertex* 2^{i+j}.

And as a corollary, Larson obtained another characterization of the unilaterality of G_x^{dir}.

Corollary 2.14 *([6, Corollary 1.6]) For* $x \in A^\vee(n)$*, the graph* G_x^{dir} *is unilateral if and only if for each* $\xi_i^{2^j} \in A^\vee(n)$ *at least one summand of* $c(\xi_i^{2^j})$ *is a factor of* x.

3 A graphical construction at odd primes

We now define a graphical construction for the mod p dual Steenrod algebra quotients $A^\vee(n)$, where p is an odd prime, with the goal of extending Wood's construction at the prime $p = 2$. Recall the mod p dual Steenrod algebra

$$A^\vee = \mathbb{F}_p[\xi_1, \xi_2, \dots] \otimes E[\tau_1, \tau_2, \dots],$$

where $|\xi_i| = 2(p^i - 1)$ and $|\tau_i| = 2p^i - 1$. Since we work one prime at a time, we suppress p from our notation simply writing $A^\vee$ for the mod p dual Steenrod algebra.

For each integer $n \geq 1$, we study graphs arising from monomials in the quotients

$$A^\vee(n) = \mathbb{F}_p[\xi_1, \xi_2, \cdots, \xi_n, \tau_o, \tau_1, \cdots, \tau_n] \Big/ (\xi_1^{p^n}, \xi_2^{p^{n-1}}, \cdots, \xi_n^p, \tau_0^2, \tau_1^2, \cdots, \tau_n^2).$$

Since addition is represented by disjoint union of graphs, it is sufficient to restrict our study to monic monomials $x \in A^\vee(n)$. Given a monic monomial $x \in A^\vee(n)$, we can write

$$x = \tau_0^{\epsilon_0} \tau_1^{\epsilon_1} \cdots \tau_n^{\epsilon_n} \xi_1^{r_1} \xi_2^{r_2} \cdots \xi_n^{r_n}$$

where $\epsilon_i \in \{0, 1\}$ and $0 \leq r_i \leq p^{n+1-i} - 1$. Following Wood's construction, we will utilize the p-adic expansion of r_i,

$$r_i = \sum_{m=0}^{n-i} c_{i,n-i-m} \cdot p^{n-i-m}.$$

Extending Definition 2.1, we make the following definition.

Definition 3.1 For a monic monomial $x \in A^\vee(n)$, the ***standard form*** of x is

$$\tau_0^{\epsilon_0} \tau_1^{\epsilon_1} \cdots \tau_n^{\epsilon_n} \prod_{i=1}^{n} \prod_{m=0}^{n-i} \xi_i^{c_{i,n-i-m} \cdot p^{n-i-m}},$$

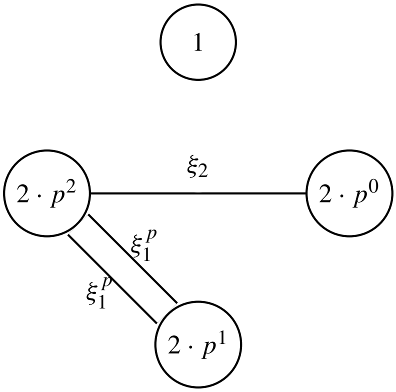
where the exponents on the ξ_i are expanded into their p-adic expansion.

Given a monic monomial $x \in A^\vee(n)$, we define the graph $G_x = (V_x, E_x)$ setting the ordered vertex set $V_x = \{1, 2\cdot p^0, 2\cdot p^1, \ldots, 2\cdot p^n\}$ and the edge set

$$E_x = \left\{\tau_i : 1 \to 2p^i\right\}_{0\le i\le n} \cup \left\{\xi_i^{c_{i,j}\cdot p^j} : 2p^j \xrightarrow{c_{i,j}} 2p^{i+j}\right\}_{\substack{1\le i\le n\\ 0\le j\le n-i}}$$

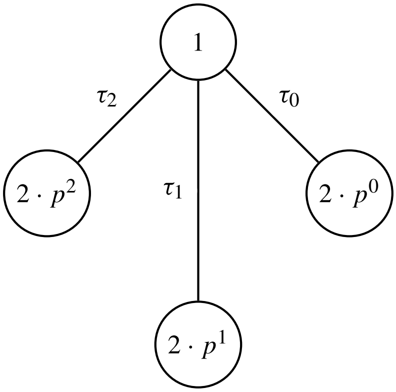
where each τ_i denotes an edge connecting vertices 1 and $2p^i$ and each $\xi_i^{c_{i,j}\cdot p^j}$ denotes $c_{i,j}$ edges connecting vertices $2p^j$ and $2p^{i+j}$. Notice that, similarly to the $p = 2$ case, the degree $|\xi_i^{2^j}|$ of the edge $\xi_i^{2^j}$, that is, the grading of $\xi_i^{2^j}$ when viewed as an element of $A^\vee(n)$, is the absolute value of the difference of its ends. Additionally, the degree $|\tau_i|$ of the edge τ_i is the absolute value of the difference of its ends.

Example 3.2 The monomial $x = \xi_1^{2p}\xi_2 \in A^\vee(2)$ has associated graph G_x:



Notice that in the graph G_x, there are $2 = c_{1,1}$ edges connecting vertices $2p^1$ and $2p^2$. These correspond to the two factors of ξ_1^p. In general, our graphs may have up to $p-1$ factors of ξ_1^p, in which case there would be $p-1$ edges connecting the vertices $2p^1$ and $2p^2$.

Example 3.3 The monomial $x = \tau_0\tau_1\tau_2 \in A^\vee(2)$ has associated graph $G_{\tau_0\tau_1\tau_2}$:



Consider a monic monomial $x \in A^\vee(n)$. We observe that the entries in the upper triangle of the adjacency matrix A_x, that is $a_{l,k}$ where $l < k$, are given by

$$a_{l,k} = \begin{cases} \epsilon_{k-2} & l = 1 \\ c_{k-l,l-2} & 1 < l < k \end{cases}. \tag{3.1}$$

It may be helpful to recall that the values of ϵ_{k-2} and $c_{k-l,l-2}$ come from the exponents of x in standard form (Definition 3.1). Therefore,

$$A_x = \begin{bmatrix} 0 & \epsilon_0 & \epsilon_1 & \epsilon_2 & & \epsilon_{n-1} & \epsilon_n \\ \epsilon_0 & 0 & c_{1,0} & c_{2,0} & & c_{n-1,0} & c_{n,0} \\ \epsilon_1 & c_{1,0} & 0 & c_{1,1} & & c_{n-2,1} & c_{n-1,1} \\ \epsilon_2 & c_{2,0} & c_{1,1} & 0 & & c_{n-3,2} & c_{n-2,2} \\ & & & & \ddots & & \\ \epsilon_{n-1} & c_{n-1,0} & c_{n-2,1} & c_{n-3,2} & & 0 & c_{1,n-1} \\ \epsilon_n & c_{n,0} & c_{n-1,1} & c_{n-2,2} & & c_{1,n-1} & 0 \end{bmatrix} \tag{3.2}$$

Notice that the i^{th} superdiagonal, excluding the first row, contains the coefficients of the p-adic expansion for ξ_i. Reading i^{th} superdiagonal from bottom to top yields the base p expansion of r_i:

$$r_i = \left(c_{i,n-i}\, c_{i,n-i-1} \, \dots \, c_{i,1}\, c_{i,0}\right)_p.$$

3.1 Examples

In this section, all examples have prime p set equal to 3.

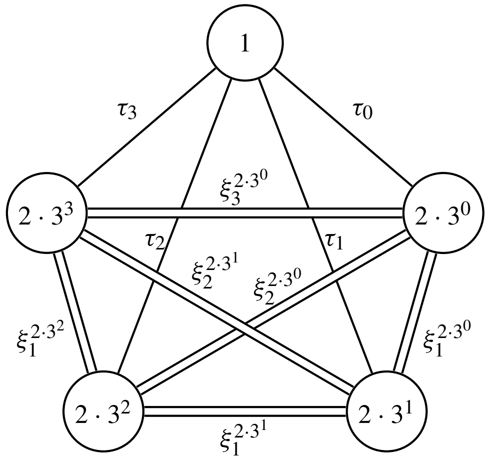
Example 3.4 (A maximally complete graph) Consider the monomial

$$x = \tau_0\tau_1\tau_2\tau_3\xi_1^{26}\xi_2^8\xi_3^2 \in A^\vee(3).$$

Rewriting x in standard form yields

$$x = \tau_0\tau_1\tau_2\tau_3\xi_1^{2\cdot 3^2}\xi_1^{2\cdot 3^1}\xi_1^{2\cdot 3^0}\xi_2^{2\cdot 3^1}\xi_2^{2\cdot 3^0}\xi_3^{2\cdot 3^0} \in A^\vee(3)$$

with associated graph and adjacency matrix depicted below.



$$A_x = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & 2 & 0 & 2 \\ 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

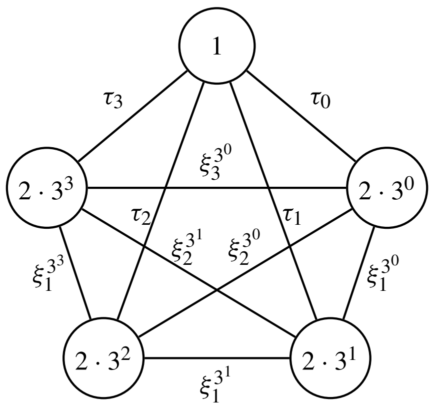
Example 3.5 (A minimally complete graph) Consider the monomial

$$x = \tau_0\tau_1\tau_2\tau_3\xi_1^{13}\xi_2^4\xi_3 \in A^\vee(3).$$

Rewriting x in standard form yields

$$\tau_0\tau_1\tau_2\tau_3\xi_1^{3^2}\xi_1^{3^1}\xi_1^{3^0}\xi_2^{3^1}\xi_2^{3^0}\xi_3^{3^0} \in A^\vee(3)$$

with associated graph and adjacency matrix depicted below.

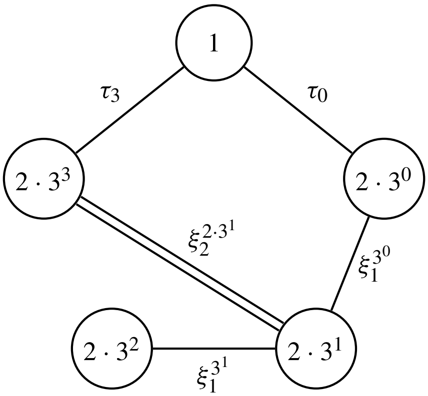


$$A_x = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Example 3.6 (A connected graph) Consider the monomial $x = \tau_0\tau_3\xi_1^4\xi_2^6 \in A^\vee(3)$. Rewriting x in standard form yields

$$\tau_0\tau_3\xi_1^{3^1}\xi_1^{3^0}\xi_2^{2\cdot 3^1} \in A^\vee(3)$$

with associated graph and adjacency matrix depicted below.

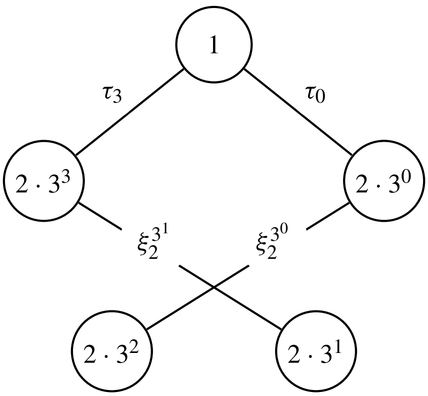


$$A_x = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \end{bmatrix}$$

Example 3.7 (A tree graph) Consider the monomial $x = \tau_0 \tau_3 \xi_2^4 \in A^\vee(3)$. Rewriting x in standard form yields

$$\tau_0 \tau_3 \xi_2^{3^1} \xi_2^{3^0} \in A^\vee(3)$$

with associated graph and adjacency matrix depicted below.

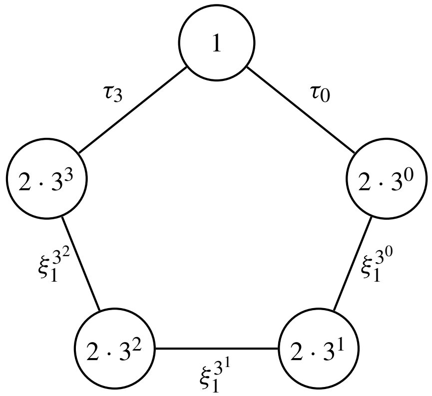


$$A_x = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Example 3.8 (A graph with a Hamilton path) Consider the monomial $x = \tau_0 \tau_3 \xi_1^{13} \in A^\vee(3)$. Rewriting x in standard form yields

$$\tau_0 \tau_3 \xi_1^{3^2} \xi_1^{3^1} \xi_1^{3^0} \in A^\vee(3)$$

with associated graph and adjacency matrix depicted below.



$$A_x = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

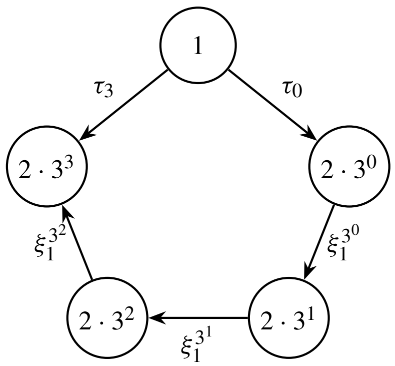
Example 3.9 (Viewing the graph as a directed graph) Consider the monomial

$$x = \tau_0 \tau_3 \xi_1^{13} \in A^\vee(3).$$

Rewriting x in standard form yields

$$\tau_0 \tau_3 \xi_1^{3^2} \xi_1^{3^1} \xi_1^{3^0} \in A^\vee(3)$$

with associated graph and adjacency matrix depicted below.



$$A_{G_x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In general, with the vertex set being ordered we can always consider the graphs to be directed graphs. The monomial information can be read from both the directed and undirected graphs. Notice that when the graph *is* viewed as a directed graph, the adjacency matrix is upper triangular and no longer symmetric (for $l > k$, $a_{l,k} = 0$).

3.2 Connectedness and Unilaterality

We begin by considering the subgraph $G_x\backslash\{1\}$ given by omitting the vertex 1. This subgraph has adjacency matrix

$$\begin{bmatrix} 0 & c_{1,0} & c_{2,0} & & c_{n-1,0} & c_{n,0} \\ c_{1,0} & 0 & c_{1,1} & & c_{n-2,1} & c_{n-1,1} \\ c_{2,0} & c_{1,1} & 0 & & c_{n-3,2} & c_{n-2,2} \\ & & & \ddots & & \\ c_{n-1,0} & c_{n-2,1} & c_{n-3,2} & & 0 & c_{1,n-1} \\ c_{n,0} & c_{n-1,1} & c_{n-2,2} & & c_{1,n-1} & 0 \end{bmatrix}. \tag{3.3}$$

Noticing that this matrix has nearly the same form as the adjacency matrix in the $p = 2$ case (2.2), we can apply a similar argument to prove the following proposition.

Proposition 3.10 *The subgraph $G_x\backslash\{1\}$ is connected if and only if the integers*

$$C(r, s) := \sum_{t=1}^{n} \sum_{T_t'} \prod_{k=1}^{t} c_{|b_k - b_{k-1}|, \max(b_{k-1}, b_k)-1}$$

are positive for all integers r and s with $0 \le r < s \le n$.

Returning to consider the entire graph G_x, we prove the following theorem.

Theorem 3.11 *The graph G_x is connected if and only if*

(1) the integers

$$C(r, s) := \sum_{t=1}^{n} \sum_{T_t'} \prod_{k=1}^{t} c_{|b_k - b_{k-1}|, \max(b_{k-1}, b_k)-1}$$

are positive for all integers r and s with $0 \le r < s \le n$ and

(2) the sum

$$\sum_{k=0}^{n} \epsilon_k > 0.$$

Proof The first condition guarantees that the subgraph $G\backslash\{1\}$ is connected. The second condition guarantees that the vertex 1 is connected by an edge to the subgraph $G\backslash\{1\}$.

Similarly to the case where $p = 2$, the graphs G_x can also naturally be viewed as directed graphs with edges oriented in the direction of the larger vertex. We will again denote the directed version of the graph G_x by G_x^{dir}. Similar techniques of proof yield the following characterization of the unilaterality of G_x.

Theorem 3.12 *The directed graph x^{dir} is unilateral if and only if*

(1) the integers

$$U(r, s) := \sum_{t=1}^{n} \sum_{T_t''} \prod_{k=1}^{t} c_{b_{k-1}, b_k}$$

are positive for all integers r, s *with* $0 \le r < s \le n+1$*, where* T_t'' *is the set of all* $(t+1)$*-tuples* $(b_0, b_1, \cdots, b_t)$ *with* $b_i \in \{0, 1, \cdots, n+1\}$ *and* $r = b_0 < b_1 < \cdots < b_t = s$*,*

(2) and furthermore x *is divisible by* τ_0*.*

3.3 Trees and Hamiltonian cycles

Theorem 3.13 *Suppose* x *is a monic monomial in* $A^\vee(n)$ *such that* G_x *is connected. Then* G_x *is a tree if and only if*

(1) the sum

$$\sum_{i=1}^{n} \alpha_p(r_i) = n,$$

where $\alpha_p(r_i)$ *denotes the number of nonzero digits in the base-*p *expansion of* r_i*, and*

(2) exactly one τ_k *divides the monic monomial* x*, for some* $0 \le k \le n$*.*

Proof Let x be a monic monomial in $A^\vee(n)$ such that G_x is connected. Recall that in our graph construction, a digit equal to 1 in the base p expansion of r_i corresponds to having exactly one edge between two vertices of the graph. Thus,

$$\sum_{i=1}^{n} \alpha_p(r_i) = n$$

implies that there are exactly n edges contained in the subgraph $G_x \backslash \{1\}$. If additionally, a singular τ_k divides x with $0 \le k \le n$, x is connected but not cyclic because G_x consists of $n+1$ distinct edges in a graph with $n+2$ vertices. Thus the graph G_x is a tree.

Recall that a Hamiltonian directed path for a graph G is defined as a directed path containing every vertex of G.

Theorem 3.14 *Let* x *be a monic monomial in* $A^\vee(n)$*. The graph* G_x *has a Hamiltonian-directed path if and only if* $\tau_0 \xi_1^{\frac{p^n-1}{p-1}}$ *divides* x*.*

Proof Suppose the graph G_x^{dir} associated with $x \in A^\vee(n)$ has a Hamiltonian directed path. Since the tail of each edge in G_x^{dir} is less than the head, and every vertex appears in the Hamiltonian directed path, the path must proceed through the ordered vertex set $V_x = \{1, 2, 2p, 2p^2, \cdots, 2p^n\}$ in order. In order for there to be an edge between the vertices 1 and 2, τ_0 must divide x. In order for there to be an edge between the vertices $2p^k$ and $2p^{k+1}$, $\xi_1^{p^k}$ must divide x. Hence

$$\tau_0 \xi_1^{\sum_{k=0}^{n} p^k} = \tau_0 \xi_1^{\frac{p^n-1}{p-1}}$$

must divide x. On the other hand, if $\tau_0\xi_1^{\frac{p^n-1}{p-1}}$ divides x, the graph G_x^{dir} has a Hamiltonian-directed path by construction.

Next we turn to establishing a criterion for when the graph G_x has a Hamilton cycle. Our goal will be to make use of Dirac's theorem, which says that a simple graph with n vertices ($n \geq 3$) is Hamiltonian if every vertex has degree $\frac{n}{2}$ or greater. Since we are working with multigraphs G_x, our first step will be to make an observation which allows us to reduce to the simple graph setting.

Let A_x be the adjacency matrix associated to G_x with the (i, j) entry of A_x denoted $a_{i,j}$. Define

$$f_{\text{out}}(2p^j) = \#\{1 < i \leq n | a_{j+2,j+2+i} \geq 1\}$$

and

$$f_{\text{in}}(2p^j) = \#\{1 < i \leq j+1 | a_{j+2-i,j+2} \geq 1\}$$

and let G_x^{R} denote the simple graph obtained from G_x by replacing any multi-edges with a single edge between the same vertices. We will call G_x^{R} the reduced graph. Then the following proposition follows from the definitions of A_x and G_x^{R}.

Proposition 3.15 *The out-degree of the vertex $2p^j$ in G_x^{R} considered as a directed graph is given by f_{out}. The in-degree of the vertex $2p^j$ in G_x^{R} considered as a directed graph is given by $f_{\text{in}}(2p^j)$.*

In order to guarantee a Hamiltonian cycle in G_x, it is sufficient to show there is a Hamiltonian cycle in G_x^{R}. Since $f_{\text{in}}(2p^j) + f_{\text{out}}(2p^j)$ gives the total degree of the vertices $2p^j$ $(0 \leq j \leq n)$ in G_x^{R}, it only remains to consider the degree of the vertex 1 in order to apply Dirac's theorem.

Since 1 is the initial vertex in the directed graph G_x^{dir}, the total degree of the vertex 1 is

$$\sum_{k=0}^{n} \epsilon_k.$$

Then using Dirac's theorem we arrive at the following result.

Theorem 3.16 *Suppose $x = \tau_0^{\epsilon_0} \cdots \tau_n^{\epsilon_n} \xi_1^{r_1} \cdots xi_n^{r_n} \in A^\vee(n)$. The corresponding graph G_x has a Hamilton cycle if*

(1) $n > 0$,
(2) for each j with $0 \leq j \leq n$,

$$f_{\text{in}}(2p^j) + f_{\text{out}}(2p^j) \geq \frac{n}{2},$$

(3) and

$$\sum_{k=0}^{n} \epsilon_k \geq \frac{n}{2}.$$

3.4 Graph theoretic interpretation of the coproduct

Similarly to Larson's graph theoretic description of the coproduct at the prime $p = 2$ (Theorem 2.13), we also have a graphical interpretation of the coproduct on elements $\xi_i^{p^j}$ and $\tau_i^{\epsilon_i} \in A^\vee(n)$ at odd primes.

Theorem 3.17 *The coproduct $\psi(\xi_i^{p^j}) \in A^\vee(n)$ is the sum of tensors of all pairs of edges that make length 2 directed paths from the vertex $2p^j$ to the vertex $2p^{i+j}$ in the complete graph on the vertices $\{1, 2, 2p, 2p^2, \cdots, 2p^n\}$ considered as a directed graph.*

Similarly, coproduct $\psi(\tau_i) \in A^\vee(n)$ is the sum of tensors of all pairs of edges that make length 2 directed paths from the vertex 1 to the vertex $2p^i$.

Proof Our proof closely follows that of [6, Theorem 1.5] in the $p = 2$ case. We first prove the statement for $x = \xi_i^{p^j}$. Recall that the coproduct on the dual Steenrod algebra $A^\vee$ is given by

$$\psi(\xi_n) = \xi_n \otimes 1 + 1 \otimes \xi_n + \sum_{k=1}^{n-1} \xi_{n-k}^{p^k}. \tag{3.4}$$

The first two summands on the right-hand side of (3.4) represent degenerate length 2 directed paths from $2p^j$ to $2p^{i+j}$. A non-degenerate length 2 directed path from $2p^j$ to $2p^{i+j}$ corresponds to a choice of an intermediate vertex, which is of the form $2p^{j+k}$ for some k where $1 \le k \le i-1$. Given this choice of k, the edge from $2p^j$ to $2p^{j+k}$ is $\xi_{j+k-j}^{p^j} = \xi_k^{p^j}$ and the edge from $2p^{j+k}$ to $2p^{i+j}$ is $\xi_{i+j-(j+k)}^{p^{i+k}}$. The terms of the sum indexed by k on the far-right side of (3.4) correspond precisely to the pairs of edges just described. Similarly,

$$\psi(\tau_n) = \tau_n \otimes 1 + 1 \otimes \tau_n + \sum_{k=0}^{n-1} \xi_{n-k}^{p^k} \otimes \tau_k$$

and a similar argument gives the desired statement.

Example 3.18 Suppose $p = 3$ and consider $\xi_2 \in A^\vee(3)$. Then

$$\psi(\xi_2) = \xi_2 \otimes 1 + \xi_1^3 \otimes \xi_1 + 1 \otimes \xi_2$$

as depicted in Figure 1. Note that the loops in Figure 1 denote degenerate paths corresponding to the factors of 1 appearing in the tensor product. The dashed arrows are directed edges between vertices that do not form directed paths of length 2 between the vertices $2\cdot3^0$ and $2\cdot3^2$ in the complete graph on the vertices $\{1,\ 2\cdot3^0,\ 2\cdot3,\ 2\cdot3^2,\ 2\cdot3^3\}$

Fig. 1 Coproduct construction of ξ_2 in $A_3^\vee(3)$

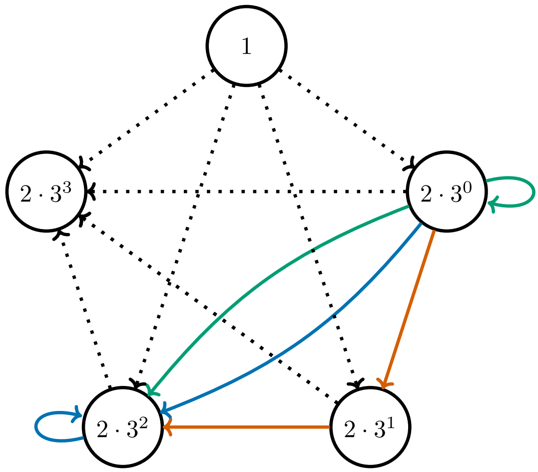
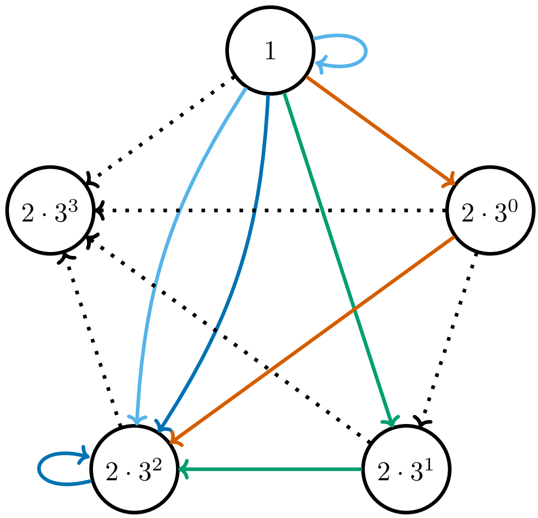


Fig. 2 Coproduct construction of τ_2 in $A_3^\vee(3)$



considered as a directed graph. The solid arrows are edges that do form directed paths of length 2 between the vertices $2 \cdot 3^0$ and $2 \cdot 3^2$.

Example 3.19 Suppose $p = 3$ and consider $\tau_2 \in A^\vee(3)$. Then

$$\psi(\tau_2) = \tau_2 \otimes 1 + \xi_2 \otimes \tau_0 + \xi_1^3 \otimes \tau_1 + 1 \otimes \tau_2$$

as depicted in Figure 2. Again loops denote degenerate paths corresponding to factors of 1 appearing in the tensor product.

3.5 Graph theoretic interpretation of the antipode

Similarly to Larson's graph theoretic description of the antipode at the prime $p = 2$ (Theorem 2.13), we also have a graphical interpretation of the antipode $c : A^\vee(n) \to A^\vee(n)$ on the elements $\xi_i^{p^j}$ and τ_i at odd primes.

Theorem 3.20 *The antipode $c(\xi_i^{p^j}) \in A^\vee(n)$ is the signed sum of all directed paths from the vertex $2p^j$ to the vertex $2p^{i+j}$ in the complete graph on the vertices $\{1, 2, 2p, 2p^2, \cdots, 2p^n\}$ considered as a directed graph. The positive terms in the sum correspond exactly to paths consisting of an even number of edges from the vertex 2 to the vertex $2p^{i+j}$.*

Similarly, the antipode $c(\tau_i) \in A^\vee(n)$ is the signed sum of all directed paths from the vertex 1 to the vertex $2p^i$. The positive terms correspond exactly to paths consisting of an even number of edges from the vertex 1 to the vertex $2p^i$.

Proof Recall that the antipode map $c : A^\vee(n) \to A^\vee(n)$ of the Hopf algebra structure sends

$$c(\xi_n) = -\xi_n - \sum_{k=0}^{n-1} \xi_{n-k}^{p^k} c(\xi_k),$$

and

$$c(\tau_n) = -\tau_n - \sum_{k=0}^{n-1} \xi_{n-k}^{p^k} c(\tau_k).$$

We will prove the statement for $c(\xi_i^{p^j})$ by induction on i. The proof for $c(\tau_i)$ is similar.

To begin the induction, consider $c(\xi_1^{p^j}) = c(\xi_1)^{p^j} = (-\xi_1)^{p^j} = -\xi_1^{p^j}$ since p is an odd prime. In the complete graph on the vertices $\{1, 2, 2p, 2p^2, \cdots, 2p^n\}$, the monomial $\xi_1^{p^j}$ corresponds exactly to the a path from $2p^j$ to $2p^{j+1}$. This is a path of length one so we observe the desired statement is true when $i = 1$.

Now suppose $c(\xi_\ell^{p^j})$ is the signed sum of all directed paths from $2p^j$ to $2p^{\ell+j}$ for all $\ell \le i$ where paths of an even length correspond to positive terms and paths of an odd length correspond to negative terms. We must show that

$$c(\xi_{i+1}^{p^j}) = -\xi_{i+1}^{p^j} - \sum_{k=0}^{i} \xi_{i+1-k}^{p^{k+j}} c(\xi_k^{p^j}) \tag{3.5}$$

is the signed sum of all directed paths from $2p^j$ to $2p^{i+1+j}$ where directed paths of an even length correspond to positive terms and directed paths of an odd length correspond to negative terms.

Consider the term $-\xi_{i+1-k}^{p^{k+j}} c(\xi_k^{p^j})$ in Equation 3.5. The $c(\xi_k^{p^j})$ factor in this term consists of a signed sum of all paths from $2p^j$ to $2p^{j+k}$ while the $\xi_{i+1-k}^{p^{k+j}}$ factor consists of the edge from $2p^{j+k}$ to $2p^{j+i+1}$. Thus $\xi_{i+1-k}^{p^{k+j}} c(\xi_k^{p^j})$ is a signed sum of all directed paths from $2p^j$ to $2p^{j+k}$ followed by the edge from $2p^{j+k}$ to $2p^{j+i+1}$ in the reduced graph G_x^{R}. Observe that the length of each directed path in $-\xi_{i+1-k}^{p^{k+j}} c(\xi_k^{p^j})$ is one step longer than in $c(\xi_k^{p^j})$. Moreover, the sign in front of each term appearing in $-\xi_{i+1-k}^{p^{k+j}} c(\xi_k^{p^j})$ is the opposite of that appearing in $c(\xi_k^{p^j})$. Hence directed paths of even length still correspond to positive terms in the sum and directed paths of odd length still correspond to negative terms in the sum. Thus summing over all k where $0 \leq k \leq i$ gives the desired signed sum of all paths from $2p^j$ to $2p^{i+1+j}$ passing through at least one intermediate vertex. The term $-\xi_{i+1}^{p^j}$ gives the remaining path from $2p^j$ to $2p^{i+1+j}$ passing through no intermediate vertices. Hence $c(\xi_{i+1}^{p^j})$ is the signed sum of all paths from $2p^j$ to $2p^{i+1+j}$ where directed paths of an even length correspond to positive terms and directed paths of an odd length correspond to negative terms.

Example 3.21 Let p be an odd prime and consider $\xi_2 \in A^\vee(3)$. Then

$$c(\xi_2) = -\xi_2 + \xi_1^{p+1}$$

as depicted in Figure 3.

Fig. 3 Antipode of ξ_2 in $A_3^\vee(3)$

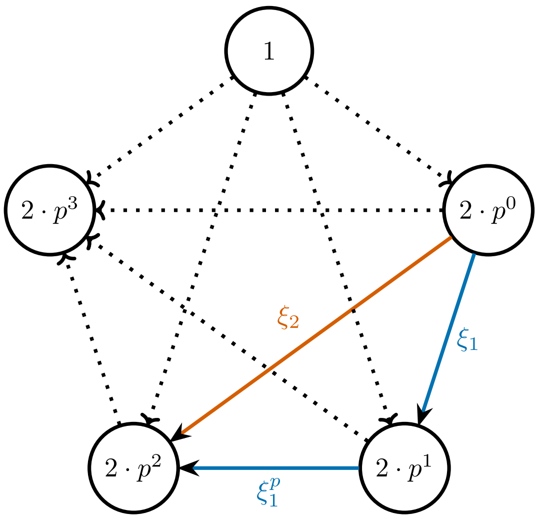
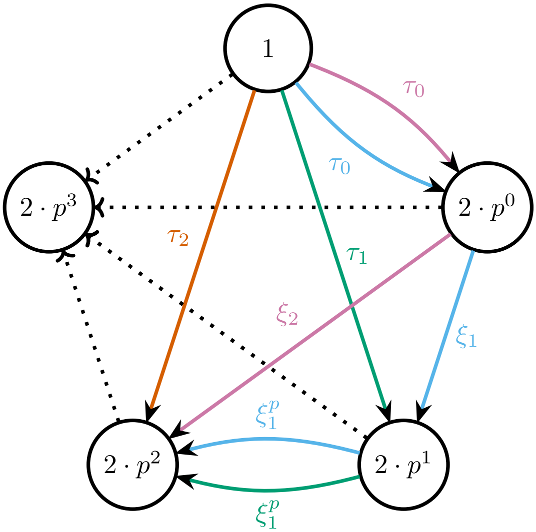


Fig. 4 Antipode of τ_2 in $A^\vee(3)$



Example 3.22 Let p be an odd prime and consider $\tau_2 \in A^\vee(3)$. Then

$$c(\tau_2) = -\tau_2 + \tau_1\xi_1^p + \tau_0\xi_2 - \tau_0\xi_1^{p+1}$$

as depicted in Figure 4.

Corollary 3.23 *For $x \in A^\vee(n)$, the directed graph G_x^{dir} is unilateral if and only if*

(1) for each $\xi_i^{p^j} \in A^\vee(n)$, at least one summand of $c(\xi_i^{p^i})$ is a factor of x
(2) and τ_0 is a factor of x.

Proof Note that the directed graph G_x^{dir} is unilateral if and only if there is a directed path connecting any two of its vertices, say $2p^i$ and $2p^{i+j}$ or 1 and $2p^{i+j}$. Theorem 3.20 shows this is equivalent to the demand that at least one summand of $c(\xi_i^{p^j})$ and τ_0 appear as factors of x.

4 A C_2-equivariant graphical construction

4.1 The coefficients $\mathbb{M}_2$

Before describing the C_2-equivariant dual Steenrod algebra, we give a more detailed description of the bigraded coefficient ring

$$\mathbb{M}_2 \cong \mathbb{F}_2[a, u] \oplus \frac{\mathbb{F}_2[a, u]}{(a^\infty, u^\infty)}\{\theta\},$$

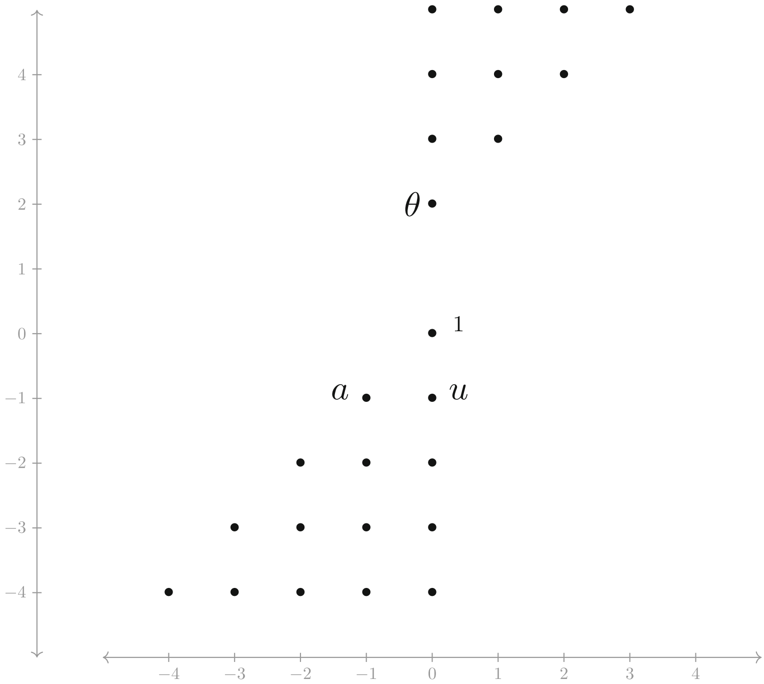


Fig. 5 The coefficients $\mathbb{M}_2$

where $|a| = (-1, -1)$, $|u| = (0, -1)$. A bigraded plot of $\mathbb{M}_2$ appears in Figure 5 where each dot represents a copy of $\mathbb{F}_2$. The topological dimension is given on the horizontal axis while the vertical axis is the weight.

Remark 4.1 The coefficients $\mathbb{M}_2$ are the $RO(C_2)$-graded homology of a point with coefficients in the C_2-constant Mackey functor $\underline{\mathbb{F}}_2$.

4.2 The C_2-equivariant dual Steenrod algebra

Recall that the C_2-equivariant dual Steenrod algebra is given by

$$A^{\vee}_{C_2} = \mathbb{M}_2[\xi_1, \xi_2, \ldots, \tau_0, \tau_1, \ldots] \Big/ (\tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1}),$$

where $|\xi_i| = \rho(2^i - 1)$ and $|\tau_i| = 2^i\rho - \sigma$ (see [4, Theorem 6.41] or [7, Theorem 2.14] for the full Hopf algebroid structure). Also recall that given an integer $n \geq 0$ the quotient $A^{\vee}_{C_2}(n)$ is

$$A^{\vee}_{C_2}(n) \cong \mathbb{M}_2[\xi_1, \xi_2, \ldots, \xi_n, \tau_0, \tau_1, \ldots, \tau_n] \Big/ (\xi_i^{2^{n-i+1}}, \ \tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1}).$$

4.3 A graphical interpretation

Given a monomial $x \in A^{\vee}_{C_2}(n)$ we can use the relation $\tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1}$ to write x as a finite sum consisting of terms of the form

$$w\tau_0^{\epsilon_0}\tau_1^{\epsilon_1} + \cdots + \tau_n^{\epsilon_n}\xi_1^{r_1}\xi_2^{r_2}\cdots\xi_n^{r_n} \tag{4.1}$$

where $w \in \mathbb{M}_2$, $\epsilon_i \in \{0, 1\}$ and $0 \le r_i \le 2^{n+1-i} - 1$. Because addition is represented as disjoint union in our graphical interpretation, we will focus on monic monomials of the form (4.1) throughout this section. Similarly to Wood's construction in the nonequivariant setting at the prime 2, we will also utilize the 2-adic expansion of r_i,

$$r_i = \sum_{m=0}^{n-i} c_{i,n-i-m} \cdot 2^{n-i-m}$$

and extend Definitions 2.1 and 3.1 to make the following definition.

Definition 4.2 Given a monic monomial $x = \tau_0^{\epsilon_0}\tau_1^{\epsilon_1}\cdots\tau_n^{\epsilon_n}\xi_1^{r_1}\xi_2^{r_2}\cdots\xi_n^{r_n}$ where $\epsilon_i \in \{0, 1\}$ and $0 \le r_i \le 2^{n+1-i} - 1$, the ***standard form*** of x is

$$\tau_0^{\epsilon_0}\tau_1^{\epsilon_1}\cdots\tau_n^{\epsilon_n}\left(\prod_{i=1}^{n}\prod_{m=0}^{n-i}\xi_i^{c_{i,n-i-m}\cdot 2^{n-i-m}}\right)$$

where the exponents on the ξ_i are expanded into their 2-adic expansion.

Given such a monic monomial $x \in A^{\vee}(n)$, we define the graph $G_x = (V_x, E_x)$ by setting the ordered vertex set $V_x = \{\sigma, \rho, 2\rho, 2^2\rho, \cdots 2^n\rho\}$ and the edge set

$$E_x = \{\tau_i : \sigma \to 2^i\rho\}_{0\le i\le n} \bigcup \left\{\xi_i^{c_{i,j}\cdot 2^j} : 2^j\rho \to 2^{i+j}\rho\right\}_{\substack{1\le i\le n\\ 0\le j\le n-i}}$$

where each τ_i denotes an edge connecting vertices σ and $2^i\rho$, and each $\xi_i^{c_{i,j}\cdot 2^j}$ denotes $c_{i,j}$ edges connecting vertices $2^j\rho$ and $2^{i+j}\rho$. Notice that, similarly to nonequivariant setting, $|\xi_i^{2^j}|$, the degree of the edge $\xi_i^{2^j}$ when viewed as an element of $A^{\vee}_{C_2}(n)$ corresponds to the absolute value of the difference of the ends of $\xi_i^{2^j}$ and the degree of the edge τ_i when viewed as an element of $A^{\vee}_{C_2}(n)$, $|\tau_i|$, also corresponds to the absoslute value of the difference of the ends of τ_i.

Since $\mathbb{M}_2$ is a bigraded ring, and the coefficient $w \in \mathbb{M}_2$ may live in a nonzero bidegree, multiplication by $w \in \mathbb{M}_2$ may change the degree of $\tau_0^{\epsilon_0}\tau_1^{\epsilon_1}\cdots\tau_n^{\epsilon_n}\xi_1^{r_1}\xi_2^{r_2}\cdots\xi_n^{r_n}$. To account for this in our graphical interpretation, we view the vertices V_x of G_x as embedded in $\mathbb{R}^2$ with the embedding given by the bidegree of the vertices plus the bidegree of w as illustrated in the next two examples.

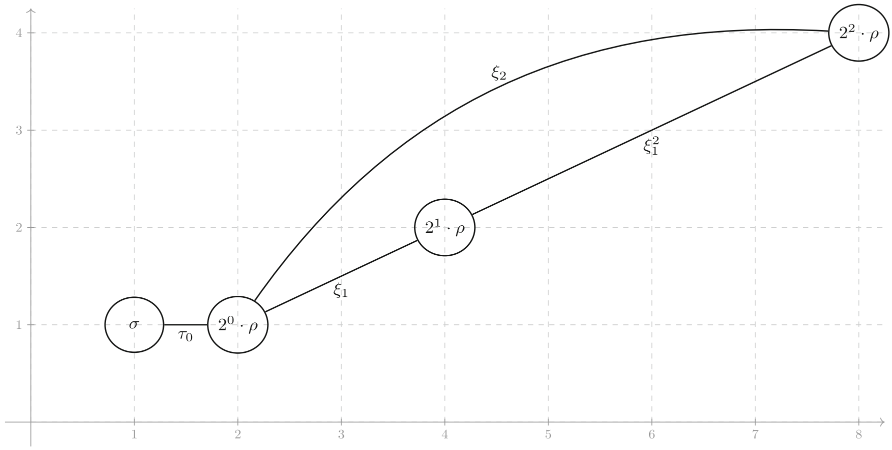


Fig. 6 The graph G_x associated to $x = \xi_1^3\xi_2\tau_0 \in A_{C_2}^{\vee}(2)$

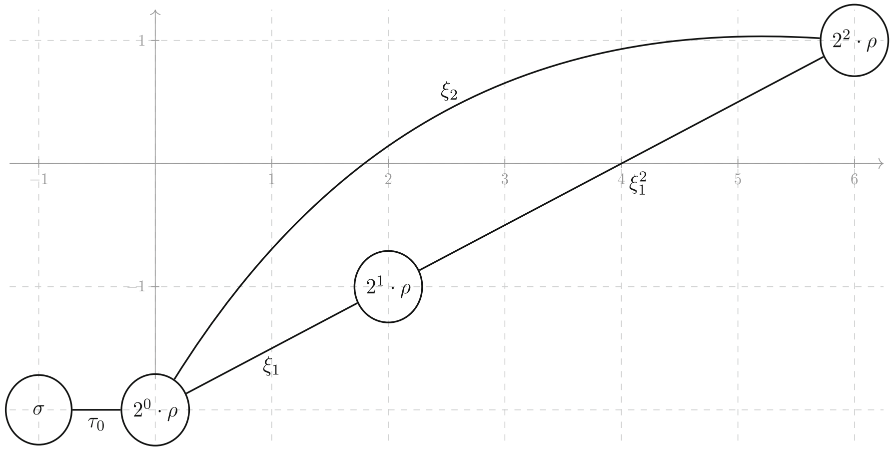


Fig. 7 The graph G_x associated to $x = a^2u\xi_1^3\xi_2\tau_0 \in A_{C_2}^{\vee}(2)$

Example 4.3 The monomial $x = \xi_1^3\xi_2\tau_0$ in $A_{C_2}^{\vee}(2)$ has associated graph G_x as depicted in Figure 6. Here, and throughout this section, we use the motivic grading convention where the topological degree is plotted along the horizontal axis while the weight is given by the vertical axis.

Example 4.4 The monomial $x = a^2u\xi_1^3\xi_2\tau_0 \in A_{C_2}^{\vee}(2)$ has associated graph G_x as depicted in Figure 7. It may be helpful to recall that $|a| = (-1, -1)$ and $|u| = (0, -1)$.

Finally, the adjacency matrix A_x associated to G_x has exactly the same form as in the odd primary case (3.2).

4.4 Connectivity and Unilaterality

Similarly to the nonequivariant odd primes case (§3.2), we begin by considering the subgraph $G_x \backslash \{\sigma\}$. This subgraph has adjacency matrix

$$\begin{bmatrix} 0 & c_{1,0} & c_{2,0} & & c_{n-1,0} & c_{n,0} \\ c_{1,0} & 0 & c_{1,1} & & c_{n-2,1} & c_{n-1,1} \\ c_{2,0} & c_{1,1} & 0 & & c_{n-3,2} & c_{n-2,2} \\ & & & \ddots & & \\ c_{n-1,0} & c_{n-2,1} & c_{n-3,2} & & 0 & c_{1,n-1} \\ c_{n,0} & c_{n-1,1} & c_{n-2,2} & & c_{1,n-1} & 0 \end{bmatrix}.$$

Since this matrix has the same form as the adjacency matrix in the nonequivariant p and odd prime case (3.2), we can apply a similar argument to prove the following proposition.

Proposition 4.5 *The subgraph $G_x \backslash \{\sigma\}$ is connected if and only if the integers*

$$C(r,s) := \sum_{t=1}^{n} \sum_{T_t'} \prod_{k=1}^{t} c_{|b_k - b_{k-1}|, \max(b_{k-1}, b_k) - 1}$$

are positive for all integers r and s with $0 \le r < s \le n$.

Returning to consider the entire graph G_x, we prove the following theorem, which has the same statement as in the nonequivariant odd-primary case (Theorem 3.11).

Theorem 4.6 *Suppose $x \in A^{\vee}_{C_2}(n)$ is a monic monomial in standard form. Then the graph G_x is connected if and only if*

(1) the integers

$$C(r,s) := \sum_{t=1}^{n} \sum_{T_t'} \prod_{k=1}^{t} c_{|b_k - b_{k-1}|, \max(b_{k-1}, b_k) - 1}$$

are positive for all integers r and s with $0 \le r < s \le n$ and

(2) the sum

$$\sum_{k=0}^{n} \epsilon_k > 0.$$

Proof The first condition guarantees that the subgraph $G \backslash \{\sigma\}$ is connected. The second condition guarantees that the vertex 1 is connected by an edge to the subgraph $G \backslash \{\sigma\}$.

Similarly to the nonequivariant setting, the graphs G_x can naturally be viewed as directed graphs with edges oriented in the direction of the vertex of larger dimension. We will again denote the directed version of the graph G_x by G_x^{dir}. Similar techniques of proof yield the following characterization of the unilaterality of G_x.

Theorem 4.7 *Suppose $x \in A^{\vee}_{C_2}(n)$ is a monic monomial in standard form. Then the directed graph G_x^{dir} is unilateral if and only if*

(1) the integers

$$U(r, s) := \sum_{t=1}^{n} \sum_{T_t''} \prod_{k=1}^{t} c_{b_{k-1}, b_k}$$

are positive for all integers r, s with $0 \le r < s \le n+1$, where T_t'' is the set of all $(t+1)$-tuples $(b_0, b_1, \cdots, b_t)$ with $b_i \in \{0, 1, \cdots, n+1\}$ and $r = b_0 < b_1 < \cdots < b_t = s$.

(2) And x is divisible by τ_0.

4.5 Trees and Hamiltonian Cycles

Theorem 4.8 *Suppose x is a monic monomial in standard form in $A^{\vee}_{C_2}(n)$ such that G_x is connected. Then G_x is a tree if and only if*

(1) the sum

$$\sum_{i=1}^{n} \alpha_p(r_i) = n,$$

where $\alpha_p(r_i)$ denotes the number of nonzero digits in the base-p expansion of r_i, and

(2) exactly one τ_k divides the connected graph x, for some $0 \le k \le n$.

Proof Let x be a monic monomial in standard form in $A^{\vee}_{C_2}(n)$ such that G_x is connected. Recall that in our graph construction, a digit equal to 1 in the base 2 expansion of r_i corresponds exactly to having one edge between two vertices of the graph. Thus,

$$\sum_{i=1}^{n} \alpha_p(r_i) = n$$

implies that there are exactly n edges contained in the subgraph $G_x \backslash \{\sigma\}$. If additionally, a singular τ_k divides x with $0 \le k \le n$, x is connected but not cyclic because G_x consists of $n+1$ distinct edges in a graph with $n+2$ vertices. Thus, the graph of G_x is a tree.

Recall that a Hamiltonian directed path for a graph G is defined as a directed path containing every vertex of G.

Theorem 4.9 *Let x be a monic monomial in standard form in $A^{\vee}_{C_2}(n)$. The graph G_x has a Hamiltonian-directed path if and only if $\tau_0 \xi_1^{2^{n+1}-1}$ divides x.*

Proof Suppose the graph G_x^{dir} associated with $x \in A^{\vee}_{C_2}(n)$ has a Hamiltonian directed path. Because G_x is directed such that each tail edge end vertex has representation dimension less than that of the head edge and every vertex appears in the

Hamiltonian directed path, the path must proceed through the ordered vertex set $V_x = \{\sigma, \rho, 2\rho, 2^2\rho, \cdots, 2^n\rho\}$ in order. In order for there to be an edge between the vertices σ and ρ, τ_0 must divide x. In order for there to be an edge between the vertices $2^k\rho$ and $2^{k+1}\rho$, $\xi_1^{2^k}$ must divide x. Hence

$$\tau_0\xi_1^{\sum_{k=0}^n 2^k} = \tau_0\xi_1^{2^{n+1}-1}$$

must divide x. On the other hand, if $\tau_0\xi_1^{2^{n+1}-1}$ divides x, then the graph G_x^{dir} has a Hamiltonian-directed path by construction.

Next we will turn to establishing a criterion for when the graph G_x has a Hamilton cycle. Our goal will be to again make use of Dirac's theorem, which says that a simple graph with n vertices ($n \geq 3$) is Hamiltonian if every vertex has degree $\frac{n}{2}$ or greater. Unlike the nonequivariant odd prime setting, the graphs in the C_2-equivariant setting are all simple graph so the arguments here are simpler than those of Section 3.3.

Let A_x be the adjacency matrix associated to G_x with the (i, j) entry of A_x denoted $a_{i,j}$. Define

$$\deg_{\mathrm{out}}(2^j\rho) = \#\{1 < i \leq n | a_{j+2,j+2+i} = 1\}$$

and

$$\deg_{\mathrm{in}}(2^j\rho) = \#\{1 < i \leq j+1 | a_{j+2-i,j+2} = 1\}$$

The following proposition is immediate from the definitions of A_x and G_x.

Proposition 4.10 *The out-degree of the vertex $2^j\rho$ in G_x^{dir} is given by* $\deg_{\mathrm{out}}$. *The in-degree of the vertex $2^j\rho$ in G_x^{dir} is given by* $\deg_{\mathrm{in}}(2^j\rho)$.

Since $\deg_{\mathrm{in}}(2^j\rho)+\deg_{\mathrm{out}}(2^j\rho)$ gives the total degree of the vertex $2^j\rho$ $(0 \leq j \leq n)$ in G_x^{dir}, it only remains to consider the degree of the vertex σ in order to apply Dirac's theorem.

Since σ is the initial vertex in the directed graph G_x^{dir}, the total degree of the vertex σ is

$$\sum_{k=0}^n \epsilon_k.$$

Then using Dirac's theorem we arrive at the following result.

Theorem 4.11 *Suppose $x = \tau_0^{\epsilon_0}\cdots\tau_n^{\epsilon_n}\xi_1^{r_1}\cdots\xi_n^{r_n}$ is a monic monomial in standard form in $A_{C_2}^\vee(n)$. The corresponding graph G_x has a Hamilton cycle if*

(1) $n > 0$,

(2) for each j with $0 \leq j \leq n$,

$$\deg_{\mathrm{in}}(2^j\rho) + \deg_{\mathrm{out}}(2^j\rho) \geq \frac{n}{2},$$

(3) and

$$\sum_{k=0}^n \epsilon_k \geq \frac{n}{2}.$$

4.6 Graph theoretic interpretation of coproduct and antipode

Similarly to Larson's graph theoretic description of the coproduct at the prime $p = 2$ (Theorem 2.13) and our graph theoretic description at odd primes (Section 3.4), we also have a graphical interpretation of the coproduct on elements $\xi_i^{2^j}$ and $\tau_i \in A_{C_2}^{\vee}(n)$.

Theorem 4.12 *The coproduct $\psi(\xi_i^{p^j}) \in A_{C_2}^{\vee}(n)$ is the sum of tensors of all pairs of edges that make length 2 directed paths from the vertex $2^j\rho$ to the vertex $2^{i+j}\rho$ in the complete graph on the vertices $\{\sigma, \rho, 2\rho, 2^2\rho, \cdots 2^n\rho\}$ considered as a directed graph.*

Similarly, the coproduct $\psi(\tau_i) \in A_{C_2}^{\vee}(n)$ is the sum of tensors of all pairs of edges that make length 2 directed paths from the vertex σ to the vertex $2^i\rho$.

Proof Recall that the coproduct on $A_{C_2}^{\vee}(n)$ is given by

$$\psi(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \xi_i$$

$$\psi(\tau_n) = \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \tau_i.$$

This formulas have the same form as in the case of the odd primary nonequivariant dual Steenrod algebra and the proof is the same as in that case.

Using a similar argument as in the nonequivariant p an odd prime case, we obtain a graphical description of the antipode on elements $\xi_i^{2^j}$ and $\tau_i \in A_{C_2}^{\vee}(n)$.

Theorem 4.13 *The antipode $c(\xi_i^{p^j}) \in A_{C_2}^{\vee}(n)$ is the sum of all directed paths from the vertex $2^j\rho$ to the vertex $2^{i+j}\rho$ in the complete graph on the vertices $\{\sigma, \rho, 2\rho, 2^2\rho, \cdots 2^n\rho\}$ considered as a directed graph.*

Similarly, the antipode $c(\tau_i) \in A_{C_2}^{\vee}(n)$ is the sum of all directed paths from the vertex σ to the vertex $2^i\rho$.

As a corollary, we also obtain another characterization of the unilaterality of the directed graphs G_x^{dir}.

Corollary 4.14 *Suppose x is a monic monomial in standard form in $A_{C_2}^{\vee}(n)$. Then the directed graph G_x^{dir} is unilateral if and only if*

(1) for each $\xi_i^{2^j} \in A_{C_2}^{\vee}(n)$, at least one summand of $c(\xi_i^{2^i})$ is a factor of x
(2) and τ_0 is a factor of x.

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