

THE BIGRADED HOMOTOPY GROUPS $\pi_{i,j}X$ OF A POINTED SIMPLICIAL SPACE X

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§1. INTRODUCTION

1.1 Summary. Just as differential graded modules are an essential tool in the study of modules over a ring, so are simplicial (and cosimplicial) spaces of considerable use in ordinary homotopy theory. In fact Stover [13] recently showed that pointed simplicial spaces are in particular useful in solving problems involving $\mathbf{\Pi}$ -algebras (recall that a $\mathbf{\Pi}$ -algebra is a ≥ 1 -graded group together with an action of all primary unstable homotopy operations; an example is the homotopy $\mathbf{\Pi}$ -algebra $\pi_*T = \{\pi_j T\}_{j=1}^\infty$ of a pointed space T).

Motivated by [13] we began in [5] a study of the homotopy theory of pointed simplicial spaces. In the present paper we further develop this theory by introducing the *bigraded homotopy groups* of a pointed simplicial space and observing that these groups behave very much like the ordinary homotopy groups of a pointed space. For instance, there is a kind of *Hurewicz theorem*; there is also a *Whitehead theorem* which says that a map of pointed simplicial spaces is a weak equivalence in the sense of [5] iff it induces an isomorphism on bigraded homotopy groups.

An interesting application will be given in [4], where we consider a Postnikov decomposition of a pointed simplicial space “in the simplicial direction” and (see 1.3) then use this to obtain *obstructions to realizing a given $\mathbf{\Pi}$ -algebra B* , i.e., to finding pointed spaces T such that $\pi_*T \cong B$.

1.2 Further details. In [5] we constructed a closed simplicial model category structure on the category $s\mathbf{T}_*$ of simplicial objects over the category $\mathbf{T}_*$ of pointed (topological) spaces, in which a map $X \rightarrow Y \in s\mathbf{T}_*$ is a weak equivalence (called an E^2 weak equivalence) whenever it induces isomorphisms $\pi_i \pi_* X \cong \pi_i \pi_* Y$ (where $\pi_i \pi_* X$ and $\pi_i \pi_* Y$ ($i \geq 0$) denote the $\mathbf{\Pi}$ -algebras obtained by applying the functor π_i to the simplicial $\mathbf{\Pi}$ -algebras $\pi_* X$ and $\pi_* Y$). This E^2 terminology is motivated by the fact that (see below) the groups $\pi_i \pi_j X$ form the E^2 term of a spectral sequence which (under a mild restriction on X) converges strongly to the homotopy groups of the realization $|X|$ of X .

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In this paper we note that there are *sphere-like objects* $\Sigma^i S^j \in s\mathbf{T}_*$ ($i \geq 0, j \geq 1$) which can be used to define the (i, j) -homotopy group $\pi_{i,j}X$ of an object $X \in s\mathbf{T}_*$ (as the group of the E^2 homotopy classes of maps $\Sigma^i S^j \rightarrow X \in s\mathbf{T}_*$, when X is E^2 fibrant). These groups are related to the (ordinary) homotopy groups of the realization $|X|$ by natural maps $\pi_{i,j}X \rightarrow \pi_{i+j}|X|$ which (under a mild restriction on X) are isomorphisms when $j = 1$).

For each $i \geq 0$, the graded group $\pi_{i,*}X = \{\pi_{i,j}X\}_{j=0}^\infty$ is actually a Π -algebra, which is called the i 'th homotopy Π -algebra of X . As one would expect, every E^2 fibration in $s\mathbf{T}_*$ gives rise to a long exact sequence of these homotopy Π -algebras and there is a Whitehead theorem which says that a map $X \rightarrow Y$ in $s\mathbf{T}_*$ is an E^2 weak equivalence iff it induces isomorphisms $\pi_{i,*}X \cong \pi_{i,*}Y$ ($i \geq 0$). This last result is an immediate consequence of the existence of the surprisingly useful natural *spiral exact sequence* of Π -algebras

$$\cdots \rightarrow \pi_{i+1}\pi_*X \rightarrow \Omega\pi_{i-1,*}X \rightarrow \pi_{i,*}X \rightarrow \pi_i\pi_*X \rightarrow \cdots$$

(in which $\Omega\pi_{i-1,*}X$ denotes the loop algebra of $\pi_{i-1,*}X$, i.e., the Π -algebra obtained from $\pi_{i-1,*}X$ by lowering all degrees by 1), which, for instance, immediately implies the Whitehead theorem. This exact sequence is actually an *exact couple* and its *associated spectral sequence* (with $E_{i,j}^2 = \pi_i\pi_jX$ for $i \geq 0$ and $j \geq 1$) coincides (up to a possible sign) with the Bousfield-Friedlander spectral sequence of a pointed simplicial path-connected space [2] as well as with the delooped version of the Quillen spectral sequence of a bisimplicial group [9]. The Π -algebra maps $\Omega\pi_{i,*}X \rightarrow \pi_{i+1,*}X$ in this spiral exact sequence consist of maps $\pi_{i,j}X \rightarrow \pi_{i+1,j-1}X$ ($j > 1$) which we will call *shift maps* and which are compatible with the (see above) natural maps to $\pi_{i+j}|X|$. It follows that (under a mild restriction on X) the spectral sequence converges strongly to $\pi_*|X|$ and that in fact $E_{i,j}^\infty$ ($i \geq 0, j \geq 1$) is isomorphic to the quotient of the image of $\pi_{i,j}X$ in $\pi_{i+j}|X|$ by the image of $\pi_{i-1,j+1}X$. The spiral exact sequence also readily implies that the Π -algebra maps $\pi_{i,*}X \rightarrow \pi_i\pi_*X$ satisfy a Hurewicz theorem, i.e., that the first nontrivial one is an isomorphism and the next one is onto. We therefore call these maps *Hurewicz maps*.

1.3 Application to the realizability of Π -algebras. A Π -algebra B has a *realization* (i.e., a space $T \in \mathbf{T}_*$ such that $\pi_*T \cong B$) iff [13] there exists an object $X \in s\mathbf{T}_*$ such that $\pi_0\pi_*X \cong B$ and $\pi_i\pi_*X \cong *$ for $i > 0$. In view of the above mentioned spiral exact sequence, this is equivalent to requiring that the shift maps $\Omega\pi_{i,*}X \rightarrow \pi_{i+1,*}X$ ($i \geq 0$) be isomorphisms, which implies that $\pi_{i,*}X \cong \Omega^i B$ for all $i \geq 0$. To decide whether B can be realized one can try to construct such an X by induction on its Postnikov approximations “in the simplicial direction”. This will be done in [4], where we observe in particular that

- (i) a 0-Postnikov approximation (i.e. an $X^0 \in s\mathbf{T}_*$, such that $\pi_{0,*}X^0 \cong B$ and $\pi_{k,*}X^0 \cong *$ for $k > 0$) always exists, and
- (ii) If $X^i \in s\mathbf{T}_*$ is an i -Postnikov approximation ($i \geq 0$) (i.e., in view of the spiral exact sequences $\pi_0\pi_*X^i \cong B$, $\pi_{i+2}\pi_*X^i \cong \Omega^{i+1}B$, $\pi_k\pi_*X^i \cong *$ for $k \neq 0, i+2$ and $\pi_k\pi_*X^i \cong *$ for $k > i$), then X^i can be extended to an $(i+1)$ -Postnikov approximation iff the k -invariant of the simplicial Π -algebra π_*X is trivial. This k -invariant lies in a Quillen cohomology group $H^{i+3}(B, \Omega^{i+1}B)$ which does not depend on the Postnikov approximation X^i , but only on the Π -algebra B and the integer i .

1.4 Organization of the paper. After fixing some notation and terminology (in 1.5 and 1.6) we review (in §2) the *Reedy model category structure* on the category $s\mathbf{T}_*$ of pointed simplicial spaces in which the weak equivalences are the dimensionwise weak homotopy equivalences and then consider (in §3) the E^2 *model category structure* on $s\mathbf{T}_*$ which has more weak equivalences and fewer cofibrations and fibrations, but still the same fibrant objects. Next (in §4) we discuss the *realization functor* $s\mathbf{T}_* \rightarrow \mathbf{T}_*$ and observe that, in either model category structure, it sends weak equivalences between cofibrant objects to weak homotopy equivalences.

The *bigraded homotopy groups* of a pointed simplicial space are introduced in §5, followed by the *shift maps* (in §6), the *Hurewicz maps* (in §7) and the *spiral exact sequence* and its associated spectral sequence (in §8).

In an appendix (§9) we review some results on $\mathbf{\Pi}$ -algebras, *abelian $\mathbf{\Pi}$ -algebras* and *simplicial $\mathbf{\Pi}$ -algebras* which are needed in the paper.

1.5 Some simplicial notation and terminology.

(i) *Simplicial objects.* Given a category $\mathbf{C}$, we denote by $s\mathbf{C}$ the category of *simplicial objects* over $\mathbf{C}$, i.e. [10, I, §2]. the category which has as objects the functors $\Delta^{\text{op}} \rightarrow \mathbf{C}$ and as maps the natural transformations between them (where Δ is the category which has as objects the finite ordered sets of integers $[n] = (0, \dots, n)$, ($n \geq 0$) and as maps the weakly monotone functions between them). For an object $X \in s\mathbf{C}$ we write, when no confusion can arise, X_n instead of $X[n]$ ($n \geq 0$), and we often identify the category $\mathbf{C}$ with the full subcategory of $s\mathbf{C}$ spanned by the constant functors. If the category $\mathbf{C}$ is *pointed* (i.e., *initial object* $= *$ *= terminal object*), then so is the category $s\mathbf{C}$.

(ii) *Simplicial sets.* As usual $\mathbf{S}$ will denote the category of *simplicial sets* (i.e., $s\mathbf{Sets}$) and $\mathbf{S}_*$ the category of *pointed simplicial sets* (i.e., simplicial pointed sets), $\Delta[n] \in \mathbf{S}$ ($n \geq 0$) will be the *standard n -simplex* (which has as j -simplices ($j \geq 0$) the maps $[j] \rightarrow [n] \in \Delta$ and which has in particular as its only non-degenerate n -simplex the identity map $\iota_n : [n] \rightarrow [n]$), and $\dot{\Delta}[n]$ and $\Delta^k[n]$ ($0 \leq k \leq n$) will be the subcomplexes of $\Delta[n]$ generated by the simplices $d_0\iota_n, \dots, d_n\iota_n$ and $d_0\iota_n, \dots, d_{k-1}\iota_n, d_{k+1}\iota_n, \dots, d_n\iota_n$ respectively.

In this notation one can, for a fibration $L \rightarrow L' \in \mathbf{S}_*$ with fibre L'' and with L, L' and L'' all fibrant (2.2), describe the *boundary map* $\pi_i L' \rightarrow \pi_{i-1} L''$ ($i \geq 1$) as the function which for every commutative diagram in $\mathbf{S}_*$

$$\begin{array}{ccccc}
 \Delta[i-1]/\dot{\Delta}[i-1] & \longrightarrow & \Delta[i]/\Delta^0[i] & \longrightarrow & \Delta[i]/\dot{\Delta}[i] \\
 \downarrow & & \downarrow & & \downarrow \\
 L'' & \longrightarrow & L & \longrightarrow & L'
 \end{array}$$

in which the horizontal maps are the obvious ones, sends the homotopy class of the map on the right to the homotopy class of the map on the left.

(iii) *A simplicial structure on $s\mathbf{C}$.* Let $\mathbf{C}$ be a category which is closed under finite limits and colimits and, for every object $X \in s\mathbf{C}$ and every *finite* simplicial set A (i.e., A is generated by a *finite* number of its simplices), let $A \otimes X \in s\mathbf{C}$ be the simplicial object which, in each dimension n , consists of the coproduct of as many copies of X_n as there

are n -simplices in A and which has the face and degeneracy operators induced by those of A and X [10, II, §1]. Then one can define a *simplicial structure* [10, II, §1] on $s\mathbf{C}$ by assigning to every pair of objects $X, Y \in s\mathbf{C}$ the *function complex* $\text{hom}(X, Y) \in \mathbf{S}$ which has as n -simplices ($n \geq 0$) the maps $\Delta[n] \otimes X \rightarrow Y \in s\mathbf{C}$, with the obvious [7, 6.4] faces and degeneracies. If $\mathbf{C}$ is pointed, then so are the function complexes in $s\mathbf{C}$.

1.6 Some topological notation and terminology.

(i) *Topological standard simplices.* We denote by Δ_n ($n > 0$) the *topological n -simplex* with vertices $v_0, \dots, v_n$, by $d^k : \Delta_{n-1} \rightarrow \Delta_n$ ($0 \leq k \leq n$, $n \geq 0$) and $s^k : \Delta_{n+1} \rightarrow \Delta_n$ ($0 \leq k \leq n$) the linear maps which send $v_0, \dots, v_{n-1}$ to $v_0, \dots, \hat{v}_k, \dots, v_n$ and $v_0, \dots, v_{n+1}$ to $v_0, \dots, v_k, v_k, \dots, v_n$ and by $\dot{\Delta}_n$ and Δ_n^0 the subspaces of Δ_n which consist of the unions of the images of the maps $d^0, \dots, d^n$ and $d^1, \dots, d^n$ respectively. The spaces Δ_n , $\dot{\Delta}_n$, and Δ_n^0 so defined can, in an obvious manner, be identified with the geometric realizations (4.1) of the simplicial sets $\Delta[n]$, $\dot{\Delta}[n]$ and $\Delta^0[n]$ (1.5). Given a Serre fibration $T \rightarrow T' \in \mathbf{T}_*$ (1.2) with fibre T'' one can then (as in 1.5) describe the *boundary maps* $\pi_i T' \rightarrow \pi_{i-1} T''$ ($i \geq 1$) in terms of the Δ_n , $\dot{\Delta}_n$ and Δ_n^0 . We also need the slightly different description of these boundary maps in terms of

(ii) *Spheres and balls.* We denote by $S^0 \in \mathbf{T}_*$ the 0-sphere and by $D^1 \in \mathbf{T}_*$ the 1-ball, i.e., the contractible CW complex obtained from S^0 by attaching one 1-cell. We then define inductively the 1-sphere S^1 by $S^1 = D^1/S^0$ and the i -sphere $S^i \in \mathbf{T}_*$ and the i -ball $D^i \in \mathbf{T}_*$ ($i > 1$) by $S^i = S^{i-1} \wedge S^1$ and $D^i = S^{i-1} \wedge D^1$. In this way there are obvious inclusions $S^{i-1} \rightarrow D^i$ and homeomorphisms $D^i/S^{i-1} \cong S^i$ ($i \geq 1$) as well as homeomorphisms $S^i \wedge S^j \cong S^{i+j}$ ($i, j \geq 0$) which are “associative” in the sense that the composition $S^i \wedge S^k \wedge S^k \cong S^{i+j} \wedge S^k \cong S^{i+j+k}$ coincides with the composition $S^i \wedge S^j \wedge S^k \cong S^i \wedge S^{j+k} \cong S^{i+j+k}$ ($i, j, k \geq 0$).

In this notation one can, for a Serre fibration $T \rightarrow T' \in \mathbf{T}_*$ with fibre T'' , describe the *boundary map* $\pi_i T' \rightarrow \pi_{i-1} T''$ ($i \geq 1$) as the function which, for every commutative diagram

$$\begin{array}{ccccc} S^{i-1} & \longrightarrow & D^i & \longrightarrow & S^i \\ \downarrow & & \downarrow & & \downarrow \\ T'' & \longrightarrow & T & \longrightarrow & T' \end{array} \quad \text{in } \mathbf{T}_*$$

in which the horizontal maps are the obvious ones, sends the homotopy class of the map on the right to the homotopy class of the map on the left. In order to avoid the appearance of a sign in proposition 6.3 below, which simultaneously involves these boundary maps and the ones of 1.5(ii) and 1.6(i), we identify the present boundary maps with the ones of 1.6(i) by choosing arbitrary but *fixed* homeomorphisms $\Delta_n/\dot{\Delta}_n \cong S^n$, $\Delta[n]/\Delta_n^0 \cong D^n$ ($n \geq 1$) and $\Delta_0/\dot{\Delta}_0 \cong S^0$ (where $\Delta_0/\dot{\Delta}_0$ should be interpreted as the union of Δ_0 with a disjoint base point) in $\mathbf{T}_*$ such that for each $n \geq 1$, the following diagram in $\mathbf{T}_*$, in which

the horizontal maps are the obvious ones, commutes

$$\begin{array}{ccccc}
 \Delta_{n-1}/\dot{\Delta}_{n-1} & \xrightarrow{d^0} & \Delta_n/\Delta_n^0 & \longrightarrow & \Delta_n/\dot{\Delta}_n \\
 \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\
 S^{n-1} & \longrightarrow & D^n & \longrightarrow & S^n
 \end{array} .$$

§2. THE REEDY MODEL CATEGORY STRUCTURE ON $s\mathbf{T}_*$

In this section we briefly review the *Reedy model category structure* ([5], [11]) on the category $s\mathbf{T}_*$ of pointed simplicial spaces, in which *the weak equivalences are the dimensionwise weak equivalences*.

We start with a discussion of

2.1 Matching objects. Let $\mathbf{C}$ be a category which is closed under finite limits and colimits. Given an object $X \in s\mathbf{C}$ and a finite simplicial set A (1.5) one can [2] define the *A-matching object* M_AX of X as the object $M_AX \in \mathbf{C}$ such that

- (i) M_AX is functorial, covariant in X and contravariant in A ,
- (ii) for ever map $b : \Delta[n] \rightarrow \Delta[n'] \in \mathbf{S}$ ($n, n' \geq 0$) the resulting map

$$X_{n'} = M_{\Delta[n']}X \rightarrow M_{\Delta[n]}X = X_n \in \mathbf{C}$$

is the simplicial operator corresponding to b , and

- (iii) if $\mathbf{A}$ is a finite diagram of finite simplicial sets, then

$$M_{\varinjlim \mathbf{A}}X = \varprojlim (M_{\mathbf{A}}X) .$$

Thus, if A is empty, M_AX is the terminal object of $\mathbf{C}$.

Instead of M_AX we write M_nX when $A = \dot{\Delta}[n]$ and M_n^kX ($0 \leq k \leq n, n > 0$) when $A = \Delta^k[n]$ (1.5).

2.2 Examples. [10, II, §3] Let $\mathbf{G}$ denote the category of groups. Then a map $K \rightarrow L$ in $\mathbf{S}$ or $s\mathbf{G}$ (1.5)

- (i) is a *fibration* iff, for every pair of integers (k, n) with $0 \leq k \leq n$ and $n > 0$, the induced map $K_n \rightarrow M_n^kK \prod_{M_n^kL} L_n$ is onto, and
- (ii) is a *trivial fibration* iff, for every integer $n \geq 0$ the induced map $K_n \rightarrow M_nK \prod_{M_nL} L_n$ is onto.

Now we can describe

2.3 Reedy model category structures. ([5], [11]) If $\mathbf{C}$ is a closed model category (e.g., if $\mathbf{C} = \mathbf{T}_*$, the category of pointed topological spaces [10, II §2]) then the category $s\mathbf{C}$ admits a closed model category structure [10, II, §2] in which

- (i) a map $X \rightarrow Y \in s\mathbf{C}$ is a *weak equivalence* (called a Reedy weak equivalence) if, for every integer $n \geq 0$, the restriction $X_n \rightarrow Y_n \in \mathbf{C}$ is a weak equivalence, and

- (ii) a map $X \rightarrow Y \in s\mathbf{C}$ is a fibration (called a Reedy fibration) if, for every integer $n \geq 0$, the induced map $X_n \rightarrow M_n X \prod_{M_n Y} Y_n \in \mathbf{C}$ is a fibration (i.e., a Serre fibration if $\mathbf{C} = \mathbf{T}_*$).

In view of 2.1 this implies

2.4 Proposition. *If A is a finite simplicial set (1.5), $B \subset A$ is a subcomplex and $X \rightarrow Y \in s\mathbf{C}$ is a Reedy fibration, then the induced map $M_A X \rightarrow M_B X \prod_{M_B Y} M_A Y \in \mathbf{C}$ (2.1) is a fibration (i.e., a Serre fibration if $\mathbf{C} = \mathbf{T}_*$).*

2.5 Examples. Call an object $X \in s\mathbf{T}_*$ a *CW object* if, for each $n \geq 0$, the pair $(X_n, *)$ is a relative CW complex and the degeneracy maps $s_0, \dots, s_n : X_n \rightarrow X_{n+1}$ send cells to cells. Then it is not difficult to see that every CW object $X \in s\mathbf{T}_*$ is *Reedy cofibrant*, but not necessarily Reedy fibrant. If $T \in \mathbf{T}_*$, then a Reedy fibrant object which is Reedy weakly equivalent to the constant object $T \in \mathbf{T}_* \subset s\mathbf{T}_*$ (1.5) is provided by the *expansion* of T , i.e., the object of $s\mathbf{T}_*$ which, in each dimension $n \geq 0$, consists of the function space T^{Δ_n} and in which the face and degeneracy maps are induced by the maps $d^k : \Delta_{n-1} \rightarrow \Delta_n$ and $s^k : \Delta_{n+1} \rightarrow \Delta_n$ of 1.6.

2.6 Remark. The above Reedy model category structure on $s\mathbf{T}_*$ is *not* compatible with the simplicial structure of 1.5, in the sense that this simplicial structure does *not* turn $s\mathbf{T}_*$ into a closed *simplicial* model category [10, II, §2]. However, there is another simplicial structure on $s\mathbf{T}_*$, the one “inherited” from the usual simplicial structure on $\mathbf{T}_*$ [10, II, §3], which is compatible with the Reedy model category structure. For any two objects $X, Y \in s\mathbf{T}_*$, the *function complex* $\text{hom}^R(X, Y) \in \mathbf{S}_*$ in this inherited simplicial structure has as n -simplices the maps $(\Delta_n \times X)/(\Delta_n \times *) \rightarrow Y \in s\mathbf{T}_*$ (where $(\Delta_n \times X)/(\Delta_n \times *) \in s\mathbf{T}_*$ denotes the object which in dimension i ($i \geq 0$) consists of the pointed space $(\Delta_n \times X_i)/(\Delta_n \times *)$) and has face and degeneracy maps induced by the maps $d^k : \Delta_{n-1} \rightarrow \Delta_n$ and $s^k : \Delta_{n+1} \rightarrow \Delta_n$ of 1.6.

§3. THE E^2 MODEL CATEGORY STRUCTURE ON $s\mathbf{T}_*$

Next we discuss another model category structure on the category $s\mathbf{T}_*$ of pointed simplicial spaces (called the *E^2 model category structure*) which, unlike the Reedy model category structure (2.3 and 2.6), is *compatible with the simplicial structure* of 1.5. It has more weak equivalences than the Reedy model category structure, fewer cofibrations and fibrations, but the same fibrant objects. We recall from [5]

3.1 The E^2 model category structure on $s\mathbf{T}_*$. *The category $s\mathbf{T}_*$ of pointed simplicial spaces admits a closed simplicial model category structure [10, II, §2] in which*

- (i) *the simplicial structure is as in 1.5,*
- (ii) *a map $X \rightarrow Y \in s\mathbf{T}_*$ is a weak equivalence (called E^2 weak equivalence) if, for every integer $j \geq 1$, the induced map $\pi_j X \rightarrow \pi_j Y$ is a weak equivalence of simplicial groups (i.e., the induced maps $\pi_i \pi_j X \rightarrow \pi_i \pi_j Y$ ($i \geq 0$) are isomorphisms) or equivalently (9.5) the induced map of simplicial Π -algebras $\pi_* X \rightarrow \pi_* Y$ is a weak equivalence, and*

- (iii) a map $X \rightarrow Y \in s\mathbf{T}_*$ is a *fibration* (called E^2 fibration) if it is a Reedy fibration (2.3) and if in addition, for every integer $j \geq 1$, the induced map $\pi_j X \rightarrow \pi_j Y$ is (2.2) a fibration of simplicial groups (i.e., the image of $\pi_j X$ in $\pi_j Y$ contains the identity component of $\pi_j Y$) or equivalently (9.5) the induced map of simplicial $\mathbf{\Pi}$ -algebras $\pi_* X \rightarrow \pi_* Y$ is a fibration.

This readily implies

3.2 Proposition. *Let $X \in s\mathbf{T}_*$ and let $X^b \subset X$ denote the subobject consisting of the path components of the base points of the X_n ($n \geq 0$). Then the inclusion $X^b \subset X \in s\mathbf{T}_*$ is an E^2 weak equivalence.*

3.3 Proposition. *Every Reedy weak equivalence is an E^2 weak equivalence and every trivial Reedy cofibration (resp. fibration) is a trivial E^2 cofibration (resp. fibration).*

3.4 Proposition. *The E^2 fibrant objects are just the Reedy fibrant ones.*

We also recall from [5] the following

3.5 Description of the E^2 cofibrant objects. The E^2 cofibrant objects of $s\mathbf{T}_*$ are the retracts of the “ S^1 -free” ones, where an object $X \in s\mathbf{T}_*$ is called S^1 -free if it is a CW object (2.5) with the property that, for every integer $n \geq 0$, the inclusion in X_n of the subcomplex $sX_{n-1} \subset X_n$ which consists of the union of the images of the degeneracy maps $s_0, \dots, s_{n-1} : X_{n-1} \rightarrow X_n$, can be extended to a homotopy equivalence $sX_{n-1} \coprod V_n \rightarrow X_n \in s\mathbf{T}_*$, in which V_n is a wedge of spheres of dimensions ≥ 1 .

Thus, if $Y \in s\mathbf{T}_*$ is E^2 cofibrant, then (3.2) $Y^b = Y$.

3.6 Examples. Some useful examples of E^2 cofibrant objects are the following:

- (i) For every object $U \in \mathbf{\Pi}$ (9.1), the corresponding constant object $U \in \mathbf{T}_* \subset s\mathbf{T}_*$ (1.5) is E^2 cofibrant and so is (1.5) its i -fold suspension

$$\Sigma^i U = (\Delta[i] \otimes U) / (\dot{\Delta}[i] \otimes U) \in s\mathbf{T}_* \quad i \geq 0$$

and the cone thereon

$$C\Sigma^i U = (\Delta[i+1] \otimes U) / (\Delta^0[i+1] \otimes U) \in s\mathbf{T}_* \quad i \geq 0.$$

They are related by the E^2 cofibration $\Sigma^i U \rightarrow C\Sigma^i U \in s\mathbf{T}_*$ induced by the map $\Delta[i] \rightarrow \Delta[i+1] \in \mathbf{S}$ which sends ι_n to $d_0 \iota_{n+1}$ (1.5). Moreover, one readily verifies [5, 4.1] that $C\Sigma^i U$ is E^2 contractible, i.e., E^2 weakly equivalent to $*$.

- (ii) For every object $U \in \mathbf{\Pi}$ (9.1) and $i \geq 0$ there is also the E^2 cofibrant object (1.5)

$$\bar{\Sigma}^i(S^1 \wedge U) = \Sigma^i(S^1 \wedge U) \coprod_{\Sigma^i(D^1 \wedge U)} C\Sigma^i(D^1 \wedge U)$$

which has the same E^2 homotopy type as $\Sigma^i(S^1 \wedge U)$ (as the obvious map $\Sigma^i(S^1 \wedge U) \rightarrow \bar{\Sigma}^i(S^1 \wedge U) \in s\mathbf{T}_*$ is clearly a Reedy weak equivalence).

However, although there exists no non-trivial map $\Sigma^{i+1}U \rightarrow \Sigma^i(S^1 \wedge U) \in s\mathbf{T}_*$ which is natural in U , there exists a natural map $\Sigma^{i+1}U \rightarrow \bar{\Sigma}^i(S^1 \wedge U) \in s\mathbf{T}_*$ (namely the map which in dimension $i+1$ consists of the obvious composition

$$(\Sigma^{i+1}U)_{i+1} \cong U \rightarrow D^1 \wedge U \rightarrow (C\Sigma^i(D^1 \wedge U))_{i+1} \rightarrow (\bar{\Sigma}^i(S^1 \wedge U))_{i+1}$$

which is not only non-trivial, but also, as we will see in 6.1 and 7.3, rather useful. Note that this map fits into a *pushout square*

$$\begin{array}{ccc} C\Sigma^i U & \longrightarrow & C\Sigma^i(D^1 \wedge U) \\ \downarrow & & \downarrow \\ \Sigma^{i+1} U & \longrightarrow & \bar{\Sigma}^i(S^1 \wedge U) \end{array}$$

in which the other maps are the obvious ones.

3.7 Remark. The 0-sphere $S^0 \in \mathbf{T}_* \subset s\mathbf{T}_*$ has the somewhat unexpected property that the map $* \rightarrow S^0 \in s\mathbf{T}_*$ is an E^2 weak equivalence as well as an E^2 fibration, but not an E^2 cofibration. Thus S^0 is not E^2 cofibrant. Still, as the following proposition shows, the function complex $\text{hom}(S^0, X)$ (1.5) has E^2 homotopy meaning if X is E^2 fibrant and $X^b = X$ (3.2), i.e., each X_n is path connected.

3.8 Proposition. *Let $X \in s\mathbf{T}_*$ be E^2 fibrant and such that (3.2) $X^b = X$ (e.g., X is E^2 cofibrant). Then*

- (i) *the function complex $\text{hom}(D^1, X) \in \mathbf{S}_*$ (1.5 and 1.6) is contractible, in fact the map $\text{hom}(D^1, X) \rightarrow * \in \mathbf{S}_*$ is a trivial fibration (2.2), and*
- (ii) *the obvious map $\text{hom}(D^1, X) \rightarrow \text{hom}(S^0, X) \in \mathbf{S}_*$ (1.6) is a fibration (2.2) which is onto and which has the function complex $\text{hom}(S^1, X) \in \mathbf{S}_*$ as its fibre.*

Proof. The E^2 fibrancy of X , together with 2.2 and 2.4, readily implies (i). Moreover, it follows from the path connectedness of the X_n , 2.2(i) above and lemma B.11 of [2] that (2.4) the Serre fibrations $X_n \rightarrow M_n^k X \in \mathbf{T}_*$ ($0 \leq k \leq n$) have path connected fibres and from this one readily deduces the first part of (ii). The rest of (ii) is immediate.

We end with observing that an E^2 fibration gives rise to a long exact sequence of $\mathbf{\Pi}$ -algebras as follows. Let $f : X \rightarrow Y \in s\mathbf{T}_*$ be an E^2 fibration, let $\pi_* f : \pi_* X \rightarrow \pi_* Y$ be the induced fibration of simplicial $\mathbf{\Pi}$ -algebras, let F_f and $F_{\pi_* f}$ denote the fibres of f and $\pi_* f$ and let C be the cokernel of the induced map of abelian $\mathbf{\Pi}$ -algebras $\Omega\pi_0\pi_* X \rightarrow \Omega\pi_0\pi_* Y$ (9.4). A straightforward calculation using 9.8 then yields

3.9 Proposition. *There is a natural long exact sequence of $\mathbf{\Pi}$ -algebras*

$$\cdots \rightarrow \pi_i F_{\pi_* f} \rightarrow \pi_i \pi_* X \rightarrow \pi_i \pi_* Y \rightarrow \pi_{i-1} F_{\pi_* f} \rightarrow \cdots \rightarrow \pi_0 \pi_* X \rightarrow \pi_0 \pi_* Y.$$

Moreover, there is a natural short exact sequence of simplicial $\mathbf{\Pi}$ -algebras $* \rightarrow C \rightarrow \pi_* F_f \rightarrow F_{\pi_* f} \rightarrow *$ and hence there are natural isomorphisms of $\mathbf{\Pi}$ -algebras

$$\pi_i \pi_* F_f \cong \pi_i F_{\pi_* f} \quad \text{for } i \geq 2$$

and a natural exact sequence of $\mathbf{\Pi}$ -algebras

$$* \rightarrow \pi_1 \pi_* F_f \rightarrow \pi_1 F_{\pi_* f} \rightarrow C \rightarrow \pi_0 \pi_* F_f \rightarrow \pi_0 F_{\pi_* f} \rightarrow *.$$

3.10 Corollary. *If the map $\pi_0\pi_*X \rightarrow \pi_0\pi_*Y$ is onto (e.g., if the map $\pi_*X \rightarrow \pi_*Y$ is onto), then the long exact sequence of (3.9) becomes*

$$\cdots \rightarrow \pi_i\pi_*F_f \rightarrow \pi_i\pi_*X \rightarrow \pi_i\pi_*Y \rightarrow \pi_{i-1}\pi_*F_f \rightarrow \cdots \rightarrow \pi_0\pi_*X \rightarrow \pi_0\pi_*Y \rightarrow *.$$

§4. THE REALIZATION FUNCTOR $s\mathbf{T}_* \rightarrow \mathbf{T}_*$

In this section we consider the realization functor $s\mathbf{T}_* \rightarrow \mathbf{T}_*$ and observe that it is well behaved with respect to both the Reedy model category structure on $s\mathbf{T}_*$ (§2) and the E^2 model category structure (§3), in the sense that *in both model category structures a weak equivalence between cofibrant objects induces a weak homotopy equivalence between their realizations*. We also note that, for a Reedy fibrant and Reedy cofibrant object $X \in s\mathbf{T}_*$, its realization has the same homotopy type as the realization of the simplicial discrete space obtained from X by forgetting the topologies.

We start with recalling the definition ([11], [12]) of

4.1 The realization functor $s\mathbf{T}_* \rightarrow \mathbf{T}_*$. This is the left adjoint of the expansion functor (2.5). For an object $X \in s\mathbf{T}_*$, its *realization* $|X| \in \mathbf{T}_*$ thus is the colimit of the diagram which consists of

- (i) for every integer $n \geq 0$, a copy $(\Delta_n \times X_n)$ of $\Delta_n \times X_n$, and
- (ii) for every pair of integers (k, n) with $0 \leq k \leq n$, a copy $(\Delta_{n-1} \times X_n; k)$ of $\Delta_{n-1} \times X_n$ and a copy $(\Delta_{n+1} \times X_n; k)$ of $\Delta_{n+1} \times X_n$, together with maps

$$(\Delta_{n-1} \times X_{n-1}) \xleftarrow{d_k} (\Delta_{n-1} \times X_n; k) \xrightarrow{d^k} (\Delta_n \times X_n)$$

$$(\Delta_{n+1} \times X_{n+1}) \xleftarrow{s_k} (\Delta_{n+1} \times X_n; k) \xrightarrow{s^k} (\Delta_n \times X_n)$$

induced by the face and degeneracy maps of X and the maps $d^k : \Delta_{n-1} \rightarrow \Delta_n$ and $s^k : \Delta_{n+1} \rightarrow \Delta_n$ of 1.6. Clearly $|X|$ so defined comes with a natural map $X_0 \rightarrow |X| \in \mathbf{T}_*$.

If $X \in s\mathbf{T}_*$ is a CW object (2.5) then $|X|$ is a CW complex. It is also immediate that, if one considers an object $L \in \mathbf{S}_*$ (1.5) as a simplicial discrete space, then its realization coincides with the usual [8] *geometric realization* $|L|$ of L .

4.2 Example. Let $U \in \mathbf{\Pi}$ (9.1). Then it is not difficult to see (1.6 and 3.6) that, for every integer $i \geq 0$, there is a natural homeomorphism $|\Sigma^i U| \cong S^i \wedge U \in \mathbf{T}_*$ (which of course depends on the choice of homeomorphism $\Delta_i/\dot{\Delta}_i \cong S^i$ made in 1.6).

As mentioned above, the realization functor is well behaved with respect to the Reedy model category structure, i.e., one has ([11], [12])

4.3 Proposition. *Let $X \rightarrow Y \in s\mathbf{T}_*$ be a Reedy weak equivalence between Reedy cofibrant objects. Then the induced map $|X| \rightarrow |Y| \in \mathbf{T}_*$ is a weak homotopy equivalence.*

4.4 Corollary. *Let $X \in s\mathbf{T}_*$ be Reedy cofibrant and such that all face and degeneracy maps between the X_n ($n \geq 0$) are weak homotopy equivalences. Then the natural map $X_0 \rightarrow |X| \in \mathbf{T}_*$ (4.1) is also a weak homotopy equivalence.*

The realization functor is also well behaved with respect to the E^2 model category structure, i.e.,

4.5 Proposition. *Let $X \rightarrow Y \in s\mathbf{T}_*$ be an E^2 weak equivalence between E^2 cofibrant objects. Then the induced map $|X| \rightarrow |Y| \in \mathbf{T}_*$ is a weak homotopy equivalence.*

Proof. We may assume that X and Y are Reedy fibrant and hence (3.4) E^2 fibrant. In that case the map $X \rightarrow Y$ is an E^2 homotopy equivalence and the desired result now follows readily from the existence of a natural homeomorphism $|\Delta[1] \otimes X| \cong (\Delta_1 \times |X|)/(\Delta_1 \times *) \in \mathbf{T}_*$ [12].

We end with observing that for an object $X \in s\mathbf{T}_*$, the function complex $\text{hom}(S^0, X) \in \mathbf{S}_*$ (1.5) has as n -simplices ($n \geq 0$) exactly the points of the space X_n , i.e., $\text{hom}(S^0, X)$ is the simplicial set obtained from X by “forgetting the topologies”. We now show

4.6 Proposition. *Let $X \in s\mathbf{T}_*$ be Reedy fibrant and Reedy cofibrant. Then the (see 4.1) obvious map $|\text{hom}(S^0, X)| \rightarrow |X| \in \mathbf{T}_*$ is a weak homotopy equivalence.*

Proof. Let $Z \in s\mathbf{S}_*$ (1.5) be the simplicial object obtained from X by dimensionwise application of the singular functor (i.e., Z_n is the singular complex of X_n , $n \geq 0$), let $Z_{n,k}$ ($n, k \geq 0$) denote the set of k -simplices of Z_n , let $\text{diag } Z \in \mathbf{S}_*$ denote the diagonal of Z (i.e., $(\text{diag } Z)_n = Z_{n,n}$, $n \geq 0$), let $Z^t \in s\mathbf{S}_*$ denote the transpose of Z (i.e., $Z_{n,k}^t = Z_{k,n}$, $n, k \geq 0$) and let $\bar{Z}$ and $\bar{Z}^t \in \mathbf{S}_*$ denote the objects obtained by applying the simplicial analog of the realization functor of 4.1 to Z and Z^t . Then [2, B.1] there are natural isomorphisms

$$\bar{Z}^t \cong \text{diag } Z^t = \text{diag } Z \cong \bar{Z} \in \mathbf{S}_*.$$

Moreover, the Reedy fibrancy of X implies a similarly defined Reedy fibrancy of Z and (see [11, 4.2]) it is not difficult to deduce from this that the face and degeneracy maps between the Z_n^t ($n \geq 0$) are weak equivalences. In view of the simplicial analog of 4.4, the natural map $Z_0^t \rightarrow \bar{Z}^t \in \mathbf{S}_*$ thus is a weak equivalence and the induced map $|Z_0^t| \rightarrow |\bar{Z}^t| \in \mathbf{T}_*$ thus is a weak homotopy equivalence.

Let $Y \in s\mathbf{T}_*$ be the simplicial object obtained from Z by dimensionwise application of the (see 4.1) geometric realization functor (i.e., $Y_n = |Z_n|$, $n \geq 0$). One readily sees that the canonical map $Y \rightarrow X \in s\mathbf{T}_*$ satisfies the conditions of 4.3 and thus induces a weak homotopy equivalence $|Y| \rightarrow |X| \in \mathbf{T}_*$.

Finally one notes the existence of obvious isomorphisms $\text{hom}(S^0, X) \cong Z_0^t \in \mathbf{S}_*$ and $|\bar{Z}| \cong |Y| \in \mathbf{T}_*$ and the desired result now follows from the observation that the composition of the weak homotopy equivalences

$$|\text{hom}(S^0, X)| \cong |Z_0^t| \rightarrow |\bar{Z}^t| \cong |\text{diag } Z^t| = |\text{diag } Z| \cong |\bar{Z}| \cong |Y| \rightarrow |X|$$

is exactly the map in which we are interested.

§5. THE HOMOTOPY $\mathbf{\Pi}$ -ALGEBRAS $\pi_{i,*}X$ OF A POINTED SIMPLICIAL SPACE X

We now introduce the central notion of this paper, the *homotopy $\mathbf{\Pi}$ -algebras* $\pi_{i,*}X = \{\pi_{i,j}X\}_{j=1}^{\infty}$ ($i \geq 0$) of a pointed simplicial space X . These $\mathbf{\Pi}$ -algebras play a role in E^2 homotopy theory (§3) which is rather similar to that of the homotopy groups in ordinary homotopy theory. Thus every E^2 fibration gives rise to a long exact sequence of these *homotopy $\mathbf{\Pi}$ -algebras* and there is a *Whitehead theorem* which says that a map $X \rightarrow Y \in s\mathbf{T}_*$ is an E^2 weak equivalence iff it induces isomorphisms $\pi_{i,*}X \cong \pi_{i,*}Y$ ($i \geq 0$). We also note that the groups $\pi_{i,j}X$ are related to the homotopy groups of the realization $|X|$ by *natural maps* $\pi_{i,j}X \rightarrow \pi_{i+j}|X|$ ($i \geq 0, j \geq 1$) which are isomorphisms when X is E^2 cofibrant and $j = 1$.

We start with a definition of the homotopy $\mathbf{\Pi}$ -algebras which implies that the $\pi_{i,j}X$ consist of the E^2 homotopy classes of maps from the *sphere-like objects* $\Sigma^i S^j$ (3.6). At the end of this section we give another description, in terms of the spaces X_n ($n \geq 0$) and the face maps between them.

5.1 The homotopy $\mathbf{\Pi}$ -algebras $\pi_{i,*}X$. Let $X \in s\mathbf{T}_*$, and let $X \rightarrow Y \in s\mathbf{T}_*$ be a trivial Reedy cofibration such that Y is Reedy fibrant (2.3) and hence (3.4) E^2 fibrant. Then it is not difficult to verify that, for every integer $i \geq 0$, the functor $\mathbf{\Pi} \rightarrow \mathbf{Sets}_*$ (9.1) which sends an object $U \in \mathbf{\Pi}$ to $\pi_0 \text{hom}(\Sigma^i U, Y)$ (3.6), i.e., the pointed set of the E^2 homotopy classes of maps $\Sigma^i U \rightarrow Y \in s\mathbf{T}_*$, is a $\mathbf{\Pi}$ -algebra, which does not depend on the choice of the map $X \rightarrow Y$. We will therefore denote this $\mathbf{\Pi}$ -algebra by $\pi_{i,*}X$ and call it the *i 'th homotopy $\mathbf{\Pi}$ -algebra of X* . The group in degree j ($j \geq 1$), i.e. $(\pi_{i,*}X)S^j$, will be denoted by $\pi_{i,j}X$ and called the *(i, j) -homotopy group of X* . As $\pi_0 \text{hom}(\Sigma^i U, Y) \cong \pi_i \text{hom}(U, Y)$, the $\mathbf{\Pi}$ -algebra $\pi_{i,*}X$ is clearly (9.3) abelian if $i > 0$.

Obviously the above definition is natural in X , and the resulting functors $\pi_{i,*} : s\mathbf{T}_* \rightarrow \mathbf{\Pi}\text{-al}$ ($i \geq 0$) (9.1) send E^2 weak equivalences to isomorphisms. One has [10, I, §3]

5.2 Proposition. *Let $X \rightarrow X' \in s\mathbf{T}_*$ be an E^2 fibration (3.1) and let X'' be its fibre. Then there is a natural exact sequence of $\mathbf{\Pi}$ -algebras*

$$\cdots \rightarrow \pi_{i+1,*}X' \rightarrow \pi_{i,*}X'' \rightarrow \pi_{i,*}X \rightarrow \pi_{i,*}X' \rightarrow \cdots \rightarrow \pi_{0,*}X'.$$

There is also a

5.3 Whitehead theorem. *A map $X \rightarrow Y \in s\mathbf{T}_*$ is an E^2 weak equivalence (3.1) iff the induced map $\pi_{i,*}X \rightarrow \pi_{i,*}Y$ is an isomorphism for all $i \geq 0$. More generally, given a map $X \rightarrow Y \in s\mathbf{T}_*$ and an integer $n \geq 1$, the induced map $\pi_i \pi_j X \rightarrow \pi_i \pi_j Y$ is an isomorphism for all $j \geq n$ and $i \geq 0$ iff the induced map $\pi_{i,j}X \rightarrow \pi_{i,j}Y$ is an isomorphism for all $j \geq n$ and $i \geq 0$.*

Proof. This follows readily from 8.1.

Next we consider

5.4 The natural map $\pi_{i,j}X \rightarrow \pi_{i+j}|X|$. Let $X \in s\mathbf{T}_*$ be Reedy cofibrant (e.g., a CW object (2.5)) and let $X \rightarrow Y \in s\mathbf{T}_*$ be as in 5.1. Then the realization functor (4.1) sends each map $\Sigma^i U \rightarrow Y \in s\mathbf{T}_*$ ($U \in \mathbf{\Pi}$) to a map $S^i \wedge U \cong |\Sigma^i U| \rightarrow |Y| \in \mathbf{T}_*$ (4.2).

Thus (9.4 and 4.5) it induces a *natural map* $\pi_{i,*}X \rightarrow \Omega^i \pi_*|X| \in \mathbf{\Pi}\text{-}\mathbf{al}$ and in particular it induces *group homomorphisms* $\pi_{i,j}X \rightarrow \pi_{i+j}|X|$ for $i \geq 0$ and $j \geq 1$ (homomorphisms which of course depend on the choice of the homeomorphisms $\Delta_i/\dot{\Delta}_i \cong S^i$ in 1.6). We now prove

5.5 Proposition. If $X \in s\mathbf{T}_*$ is Reedy cofibrant and such that $X^b = X$ (3.2), then the natural map $\pi_{i,1}X \rightarrow \pi_{i+1}|X|$ is an isomorphism for all $i \geq 0$. If in addition, for some integer $n \geq 1$, the X_i ($i \geq 0$) are all $(n-1)$ -connected, then the natural map $\pi_{i,j}X \rightarrow \pi_{i+j}|X|$ is an isomorphism for all $j \leq n$ and $i \geq 0$.

Proof. We may assume that X is Reedy fibrant and hence (3.4) E^2 fibrant. The first part of the proposition then follows from 3.8 and 4.6, once we have shown that the map $\pi_{i,j}X \rightarrow \pi_{i+j}|X|$ admits a factorization

$$\pi_{i,1}X = \pi_0 \operatorname{hom}(\Sigma^i S^1, X) \xrightarrow{\cong} \pi_i \operatorname{hom}(S^1, X) \rightarrow \pi_{i+1} \operatorname{hom}(S^0, X) \rightarrow \pi_{i+1}|X|$$

in which the first map results from the fact that $\operatorname{hom}(\Sigma^i S^1, X)$ is the i -fold loops on $\operatorname{hom}(S^1, X)$, the second map is inverse to the boundary map of the fibration $\operatorname{hom}(D^1, X) \rightarrow \operatorname{hom}(S^0, X)$ of 3.8 and the last map is as in 4.6. To verify this claim, one notes (using 3.6) that a commutative diagram in $\mathbf{S}_*$ of the form (1.5)

$$\begin{array}{ccccc} \Delta[i]/\dot{\Delta}[i] & \longrightarrow & \Delta[i+1]/\Delta^0[i+1] & \longrightarrow & \Delta[i+1]/\dot{\Delta}[i+1] \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{hom}(S^1, X) & \longrightarrow & \operatorname{hom}(D^1, X) & \longrightarrow & \operatorname{hom}(S^0, X) \end{array}$$

yields a map $\bar{\Sigma}^i(S^1 \wedge S^0) \rightarrow X \in s\mathbf{T}_*$ such that the composition $\Sigma^i S^1 \rightarrow \bar{\Sigma}^i(S^1 \wedge S^0) \rightarrow X$ corresponds to the map on the left and the composition $\Sigma^{i+1} S^0 \rightarrow \bar{\Sigma}^i(S^1 \wedge S^0) \rightarrow X$ corresponds to the map on the right. Applying the realization functor and the homeomorphisms of 1.6, it now suffices to observe that the maps $|\Sigma^i S^1| \rightarrow |\bar{\Sigma}^i(S^1 \wedge S^0)|$ and $|\Sigma^{i+1} S^0| \rightarrow |\bar{\Sigma}^i(S^1 \wedge S^0)|$ correspond to the following homotopy equivalences which are readily verified to be homotopic

$$\begin{aligned} S^{i+1} &\cong S^i \wedge S^0 \wedge S^1 \cong (S^i \wedge S^0 \wedge D^1)/(S^i \wedge S^0 \wedge S^0) \rightarrow (S^i \wedge D^1 \wedge D^1)/(S^i \wedge S^0 \wedge S^0) \\ S^{i+1} &\cong S^i \wedge S^1 \wedge S^0 \cong (S^i \wedge D^1 \wedge S^0)/(S^i \wedge S^0 \wedge S^0) \rightarrow (S^i \wedge D^1 \wedge D^1)/(S^i \wedge S^0 \wedge S^0). \end{aligned}$$

The second part of the proposition now follows readily from the first part and propositions 6.3 and 8.1.

We end with a description of the $\pi_{i,*}X$ in terms of

5.6 A differential graded space. Let $Y \in s\mathbf{T}_*$ and, for every integer $i \geq 0$, let $C_i Y$ and $Z_i Y \subset Y_i$ denote the intersection of the fibres of the face maps $d_1, \dots, d_i$ and $d_0, \dots, d_i$ respectively. The restrictions of the face maps $d_0 : Y_i \rightarrow Y_{i-1}$ then give rise to a *differential graded space*

$$\cdots \rightarrow C_i Y \xrightarrow{d_0} C_{i-1} Y \rightarrow \cdots \rightarrow C_0 Y = Y_0 \rightarrow *.$$

i.e., a sequence of maps in $\mathbf{T}_*$ such that the composition of any two adjacent maps is trivial. Then 2.1 and 2.4 readily imply

5.7 Proposition. *Let $Y \in s\mathbf{T}_*$ be Reedy fibrant and let $i \geq 0$. Then the map $d_0 : C_{i+1}Y \rightarrow Z_iY \in \mathbf{T}_*$ is a Serre fibration with $Z_{i+1}Y$ as its fibre.*

The promised description of $\pi_{i,*}X$ now is contained in

5.8 Proposition. *Let $Y \in s\mathbf{T}_*$ be Reedy fibrant and let $i \geq 0$. Then there is a natural exact sequence of $\mathbf{\Pi}$ -algebras*

$$\pi_*C_{i+1}Y \xrightarrow{\pi_*d_0} \pi_*Z_iY \rightarrow \pi_{i,*}Y \rightarrow *.$$

Proof. Given an object $U \in \mathbf{\Pi}$ (9.1), the maps $U \rightarrow Z_iY \in \mathbf{T}_*$ are in a natural 1–1 correspondence with the maps $\Sigma^iU \rightarrow Y \in s\mathbf{T}_*$ (3.6) and homotopies $(\Delta_1 \times U)/(\Delta_1 \times *) \rightarrow Z_iY \in \mathbf{T}_*$ are in a similar correspondence with “homotopies” $\Sigma^i((\Delta_1 \times U)/(\Delta_1 \times *)) \rightarrow Y \in s\mathbf{T}_*$ of the kind appropriate (2.6) to the Reedy model category structure on $s\mathbf{T}_*$. Since (3.3) every Reedy weak equivalence is an E^2 weak equivalence, these correspondences give rise to a function from the homotopy classes of maps $U \rightarrow Z_iY \in \mathbf{T}_*$ to the E^2 homotopy classes of maps $\Sigma^iU \rightarrow Y \in s\mathbf{T}_*$, which is natural in U , i.e., (9.1) a map of $\mathbf{\Pi}$ -algebras $\pi_*Z_iY \rightarrow \pi_{i,*}Y$. Clearly this map is onto. Furthermore, a map $\Sigma^iU \rightarrow Y \in s\mathbf{T}_*$ is E^2 homotopic to the trivial map iff it extends to a map $C\Sigma^iU \rightarrow Y \in s\mathbf{T}_*$ (3.6) and exactness at π_*Z_iY now follows from the natural 1–1 correspondence between the maps $C\Sigma^iU \rightarrow Y \in s\mathbf{T}_*$ and the maps $U \rightarrow C_{i+1}Y \in \mathbf{T}_*$.

§6. THE SHIFT MAPS $\pi_{i,j+1}X \rightarrow \pi_{i+1,j}X$

A somewhat surprising feature of the homotopy groups $\pi_{i,j}X$ ($i \geq 0, j \geq 1$) is that they are connected by *shift maps* $\pi_{i,j+1}X \rightarrow \pi_{i+1,j}X$ which

- (i) fit together into maps of $\mathbf{\Pi}$ -algebras $\Omega\pi_{i,*} \rightarrow \pi_{i+1,*}X$ ($i \geq 0$), and
- (ii) for Reedy cofibrant X , are compatible with the natural maps $\pi_{i,j}X \rightarrow \pi_{i+j}|X|$ of 5.4 (so that (5.5) if $X^b = X$, each group $\pi_n|X|$ ($n \geq 1$) is filtered by the images of $\pi_{0,n}X, \dots, \pi_{n-1,1}X$).

We describe these shift maps in two different ways. The first involves the original definition of the homotopy $\mathbf{\Pi}$ -algebras $\pi_{i,*}X$ (5.1), while the other uses their alternate description (5.8).

6.1 The shift maps. Let $X \in s\mathbf{T}_*$, let $X \rightarrow Y \in s\mathbf{T}_*$ be as in 5.1 and let $i \geq 0$. Furthermore consider, for every object $U \in \mathbf{\Pi}$ (9.1), the trivial E^2 cofibration (3.6) $\Sigma^i(S^1 \wedge U) \rightarrow \bar{\Sigma}^i(S^1 \wedge U) \in s\mathbf{T}_*$ and the maps $\Sigma^{i+1}U \rightarrow \bar{\Sigma}^i(S^1 \wedge U) \in s\mathbf{T}_*$ described in 3.6. Composing the inverses of the resulting isomorphisms $\pi_0 \text{hom}(\bar{\Sigma}^i(S^1 \wedge U), Y) \cong \pi_0 \text{hom}(\Sigma^i(S^1 \wedge U), Y)$ with the resulting functions $\pi_0 \text{hom}(\bar{\Sigma}^i(S^1 \wedge U), Y) \rightarrow \pi_0 \text{hom}(\Sigma^{i+1}U, Y)$, one gets a natural map of $\mathbf{\Pi}$ -algebras $\Omega\pi_{i,*}X \rightarrow \pi_{i+1,*}X$ which we call the *shift maps* and which, in each degree $j \geq 1$, consists of a group homomorphism $\pi_{i,j+1}X \rightarrow \pi_{i+1,j}X$.

This definition readily implies

6.2 Proposition. *Let $Y \in s\mathbf{T}_*$ be Reedy fibrant (and hence (3.4) E^2 fibrant). Then the shift maps $\Omega\pi_{i,*}Y \rightarrow \pi_{i+1,*}Y \in \mathbf{\Pi}\text{-al}$ ($i \geq 0$) admit a factorization*

$$\Omega\pi_{i,*}Y \rightarrow \pi_*Z_{i+1} \rightarrow \pi_{i+1,*}Y$$

*in which the first map is induced (5.8) by the boundary map $\Omega\pi_*Z_iY \rightarrow \pi_*Z_{i+1}Y$ of the Serre fibration $C_{i+1}Y \rightarrow Z_iY$ (9.4 and 5.7) and the other map is as in 5.8.*

It thus remains to show that the shift maps are compatible with the natural maps of 5.4, i.e.,

6.3 Proposition. *Let X be Reedy cofibrant. Then the diagrams*

$$\begin{array}{ccc} \pi_{i,j+1}X & & \\ \downarrow & \searrow & \\ \pi_{i+1,j}X & \rightarrow & \pi_{i+j+1}|X| \end{array} \quad i \geq 0, j \geq 1$$

in which the vertical map is the shift map and the other maps are as in (5.4), commute.

Proof. As in the proof of 5.5 the maps $|\Sigma^i(S^1 \wedge U)| \rightarrow |\bar{\Sigma}^i(S^1 \wedge U)|$ and $|\Sigma^{i+1}U| \rightarrow |\bar{\Sigma}^i(S^1 \wedge U)|$ ($U \in \mathbf{\Pi}$) correspond, using the homeomorphisms of 1.6, to homotopy equivalences $S^{i+1} \wedge U \rightarrow (S^i \wedge D^1 \wedge D^1)/(S^i \wedge S^0 \wedge S^0) \wedge U$ which are homotopic. The rest of the proof is straightforward.

§7. A HUREWICZ THEOREM

Next we observe that

- (i) there exist *natural maps of $\mathbf{\Pi}$ -algebras* $\pi_{i,*}X \rightarrow \pi_i\pi_*X$ ($i \geq 0$) for which a Hurewicz theorem holds, i.e., they are an isomorphism in the first nontrivial dimension and onto in the next, and
- (ii) *the functor $\pi_i\pi_* : s\mathbf{T}_* \rightarrow s\mathbf{\Pi}\text{-al}$ are E^2 representable for $i \geq 2$, i.e., there exist functors ψ^i ($i \geq 2$) from $\mathbf{\Pi}$ to the E^2 homotopy category of $s\mathbf{T}_*$ such that $\psi^i U \in s\mathbf{T}_*$ is E^2 cofibrant for every object $U \in \mathbf{\Pi}$ and such that, for every E^2 fibrant object $Y \in s\mathbf{T}_*$, the induced functor $\pi_0 \text{hom}(\psi^i -, Y) : \mathbf{\Pi} \rightarrow \mathbf{Sets}_*$ is a $\mathbf{\Pi}$ -algebra which is naturally equivalent to $\pi_i\pi_*Y$. (The functor $\pi_0\pi_*$ is also E^2 representable as (see (ii)) $\pi_0\pi_* \cong \pi_{0,*}$, but apparently $\pi_1\pi_*$ is not.)*

We start with constructing

7.1 A Hurewicz map $\pi_{i,*}X \rightarrow \pi_i\pi_*X$ ($i \geq 0$). Let $X \in s\mathbf{T}_*$ and let $X \rightarrow Y \in s\mathbf{T}_*$ be as in 5.1. Then it is not difficult to verify that (in the notation of 9.6) $\pi_*C_iY \cong C_i\pi_*Y$ for all $i \geq 0$. Hence (5.8 and 9.6) the inclusions $Z_iY \rightarrow C_iY$ ($i \geq 0$) induce natural maps of $\mathbf{\Pi}$ -algebras $\pi_{i,*}X \rightarrow \pi_i\pi_*X$. As mentioned above, these *Hurewicz maps* satisfy

7.2 A Hurewicz theorem. *Let $X \in s\mathbf{T}_*$ and let n be an integer ≥ 0 . If $\pi_i\pi_*X = *$ for $i < n$, then the Hurewicz map $\pi_{i,*}X \rightarrow \pi_i\pi_*X \in \mathbf{\Pi}\text{-al}$ is an isomorphism for $i = n$ and is onto for $i = n+1$. In particular, for every object $X \in s\mathbf{T}_*$, one has $\pi_{0,*}X \cong \pi_0\pi_*X$, while the Hurewicz map $\pi_{1,*}X \rightarrow \pi_1\pi_*X$ is onto.*

Proof. This follows immediately from 8.1.

Another consequence of 8.1 is somewhat dual:

7.3 Proposition. *Let $X \in s\mathbf{T}_*$ and let n be an integer ≥ 1 . If $\pi_i \pi_j X = *$ for $i \geq 0$ and $j > n$, then the Hurewicz map $\pi_{i,n} X \rightarrow \pi_i \pi_n X$ is an isomorphism for all $i \geq 0$.*

Next we construct

7.4 The representing functors ψ^i ($i \geq 2$). For every integer $i \geq 2$ we denote by ψ^i the functor from the category $\mathbf{\Pi}$ (9.1) to the E^2 homotopy category of the category $s\mathbf{T}_*$ (3.1) which sends an object $U \in \mathbf{\Pi}$ to the push out

$$\psi^i U = C\Sigma^{i-2}(D^1 \wedge U) \coprod_{C\Sigma^{i-2}U} C\Sigma^{i-1}U$$

where the map $C\Sigma^{i-2}U \rightarrow C\Sigma^{i-1}U$ is induced by the map $\Delta[i-1] \rightarrow \Delta[i] \in \mathbf{S}$ which sends the non-degenerate $(i-1)$ -simplex ι_{i-1} to the simplex $d_0 \iota_i \in \Delta[i]$ (1.5). Clearly $\psi^i U$ is E^2 cofibrant and it thus remains to prove

7.5 Proposition. *Let $Y \in s\mathbf{T}_*$ be E^2 fibrant. Then*

- (i) *the functor $\pi_0 \text{hom}(\psi^i-, Y) : \mathbf{\Pi} \rightarrow \mathbf{Sets}_*$ is a $\mathbf{\Pi}$ -algebra, and*
- (ii) *there is an equivalence of $\mathbf{\Pi}$ -algebras $\pi_0 \text{hom}(\psi^i-, Y) \cong \pi_i \pi_* Y$ which is natural in Y .*

Proof. Since ψ^i preserves coproducts, the functor $\pi_0 \text{hom}^R(\psi^i-, Y) : \mathbf{\Pi} \rightarrow \mathbf{Sets}_*$ (where hom^R is as in 2.6) and the functor $\pi_0 \text{hom}(\psi^i-, Y) : \mathbf{\Pi} \rightarrow \mathbf{Sets}_*$ are both $\mathbf{\Pi}$ -algebras. Moreover there are natural epimorphisms of $\mathbf{\Pi}$ -algebras $\pi_0 \text{hom}^R(\psi^i-, Y) \rightarrow \pi_0 \text{hom}(\psi^i-, Y)$ (in view of 3.3) and $\pi_0 \text{hom}^R(\psi^i-, Y) \rightarrow \pi_i \pi_* Y$ (since a map $\psi^i U \rightarrow Y \in s\mathbf{T}_*$ corresponds to a map $U \rightarrow C_i Y \in \mathbf{T}_*$ (5.6) together with a null homotopy of its composition with the face map $d_0 : C_i Y \rightarrow C_{i-1} Y$). It thus remains to show that these two maps of $\mathbf{\Pi}$ -algebras have the same kernel.

To do this we define, for every object $U \in \mathbf{\Pi}$,

$$C' \psi^i U = \psi^i U \coprod_{C\Sigma^{i-1}U} C\Sigma^{i-1}((\Delta_1 \times U)/(\Delta_1 \times *)) \coprod_{C\Sigma^{i-1}U} C\Sigma^i U \in s\mathbf{T}_*$$

where the maps $C\Sigma^{i-1}U \rightarrow C\Sigma^{i-1}((\Delta_1 \times U)/(\Delta_1 \times *))$ are induced by the maps $d^0, d^1 : \Delta_0 \rightarrow \Delta_1$ (1.6) and note that the resulting map $\psi^i U \rightarrow C' \psi^i U$ is an E^2 cofibration and that $C' \psi^i U$ is E^2 contractible because the map $s^0 : \Delta_1 \rightarrow \Delta_0$ (1.6) induces a Reedy weak equivalence

$$C' \psi^i U \rightarrow \psi^i U \coprod_{C\Sigma^{i-1}U} C\Sigma^i U \cong C\Sigma^{i-2}(S^1 \wedge U) \coprod_{C\Sigma^{i-1}U} C\Sigma^i U.$$

This readily implies the exactness of the sequence

$$\pi_0 \text{hom}^R(C' \psi^i-, Y) \rightarrow \pi_0 \text{hom}^R(\psi^i-, Y) \rightarrow \pi_0 \text{hom}(\psi^i-, Y) \rightarrow *$$

and, using the isomorphisms $\pi_0 \text{hom}(C\Sigma^k-, Y) \cong C_{k+1} \pi_* Y$ (7.1), also the exactness of the sequence

$$\pi_0 \text{hom}^R(C' \psi^i-, Y) \rightarrow \pi_0 \text{hom}^R(\psi^i-, Y) \rightarrow \pi_i \pi_* Y \rightarrow *.$$

We end with a

7.6 Remark. It is not difficult to see that the Hurewicz map $\pi_{i,*} \rightarrow \pi_i \pi_*$ is induced by the obvious natural maps $\psi^i U \rightarrow \Sigma^i U \in s\mathbf{T}_*$. Moreover, there is a kind of *boundary map* $\pi_i \pi_* \rightarrow \Omega \pi_{i-2,*}$ ($i \geq 2$) induced by the composition $\Sigma^{i-2}(S^1 \wedge U) \rightarrow \bar{\Sigma}^{i-2}(S^1 \wedge U) \rightarrow \psi^i U \in s\mathbf{T}_*$, where the second map is the obvious one arising from the definition of $\psi^i U$ and the isomorphisms (3.6)

$$\bar{\Sigma}^{i-2}(S^1 \wedge U) \cong C\Sigma^{i-2}(D^1 \wedge U) \coprod_{C\Sigma^{i-2}U} \Sigma^{i-1}U.$$

§8. THE SPIRAL EXACT SEQUENCE

We end with noting the existence of the extremely useful spiral exact sequence, which among other things immediately implies the Whitehead theorem (5.3) and the Hurewicz theorem (7.2).

8.1 The spiral exact sequence. *Let $X \in s\mathbf{T}_*$. Then the sequence of Π -algebras and maps of Π -algebras (§9)*

$$\cdots \rightarrow \pi_{i+1}\pi_*X \rightarrow \Omega\pi_{i-1,*}X \rightarrow \pi_{i,*}X \rightarrow \pi_i\pi_*X \rightarrow \cdots \rightarrow \pi_0\pi_*X \rightarrow *$$

in which the maps are the shift maps (6.1), the Hurewicz maps (7.1) and the boundary maps of 7.5 and in which $\pi_{-1,}X$ should be interpreted as $*$, is exact.*

The exact sequence of 8.1 will be called the *spiral exact sequence* of X because, as we will see in 8.3, it is not only exact, but is actually (most of) an *exact couple*. The latter gives rise to a *first quadrant spectral sequence* with $E_{i,j}^2 = \pi_i \pi_j X$ ($i \geq 0, j \geq 1$) which strongly converges to $\pi_*|X^b|$ (3.2). In fact, $E_{i,j}^\infty$ ($i \geq 0, j \geq 1$) is the quotient of the image of $\pi_{i,j}X$ in $\pi_{i+j}|X^b|$ by the image of $\pi_{i-1,j+1}X$ (6.3). Moreover, this spectral sequence coincides (up to a possible sign) with the Bousfield-Friedlander spectral sequence of a pointed simplicial path connected space [2] and hence with the delooped version of the Quillen spectral sequence of a bisimplicial group [9].

To obtain the spiral exact sequence one notes that (8.3) the exact couple of which it is a part is itself the derived couple of

8.2 The homotopy exact couple of a pointed simplicial space. Let $X \in s\mathbf{T}_*$ and let $X \rightarrow Y \in s\mathbf{T}_*$ be as in 5.1. Then one can combine the homotopy sequences of the Serre fibrations $C_{i+1}Y \rightarrow Z_iY$ (5.7) into an *exact couple* [6, VIII] with

$$\begin{aligned} E_{i,j}^1 &= \pi_j C_i Y && \text{for } i \geq 0, j \geq 1 \\ &= * && \text{otherwise} \\ D_{i,j}^1 &= \pi_j Z_i Y && \text{for } i \geq 0, j \geq 1 \\ &= \pi_{i+j-1,1} Y && \text{for } i \geq 0, j \leq 0, i+j \geq 1 \\ &= * && \text{otherwise} \end{aligned}$$

and the obvious maps between them. Clearly this exact couple does not depend on the choice of the map $X \rightarrow Y$ and is natural in X . Furthermore it has the following properties.

8.3 Proposition.

- (i) $E_{i,j}^2 = \pi_i \pi_j X$ for $i \geq 0, j \geq 1$ and $E_{i,j}^2 = *$ otherwise.
- (ii) $D_{i,j}^2 = \pi_{i,j} X$ for $i \geq 0, j \geq 1$.
- (iii) For $i \geq 0$ and $j \geq 1$ the derived couple (D^2, E^2) consists exactly of the spiral exact sequence (8.1).
- (iv) The differentials d^2 are maps of Π -algebras $d^2 : \pi_{i+1} \pi_* X \rightarrow \Omega \pi_{i-1} \pi_* X$ ($i \geq 0$).
- (v) For Reedy cofibrant X , the resulting (first quadrant) spectral sequence converges strongly to $\pi_* |X^b|$ (3.2). In fact, $E_{i,j}^\infty$ ($i \geq 0, j \geq 1$) is the quotient of the image of $\pi_{i,j} X$ in $\pi_{i+j} |X^b|$ by that of the image of $\pi_{i-1,j+1} X$ (6.3).

Proof. Part (i) is not difficult to prove using the arguments of 5.8 and 7.1 and (ii) is an immediate consequence of 5.7 and 5.8. The proof of (iii) is straightforward using 9.4, while (iv) follows from (i), (ii) and (iii). Finally, (v) is verified by inspection, using 5.4.

We end with a

8.4 Comparison with the Bousfield-Friedlander spectral sequence. Let $X \in s\mathbf{T}_*$ be an E^2 cofibrant object. It is not difficult to construct a sequence of E^2 fibrations

$$\cdots Q^n X \rightarrow Q^{n-1} X \rightarrow \cdots \rightarrow Q^0 X = *$$

together with an E^2 weak equivalence $X \rightarrow Q^\infty X = \varprojlim Q^n X \in s\mathbf{T}_*$ such that

- (i) $\pi_i \pi_j Q^n X = *$ for $j > n$ and $i \geq 0$, and
- (ii) the induced map $\pi_i \pi_j Q^\infty X \rightarrow \pi_i \pi_j Q^n X$ is an isomorphism for $j \leq n$ and $i \geq 0$.

Let $L^n X$ and $K^n X$ ($n \geq 0$) denote the fibres of the maps $Q^\infty X \rightarrow Q^{n-1} X$ and $Q^n X \rightarrow Q^{n-1} X$ respectively. Then the induced map $L^n X \rightarrow K^n X$ ($n \geq 0$) is clearly an E^2 fibration, with $L^{n+1} X$ as the fibre and it follows from theorem B.4 of [2] and the arguments used in the proof of 4.6, that $|L^{n+1} X|$ is the homotopy fibre of the induced map $|L^n X| \rightarrow |K^n X|$. It now is not difficult to verify that the homotopy spectral sequence of the sequence $\cdots \rightarrow |L^{n+1} X| \rightarrow |L^n X| \rightarrow \cdots$ is isomorphic with the Bousfield-Friedlander spectral sequence of the bisimplicial set obtained from X by dimensionwise application of the singular functor. But on the other hand, there are, in view of 5.3, 5.5 and 7.3, natural isomorphisms

$$\begin{aligned} \pi_{i,j} X &\xleftarrow{\cong} \pi_{i,j} L^j X \xrightarrow{\cong} \pi_{i+j} |L^j X| \\ \pi_i \pi_j X &\xleftarrow{\cong} \pi_i \pi_j L^j X \xrightarrow{\cong} \pi_i \pi_j K^j X \xleftarrow{\cong} \pi_{i,j} K^j X \xrightarrow{\cong} \pi_{i+j} |K^j X| \end{aligned}$$

and a lengthy but essentially straightforward calculation shows that (up to a possible sign) these isomorphisms induce an isomorphism between the homotopy exact couples of the sequence $\cdots \rightarrow |L^{n+1} X| \rightarrow |L^n X| \rightarrow \cdots$ and the exact couple contained in the spiral exact sequence.

§9. APPENDIX ON Π -ALGEBRAS

In this appendix we review some results on Π -algebras, abelian Π -algebras and simplicial Π -algebras which were used in the paper.

First we recall from [3] two equivalent definitions of

9.1 Π -algebras. Let $\mathbf{Sets}_*$ denote the category of *pointed sets* and let Π be the category which has as objects the pointed CW complexes which have the homotopy type of a finite wedge of spheres of dimensions ≥ 1 and which has as maps the *homotopy classes* of (pointed) maps between them. A Π -algebra then is a *contravariant* functor $\Pi \rightarrow \mathbf{Sets}_*$ which sends *coproducts* (i.e., wedges) to *products* and a *map of Π -algebras* is a natural transformation between two such functors. The resulting category of Π -algebras will be denoted by $\Pi\text{-al}$.

This definition implies that, for every object $B \in \Pi\text{-al}$

- (i) $B* = *$, where $*$ denote the “point” in both categories Π and $\mathbf{Sets}_*$, and
- (ii) the values of B on the objects of Π are, up to isomorphism, determined by the values of B on the spheres S^j ($j \geq 1$). These are groups which, when no confusion can arise, will be denote by B_j and will be referred to as the *value of B in degree j* .

In view of Hilton’s analysis of the homotopy groups of wedges of spheres [14, XI], one can thus equivalently define a Π -algebra as a (≥ 1) -graded group $B = \{B_j\}_{j=1}^\infty$, with B_j abelian for $j > 1$, together with *Whitehead product homomorphisms* $[-, -] : B_p \otimes B_q \rightarrow B_{p+q-1}$ ($p, q \geq 1$) and *composition functions*, $(-\cdot\alpha) : B_p \rightarrow B_r$ ($\alpha \in \pi_r S^p$, $1 < p < r$) which satisfy all of the identities which hold for the Whitehead product and compositions operations on the higher homotopy groups of pointed spaces, and a *left action of B_1 on the B_j ($j > 1$)* which commutes with these operations.

An obvious example of a Π -algebra is thus provided by

9.2 The homotopy Π -algebra π_*T of a pointed space T . Let $T \in \mathbf{T}_*$ (1.6). Then the functor $\Pi \rightarrow \mathbf{Sets}_*$ which sends an object $U \in \Pi$ to the set of homotopy classes of maps $U \rightarrow T \in \mathbf{T}_*$ is easily seen to be a Π -algebra. As (9.1) this Π -algebra is completely determined by the homotopy groups $\pi_j T$ ($j \geq 1$) and the actions of the primary unstable homotopy operations on them, we denote this Π -algebra by π_*T .

Next we consider

9.3 Abelian Π -algebras. Let $\mathbf{ab-Grp}$ denote the category of *abelian groups*. A Π -algebra $A : \Pi \rightarrow \mathbf{Sets}_*$ then will be called *abelian* if it admits a factorization $\Pi \rightarrow \mathbf{ab-Grp} \rightarrow \mathbf{Sets}_*$ in which the second functor is the forgetful one. Such a factorization, if it exists, is unique and hence the resulting category $\mathbf{ab-}\Pi\text{-al}$ of abelian Π -algebras is a full subcategory of $\Pi\text{-al}$. It is also not difficult to verify that a Π -algebra $A = \{A_j\}_{j=1}^\infty$ is *abelian iff*

- (i) A_1 is abelian and acts trivially on the A_j ($j > 1$),
- (ii) all Whitehead products are trivial, and
- (iii) all composition functions are homomorphisms.

9.4 Example. An example of an abelian Π -algebra is the *loop algebra* ΩB of a Π -algebra B . This is the functor $\Pi \rightarrow \mathbf{Sets}_*$ which sends an object $U \in \Pi$ to the set $B(S^1 \wedge U)$, or equivalently, the Π -algebra with $(\Omega B)_j = B_{j+1}$ for all $j \geq 1$, with trivial $(\Omega B)_1$ -action and trivial Whitehead products and with composition given by the formulas $(\Omega b) \cdot \alpha = \Omega(b \cdot \Sigma\alpha)$ ($b \in B_{p+1}$, $\alpha \in \pi_r S^p$) in which $\Sigma\alpha \in \pi_{r+1} S^{p+1}$ is the suspension of

α and $\Omega b \in (\Omega B)_p$ and $\Omega(b \cdot \Sigma \alpha) \in (\Omega B)_r$ denote the elements which correspond to the elements $b \in B_{p+1}$ and $b \cdot \Sigma \alpha \in B_{r+1}$ respectively.

Of course, if $T \in \mathbf{T}_*$ (1.6), then the loop algebra $\Omega \pi_* T$ is canonically isomorphic to the homotopy $\mathbf{\Pi}$ -algebra $\pi_* \Omega T$ (π_0 is not included (9.2)) of the loop space ΩT of T . Moreover, if $T \rightarrow T' \in \mathbf{T}_*$ is a Serre fibration with fibre T'' , then the boundary maps $\pi_{j+1} T' \rightarrow \pi_j T''$ ($j \geq 1$) (1.6) give rise to a map of $\mathbf{\Pi}$ -algebras $\Omega \pi_* T' \rightarrow \pi_* T''$.

Finally we briefly discuss simplicial $\mathbf{\Pi}$ -algebras. We begin with recalling from [10, II, §4] the existence of

9.5 A closed simplicial model category structure on the category $s\mathbf{\Pi}$ -al of simplicial $\mathbf{\Pi}$ -algebras.. *The category $s\mathbf{\Pi}$ -al admits a closed simplicial model category structure [10, II, §2] in which*

- (i) *the simplicial structure is as in 1.5,*
- (ii) *a map $D \rightarrow E \in s\mathbf{\Pi}$ -al is a weak equivalence if, for every integer $j \geq 1$, the induced map of simplicial groups $DS^j \rightarrow ES^j$ (9.1) is a weak equivalence, i.e., induces isomorphisms $\pi_i(DS^j) \cong \pi_i(ES^j)$ ($i \geq 0$), and*
- (iii) *a map $D \rightarrow E$ in $s\mathbf{\Pi}$ -al is a fibration if, for every integer $j \geq 1$, the induced map of simplicial groups $DS^j \rightarrow ES^j$ is a fibration, i.e., the image of DS^j in ES^j contains the identity component of ES^j .*

Important homotopy invariants of a simplicial $\mathbf{\Pi}$ -algebra are its

9.6 Homotopy $\mathbf{\Pi}$ -algebras. Let $D \in s\mathbf{\Pi}$ -al and let i be an integer ≥ 0 . Then it is not difficult to verify that the functor $\pi_i D : \mathbf{\Pi} \rightarrow \mathbf{Sets}_*$ which sends an object $U \in \mathbf{\Pi}$ to the set $\pi_i(DU)$ is a $\mathbf{\Pi}$ -algebra, which is *abelian* if $i > 0$. Of course one can also use the analog of Moore's description of the homotopy groups of a simplicial group [7, §17], i.e., construct the differential graded $\mathbf{\Pi}$ -algebra

$$\cdots \rightarrow C_i D \xrightarrow{d_0} C_{i-1} D \rightarrow \cdots \rightarrow C_0 D = D[0] \rightarrow *$$

in which $C_i D = D[i] \cap \ker d_1 \cap \cdots \cap \ker d_n$ (1.5), observe that $\text{im}(C_{i+1} D \rightarrow C_i D)$ is, in an appropriate sense, normal in $\ker(C_i D \rightarrow C_{i-1} D)$ and define $\pi_i D$ as the resulting quotient.

As one would expect, there are a *Whitehead theorem* and a *long exact homotopy sequence for a fibration*.

9.7 Proposition. *A map $D \rightarrow E \in s\mathbf{\Pi}$ -al is a weak equivalence (9.5) if it induces isomorphisms $\pi_i D \cong \pi_i E$ ($i \geq 0$).*

9.8 Proposition. *Let $D \rightarrow D' \in s\mathbf{\Pi}$ -al be a fibration (9.5) with fibre D'' . Then there is a natural long exact sequence of $\mathbf{\Pi}$ -algebras*

$$\cdots \rightarrow \pi_{i+1} D' \rightarrow \pi_i D'' \rightarrow \pi_i D \rightarrow \pi_i D' \rightarrow \cdots \rightarrow \pi_0 D \rightarrow \pi_0 D'$$

in which the last map is onto iff the map $D \rightarrow D'$ is so.

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