MANY NON-EQUIVALENT REALIZATIONS OF THE ASSOCIAHEDRON

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Hohlweg and Lange (2007) and Santos (2004, unpublished) have found two different ways of constructing exponential families of realizations of the n-dimensional associahedron with normal vectors in $\{0, \pm 1\}^n$, generalizing the constructions of Loday (2004) and Chapoton–Fomin–Zelevinsky (2002). We classify the associahedra obtained by these constructions modulo linear equivalence of their normal fans and show, in particular, that the only realization that can be obtained with both methods is the Chapoton–Fomin–Zelevinsky (2002) associahedron.

For the Hohlweg–Lange associahedra our classification is a priori coarser than the classification up to isometry of normal fans, by Bergeron–Hohlweg–Lange–Thomas (2009). However, both yield the same classes. As a consequence, we get that two Hohlweg–Lange associahedra have linearly equivalent normal fans if and only if they are isometric.

The Santos construction, which produces an even larger family of associahedra, appears here in print for the first time. Apart of describing it in detail we relate it with the c-cluster complexes and the denominator fans in cluster algebras of type A.

A third classical construction of the associahedron, as the secondary polytope of a convex n-gon (Gelfand–Kapranov–Zelevinsky, 1990), is shown to never produce a normal fan linearly equivalent to any of the other two constructions.

1. Introduction

The n-dimensional associahedron is a simple polytope with C_{n+1} (the Catalan number) vertices, corresponding to the triangulations of a convex $(n+3)$-gon, and $n(n+3)/2$ facets, in bijection with the diagonals of the $(n+3)$-gon. It appears for the first time in Dov Tamari's 1951 thesis [43], where Tamari draws it as a convex polytope for $n = 1, 2, 3$ (the original drawing is reproduced in [24,41]) and he remarks that *généralement, on aura des hyperpolyèdres*.[1] As a cell complex, it was described and shown to be a topological ball by Jim Stasheff in 1963, in his work on the associativity of H-spaces [39]. A realization as a polytope by John Milnor from the 1960s is lost; Huguet & Tamari claimed in 1978 that the associahedron can be realized as a convex polytope [19]. The first such construction, via an explicit inequality system, was provided in a manuscript by Mark Haiman from 1984 that remained unpublished, but is available as [16]. The first constructions in print are due to Carl Lee, from 1989 [22].
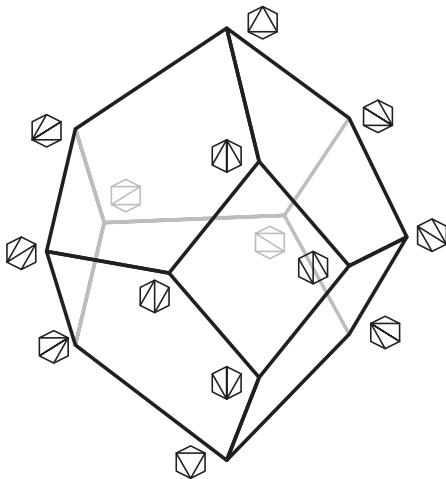

Figure 1. An associahedron, as the secondary polytope of a regular hexagon.

Subsequently three systematic approaches were developed that produce realizations of the associahedra in more general frameworks:

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[1] The (abridged) version of the thesis published in *Bulletin de la S. M. F.* 82 (1954), pp. 53–96, does not contain the picture or the remark.

○ the associahedron as the *secondary polytope* of a convex polygon, due to Gelfand, Kapranov and Zelevinsky [14,15] (see also [13, Chap. 7]), depicted in Figure 1.

○ the associahedron as one of the *generalized permutahedra* introduced by Postnikov in [29]. The history of this construction begins with Shnider and Sternberg [37] (compare Stasheff and Shnider [40, Appendix B]), who show that the associahedron can be obtained by cutting certain faces in a simplex (this is polar to the construction by Carl Lee in [22, Sec. 3], which produces the normal fan of the associahedron as a stellar subdivision of the central fan of the simplex). Loday [23], shows how to obtain explicit and nice vertex coordinates for this associahedron using combinatorics of binary trees. Postnikov [29] then puts Loday's construction in context, regarding this associahedron as a special case of a *generalized permutahedron*; a polytope lying in (the closure of) the deformation cone of the standard permutahedron. Rote, Santos and Streinu [35] and, more recently, Buchstaber [4] found constructions of essentially the same associahedron but described quite differently. Following [17,29] we reference this associahedron as the "Loday realization".

○ the associahedron as the polar of the *cluster complex* of type A_n, conjectured by Fomin and Zelevinsky [12] and constructed by Chapoton, Fomin and Zelevinsky [7].

We review these three constructions in Section 3, after some preliminaries in Section 2. The last two of them have the following properties in common:

1. They both have exactly n pairs of parallel facets.
2. In the basis given by the normals to those n pairs, all facet normals have coordinates in $\{0, \pm 1\}$.

This was generalized by Hohlweg and Lange [17] and by Santos [36], who showed that the Loday and Chapoton–Fomin–Zelevinsky constructions are particular cases of *exponentially many* constructions of the associahedron, all of them with properties (1) and (2). That is, all these associahedra are (normally isomorphic to) polytopes obtained from the regular n-cube by cutting certain $\binom{n}{2}$ faces, as seen in Figure 2. Note, however, that the last example of Figure 2 cannot be obtained by cutting faces one after the other; that is to say, its normal fan is not a stellar subdivision of the normal fan of the cube.

The goal of this paper is to compare the Hohlweg-Lange and Santos constructions, which we discuss in Sections 4 and 5. The construction by Santos appears here in print for the first time, so we prove things in detail. For the Hohlweg–Lange realizations we rely on the original papers for most of
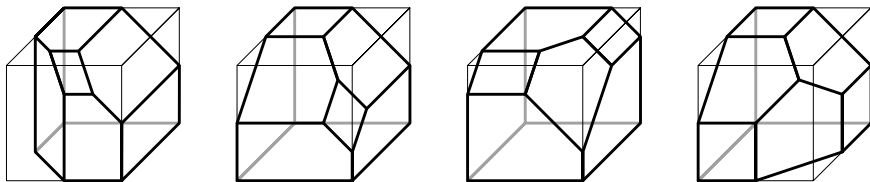
Figure 2. Four normally non-isomorphic 3–dimensional associahedra. From left to right: The Loday associahedron (which is a special case of the Hohlweg–Lange associahedra), the Chapoton–Fomin–Zelevinsky associahedron (a special case of both Hohlweg–Lange and Santos) and the other two Santos associahedra. Since they all have three pairs of parallel facets, we used a linear transformation to draw them fitting in the same cube.

the details. Our main conclusion is that the exponentially many realizations obtained by the two procedures are essentially different.

Let us explain what we exactly mean by *different* (see more details in Section 2). Since the associahedron is simple, its realizations form an open subset in the space of $\frac{(n+3)n}{2}$-tuples of half-spaces in $\mathbb{R}^n$. Hence, classifying them by affine or projective equivalence does not seem the right thing to do. But for the Hohlweg–Lange and Santos constructions, with normals in $\{-1, 0, 1\}^n$, the set of possible normal fans obtained is finite. This suggests a natural classification to be by *linear isomorphism of normal fans* or, as we call it, *normal isomorphism.* In this language:

- the (normal isomorphism classes of) Hohlweg–Lange associahedra are in bijection with the sequences in $\{+, -\}^{n-1}$, modulo reflection and reversal (Theorem 4.9, see also [2, Cor. 2.6]).
- the (normal isomorphism classes of) Santos associahedra are in bijection with the triangulations of the $(n+3)$-gon, modulo dihedral symmetries of the polygon (Corollary 5.7).

The numbers of distinct associahedra obtained by the two constructions are, thus, roughly 2^{n-3} and $\frac{1}{2(n+3)}C_{n+1} \approx 2^{2n+1}/\sqrt{\pi n^5}$; exact counts are in Sections 4 and 5, see also Table 1.

Although a classification of the Hohlweg–Lange associahedra appears already in [2], we think our new derivation has some novelty. On the one hand, the classification in [2] is only up to isometry of linear fans; it left the door open for two associahedra classified as different still being equivalent if a linear transformation of the normal fan is allowed (see Remark 4.10). On the other hand, we show that two Hohlweg–Lange associahedra coming from non-equivalent sign sequences can be distinguished by their pairs of parallel facets (see the proof of Theorem 4.9). The fact that Hohlweg–Lange asociahedra have parallel facets is obvious from the definitions, but was not used in [2].

The same works for Santos asociahedra: if two of them are produced by non-equivalent triangulations, then they are not normally isomorphic (Lemma 5.6). Even more so, the only Hohlweg–Lange associahedron with the same pairs of parallel facets as a Santos associahedron is the Chapoton–Fomin–Zelevinsky associahedron. That is to say (Theorem 6.1):

Theorem 1.1. *The Hohlweg–Lange and Santos families of associahedra are almost disjoint, the only common element being the Chapoton–Fomin–Zelevinsky associahedron.*

The secondary polytope construction of the associahedron has a completely different flavor, since a continuous deformation of the polygon produces a continuous deformation of the associahedron obtained and of its normal fan. Moreover, the secondary polytope of a convex polygon never has parallel facets (Proposition 3.5, already noticed in [35, Sec. 5.3]). This difference is apparent comparing Figures 1 and 2.

In Section 5.4 we relate the Santos construction with the c-cluster complexes and the denominator fans in cluster algebras of type A. The c-cluster complexes are simplicial complexes defined by Reading in [32] following ideas from [25]. We obtain a simple combinatorial description of these complexes for Coxeter groups of type A, and show that they are the normal fans of some of the Santos associahedra (Proposition 5.10). In the general case, the normal fans of the Santos associahedra can be interpreted as the denominator fans in cluster algebras of type A (Proposition 5.12). This connection suggests a natural generalization of the Santos construction to a construction of generalized cluster-associahedra in arbitrary finite Coxeter groups (Question 5.13).

Let us remark that, even if both the Hohlweg–Lange and the Santos constructions have very natural interpretations (and generalizations, modulo the question above) in the context of finite Coxeter groups, they have a significant difference; their normal fans lie in the *root space* and the *weight space* respectively. This is a bit hidden in Figure 2, where we have performed a linear transformation to draw the polytopes inscribed in the same cube.

One of the questions that remains is whether there is a common generalization of the Hohlweg–Lange and the Santos construction, which may perhaps produce even more examples of "combinatorial" associahedra. An exhaustive search produces, besides the four 3-associahedra of Figure 2, another four 3-associahedra that arise by cutting three faces of a 3-cube (see Figure 3). Do these admit a natural combinatorial interpretation as well?

As a final remark, part of the motivation of this paper was to try to find out *what is the most "natural" or "canonical" realization of the associahedron.* The answer is not clear. If one wants to realize all the combinatorial
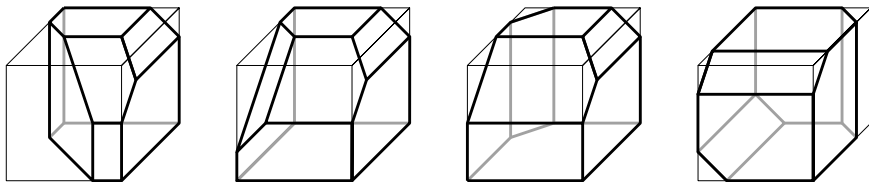
Figure 3. More 3-associahedra inscribed in a 3-cube. The 3-associahedron is the only simple 3-polytope with nine facets all of which are quadrilaterals or pentagons.

symmetries of the polytope (which, as we show in Lemma 2.2, are precisely the dihedral symmetries of the $(n+3)$-gon) then the best candidate is the secondary polytope of a regular polygon (Figure 1). But if small integer coordinates for vertices and facet normals are sought then you certainly want one of the Hohlweg–Lange or Santos associahedra (Figure 2). Among them, the Loday associahedron sticks out as the most ubiquitous in the literature [4,17,22,23,27,29,35,37], but the Chapoton–Fomin–Zelevinsky associahedron is the only one produced both by the Hohlweg–Lange and the Santos constructions.

2. Two preliminaries

Let P_{n+3} be a convex $(n+3)$-gon. An *associahedron* Ass_n is an n-dimensional simple polytope whose poset of non-empty faces is isomorphic to the poset of non-crossing sets of diagonals of P_{n+3}, ordered by reverse inclusion. In particular, the vertices of the associahedron correspond to the *triangulations* of P_{n+3} and its facets to the internal diagonals.

2.0.1. Normal isomorphism.   The goal of this paper is to compare different types of constructions of the associahedron, saying which ones produce equivalent polytopes, in a suitable sense. The following notion reflects the fact that the main constructions that we are going to discuss produce associahedra whose normal vectors have small integer coordinates, usually 0 or ± 1. In these constructions the normal fan of the associahedron can be considered canonical, while there is still freedom in the right-hand sides of the inequalities. Recall that the normal fan of a polytope $P \subset \mathbb{R}^n$ is the partition of $(\mathbb{R}^*)^n$ into the normal cones of the different faces of P. Each 1-dimensional cone (ray) in the normal fan is generated by the (exterior) normal vector to a facet of P and the n-dimensional cones are normal to the vertices of P. Since all the polytopes in this paper are simple, their normal fans are *simplicial*: every cone is generated by an independent set of vectors. (See [44, Sec. 7.1] for further discussion of fans and of normal fans.)

This leads us to use the following notion of equivalence.

Definition 2.1. Two complete fans in real vector spaces V and V' of the same dimension are *linearly isomorphic* if there is a linear isomorphism $V \to V'$ sending each cone of one to a cone of the other. Two polytopes P and P' are *normally isomorphic* if they have linearly isomorphic normal fans.

Normal isomorphism is weaker than the usual notion of *normal equivalence*, in which the two polytopes P and P' are assumed embedded *in the same space* and their normal fans are required to be exactly the same, not only linearly isomorphic. One easy way to check that two polytopes are *not* normally isomorphic is to show that no combinatorial isomorphism sends parallel facets of one to parallel facets of the other.

2.0.2. Automorphisms of the associahedron. The following lemma is very useful in order to disprove that two associahedra are normally isomorphic. It implies that all normal (or combinatorial, for that matter) isomorphisms between associahedra come from isomorphisms between the $(n+3)$-gons defining them.

Lemma 2.2. *All automorphisms of the face lattice of the associahedron* Ass_n *are induced by symmetries of the* $(n+3)$-gon. *In particular, for* $n \geq 2$, *the automorphism group of the face lattice of the associahedron* Ass_n *is isomorphic to the dihedral group of order* $2n+6$.

Proof. The second sentence follows from the first one since the only symmetry of the k-gon sending every diagonal to itself is the identity, for $k \geq 5$. For the first sentence, suppose φ is an automorphism of the face lattice of the associahedron Ass_n, and let D be the set of all diagonals of a convex $(n+3)$-gon. φ induces a natural bijection

$$\widetilde{\varphi} \colon D \longrightarrow D$$

such that for any two diagonals $\delta, \delta' \in D$ we have:

$$\delta \text{ crosses } \delta' \Longleftrightarrow \widetilde{\varphi}(\delta) \text{ crosses } \widetilde{\varphi}(\delta').$$

Call *length* of a diagonal $\delta \in D$ the minimum between the lengths of the two paths that connect the two end points of δ on the boundary of the $(n+3)$-gon. Since the length of δ is determined by the number of diagonals that cross δ,

$$\text{length}(\delta) = \text{length}(\widetilde{\varphi}(\delta)).$$

Now, two diagonals of the $(n+3)$-gon have a common vertex if and only if there is a diagonal of length 2 crossing both of them. In particular, $\widetilde{\varphi}$ sends diagonals with a common vertex to diagonals with common vertex and can thus be understood as a graph automorphism of D, when D is regarded as the graph $K_{n+3} \setminus C_{n+3}$ (the complete graph minus the Hamiltonian cycle along the boundary of the $(n+3)$-gon). The only such automorphisms are clearly the dihedral automorphisms of the cycle C_{n+3}. ∎

3. Three constructions of the associahedron

In this section we review three very nice constructions in geometric combinatorics that have the associahedron as particular cases.

3.1. The associahedron as a secondary polytope

The secondary polytope is an ingenious construction motivated by the theory of hypergeometric functions as developed by Gelfand, Kapranov and Zelevinsky [13], later generalized and explained in terms of fiber polytopes by Billera and Sturmfels [3]. In this section we recall the basic definitions and main results related to this topic, which yield in particular that the secondary polytope of any convex $(n+3)$-gon is an n-dimensional associahedron. For more detailed presentations we refer to [8, Ch. 5] and [44, Lect. 9]. All the subdivisions and triangulations of polytopes that appear in the following are understood to be without new vertices.

3.1.1. The secondary polytope construction.

Definition 3.1 (GKZ vector/secondary polytope). Let Q be a d-dimensional convex polytope with $n + d + 1$ vertices. The *GKZ vector* $v(t) \in \mathbb{R}^{n+d+1}$ of a triangulation t of Q is

$$v(t) := \sum_{i=1}^{n+d+1} \mathrm{vol}(\mathrm{star}_t(i))e_i = \sum_{i=1}^{n+d+1} \sum_{\sigma \in t:\, i \in \sigma} \mathrm{vol}(\sigma)e_i.$$

The *secondary polytope* of Q is defined as

$$\Sigma(Q) := \mathrm{conv}\{v(t): t \text{ is a triangulation of } Q\}.$$

Theorem 3.2 (Gelfand–Kapranov–Zelevinsky [14]). *Let Q be a d-dimensional convex polytope with $m = n+d+1$ vertices. The secondary polytope $\Sigma(Q)$ has the following properties:*

(i) $\Sigma(Q)$ is an n-dimensional polytope.
(ii) The vertices of $\Sigma(Q)$ are in bijection with the regular triangulations of Q.
(iii) The faces of $\Sigma(Q)$ are in bijection with the regular subdivisions of Q.
(iv) The face lattice of $\Sigma(Q)$ is isomorphic to the lattice of regular subdivisions of Q, ordered by refinement.

3.1.2. The associahedron as the secondary polytope of a convex $(n+3)$-gon.

Definition 3.3. The *Gelfand–Kapranov–Zelevinsky associahedron* $\mathrm{GKZ}_n(Q) \subset \mathbb{R}^{n+3}$ is defined as the (n-dimensional) secondary polytope of a convex $(n+3)$-gon $Q \subset \mathbb{R}^2$:

$$\mathrm{GKZ}_n(Q) := \Sigma(Q).$$

Since every triangulation or subdivision of convex polygon is regular, one obtains:

Corollary 3.4 (Gelfand–Kapranov–Zelevinsky [14]). $\mathrm{GKZ}_n(Q)$ *is an n-dimensional associahedron.*

There is one feature that distinguishes the associahedron as a secondary polytope from all the other constructions that we mention in this paper: the absence of parallel facets.

Proposition 3.5 (Rote–Santos–Streinu [35, Sec. 5.3]). *Let Q be a convex $(n+3)$-gon. Then $\mathrm{GKZ}_n(Q)$ has no parallel facets for $n \geq 2$.*

This was stated without proof by Rote, Santos and Streinu [35, Sec. 5.3]. Here we offer a proof, based on the understanding of the facet normals in secondary polytopes. Let Q be an arbitrary d-polytope with $n+d+1$ vertices $\{q_1, \ldots, q_{n+d+1}\}$, so that $\mathrm{GKZ}_n(Q)$ lives in $\mathbb{R}^{n+d+1}$, although it has dimension n. In the theory of secondary polytopes one thinks of each linear functional $\mathbb{R}^{n+d+1} \to \mathbb{R}$ as a function $\omega \colon \mathrm{vertices}(Q) \to \mathbb{R}$ assigning a value $\omega(q_i)$ to each vertex q_i. In turn, to each triangulation t of Q (with no additional vertices) and any such ω one associates the function $g_{\omega,t} \colon Q \to \mathbb{R}$, which takes the value $\omega(q_i)$ at each q_i and is affine linear on each simplex of t. That is, we use t to piecewise linearly interpolate a function whose values $(\omega(q_1), \ldots, \omega(q_n))$ we know on the vertices of Q. The main result we need is the following equality for every ω and every triangulation t (see, e.g., [8, Thm. 5.2.16]):

$$\langle \omega, v(t) \rangle = (d+1) \int_Q g_{\omega,t}(x)\,dx.$$

In particular:

○ If ω is affine-linear (that is, if the points $\{(q_1, \omega_1), \ldots, (q_{n+d+1}, \omega_{n+d+1})\} \subset \mathbb{R}^{n+d+1} \times \mathbb{R}$ lie in a hyperplane), then $\langle \omega, v(t) \rangle$ is the same for all t. Moreover, the converse is also true: The affine-linear ω's form the lineality space of the normal fan of $\mathrm{GKZ}_n(Q)$.

○ An ω lies in the linear cone of the (inner) normal fan of $\mathrm{GKZ}_n(Q)$ corresponding to a certain triangulation t (that is, $\langle \omega, v(t) \rangle \leq \langle \omega, v(t') \rangle$ for every other triangulation t') if and only if the function $g_{\omega,t}$ is convex; that is to say, if its graph is a convex hypersurface.

Proof of Proposition 3.5. With the previous description in mind we can identify the facet normals of the secondary polytope of a polygon Q. For this we use the correspondence:

$$\begin{aligned} \text{vertices} &\longleftrightarrow \text{triangulations of } Q, \\ \text{facets} &\longleftrightarrow \text{diagonals of } Q. \end{aligned}$$

For a given diagonal δ of Q, denote by F_δ the facet of $\mathrm{GKZ}_n(Q)$ corresponding to δ. The vector normal to F_δ is not unique, since adding to any vector normal to F_δ an affine-linear ω_0 we get another one. One natural choice is

$$\omega_\delta(q_i) := \mathrm{dist}(q_i, l_\delta),$$

where l_δ is the line containing δ and $\mathrm{dist}(\cdot, \cdot)$ is the Euclidean distance. Indeed, ω_δ lifts the vertices of Q on the same side of δ to lie in a half-plane in $\mathbb{R}^3$, with both half-planes having δ as their common intersection. That is, $g_{\omega_\delta, t}$ is convex for every t that uses δ. But another choice of normal vector is better for our purposes: choose one side of l_δ to be called positive and take

$$\omega_\delta^+(q_i) := \begin{cases} \mathrm{dist}(q_i, l_\delta) & \text{if } q_i \in l_\delta^+ \\ 0 & \text{if } q_i \in l_\delta^- \end{cases}.$$

For the end-points of δ, which lie in both l_δ^+ and l_δ^-, there is no ambiguity since both definitions give the value 0. Again, ω_δ^+ is a normal vector to F_δ since it lifts points on either side of l_δ to lie in a plane.

We are now ready to prove the proposition. If two diagonals δ and δ' of Q do not cross, then they can simultaneously be used in a triangulation. Hence, the corresponding facets F_δ and $F_{\delta'}$ meet, and they cannot be parallel. So, assume in what follows that δ and δ' are two crossing diagonals. Let $\delta = pr$ and $\delta' = qs$, with $pqrs$ being cyclically ordered along Q. Since $n \geq 2$ there is at least another vertex a in Q. Without loss of generality suppose a lies between s and p. Now, we call negative the side of l_δ and the side of $l_{\delta'}$

containing a, and consider the normal vectors ω_δ^+ and $\omega_{\delta'}^+$ as defined above. They take the following values on the five points of interest:

$$\omega_\delta^+(a) = 0, \quad \omega_\delta^+(p) = 0, \quad \omega_\delta^+(q) > 0, \quad \omega_\delta^+(r) = 0, \quad \omega_\delta^+(s) = 0,$$
$$\omega_{\delta'}^+(a) = 0, \quad \omega_{\delta'}^+(p) = 0, \quad \omega_{\delta'}^+(q) = 0, \quad \omega_{\delta'}^+(r) > 0, \quad \omega_{\delta'}^+(s) = 0.$$

Suppose that F_δ and $F_{\delta'}$ were parallel. This would imply that δ and δ' are linearly dependent or, more precisely, that there is a linear combination of them that gives an affine-linear ω (in the lineality space of the normal fan). But any (non-trivial) linear combination ω of ω_δ^+ and $\omega_{\delta'}^+$ necessarily takes the following values on our five points, which implies that ω is not affine-linear:

$$\omega(a) = 0, \quad \omega(p) = 0, \quad \omega(q) \neq 0, \quad \omega(r) \neq 0, \quad \omega(s) = 0. \qquad \blacksquare$$

Remark 3.6. The secondary polytope of points $\{q_1, \ldots, q_{n+3}\}$ in the plane that are not the vertices of a convex polygon is, in general, not an associ-ahedron. But there is a case in which it is: if the points are placed on the boundary of an m-gon (with $m \leq n+3$) in such a way that no four of them lie on the same edge. By the arguments in the proof above, a necessary condi-tion for the associahedron obtained to have parallel facets is that $m \leq 4$. For $m = 4$ we can obtain associahedra up to dimension 4 with exactly one pair of parallel facets (those corresponding to the main diagonals of the quadrilat-eral). For $m = 3$, we can obtain 2-dimensional associahedra with two pairs of parallel facets, and 3-dimensional associahedra with three pairs of parallel facets. The latter is obtained for six points $\{p, q, r, a, b, c\}$ with p, q and r be-ing the vertices of a triangle and $a \in pq$, $b \in qr$ and $c \in ps$ intermediate points in the three sides. The associahedron obtained has the following three pairs of parallel facets:

$$F_{pq} \parallel F_{ar}, \quad F_{qr} \parallel F_{bs}, \quad F_{ps} \parallel F_{cq}.$$

It is normally isomorphic to the right-most associahedron of Figure 2.

Remark 3.7. Rote, Santos and Streinu [35] introduce a *polytope of pseudo-triangulations* associated to each finite set A of m points (in general position) in the plane. This polytope lives in $\mathbb{R}^{2m}$ and has dimension $m+3+i$, where i is the number of points interior to $\mathrm{conv}(A)$. They show that for points in convex position their polytope is affinely isomorphic to the secondary polytope for the same point set. Their constructions uses rigidity theoretic ideas: the edge-direction joining two neighboring triangulations t and t' is the vector of velocities of the (unique, modulo translation and rotation) infinitesimal flex of the embedded graph of $t \cap t'$.

3.2. The associahedron as a generalized permutahedron

We here review two constructions of the associahedron: one by Postnikov [29] and one by Rote–Santos–Streinu [35] (different from the one in Remark 3.7). The main goal of this section is to prove that these two constructions produce affinely equivalent results. In both constructions only the normal fan is fixed. Equivalently, there is freedom in the construction for the right-hand sides of facet-defining inequalities, and the space of valid right-hand sides is explicitly described. Specific right-hand sides produce, respectively, the realizations by Loday [23] and Buchstaber [4], which turn out to be affinely equivalent as well.

3.2.1. The Postnikov associahedron. The Postnikov associahedron is a special case of the family of generalized permutahedra studied in [29]. Recall that the standard n-dimensional permutahedron is the polytope

$$(1) \quad \left\{ (x_1, \ldots x_{n+1}) \in \mathbb{R}^{n+1} \colon \sum_{i \in S} x_i \geq \binom{|S| + 1}{2} \right.$$
$$\left. \text{for all } S \subsetneq [n+1], \quad \sum_{i \in [n+1]} x_i = \binom{n+2}{2} \right\}.$$

Equivalently, it equals the convex hull of the $n!$ points in $\mathbb{R}^{n+1}$ obtained by permuting coordinates in $(1, \ldots, n+1)$ and also the Minkowski sum of the edges of the standard simplex $\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \colon \sum x_i = 1, \, x_i \geq 0\}$. A generalized permutahedron is a polytope with facet normals contained in those of the standard permutahedron and such that the collection of right hand side parameters of the defining inequalities belongs to (the closure of) the deformation cone of the standard permutahedron. Besides associahedra, generalized permutahedra include many interesting polytopes such as cyclohedra, graph associahedra and nestohedra.

   It follows from the Minkowski sum description of the permutahedron that every positively weighted Minkowski sum of arbitrary faces of the standard simplex is a generalized permutahedron (the converse is only partially true; see Remark 4.5). Following Postnikov [29] we use this to define the Loday and Postnikov associahedra:

Definition 3.8. The *Postnikov associahedron* is the polytope

$$\text{Post}_n(\mathbf{a}) := \sum_{1 \leq i \leq j \leq n+1} a_{ij} \Delta_{[i, \ldots, j]},$$

where $\mathbf{a} = \{a_{ij} > 0: 1 \leq i \leq j \leq n+1\}$ is a positive vector and $\Delta_{[i,\ldots,j]}$ denotes the simplex $\mathrm{conv}\{e_i, e_{i+1}, \ldots, e_j\}$ in $\mathbb{R}^{n+1}$. The special case where $a_{ij} = 1$ for all i, j is the *Loday associahedron*.

Proposition 3.9 (Postnikov [29, Sec. 8.2]). $\mathrm{Post}_n(\mathbf{a})$ *is an n-dimensional associahedron.*
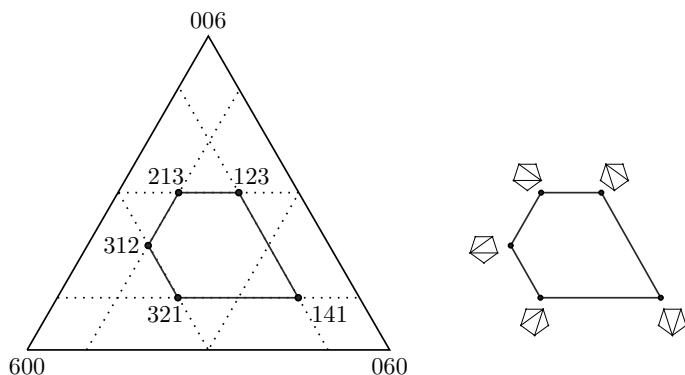

Figure 4. The Loday associahedron $\mathrm{Post}_n(\mathbf{1})$ with the coordinates of its vertices.

Using a special case of [29, Prop. 6.3], the Postnikov associahedron can be described in terms of inequalities as follows.

Lemma 3.10 (Postnikov [29]).

$$\mathrm{Post}_n(\mathbf{a}) = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{p < i < q} x_i \geq f_{p,q} \right.$$

$$\left. \text{for } 0 \leq p < q \leq n+2, x_1 + \cdots + x_{n+1} = f_{0,n+2} \right\},$$

where $f_{p,q} = \sum_{p < i \leq j < q} a_{i,j}$.

Conversely, the Minkowski weights $a_{i,j}$ of a Postnikov associahedron defined by right-hand sides $f_{p,q}$ can be obtained by Möbius inversion. This is thoroughly analyzed in [20].

The facet of $\mathrm{Post}_n(\mathbf{a})$ labeled by a pair (p, q) corresponds to the diagonal pq of an $(n+3)$-gon with vertices labeled in counterclockwise direction from 0 to $n+2$. It is obvious from the description in Lemma 3.10 that $\mathrm{Post}_n(\mathbf{a})$ has exactly n pairs of parallel facets. These correspond to the pairs of diagonals

($\{0, i+1\}, \{i, n+2\}$) for $1 \le i \le n$, as illustrated in Figure 5. This is a particular case of Proposition 4.7.
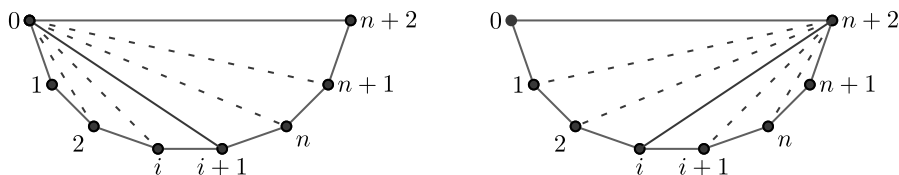

Figure 5. Diagonals of the $(n+3)$-gon that correspond to the pairs of parallel facets of both $\mathrm{Post}_n(\mathbf{a})$ and $\mathrm{RSS}_n(\mathbf{g})$.

3.2.2. The Rote–Santos–Streinu associahedron. By "generalizing" the construction of Remark 3.7 to sets of points along a line, Rote, Santos and Streinu [35] obtain a second realization of the associahedron.

Definition 3.11. The *Rote–Santos–Streinu associahedron* is the polytope

$$\mathrm{RSS}_n(\mathbf{g}) = \{(y_0, y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+2} : y_j - y_i \ge g_{i,j}$$
$$\text{for } j > i, y_0 = 0, y_{n+1} = g_{0,n+1}\},$$

where $\mathbf{g} = (g_{i,j})_{0 \le i < j \le n+1}$ is any vector with real coordinates satisfying

(2) $$g_{i,l} + g_{j,k} > g_{i,k} + g_{j,l} \text{ for all } i < j \le k < l,$$
(3) $$g_{i,l} > g_{i,k} + g_{k,l} \text{ for all } i < k < l.$$

Proposition 3.12 (Rote–Santos–Streinu [35, Sec. 5.3]). *If the vector $\mathbf{g}$ satisfies inequalities (2) and (3) then $\mathrm{RSS}_n(\mathbf{g})$ is an n-dimensional associahedron.*

A particular example of valid parameters $\mathbf{g}$ is given by $\mathbf{g_0}$: $g_{i,j} = i(i-j)$. In this case we get the realization of the associahedron introduced by Buchstaber in [4, Lect. II Sec. 5].

The facet of $\mathrm{RSS}_n(\mathbf{g})$ defined by $y_j - y_i \ge g_{i,j}$ corresponds to the diagonal $\{i, j+1\}$ of an $(n+3)$-gon with vertices labeled in counterclockwise direction from 0 to $n+2$. Rote, Santos and Streinu [35, Sec. 5.3] notice that $\mathrm{RSS}_n(\mathbf{g})$ has exactly n pairs of parallel facets, corresponding to the pairs of diagonals ($\{0, i+1\}, \{i, n+2\}$) for $1 \le i \le n$, as illustrated in Figure 5.
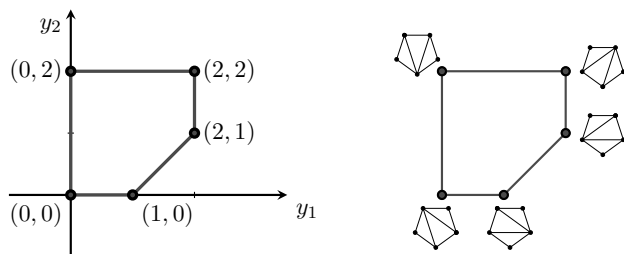
Figure 6. The Rote–Santos–Streinu associahedron $\mathrm{RSS}_2(\mathbf{g_0})$ with the coordinates of the vertices. This coincides with the realization of Buchstaber.

3.2.3. Affine equivalence. Rote, Santos and Streinu stated in [35, Sec. 5.3] that $\mathrm{RSS}_n(\mathbf{g})$ is not affinely equivalent to neither the associahedron as a secondary polytope nor the Chapoton–Fomin–Zelevinsky associahedron of Section 3.3. Next we prove that $\mathrm{RSS}_n(\mathbf{g})$ is normally isomorphic to $\mathrm{Post}_n(\mathbf{a})$ and that this isomorphism induces an affine isomorphism between the Loday and Buchstaber specific realizations.

Theorem 3.13. *Let φ be the affine transformation*

$$\varphi\colon \quad \mathbb{R}^{n+1} \quad \rightarrow \quad \mathbb{R}^n$$
$$(x_1,\ldots,x_{n+1}) \rightarrow (y_1,\ldots,y_n)$$

defined by $y_k = \sum_{i=1}^{k}(x_i - i)$. Then φ maps $\mathrm{Post}_n(\mathbf{a})$ bijectively to $\mathrm{RSS}_n(\mathbf{g})$, for $\mathbf{g}$ given by $g_{i,j} - \frac{(i+j+1)(j-i)}{2} = f_{i,j+1}(\mathbf{a})$. In particular, φ maps the Loday associahedron $\mathrm{Post}_n(\mathbf{1})$ to the Buchstaber associahedron $\mathrm{RSS}_n(\mathbf{g_0})$.

Proof. The result follows from the following computation

$$y_j - y_i \geq g_{i,j}$$
$$(x_{i+1} + \cdots + x_j) + ((i+1) + \cdots + j) \geq g_{i,j}$$
$$x_{i+1} + \cdots + x_j \geq g_{i,j} - \frac{(i+j+1)(j-i)}{2}. \qquad \blacksquare$$

3.3. The associahedron as a cluster polytope of type A

Cluster complexes are simplicial complexes associated to root systems and arose in the theory of cluster algebras initiated by Fomin and Zelevinsky [10,11]. In the initial papers by these two authors cluster complexes

were realized only as complete fans, but these fans were shown to be poly-topal in their subsequent work with Chapoton [7]. The polytopes obtained are called *generalized associahedra* because the case of type A_n yields to an associahedron. We refer to [7], [12] and [9] for more detailed presentations.

3.3.1. The cluster complex of type A_n. The *root system of type A_n* is the set $\Phi := \Phi(A_n) = \{e_i - e_j, 1 \leq i \neq j \leq n+1\} \subset \mathbb{R}^{n+1}$. The *simple roots* of type A_n are the elements of the set $\Pi = \{\alpha_i = e_i - e_{i+1}, i \in [n]\}$, the set of *positive roots* is $\Phi_{>0} = \{e_i - e_j : i < j\}$, and the set of *almost positive roots* is $\Phi_{\geq -1} := \Phi_{>0} \cup -\Pi$.

In the theory of cluster algebras, a compatibility relation is introduced in the set of almost positive roots of a finite crystallographic root system and the *cluster complex* is defined as the simplicial complex of pairwise compat-ible roots [11,12]. For the root system of type A_n, there is a natural corre-spondence between the set $\Phi_{\geq -1}$ and the diagonals of the $(n+3)$-gon P_{n+3} that sends compatible roots to non-crossing diagonals, and vice-versa [12, Prop. 3.14]. We take this property, which makes the cluster complex anti-isomorphic to the face complex of the associahedron, as a definition.

Definition 3.14 (Cluster complex of type A_n). Identify the negative simple roots $-\alpha_i$ with the diagonals on the *snake* of P_{n+3} illustrated in Figure 7. Each positive root is a consecutive sum

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \qquad 1 \leq i \leq j \leq n,$$

and thus can be identified with the unique diagonal of P_{n+3} crossing the (consecutive) diagonals that correspond to $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$, and no oth-ers. Two roots α and β in $\Phi_{\geq -1}$ are called *compatible* if their corresponding diagonals do not cross. The *cluster complex* $\Delta(\Phi)$ of type A_n is the clique complex of the compatibility relation on $\Phi_{\geq -1}$, i.e., the complex whose sim-plices correspond to the sets of almost positive roots that are pairwise com-patible. Maximal simplices of $\Delta(\Phi)$ are called *clusters*.

Theorem 3.15 (Fomin–Zelevinsky [12, Thms. 1.8, 1.10]). *The sim-plicial cones $\mathbb{R}_{\geq 0} C$ generated by all clusters C of type A_n form a complete simplicial fan in the ambient space*

$$\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 + \cdots + x_{n+1} = 0\}.$$

Theorem 3.16 (Chapoton–Fomin–Zelevinsky [7, Thm. 1.4]). *The simplicial fan in Theorem 3.15 is the normal fan of a simple n-dimensional polytope.*
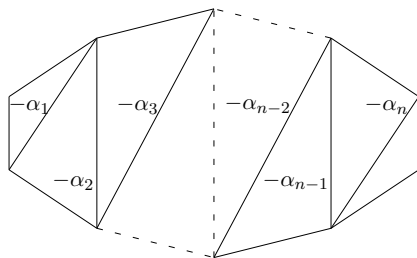
Figure 7. Snake and negative roots of type A_n.

Definition 3.17. We call *Chapoton–Fomin–Zelevinsky associahedron* $\mathrm{CFZ}(A_n)$ any polytope whose normal fan is the fan with maximal cones $\mathbb{R}_{\geq 0}C$ generated by all clusters C of type A_n.

A realization $\mathrm{CFZ}(A_2)$ is illustrated in Figure 8; note how the facet normals correspond to the almost positive roots of A_2. It is obvious from the definition of cluster complexes that $\mathrm{CFZ}(A_n)$ has exactly n pairs of parallel facets. These correspond to the pairs of roots $\{\alpha_i, -\alpha_i\}$, for $i=1,\ldots,n$, or, equivalently, to the pairs of diagonals $\{\alpha_i, -\alpha_i\}$ as indicated in Figure 9. This is a particular case of Proposition 5.5.
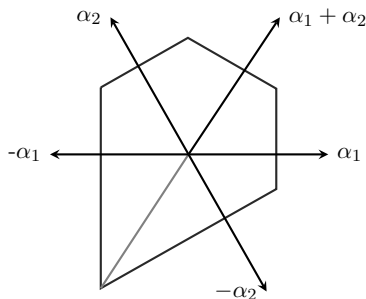
Figure 8. The complete simplicial fan of the cluster complex of type A_2 and an associahedron $\mathrm{CFZ}(A_2)$.

Theorem 3.15 is the case of type A_n of [12, Thm. 1.10]. Theorem 3.16 was conjectured by Fomin and Zelevinsky [12, Conj. 1.12] and subsequently proved by Chapoton, Fomin, and Zelevinsky [7]. For an explicit description by inequalities see [7, Cor. 1.9]. These two theorems (for type A) are special cases of our Theorems 5.1 and 5.2, proved in Section 5.

Figure 9. The diagonals of the $(n+3)$-gon that correspond to the pairs of parallel facets of CFZ(A_n).

4. Exponentially many realizations, by Hohlweg–Lange

4.1. The Hohlweg–Lange construction

In this section we give a short description of the first, which we call "type I", exponential family of realizations of the associahedron, obtained by Hohlweg and Lange in [17]. These associahedra are also generalized permutahedra and their construction uses ideas from Shnider–Sternberg and Loday's construction [37,23]. It was shown in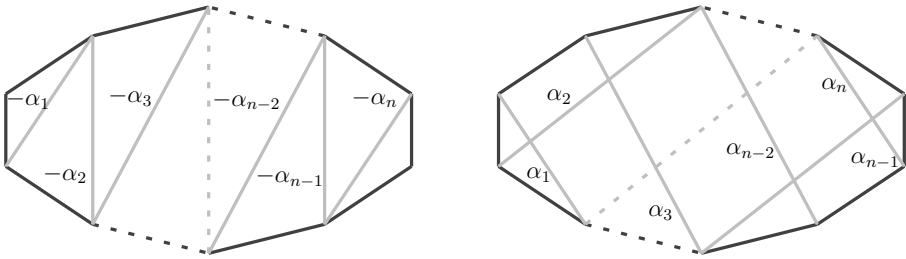 [2] that the number of normally non-isometric realizations obtained this way is equal to the number of sequences $\{+,-\}^{n-1}$ modulo reflection and reversal, which equals $2^{n-3}+2^{\lfloor \frac{n-3}{2} \rfloor}$ for $n \geq 3$ (see [38, Sequence A005418]). We show that classification by normal isomorphism yields the same number. See the recent paper [21] for another perspective on the Hohlweg–Lange construction.

Let $\sigma \in \{+,-\}^{n-1}$ be a sequence of signs on the edges of a horizontal path on n nodes. We identify $n+3$ vertices $\{0,1,\ldots,n+1,n+2\}$ with the signs of the sequence $\widetilde{\sigma} = \{+,-,\sigma,-,+\}$, and place them in convex position from left to right so that all positive vertices are above the horizontal path, and all negative vertices are below it. These vertices form a convex $(n+3)$-gon that we call $P_{n+3}(\sigma)$. Figure 10 illustrates the example $P_7(\{+,-,+\})$, where $n=4$.

The Hohlweg–Lange associahedra are obtained by removing certain facets of the standard n-dimensional permutahedron (1). The facets that are removed depend on the choice of σ, as follows.

Definition 4.1. For a diagonal ij $(i<j)$ of $P_{n+3}(\sigma)$, we denote by $R_{ij}(\sigma)$ the set of vertices strictly below it. We define the set $S_{ij}(\sigma)$ as the result of replacing 0 by i in $R_{ij}(\sigma)$ if $0 \in R_{ij}(\sigma)$, and replacing $n+2$ by j if $n+2 \in R_{ij}(\sigma)$.
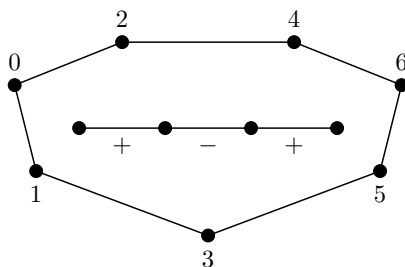
Figure 10. $P_7(\{+,-,+\})$.

The *Hohlweg–Lange associahedron* $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ is the polytope

$$\mathrm{Ass}_n^{\mathrm{I}}(\sigma) = \left\{ (x_1, \ldots x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i \in S_\delta(\sigma)} x_i \geq \tfrac{1}{2}|S_\delta(\sigma)|(|S_\delta(\sigma)| + 1) \right.$$

$$\left. \text{for all diagonals } \delta, \ x_1 + \cdots + x_{n+1} = \tfrac{(n+1)(n+2)}{2} \right\}.$$

Remark 4.2. If we interchange the first two signs and/or the last two signs in $\widetilde{\sigma} = \{+,-,\sigma,-,+\}$ the sets $S_\delta(\sigma)$ do not change and the construction produces the same associahedron $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$.

Proposition 4.3 (Hohlweg–Lange [17, Prop. 1.3]). $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ *is an n-dimensional associahedron.*

Hohlweg and Lange ([17, Thm. 1.1]) describe also the vertices of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$, extending Loday's rule ([23, Thm. 1.1], [29, Cor. 8.2]): To compute the i-th component of the vertex $(x_1, \ldots, x_{n+1})$ corresponding to a triangulation T, look at the unique triangle τ_i of T incident to vertex i and whose interior intersects the vertical line through vertex i. The n vertices in $T \setminus \tau$ fall into three components: those to the left of τ, those to the right, and those above τ (if i is a negative vertex) or below τ (if i is a positive vertex). Let l_i and r_i be the numbers of vertices to the left and right of τ. Set:

$$x_i = \begin{cases} (l_i + 1)(r_i + 1) & \text{if } \widetilde{\sigma}(i) = + \\ n + 2 - (l_i + 1)(r_i + 1) & \text{if } \widetilde{\sigma}(i) = -. \end{cases}$$

The reader can verify this rule for the Loday associahedron of Figure 4.

Proposition 4.4 (Hohlweg–Lange [17, Remarks 1.2, 4.3]). $\mathrm{Ass}_n^{\mathrm{I}}(\{-,-,\ldots,-\})$ *produces the Loday associahedron* $\mathrm{Post}_n(\mathbf{1})$, *and* $\mathrm{Ass}_n^{\mathrm{I}}(\{+,-,+,-,\ldots\})$ *is normally isomorphic to the Chapoton–Fomin–Zelevinsky associahedron* $\mathrm{CFZ}(A_n)$.

Remark 4.5. We defined the Loday associahedron as a Minkowski sum of certain faces of the standard simplex $\Delta_{[n+1]}$. The question arises whether such Minkowski sum descriptions exist for $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ in general. A partial answer is as follows: The associahedra $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ are examples of generalized permutahedra (recall that a *generalized permutahedron* is a polytope with facet normals contained in those of the standard permutahedron such that the collection of right hand side parameters of the defining inequalities belongs to the deformation cone of the standard permutahedron; compare with the appendix by Postnikov, Reiner and Williams in [30]). Generalized permutahedra include all the Minkowski sums $\sum_{S \subseteq [n+1]} a_S \Delta_S$ for which the coefficients a_S are non-negative. Ardila, Benedetti and Doker [1] have shown that every generalized permutahedron admits a (unique) expression as a *Minkowski sum and difference* of faces of the standard simplex. These decompositions, for the case of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$, are studied by Lange in [20]. A different decomposition arises from the work of Pilaud and Santos [27], who show that the associahedra $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ are the "brick polytopes" of certain sorting networks. As such, they admit a decomposition as the Minkowski sum of the $\binom{n}{2}$ polytopes of the individual "bricks". However, these summands need not be simplices.

Remark 4.6. Hohlweg–Lange–Thomas [18] provide a generalization of the Hohlweg–Lange construction to all finite Coxeter groups; for each Coxeter element c (equivalently, for each orientation of the Coxeter graph) in a finite Coxeter system, they construct a realization of the corresponding generalized associahedron having as normal fan the c-Cambrian fan introduced earlier by Reading [31] and discussed by Reading and Speyer [33]. They call this polytope the c-generalized associahedron. For types A and B, this specializes to the Hohlweg–Lange associahedra and cyclohedra.

A common generalization of c-generalized associahedra and brick polytopes (see previous remark) is introduced by Pilaud–Stump [28]. Another interesting construction of the Hohlweg–Lange–Thomas c-generalized associahedra is obtained by Stella in [42].

4.2. Normal facet vectors, and normal isomorphism

The Hohlweg–Lange associahedra satisfy properties (1) and (2) mentioned in the introduction: they have n pairs of parallel facets and in the basis given by the normals to those facets all normal facet vectors are in $\{-1, 0, 1\}^n$. To see this, we denote e_S the characteristic vector of each subset $S \subset [n+1]$. By definition, the normal vectors of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ are the characteristic vectors

of the sets $S_{ij}(\sigma)$ associated to the different diagonals of $P_{n+3}(\sigma)$, but these vectors are to be considered modulo $e_{[n+1]} = (1,\ldots,1)$. In particular, we have $e_S + e_{\overline{S}} = 0$ for $\overline{S} = [n+1] \setminus S$.

Although the following result is implicit in [17], we include a proof because our description of the diagonals corresponding to parallel facets is a bit more explicit and will be used in the proofs of Theorems 4.9 and 6.1.

Proposition 4.7 (Hohlweg–Lange [17, Lem. 2.2 and Cor. 2.3]). $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ *has exactly n pairs of parallel facets, whose normal vectors are $e_{[j]}$ and $e_{\overline{[j]}}$ for $j = 1,\ldots,n$. They correspond to the diagonals of the quadrilaterals with vertices $\{i,j,j+1,k\}$ for $j = 1,\ldots,n$, where*

$$i = \max\{0 \le r < j \colon \widetilde{\sigma}(r) \cdot \widetilde{\sigma}(j) = -\}$$

$$k = \min\{j + 1 < r \le n + 2 \colon \widetilde{\sigma}(r) \cdot \widetilde{\sigma}(j+1) = -\}$$

Hohlweg and Lange [17, Cor. 2.3] also remark that the facets with normals $e_{[j]}$, $j \in [n+1]$ intersect at a vertex of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ and the ones with normals $e_{\overline{[j]}}$, $j \in [n+1]$ intersect at another (opposite) one. That is to say, the $2n$ corresponding diagonals of $P_{n+3}(\sigma)$ form two triangulations, as was the case in Figure 5.

Proof. Two diagonals δ and δ' correspond to two parallel facets of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ if and only if the sets S_δ and $S_{\delta'}$ are complementary. By the definition of S_δ, the only way this can happen is when δ and δ' are two crossing diagonals of opposite slope signs and such that the quadrilateral containing them uses an edge from the lower chain of $P_{n+3}(\sigma)$ and an edge of the upper chain. This is the case described in the statement, and it is easy to check that the corresponding S_δ and $S_{\delta'}$ are, respectively, $\{1,\ldots,j\}$ and $\{j+1,\ldots,n+1\}$. ∎

That all other normals have coordinates in $\{-1,0,+1\}^n$ when expressed in the basis $\{e_{[j]}, j \in [n]\}$ follows trivially from the following equation, valid for every $S \subset [n+1]$:

$$e_S = \sum_{\substack{j \in S \\ j+1 \notin S}} e_{[j]} - \sum_{\substack{j+1 \in S \\ j \notin S}} e_{[j]}.$$

Corollary 4.8. *With respect to the basis $\{e_{[1]},\ldots,e_{[n]}\}$, (and considered modulo $e_{[n+1]}$), the normal vectors of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ are all in $\{0,+1,-1\}^n$ and include $\{\pm e_{[1]},\ldots,\pm e_{[n]}\}$.*

As was pointed out to us by one of the referees, this result can be stated in the language of Coxeter combinatorics as follows: the normal vectors of the Hohlweg–Lange associahedra are weights of the root system of type A; the parallel facets correspond to fundamental weights and every weight of type A can be written as a linear combination, with coefficients in $\{0, -1, 1\}$, of the fundamental weights.

We now use parallel facets to classify Hohlweg–Lange associahedra. For a sequence $\sigma \in \{-, +\}^{n-1}$ we define the reflection of σ as the sequence $-\sigma$, and the reversal σ^t as the result of reversing the order of coordinates in σ.

Theorem 4.9. *Let $\sigma_1, \sigma_2 \in \{+, -\}^{n-1}$. Then $\mathrm{Ass}_n^I(\sigma_1)$ and $\mathrm{Ass}_n^I(\sigma_2)$ are normally isomorphic if and only if σ_2 can be obtained from σ_1 by reflections and reversals.*

Proof. Suppose there is a linear isomorphism between the normal fans of $\mathrm{Ass}_n^I(\sigma_1)$ and $\mathrm{Ass}_n^I(\sigma_2)$. It induces an automorphism of the face lattice of the associahedron that, by Lemma 2.2, corresponds to a certain reflection-rotation of the polygon. We denote this reflection-rotation by $\varphi \colon P_{n+3}(\sigma_1) \to P_{n+3}(\sigma_2)$. Any linear isomorphism of the normal fans preserves the property of a pair of facets being parallel, so φ maps the "parallel" pairs of diagonals of $P_{n+3}(\sigma_1)$, to the "parallel" pairs of diagonals of $P_{n+3}(\sigma_2)$. Furthermore, for both realizations there are exactly four diagonals that cross at least one diagonal of every parallel pair; they are $\{0, n+1\}, \{0, n+2\}, \{1, n+1\}$ and $\{1, n+2\}$. The set of these four diagonals is also preserved under φ. This is possible only if φ is a reflection-rotation of $P_{n+3}(\sigma_1)$, which corresponds to a reflection-reversal of the sequence $\widetilde{\sigma_1} = \{+, -, \sigma_1, -, +\}$.

It remains to be proved that $\mathrm{Ass}_n^I(\sigma)$ is normally-isomorphic to both $\mathrm{Ass}_n^I(-\sigma)$ and $\mathrm{Ass}_n^I(\sigma^t)$. The isomorphism between the normal fans of $\mathrm{Ass}_n^I(\sigma)$ and $\mathrm{Ass}_n^I(-\sigma)$ is given by multiplication by -1, since $S_\delta(-\sigma) = [n] - S_\delta(\sigma)$. The isomorphism between the normal fans of $\mathrm{Ass}_n^I(\sigma)$ and $\mathrm{Ass}_n^I(\sigma^t)$ is given by the permutation of coordinates $\tau(i) = n + 1 - i$, as $S_\delta(\sigma^t) = \tau(S_\delta(\sigma))$. ∎

In particular, putting together Proposition 4.4 and Theorem 4.9, one obtains that the Loday associahedron is not normally isomorphic to the Chapoton–Fomin–Zelevinsky associahedron, for $n \geq 3$.

Remark 4.10. Bergeron, Hohlweg, Lange and Thomas [2, Thm. 2.3] classify the Hohlweg–Lange–Thomas c-generalized associahedra up to isometry, and also up to isometry of normal fans [2, Cor. 2.6]. Even if those classifications yield the same result as ours, they do not automatically imply it. As an example of why these classifications are potentially different, consider

the rhombus obtained by removing two opposite facets of a regular hexagon. This generalized permutahedron admits a normal automorphism that is not an isometry of normal fans: the affine map that interchanges acute and obtuse angles. That is to say, the fact that Theorem 4.9 yields the same classes as the classification in [2] implies the following result for Hohlweg–Lange associahedra, which fails for other generalized permutahedra:

Proposition 4.11. $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ and $\mathrm{Ass}_n^{\mathrm{I}}(\sigma')$ are normally isomorphic if and only if they are isometric.

We do not know whether the same is true for c-generalized associahedra in other types.

5. Catalan many realizations, by Santos

In this section we describe a generalization of the Chapoton–Fomin–Zelevinsky construction of the associahedron (Section 3.3). It was originally presented at a conference in 2004 [36], but unpublished until now. We prove that the number of normally non-isomorphic realizations obtained this way, our "type II exponential family", is equal to the number of triangulations of an $(n+3)$-gon modulo reflections and rotations. This number equals

$$\tfrac{1}{2(n+3)}C_{n+1} + \tfrac{1}{4}C_{(n+1)/2} + \tfrac{1}{2}C_{\lfloor (n+1)/2 \rfloor} + \tfrac{1}{3}C_{n/3},$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \in \mathbb{Z}$ and $C_n = 0$ otherwise. Interest in this sequence goes back to Motzkin (1948) [26], [38, Sequence A000207].

Let $\alpha_1, \ldots, \alpha_n$ denote a linear basis of an n-dimensional real vector space $V \cong \mathbb{R}^n$, and let T_0 be a certain triangulation of the $(n+3)$-gon, fixed once and for all throughout the construction. We call T_0 the *seed triangulation*. The CFZ associahedron will arise as the special case where $V = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum x_i = 0\}$, $\alpha_i = e_i - e_{i+1}$, and T_0 is the *snake triangulation* of Figure 7.

Let $\{\delta_1, \ldots, \delta_n\}$ denote the n diagonals present in the seed triangulation T_0. To each diagonal pq out of the $\frac{n(n+3)}{2}$ possible diagonals of the $(n+3)$-gon we associate a vector v_{pq} as follows:

○ If $pq = \delta_i$ for some i (that is, if pq is used in T_0) then let $v_{pq} = -\alpha_i$.
○ If $pq \notin T_0$ then let

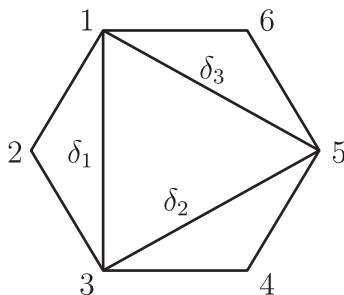$$v_{pq} := \sum_{pq \text{ crosses } \delta_i} \alpha_i.$$

Figure 11. A seed triangulation for Santos' construction.

As a running example, consider the triangulation $\{123, 345, 156, 135\}$ of a hexagon with its vertices labelled cyclically. Let $\delta_1 = 13$, $\delta_2 = 35$ and $\delta_3 = 15$. Written with respect to the basis $\{\alpha_1, \alpha_2, \alpha_3\}$ the nine vectors v_{pq} that we get are as follows (see Figure 11):

$$
\begin{aligned}
v_{13} &= -\alpha_1 = (-1, 0, 0), & v_{35} &= -\alpha_2 = (0, -1, 0), \\
v_{15} &= -\alpha_3 = (0, 0, -1), & v_{25} &= \alpha_1 = (1, 0, 0), \\
v_{14} &= \alpha_2 = (0, 1, 0), & v_{36} &= \alpha_3 = (0, 0, 1), \\
v_{46} &= \alpha_2 + \alpha_3 = (0, 1, 1), & v_{26} &= \alpha_1 + \alpha_3 = (1, 0, 1), \\
v_{24} &= \alpha_1 + \alpha_2 = (1, 1, 0).
\end{aligned}
$$

With a slight abuse of notation we denote with the same symbol a subset of diagonals of the polygon and the set of vectors associated with them. For example, $\mathbb{R}_{\geq 0} T_0 = \mathbb{R}_{\geq 0}\{-\alpha_1, \ldots, -\alpha_n\}$ is the negative orthant in V (with respect to the basis $[\alpha_i]_i$). More generally, for each triangulation T of the $(n+3)$-gon consider the cone $\mathbb{R}_{\geq 0} T$. We claim the following generalizations of Theorems 3.15 and 3.16:

Theorem 5.1. *The simplicial cones $\mathbb{R}_{\geq 0} T$ generated by all triangulations T of the $(n+3)$-gon form a complete simplicial fan $\mathcal{F}_{T_0}$ in the ambient space V.*

Theorem 5.2. *This fan $\mathcal{F}_{T_0}$ is the normal fan of an n-dimensional associahedron.*

Our proofs are based on the understanding of a complete simplicial fan as a *triangulation of a totally cyclic vector configuration*, which makes *regular triangulations* correspond to normal fans of simple polytopes (see [8, Sects. 2.5, 9.5], and compare our two statements to steps (1) and (2) in [8, p. 503]). Incidentally, this method is illustrated there by constructing the normal fan of the Loday associahedron and showing its polytopality.

5.1. Proof of Theorem 5.1

The statement follows from the following two claims:

(1) $\mathbb{R}_{\geq 0}T_0$ is a simplicial cone and is the only cone in $\mathcal{F}_{T_0}$ that intersects (the interior of) the negative orthant.
(2) If T_1 and T_2 are two triangulations that differ by a flip, let $v_1 \in T_1$ and $v_2 \in T_2$ be the diagonals removed and inserted by the flip. That is, $T_1 \setminus T_2 = \{v_1\}$ and $T_2 \setminus T_1 = \{v_2\}$. Then there is a linear dependence in $T_1 \cup T_2$, which has coefficients of the same sign (and different from zero) in the elements v_1 and v_2.

The first assertion is obvious, and the second one is Lemma 5.3 below. Before proving it let us argue why these two assertions imply Theorem 5.1. Suppose that we have two triangulations T_1 and T_2 related by a flip as in the second assertion, and suppose that we already know that one of them, say T_1, spans a full-dimensional cone (that is, we know that T_1 considered as a set of vectors is independent). Then assertion (2) implies that T_2 spans a full-dimensonal cone as well and that $\mathbb{R}_{\geq 0}T_1$ and $\mathbb{R}_{\geq 0}T_2$ lie in opposite sides of their common facet $\mathbb{R}_{\geq 0}(T_1 \cap T_2)$. This, together with the fact that there is some part of V covered by exactly one cone (which is why we need assertion (1)) implies that we have a complete fan. (See, for example, [8, Cor. 4.5.20], where assertion (2) is a special case of "property (ICoP)" and assertion (1) a special case of "property (IPP)".)

Lemma 5.3. *Let T_1 and T_2 be two triangulations that differ by a flip, and let v_1 and v_2 be the diagonals removed and inserted by the flip from T_1 to T_2, respectively (that is, $T_1 \setminus T_2 = \{v_1\}$ and $T_2 \setminus T_1 = \{v_2\}$). Then there is a linear dependence in $T_1 \cup T_2$, which has coefficients of the same sign in the elements v_1 and v_2.*

Proof. Let p, q, r and s be the four points involved by the two diagonals v_1 and v_2, in cyclic order. That is, the diagonals removed and inserted are pr and qs. We claim that one (and exactly one) of the following things occurs (see Figure 12):

(a) There is a diagonal in the seed triangulation T_0 that crosses two opposite edges of the quadrilateral $pqrs$.
(b) One of pr and qs is used in the seed triangulation T_0.
(c) There is a triangle abc in T_0 with a vertex in $pqrs$ and the opposite edge crossing two sides of $pqrs$ (that is, without loss of generality $p = a$ and bc crosses both qr and rs).

(d) There is a triangle abc in T_0 with an edge in common with $pqrs$ and with the other two edges of the triangle crossing the opposite edge of the quadrilateral (that is, without loss of generality, $p=a$, $q=b$ and rs crosses both ac and bc).
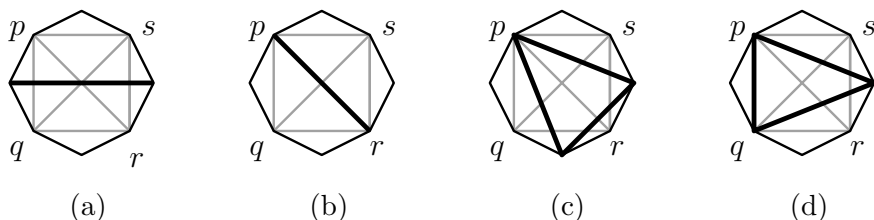

Figure 12. The four cases in the proof of Lemma 5.3.

To prove that one of the four things occurs we argue as follows. It is well-known that in any triangulation of a k-gon one can "contract a boundary edge" to get a triangulation of a $(k-1)$-gon. Doing that in all the boundary edges of the seed triangulation T_0 except those incident to either p, q, r or s we get a triangulation $\widetilde{T_0}$ of a polygon $\widetilde{P}$ with at most eight vertices: the four vertices p, q, r and s and at most one extra vertex between each two of them. We embed $\widetilde{P}$ having as vertices a subset of the vertices of a regular octagon, with $pqrs$ forming a square. We now look at the position of the center of the octagon $\widetilde{P}$ with respect to the triangulation $\widetilde{T_0}$: If it lies in the interior of an edge, then this edge is a diameter of the octagon and we are in cases (a) or (b). If it lies in the interior of a triangle of $\widetilde{T_0}$, then we are in cases (c) or (d). See Figure 12 again.

Now we show explicitly the linear dependences involved in $T_1 \cup T_2$ in each case.

(a) Suppose T_0 has a diagonal crossing pq and rs. Then

$$(4) \qquad v_{pr} + v_{qs} = v_{pq} + v_{rs},$$

because every diagonal of T_0 intersecting the two (respectively, one; respectively none) of pr and qs intersects also the two (respectively, one; respectively none) of pq and rs.

(b) If T_0 contains the diagonal pr, let a and b be vertices joined to pr in T_0, with a on the side of q and b on the side of s. We define the following vectors w_a and w_b:

  ○ w_a equals 0, v_{pq} or v_{qr} depending on whether a equals q, lies between p and q, or lies between q and r.

    ◦ w_b equals 0, v_{ps} or v_{rs} depending on whether a equals s, lies between p and s, or lies between s and r.

We claim that in the nine cases we have the equality

$$(5) \qquad\qquad v_{pr} + v_{qs} = w_a + w_b.$$

This is so because $v_{pr}+v_{qs}$ now equals the sum of the α_i's corresponding to the diagonals of $T_0 \setminus \{pr\}$ crossing qs, and we have that:

    ◦ The diagonals of T_0 crossing qs in the q-side of pr are none, the same as those crossing pq, or the same as those crossing qr in the three cases of the definition of w_a, and

    ◦ The diagonals of T_0 crossing qs in the s-side of pr are none, the same as those crossing ps, or the same as those crossing rs in the three cases of the definition of w_b

(c) If T_0 contains a triangle pbc with bc crossing both qr and rs then we have the equality

$$(6) \qquad\qquad 2v_{pr} + v_{qs} = v_{qr} + v_{rs},$$

because in this case the diagonals of T_0 crossing pr are those crossing both qr and rs, while the ones crossing qs are those crossing one, but not both, of qr and rs.

(d) If T_0 contains a triangle pqc with rs crossing both pc and qc then we have the equality

$$(7) \qquad\qquad v_{pr} + v_{qs} = v_{rs}$$

because the diagonals of T_0 crossing rs are the disjoint union of those crossing pr and those crossing qs. ∎

Observe that when T_0 is a snake triangulation (the CFZ case) or, more generally, when the dual tree of T_0 is a path, cases (c) and (d) do not occur.

5.2. Proof of Theorem 5.2

Once we know $\mathcal{F}_{T_0}$ is a complete simplicial fan, its being the normal fan of a simple polytope can be expressed as the feasibility of a system of linear inequalities. This can be done in several ways (compare, e.g., Theorem 3.1 in [18]). We choose the following one, related to the understanding of complete simplicial fans as *triangulations of vector configurations* (see [8, Sec. 9.5]).

Lemma 5.4. *Let $\mathcal{F}$ be a complete simplicial fan in a real vector space V and let A be the set of generators of $\mathcal{F}$ (more precisely, A has one generator of each ray of $\mathcal{F}$). Then the following conditions are equivalent:*

(1) *$\mathcal{F}$ is the normal fan of a polytope.*
(2) *There is a map $\omega\colon A\to\mathbb{R}_{>0}$ such that for every pair (C_1,C_2) of maximal adjacent cones of $\mathcal{F}$ the following happens: Let $\lambda\colon A\to\mathbb{R}$ be the (unique, up to a scalar multiple) linear dependence with support in $C_1\cup C_2$, with its sign chosen so that λ is positive in the generators of $C_1\setminus C_2$ and $C_2\setminus C_1$. Then the scalar product $\lambda\cdot\omega=\sum_v\lambda(v)\omega(v)$ is strictly positive.*

Proof. One short proof of the lemma is that both conditions are equivalent to "$\mathcal{F}$ is a regular triangulation of the vector configuration A" (See, e. g., Corollary 9.5.3 [8]). But let us show a more explicit proof of the implication from (2) to (1), which is the one we need. What we are going to show is that if such an ω exists and if we consider the set of points

$$\widetilde{A} := \left\{ \tfrac{v}{\omega(v)} : v \in A \right\},$$

then the convex hull of $\widetilde{A}$ is a simplicial polytope having $\mathcal{F}$ as its *central fan*. (We think of $\widetilde{A}$ as points in an affine space, rather than as vectors in a vector space.) Hence $\mathcal{F}$ is the normal fan of the polar of $\mathrm{conv}(\widetilde{A})$ (see, e. g., [44, Sec. 7.1]).

To show the claim on $\mathrm{conv}(\widetilde{A})$ we argue as follows. Consider the simplicial complex Δ with vertex set $\widetilde{A}$ obtained by embedding the face lattice of $\mathcal{F}$ in it. That is, for each cone C of $\mathcal{F}$ we consider the simplex with vertex set in $\widetilde{A}$ corresponding to the generators of C. Since $\mathcal{F}$ is a complete fan and since the elements of $\widetilde{A}$ are generators for its rays (they are positive scalings of the elements of A), Δ is the boundary of a star-shaped polyhedron with the origin in its kernel. The only thing left to be shown is that this polyhedron is strictly convex, that is, that for any two adjacent maximal simplices σ_1 and σ_2 the origin lies in the same side of σ_1 as $\sigma_2\setminus\sigma_1$. Equivalently, if we understand (the vertices of) σ_1 and σ_2 as subsets of $\widetilde{A}$, we have to show that the unique affine dependence between the points $\{O\}\cup\sigma_1\cup\sigma_2$ has opposite sign in O than in $\sigma_1\setminus\sigma_2$ and $\sigma_2\setminus\sigma_1$. The proof of this is easy. The coefficients in the *linear* dependence among the *vectors* in $\sigma_1\cup\sigma_2$ are the vector

$$(\lambda(v)\omega(v))_{v\in A}.$$

To turn this into an *affine* dependence of points involving the origin we simply need to give the origin the coefficient $-\sum_v\lambda(v)\omega(v)$ which is, by hypothesis, negative. ∎

So, in the light of Lemma 5.4, to prove Theorem 5.2 we simply need to choose weights ω_{ij} for the diagonals of the polygon with the property that, for each of the linear dependences exhibited in equations (4), (5), (6), and (7), the equation $\sum_{ij} \omega_{ij}\lambda_{ij} > 0$ holds.

As a first approximation, let $\omega_{ij} = 2$ if ij is in T_0 and $\omega_{ij} = 1$ otherwise. This is good enough for equations (6) and (7) in which all the ω's in the dependence are 1 and the sum of the coefficients in the left-hand side is greater than in the right-hand side. It also works for equations (5), in which we have

$$\omega_{pr} = 2, \qquad \omega_{qs} = 1, \qquad \lambda_{pr} = 1, \qquad \lambda_{qs} = 1,$$

so that the sum $\sum_{ij} \omega_{ij}\lambda_{ij}$ for the left-hand side is three, while that of the right-hand side can be 0, -1 or -2 depending on the cases for the points a and b.

The only (weak) failure is that in equation (4) we have

$$\lambda_{pr} = 1, \qquad \lambda_{qs} = 1, \qquad \lambda_{pq} = -1, \qquad \lambda_{rs} = -1$$

and all the ω's are 1, so we get $\sum_{ij} \omega_{ij}\lambda_{ij} = 0$. We solve this by slightly perturbing the ω's. A slight perturbation will not change the correct signs we got for equations (5), (6), and (7). For example, for each ij not in T_0 change ω_{ij} to

$$\omega_{ij} = 1 + \varepsilon g_{ij}$$

for a sufficiently small $\varepsilon > 0$ and for a vector $(g_{ij})_{ij}$ satisfying

$$g_{ik} + g_{jl} > \max\{g_{ij} + g_{kl}, g_{il} + g_{jk}\}$$
$$\text{for all } i, j, k, l, \ 1 \leq i < j < k < l \leq n+3.$$

This holds (for example) for $g_{ij} := (j-i)(n+3+i-j)$.

5.3. Distinct seed triangulations produce distinct realizations

Let $\mathrm{Ass}_n^{\mathrm{II}}(T)$ denote the n-dimensional associahedron obtained with the construction of the previous section starting with a certain triangulation T. (This is a slight abuse of notation, since the associahedron depends also in the weight vector ω that gives the right-hand sides for its inequality definition. Put differently, by $\mathrm{Ass}_n^{\mathrm{II}}(T)$ we here denote the normal fan rather than the associahedron itself.) We want to classify the associahedra $\mathrm{Ass}_n^{\mathrm{II}}(T)$ by normal isomorphism.

In principle, it looks like we have as many associahedra as there are triangulations (that is, Catalan-many) but that is not the case because, clearly,

changing T by a rotation or a reflection does not change the associahedron obtained. The question is whether this is the only operation that preserves $\mathrm{Ass}_n^{\mathrm{II}}(T)$, modulo normal isomorphism. To answer this, we look at parallel facets.

Proposition 5.5. $\mathrm{Ass}_n^{\mathrm{II}}(T_0)$ *has exactly n pairs of parallel facets, each pair consisting of (the facet of) one diagonal in T_0 and the diagonal obtained from it by a flip in T_0.*

Proof. $\mathrm{Ass}_n^{\mathrm{II}}(T)$ is full-dimensional, so two facets are parallel only if their defining normals are opposite. Since all normals except the ones from the seed triangulation T_0 lie in the positive orthant, in every pair of opposite normals one of them comes from the seed triangulation. This is the case of the statement. ∎

Lemma 5.6. *Let Q be an $(n+3)$-gon, with $n \geq 2$. For each triangulation T of Q let B_T denote the set consisting of the n diagonals in T plus the n diagonals that can be introduced by a single flip from T. Then for every $T_1 \neq T_2$ we have $B_{T_1} \neq B_{T_2}$.*

Proof. Suppose that T_1 and T_2 had $B_{T_1} = B_{T_2}$. We claim that T_2 is obtained from T_1 by a set of "parallel flips". That is, by choosing a certain subset of diagonals of T_1 such that no two of them are incident to the same triangle and flipping them simultaneously. This is so because every diagonal pr in T_2 but not in T_1 intersects a single diagonal qs of T_1. If pqr and prs were not triangles in T_2, then let a be a vertex joined to pr in T_2, different from q or s. One of pa and ra intersects the diagonal qs of T_1 and one of the edges pq, qr, rs and pr of T_1.

   Once we have proved this for T_2, the statement is obvious. For every T_2 different from T_1 but with all its diagonals in B_{T_1} there is a diagonal that we can flip to get one that is not in B_{T_1} (same argument, let pr be a diagonal in T_2 but not in T_1; let pq, qr, rs and pr be the other sides of the two triangles of T_2 containing pq. Flipping any of them, say pq, gives a diagonal that crosses pq and qs, which are both in T_1). ∎

Corollary 5.7. *Let T_1 and T_2 be two triangulations of a convex $(n+3)$-gon. Then $\mathrm{Ass}_n^{\mathrm{II}}(T_1)$ and $\mathrm{Ass}_n^{\mathrm{II}}(T_2)$ are normally isomorphic if and only if T_1 and T_2 are equivalent under rotation-reflection.*

Proof. If T_1 and T_2 are equivalent under rotation-reflection then the resulting associahedra are clearly the same. Now suppose that $\mathrm{Ass}_n^{\mathrm{II}}(T_1)$ and $\mathrm{Ass}_n^{\mathrm{II}}(T_2)$ are normally isomorphic. By Lemma 2.2 the automorphism of

the associahedron face lattice induced by the isomorphism corresponds to a rotation-reflection of the polygon. Now, normal isomorphism preserves the property of a pair of facets being parallel, so this rotation-reflection sends B_{T_1} to B_{T_2}, and thus T_1 to T_2. ∎

However, the same is not true if we only look at the set of normal vectors of $\mathrm{Ass}_n^{\mathrm{II}}(T)$:

Proposition 5.8. *Let T_1 and T_2 be two triangulations of a convex $(n+3)$-gon. Let $A(T_1)$ and $A(T_2)$ be the sets of normal vectors of $\mathrm{Ass}_n^{\mathrm{II}}(T_1)$ and $\mathrm{Ass}_n^{\mathrm{II}}(T_2)$. Then $A(T_1)$ and $A(T_2)$ are linearly equivalent if and only if T_1 and T_2 have isomorphic dual trees.*

Proof. Let $\mathcal{T}$ be the dual tree of a triangulation T. Observe that the edges of $\mathcal{T}$ correspond bijectively to the inner diagonals in T. Moreover, the diagonals of the polygon not used in T correspond bijectively to the possible paths in $\mathcal{T}$. More precisely: for every pair of nodes of $\mathcal{T}$ (that is, triangles t_1 and t_2 of T) let p_1 (resp. p_2) be the vertex of t_1 (resp. of t_2) not visible from t_2 (resp. from t_1). Then the diagonals of T crossed by p_1p_2 correspond to the path in $\mathcal{T}$ joining t_1 to t_2.

This means that, if we label the edges of $\mathcal{T}$ with the numbers 1 through n in the same manner as we labelled the diagonals of T we have that

$$A(T) = \{-\alpha_i : i \in [n]\} \cup \left\{ \sum_{i \in p} \alpha_i : p \text{ is a path in } \mathcal{T} \right\}.$$

So, $A(T)$ can be recovered knowing only $\mathcal{T}$ as an abstract graph. For the converse, observe that if two trees are not isomorphic then there is no bijection between their edges that sends paths to paths. For example, knowing only the sets of edges that form paths we can identify the (stars of) vertices of the tree as the sets of edges such that every two of them form a path. ∎

5.4. Path triangulations, *c*-cluster complexes, and denominator fans in type A_n

5.4.1. Associahedra from path triangulations. Let us call a triangulation of P_{n+3} whose dual tree is a path a *path triangulation*. By Proposition 5.8, for a path triangulation T the set of normal vectors to the facets of $\mathrm{Ass}_n^{\mathrm{II}}(T)$ equals the almost positive roots in the root system A_n, exactly as in the Chapoton–Fomin–Zelevinsky associahedron. However, these associahedra are not normally equivalent to one another. To analyze this, we

encode each path triangulation of the $(n+3)$-gon as a sequence of signs $c\in\{+,-\}^{n-1}$, as follows: the coordinates of c correspond to the $n-1$ triangles that are not *ears* (that is, are not leaves in the dual path) and we make it a $+$ or a $-$ depending on whether the dual path turns right or left at that vertex (see Figure 13). We denote T_c the triangulation obtained in this way, for each sequence c.
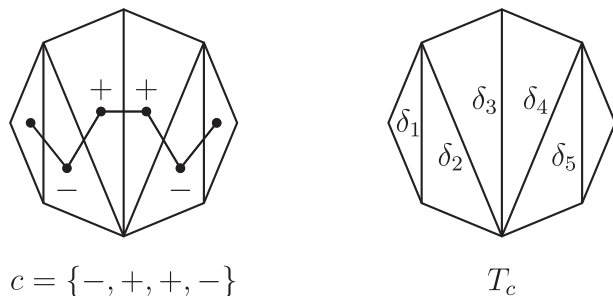

$$c = \{-, +, +, -\}$$

Figure 13. The triangulation T_c corresponding to the sequence of signs $c=\{-,+,+,-\}$.

It is clear that every path triangulation can be encoded in this way and that T_{c_1} and T_{c_2} are the same (modulo symmetries of the $(n+3)$-gon) if and only if c_1 and c_2 are the same modulo reflection and reversal. In particular, this gives us exponentially many realizations of the n-associahedron with the same set of facet normals as the Chapoton–Fomin–Zelevinsky associahedron:

Corollary 5.9. *Let T_0 be a triangulation whose dual tree is a path. Let its diagonals be numbered from 1 to n in the order they appear in the path. Then,*

(i) *taking $\alpha_i = e_{i+1} - e_i$, we have that the set of normal vectors to the facets of $\mathrm{Ass}_n^{\mathrm{II}}(T_0)$ is the set of almost positive roots in the root system A_n.*

(ii) *The number of normally non-isomorphic classes of associahedra obtained in this way is equal to the number of sequences $\{+,-\}^{n-1}$ modulo reflection and reversal.*

The number of realizations that we get in this way is exactly the same as the number of Hohlweg–Lange associahedra (see Theorem 4.9). The explanation for this coincidence is in Remark 5.11. Nevertheless, the two sets of realizations are almost disjoint; the only common one is the Chapoton–Fomin–Zelevinsky associahedron, obtained in both cases for the sequence that alternates pluses and minuses (Theorem 6.1).

5.4.2. c-cluster complexes in type A_n. It turns out that the associ-ahedra $\mathrm{Ass}_n^{\mathrm{II}}(T_0)$ of path triangulations provide a simple combinatorial de-scription of c-cluster complexes in type A_n as described by Reading in [32]. These complexes are more general than the cluster complexes of Fomin and Zelevinsky [12], and have an extra parameter c corresponding to a *Coxeter element*.

In type A_n, Coxeter elements can be represented by a sequence of signs $c \in \{+,-\}^{n-1}$; the corresponding Coxeter element is given by a product of generators $s_1, \ldots, s_n$ in some order such that s_{i+1} comes after s_i if the i-th sign in the sequence is positive, and s_{i+1} comes before s_i if the i-th sign is negative. Each sequence c induces a single Coxeter element because generators s_i and s_j commute when $|i-j| \geq 2$.

As in the description of the cluster complex of type A_n in Section 3.3.1, consider the root system of type A_n and the set of almost positive roots $\Phi_{\geq -1}$. In addition, consider a sequence of signs $c \in \{+,-\}^{n-1}$ and let T_c be the corresponding path triangulation. Label the diagonals of T_c by $\{\delta_1, \ldots, \delta_n\}$ in the order they appear in the dual path. As in the CFZ con-struction, this gives a natural correspondence between the set $\Phi_{\geq -1}$ and the diagonals of P_{n+3}: We identify the negative simple roots $\{-\alpha_1, \ldots, -\alpha_n\}$ with the diagonals $\{\delta_1, \ldots, \delta_n\}$, and each positive root

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \qquad 1 \leq i \leq j \leq n,$$

with the unique diagonal of P_{n+3} crossing the (consecutive) diagonals $-\delta_i, -\delta_{i+1}, \ldots, -\delta_j$.

We say that two roots α and β in $\Phi_{\geq -1}$ are c-*compatible* if their corre-sponding diagonals do not cross. The c-*cluster complex* can then be defined as the simplicial complex whose faces correspond to sets of almost positive roots that are pairwise c-compatible. The maximal simplices in it are called c-*clusters* and correspond naturally to triangulations of the polygon. For instance, the set

$$\{\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_3, -\alpha_5\}$$

is a c-cluster of type A_5 for $c = (-,+,+,-)$ corresponding to the Coxeter element $s_2 s_1 s_3 s_5 s_4$. The reason is that its corresponding diagonals in Fig-ure 13 form a triangulation of the polygon. This algorithm gives a simple combinatorial way of computing c-cluster complexes in type A. The proof that this description of c-cluster complexes actually coincides with the origi-nal description by Reading follows the two steps (i) and (ii) in the definition of the c-compatibility relation in [34, Sec. 5]. As a consequence we obtain:

Proposition 5.10. *The normal fan of the associahedron* $\mathrm{Ass}_n^{\mathrm{II}}(T_c)$ *coincides with the c-cluster fan of type* A_n.

Remark 5.11. As mentioned in Remark 4.6, the Hohlweg–Lange construction was generalized by Hohlweg–Lange–Thomas to a construction of c-generalized associedra (later described in different contexts by Stella [42] and Pilaud–Stump [28]). There is one c-generalized associahedron for each Coxeter element c, and it has the c-Cambrian fan [31,33] as its normal fan.

Proposition 5.10 is an analogous result for the c-cluster fans, which again exist for each Coxeter element in a finite Coxeter group. In particular, the proposition shows that in type A the c-cluster fans are the normal fans of polytopes. As far as we know, the same is not known in other types, except when c is the bipartite Coxeter element. (The c-cluster fan is, in this case, the normal fan of the Chapoton–Fomin–Zelevinsky generalized associahedron [7].) In fact, in the bipartite case Reading and Speyer have shown that the c-Cambrian fan and the c-cluster fan are linearly isomorphic [33, Thm. 9.1]. In the general case they only show combinatorial isomorphism between them [33, Thm. 1.1 and Sec. 5].

5.4.3. Denominator fans in type A_n.

For an arbitrary seed triangulation T, the normal fan of $\mathrm{Ass}_n^{\mathrm{II}}(T)$ can also be interpreted in the language of cluster algebras, as *denominator fans*. For each choice of seed cluster in a cluster algebra, the denominator fan has as rays the denominator vectors of the cluster variables with respect to the seed cluster, and it has maximal cones spanned by the denominator vectors of variables that form clusters. The c-cluster fan arises as the particular case where the cluster seed corresponds to an *acyclic quiver* associated to a Coxeter element c.

Notice that the name "denominator fan" is a slight abuse of notation since we do not know, a priori, if they are fans. But in type A, clusters correspond to triangulations and the denominator fan with seed triangulation T is nothing but the normal fan of the associahedron $\mathrm{Ass}_n^{\mathrm{II}}(T)$ (see, e.g., [6, Sec. 7]). Hence, as a consequence of Theorem 5.1 and Theorem 5.2 we obtain:

Proposition 5.12. *For cluster algebras of type* A_n,

(i) *The denominator fan associated to any triangulation T of a convex $(n+3)$-gon is a complete simplicial fan, and*

(ii) *it is the normal fan of a polytope (the associahedron* $\mathrm{Ass}_n^{\mathrm{II}}(T)$*).*

This result suggests a natural generalization of the Santos construction of associahedra to arbitrary finite Coxeter groups:

Question 5.13. Given an arbitrary cluster seed in a cluster algebra of finite type

○ is the associated denominator fan a complete simplicial fan?
○ if so, is it the normal fan of a polytope (a generalized associahedron)?

Although this question is phrased in terms of cluster algebras, which deal only with crystallographic root systems, both denominator fans and generalized associahedra make sense in the slightly more general context of finite Coxeter groups. (See [6] for an alternative description of denominator vectors in this general set up).

Question 5.14. If the answer to Question 5.13 is positive, is the classification up to normal isomorphism of Corollary 5.7 still valid for the generalized associahedra obtained this way? Note that the rotation map on convex polygons can be naturally generalized in the context of finite Coxeter groups (see, e.g., [6, Sec. 2.2] or [5, Sec. 8.3]).

6. How many associahedra?

We have presented several constructions of the associahedron. We call associahedra of types I and II the associahedra $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ and $\mathrm{Ass}_n^{\mathrm{II}}(T)$ studied in the previous two sections. Associahedra of type I include the Loday (or Shnider–Sternberg, or Rote–Santos–Streinu, or Postnikov, or Buchstaber) associahedron, and both types I and II include the Chapoton–Fomin–Zelevinsky associahedron. They all have pairs of parallel facets while the secondary polytope on an n-gon does not (Proposition 3.5). This implies that the associahedron as a secondary polytope is never normally isomorphic to any associahedron of type I or type II. In particular, it is not normally isomorphic to the Postnikov associahedron or the Chapoton–Fomin–Zelevinsky associahedron.

Both types I and II produce exponentially many normally non-isomorphic realizations. The number of normally non-equivalent associahedra of type I is asymptotically 2^{n-3}, while for type II is asymptotically $2^{2n+1}/\sqrt{\pi n^5}$. Explicit computations up to dimension 15 are given in Table 1.

Surprisingly, the realizations of types I and II are almost disjoint:

Theorem 6.1. *The only associahedron that is normally isomorphic to both one of type I and one of type II is the Chapoton–Fomin–Zelevinsky associahedron.*

Proof. Suppose that a sequence $\sigma \in \{+,-\}^{n-1}$ and a triangulation T produce normally isomorphic associahedra $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ and $\mathrm{Ass}_n^{\mathrm{II}}(T)$. By Lemma

$n=$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mathrm{Ass}_n^{\mathrm{I}}$	1	1	1	2	3	6	10	20	36	72	136	272	528	1056	2080	4160
$\mathrm{Ass}_n^{\mathrm{II}}$	1	1	1	3	4	12	27	82	228	733	2282	7528	24834	83898	285357	983244

Table 1. The number of normally non-isomorphic associahedra of types I and II up to dimension 15.

2.2 there is no loss of generality in assuming that the bijection between facets induced by this isomorphism corresponds to the identity map on the diagonals of the $(n{+}3)$-gon. Also, since normal isomorphism preserves parallelism of facets, the $2n$ diagonals corresponding to the n pairs of parallel facets are the same in $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ and $\mathrm{Ass}_n^{\mathrm{II}}(T)$. Denote the set of them B.

From the perspective of $\mathrm{Ass}_n^{\mathrm{II}}(T)$, B consists of the diagonals of T together with its flips. To analyze B from the perspective of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$, we consider the $(n{+}3)$-gon drawn in the Hohlweg–Lange fashion (with vertices placed along two x-monotone chains, the positive and the negative one, placed in the x-order indicated by σ). By Proposition 4.7, B contains only diagonals between vertices of opposite signs. Knowing this we conclude:

○ *Every triangle in T contains a boundary edge in one of the chains. (That is, the dual tree of T is a path).* Indeed, every triangle contains at least two vertices of the same sign in σ. The edge joining those two vertices cannot be in B, so it is a boundary edge.

○ *The third vertex of each triangle is in the opposite chain. (That is, the dual path of T separates the two chains).* Otherwise the three vertices of a certain triangle lie in the same chain. This is impossible, because (at least) one of the three edges of each triangle is a diagonal, hence it is in B.

○ *No two consecutive boundary edges in one chain are joined to the same vertex in the opposite chain. (That is, the dual tree of T alternates left and right turns).* Otherwise, let abp and bcp be two triangles in T with ab and bc consecutive boundary edges in one of the chains. Then the flip in bp inserts the edge ac, so that $ac \in B$. This is impossible, since a and c are in the same chain.

These three properties imply that T is the snake triangulation, so $\mathrm{Ass}_n^{\mathrm{II}}(T)$ is the Chapoton–Fomin–Zelevinsky associahedron. ∎

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