

## The Boolean algebra of spectra

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### Introduction

Let  $\mathbf{Ho}^s$  denote the stable homotopy category of CW-spectra (cf. [Adams 2]), and for  $E \in \mathbf{Ho}^s$  let  $E_*$  be the associated homology theory. For  $E, G \in \mathbf{Ho}^s$  we say  $E_*$  and  $G_*$  have the same acyclic spectra if the following equivalent conditions hold:

- (i) For  $X \in \mathbf{Ho}^s$ ,  $E_*X = 0 \Leftrightarrow G_*X = 0$ .
- (ii) For  $f: X \rightarrow Y \in \mathbf{Ho}^s$ ,  $f_*: E_*X \approx E_*Y \Leftrightarrow f_*: G_*X \approx G_*Y$ .

This gives a very coarse equivalence relation for spectra, and we let  $\mathbf{A}(\mathbf{Ho}^s)$  consist of all the equivalence classes  $\langle E \rangle$  for  $E \in \mathbf{Ho}^s$ , where  $\langle E \rangle$  is given by all  $G \in \mathbf{Ho}^s$  such that  $E_*$  and  $G_*$  have the same acyclic spectra. We partially order  $\mathbf{A}(\mathbf{Ho}^s)$  by writing  $\langle E \rangle \leq \langle G \rangle$  when each  $G_*$ -acyclic spectrum is  $E_*$ -acyclic. Our purpose in this note is to study the algebraic structure of  $\mathbf{A}(\mathbf{Ho}^s)$  when it is equipped with the relation  $\leq$  and the operations  $\vee$  and  $\wedge$  induced from the usual wedge and smash product for spectra.

We say that  $\langle E \rangle \in \mathbf{A}(\mathbf{Ho}^s)$  has a complement  $\langle E \rangle^c \in \mathbf{A}(\mathbf{Ho}^s)$  if  $\langle E \rangle \wedge \langle E \rangle^c = \langle 0 \rangle$  and  $\langle E \rangle \vee \langle E \rangle^c = \langle S \rangle$  where  $S$  is the sphere spectrum, and we note that  $\langle E \rangle^c$  is unique when it exists. We let  $\mathbf{BA}(\mathbf{Ho}^s) \subset \mathbf{A}(\mathbf{Ho}^s)$  consist of those  $\langle E \rangle \in \mathbf{A}(\mathbf{Ho}^s)$  with complements, and we observe that  $\mathbf{BA}(\mathbf{Ho}^s)$  is a Boolean algebra. We prove that  $\langle E \rangle \in \mathbf{BA}(\mathbf{Ho}^s)$  whenever  $E$  is a (possibly infinite) wedge of finite CW-spectra. It would be most interesting to determine the sublattice of  $\mathbf{BA}(\mathbf{Ho}^s)$  given by such  $\langle E \rangle$ . We show that  $\langle S^0 \cup_\alpha e^n \rangle = \langle S^0 \rangle$  for each  $\alpha \in [S^{n-1}, S^0]$  with  $n \neq 1$ , and that  $\langle DE \rangle = \langle E \rangle$  when  $E$  is a finite CW-spectrum and  $DE$  is its Spanier-Whitehead dual. This incidentally implies that  $G_*E = 0 \Leftrightarrow G^*E = 0$ , for  $G, E \in \mathbf{Ho}^s$  with  $E$  finite. Some other members of  $\mathbf{BA}(\mathbf{Ho}^s)$  are  $\langle K \rangle$  and  $\langle SZ_{(J)} \rangle$  where  $K$  is the spectrum of complex  $K$ -theory and  $SZ_{(J)}$  is the Moore spectrum associated with a subring  $Z_{(J)} \subset Q$ . Indeed,  $\langle K \rangle$  and  $\langle SZ_{(J)} \rangle$  are of the form  $\langle E \rangle^c$  where  $E$  is an appropriate wedge of finite CW-spectra, though the proof for  $K$  will be postponed to [Bousfield 3].

We also introduce a distributive lattice  $\mathbf{DL}(\mathbf{Ho}^s)$  given by all  $\langle E \rangle \in \mathbf{A}(\mathbf{Ho}^s)$  with  $\langle E \rangle \wedge \langle E \rangle = \langle E \rangle$ , and we show that  $\mathbf{BA}(\mathbf{Ho}^s) \subset \mathbf{DL}(\mathbf{Ho}^s) \subset \mathbf{A}(\mathbf{Ho}^s)$  where both

containments are proper. It turns out that  $\langle E \rangle \in \mathbf{DL}(\mathbf{Ho}^s)$  whenever  $E$  is a (possibly infinite) wedge of ring spectra and finite CW-spectra. In fact, most familiar spectra represent elements of  $\mathbf{DL}(\mathbf{Ho}^s)$ .

The class  $\mathbf{A}(\mathbf{Ho}^s)$  has applications to the homological localization theory of spectra, cf. [Bousfield 3], [Ravenel]. In particular, the  $E_*$ -localization is equivalent to the  $G_*$ -localization iff  $\langle E \rangle = \langle G \rangle$ , and a determination of  $\mathbf{A}(\mathbf{Ho}^s)$  would provide an inventory of the possible homological localization functors.

Our results on the structure of  $\mathbf{A}(\mathbf{Ho}^s)$ ,  $\mathbf{BA}(\mathbf{Ho}^s)$ , and  $\mathbf{DL}(\mathbf{Ho}^s)$  are established in §2. Some of our proofs involve  $[E, ]_*$ -colocalizations of spectra, and we develop the required theory in §1.

We essentially use the notation and terminology of [Adams 2]. However, we let  $\mathbf{Ho}^s$  be the category of CW-spectra and homotopy classes of maps of degree 0, cf. [Adams 2, p. 144]. Thus  $\mathbf{Ho}^s$  is an additive category equipped with an equivalence  $\Sigma : \mathbf{Ho}^s \rightarrow \mathbf{Ho}^s$  induced by the “shift” suspension  $\Sigma$  of CW-spectra. We write  $[X, Y]$  for the group of morphisms  $X \rightarrow Y \in \mathbf{Ho}^s$ , and write  $[X, Y]_n$  for  $[\Sigma^n X, Y]$  where  $n \in \mathbb{Z}$ . By a cofibre sequence we mean a sequence in  $\mathbf{Ho}^s$  equivalent to  $X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX$  for some cellular map  $f$  of degree 0 between CW-spectra, cf. [Adams 2, p. 155]. Recall that  $\mathbf{Ho}^s$  has arbitrary coproducts induced by the wedge  $\vee$  for CW-spectra, and for  $X, Y \in \mathbf{Ho}^s$  there is a natural smash product  $X \wedge Y \in \mathbf{Ho}^s$  which is associative, commutative, and unitary (with the sphere spectrum  $S$  as unit) up to coherent natural equivalences, cf. [Adams 2, p. 158]. We call  $E \in \mathbf{Ho}^s$  a ring spectrum if it is equipped with an associative (but not necessarily commutative) multiplication  $E \wedge E \rightarrow E$  and a two sided unit  $S \rightarrow E$  in  $\mathbf{Ho}^s$ . As usual, we let  $X * Y = \pi_* X \wedge Y = [S, X \wedge Y]_*$  for  $X, Y \in \mathbf{Ho}^s$ .

**§1.  $[E, ]_*$ -colocalizations of spectra**

In preparation for §2 and for [Bousfield 3], we now develop the  $[E, ]_*$ -colocalization theory of spectra. Some of the concepts here have previously been developed by J. P. May (unpublished) and in [Bousfield 2].

For  $E \in \mathbf{Ho}^s$ , a map  $f : A \rightarrow B \in \mathbf{Ho}^s$  is called an  $[E, ]_*$ -equivalence if  $f_* : [E, A]_* \approx [E, B]_*$ , and a spectrum  $C \in \mathbf{Ho}^s$  is called  $[E, ]_*$ -colocal if  $g_* : [C, X]_* \approx [C, Y]_*$  whenever  $g : X \rightarrow Y$  is an  $[E, ]_*$ -equivalence. It is easy to check:

- (1.1)  $E$  is  $[E, ]_*$ -colocal.
- (1.2) If  $\{X_\alpha\}$  is a set of  $[E, ]_*$ -colocal spectra, then  $\vee_\alpha X_\alpha$  is  $[E, ]_*$ -colocal.
- (1.3) If  $W \rightarrow X \rightarrow Y$  is a cofibre sequence in  $\mathbf{Ho}^s$  and any two of  $W, X, Y$  are  $[E, ]_*$ -colocal, then so is the third.
- (1.4) If  $X$  is  $[E, ]_*$ -colocal, then so is  $X \wedge Y$  for all  $Y \in \mathbf{Ho}^s$ .

A map  $\varphi : X \rightarrow A \in \mathbf{Ho}^*$  is called an  $[E, ]_*$ -colocalization of  $A$  if  $X$  is  $[E, ]_*$ -colocal and  $\varphi$  is an  $[E, ]_*$ -equivalence. Note that the  $[E, ]_*$ -colocalizations of  $A$  are initial among the  $[E, ]_*$ -equivalences with target  $A$ , and are terminal among the maps from  $[E, ]_*$ -colocal spectra to  $A$ .  $[E, ]_*$ -colocalizations are clearly unique up to equivalence and

**PROPOSITION 1.5.** *Each spectrum  $A \in \mathbf{Ho}^*$  has an  $[E, ]_*$ -colocalization.*

*Proof.* We inductively construct a transfinite sequence of inclusions of CW-spectra

$$A = B_0 \subset B_1 \subset \dots \subset B_s \subset B_{s+1} \subset \dots$$

where  $B_\lambda = \bigcup_{s < \lambda} B_s$  for each limit ordinal  $\lambda$  and where  $B_s \subset B_{s+1}$  is given by the push-out square

$$\begin{array}{ccc} \bigvee_{\alpha \in I} & \bigvee_{f: M_\alpha \rightarrow B_s} & M_\alpha \longrightarrow B_s \\ \downarrow & & \downarrow \\ \bigvee_{\alpha \in I} & \bigvee_{f: M_\alpha \rightarrow B_s} & \text{Cone}(M_\alpha) \longrightarrow B_{s+1} \end{array}$$

in which  $\{M_\alpha\}_{\alpha \in I}$  consists of all cofinal subspectra of the spectra  $\Sigma^n E$  for  $n \in \mathbb{Z}$ , and  $f$  ranges over all cellular functions  $M_\alpha \rightarrow B_s$  of degree 0, cf. [Adams, p. 140, 154]. Now let  $\sigma$  be the number of stable cells in  $E$  and let  $\gamma$  be the first infinite ordinal of cardinality greater than  $\sigma$ . Then for each  $\alpha \in I$ , each cellular function  $M_\alpha \rightarrow B_\gamma$  of degree 0 extends over  $\text{Cone}(M_\alpha)$  because the image of  $M_\alpha$  is contained in  $B_s$  for some  $s < \gamma$ . Thus  $[E, B_\gamma]_* = 0$ . Since  $A$  is a closed subspectrum of  $B_\gamma$  (cf. [Adams 2, p. 154]), there is an associated cofibre sequence

$$\Sigma^{-1}(B_\gamma/A) \rightarrow A \rightarrow B_\gamma$$

in  $\mathbf{Ho}^*$ . The morphism  $\Sigma^{-1}(B_\gamma/A) \rightarrow A$  is clearly an  $[E, ]_*$ -equivalence, so it suffices to show  $\Sigma^{-1}(B_\gamma/A)$  is  $[E, ]_*$ -colocal. For this it suffices to show inductively that  $B_s/A$  is  $[E, ]_*$ -colocal for all  $s$ . If  $B_s/A$  is  $[E, ]_*$ -colocal, then so is  $B_{s+1}/A$  because there is a cofibre sequence

$$B_s/A \rightarrow B_{s+1}/A \rightarrow B_{s+1}/B_s \in \mathbf{Ho}^*$$

where  $B_{s+1}/B_s$  is equivalent to a wedge of iterated (de)-suspensions of  $E$ . If  $B_s/A$

is  $[E, ]_*$ -colocal for all  $s < \lambda$  where  $\lambda$  is a limit ordinal, then  $B_\lambda/A$  is  $[E, ]_*$ -colocal because there is a cofibre sequence

$$\bigvee_{s < \lambda} B_s/A \xrightarrow{1-g} \bigvee_{s < \lambda} B_s/A \rightarrow B_\lambda/A \in \mathbf{Ho}^s$$

where  $g$  is induced by the maps  $B_s/A \rightarrow B_{s+1}/A$ . This completes the induction and the proof 1.5.

For each  $A \in \mathbf{Ho}^s$  let  $\varphi: {}^E A \rightarrow A \in \mathbf{Ho}^s$  denote the  $[E, ]_*$ -colocalization given by  $\sum^{-1}(B_\gamma/A) \rightarrow A$  above, and note that it is functorial and idempotent in the obvious sense. To clarify the nature of  $[E, ]_*$ -colocal spectra, we let *Class-E* denote the smallest class of spectra in  $\mathbf{Ho}^s$  such that: (i)  $E \in \text{Class-E}$ ; (ii) if  $\{X_\alpha\}$  is a set of spectra in *Class-E*, then  $\bigvee_\alpha X_\alpha \in \text{Class-E}$ ; and (iii) if  $W \rightarrow X \rightarrow Y$  is a cofibre sequence in  $\mathbf{Ho}^s$  and any two of  $W, X, Y$  are in *Class-E*, then so is the third.

**PROPOSITION 1.6.** *Class-E equals the class of  $[E, ]_*$ -colocal spectra in  $\mathbf{Ho}^s$ .*

*Proof.* *Class-E* is contained in the class of  $[E, ]_*$ -colocal spectra by (1.1)–(1.3). Conversely, if  $X$  is  $[E, ]_*$ -colocal, then  $X \in \text{Class-E}$  because  ${}^E X \simeq X$  and  ${}^E X \in \text{Class-E}$  by the proof of 1.5.

We call a spectrum  $W \in \mathbf{Ho}^s$   $[E, ]_*$ -trivial if  $[E, W]_* = 0$ , and we note that  $[V, W]_* = 0$  whenever  $V$  is  $[E, ]_*$ -colocal and  $W$  is  $[E, ]_*$ -trivial. Each spectrum  $A$  can be canonically built from  $[E, ]_*$ -colocal and  $[E, ]_*$ -trivial spectra as follows. Extend  $\varphi: {}^E A \rightarrow A$  to the cofibre sequence

$$(1.7) \quad {}^E A \xrightarrow{\varphi} A \xrightarrow{\nu} A^E \in \mathbf{Ho}^s$$

given by  $\sum^{-1}(B_\gamma/A) \rightarrow A \rightarrow B_\gamma$  above, and observe that  $A^E$  is  $[E, ]_*$ -trivial. Indeed,  $\nu$  is clearly the  $[E, ]_*$ -trivialization of  $A$ , i.e.  $\nu$  is the initial example of a map from  $A$  to an  $[E, ]_*$ -trivial spectrum. It is useful to observe:

(1.8) If  $V \rightarrow X \rightarrow W$  is a cofibre sequence in  $\mathbf{Ho}^s$  with  $V$   $[E, ]_*$ -colocal and with  $W$   $[E, ]_*$ -trivial, then  $V \rightarrow X \rightarrow W$  is equivalent to the cofibre sequence

$${}^E X \xrightarrow{\varphi} X \xrightarrow{\nu} X^E$$

It is straightforward to check that the  $[E, ]_*$ -colocalization and  $[E, ]_*$ -trivialization functors on  $\mathbf{Ho}^s$  commute with suspension and preserve cofibre sequences. In [Bousfield 3] we will show that for each  $E \in \mathbf{Ho}^s$  there exists a spectrum  $aE \in \mathbf{Ho}^s$  such that the  $E_*$ -localization and  $E_*$ -acyclization functors are respectively equivalent to the  $[aE, ]_*$ -trivialization and  $[aE, ]_*$ -colocalization functors on  $\mathbf{Ho}^s$ . Thus, many examples of trivialization and colocalization functors will be implicitly studied in [Bousfield 3].

**§2. On the structure of  $\mathbf{A}(\mathbf{Ho}^s)$**

We now examine the structure of the class  $\mathbf{A}(\mathbf{Ho}^s)$  of “acyclicity types” of spectra, and we establish the results mentioned in the introduction concerning the distributive lattice  $\mathbf{DL}(\mathbf{Ho}^s) \subset \mathbf{A}(\mathbf{Ho}^s)$  and the Boolean algebra  $\mathbf{BA}(\mathbf{Ho}^s) \subset \mathbf{DL}(\mathbf{Ho}^s)$ .

$\mathbf{A}(\mathbf{Ho}^s)$  has the following relations and operations:

(2.1) For  $\langle X \rangle, \langle Y \rangle \in \mathbf{A}(\mathbf{Ho}^s)$ , define  $\langle X \rangle \leq \langle Y \rangle$  if each  $Y_*$ -acyclic spectrum is  $X_*$ -acyclic. This is a partial order relation. Clearly  $\langle 0 \rangle$  is the smallest element of  $\mathbf{A}(\mathbf{Ho}^s)$  and  $\langle S \rangle$  is the largest. Note that if  $X$  is  $[Y, ]_*$ -colocal (or equivalently, if  $X \in \text{Class-}Y$ ), then  $\langle X \rangle \leq \langle Y \rangle$ .

(2.2) For a set  $\{\langle X_\alpha \rangle\}$  of elements in  $\mathbf{A}(\mathbf{Ho}^s)$ , define  $\vee_\alpha \langle X_\alpha \rangle \in \mathbf{A}(\mathbf{Ho}^s)$  by  $\vee_\alpha \langle X_\alpha \rangle = \langle \vee_\alpha X_\alpha \rangle$ . Note that  $\vee_\alpha \langle X_\alpha \rangle$  is the least upper bound of  $\{\langle X_\alpha \rangle\}$  in  $\mathbf{A}(\mathbf{Ho}^s)$ , and  $\vee$  is associative, commutative, and idempotent. Of course,  $\langle 0 \rangle \vee \langle X \rangle$  and  $\langle S \rangle \vee \langle X \rangle = \langle S \rangle$ .

(2.3) For  $\langle X \rangle, \langle Y \rangle \in \mathbf{A}(\mathbf{Ho}^s)$  define  $\langle X \rangle \wedge \langle Y \rangle \in \mathbf{A}(\mathbf{Ho}^s)$  by  $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$ . This is well-defined: if  $\langle X \rangle = \langle X_1 \rangle$  and  $\langle Y \rangle = \langle Y_1 \rangle$ , then clearly  $\langle X \wedge Y \rangle = \langle X_1 \wedge Y \rangle = \langle X_1 \wedge Y_1 \rangle$ . Note that  $\langle X \rangle \wedge \langle Y \rangle$  is a lower bound of  $\langle X \rangle, \langle Y \rangle \in \mathbf{A}(\mathbf{Ho}^s)$ , and that if  $\langle X \rangle \leq \langle X_1 \rangle$  and  $\langle Y \rangle \leq \langle Y_1 \rangle$  then  $\langle X \rangle \wedge \langle Y \rangle \leq \langle X_1 \rangle \wedge \langle Y_1 \rangle$ . Clearly  $\wedge$  is associative and commutative, with  $\langle S \rangle \wedge \langle X \rangle = \langle X \rangle$  and  $\langle 0 \rangle \wedge \langle X \rangle = \langle 0 \rangle$ . Also the distributive law  $\langle X \rangle \wedge (\vee_\alpha \langle Y_\alpha \rangle) = \vee_\alpha (\langle X \rangle \wedge \langle Y_\alpha \rangle)$  and absorption law  $\langle X \rangle \vee (\langle X \rangle \wedge \langle Y \rangle) = \langle X \rangle$  hold.

(2.4) For each  $\langle X \rangle \in \mathbf{A}(\mathbf{Ho}^s)$  there is an element  $a\langle X \rangle \in \mathbf{A}(\mathbf{Ho}^s)$  such that  $a\langle X \rangle$  is the greatest member of  $\mathbf{A}(\mathbf{Ho}^s)$  with  $\langle X \rangle \wedge a\langle X \rangle = \langle 0 \rangle$ . Moreover,  $aa\langle X \rangle = \langle X \rangle$  for each  $\langle X \rangle \in \mathbf{A}(\mathbf{Ho}^s)$ , and  $\langle X \rangle \leq \langle Y \rangle$  if and only if  $a\langle Y \rangle \leq a\langle X \rangle$ . This will be shown in [Bousfield 3], and we remark that  $a\langle X \rangle = \langle aX \rangle$  where  $aX$  is the spectrum mentioned at the end of §1. It turns out that  $\mathbf{DL}(\mathbf{Ho}^s)$  is not closed under  $a( )$ , although  $a( )$  gives the complement in  $\mathbf{BA}(\mathbf{Ho}^s)$ . We won't use  $a( )$  in this paper.

So far,  $\mathbf{A}(\mathbf{Ho}^s)$  resembles a Boolean algebra with complement  $a( )$ , but the following lemma shows that  $\wedge$  is not idempotent in  $\mathbf{A}(\mathbf{Ho}^s)$ .

**LEMMA 2.5.** *Let  $X \in \mathbf{Ho}^s$  be a finite CW-spectrum with  $H_*X$  finite, and let  $cX \in \mathbf{Ho}^s$  be the Brown-Comenetz dual of  $X$ . If  $X \neq 0$ , then  $\langle cX \rangle \wedge \langle cX \rangle = \langle 0 \rangle \neq \langle cX \rangle$ .*

*Proof.* Using [Brown-Comenetz, 1.14] it is easy to show  $H_*(cX; Z) = 0$ , and thus  $\langle H \rangle \wedge \langle cX \rangle = \langle 0 \rangle$  where  $H$  is the spectrum for integral homology. Since  $\pi_i cX$  is the Pontrjagin dual of  $\pi_{-i} X$ , it vanishes for sufficiently large  $i$ . Hence

$\langle cX \rangle(n, \infty) \in \text{Class-}H$  for each  $n$  where  $\langle cX \rangle(n, \infty)$  is the  $(n - 1)$ -connected section of  $cX$ . The cofibre sequence

$$\bigvee_{n \leq 0} \langle cX \rangle(n, \infty) \rightarrow \bigvee_{n \leq 0} \langle cX \rangle(n, \infty) \rightarrow cX \in \mathbf{Ho}^*$$

now shows  $cX \in \text{Class-}H$ , and thus  $\langle cX \rangle \leq \langle H \rangle$ . The lemma now follows since  $\langle cX \rangle \wedge \langle cX \rangle \leq \langle H \rangle \wedge \langle cX \rangle = \langle 0 \rangle$  and since  $\langle cX \rangle_*(S) \neq 0 = 0_*(S)$ .

To avoid the pathological spectra revealed by 2.5, we introduce

### 2.6 The distributive lattice of spectra $\mathbf{DL}(\mathbf{Ho}^*)$

Let  $\mathbf{DL}(\mathbf{Ho}^*)$  consist of all  $\langle X \rangle \in \mathbf{A}(\mathbf{Ho}^*)$  with  $\langle X \rangle \wedge \langle X \rangle = \langle X \rangle$ . For instance, if  $E$  is a ring spectrum, then  $\langle E \rangle \in \mathbf{DL}(\mathbf{Ho}^*)$  because  $E$  is a retract of  $E \wedge E$  in  $\mathbf{Ho}^*$ . Also, if  $E$  a Moore spectrum or a finite CW spectrum, then  $\langle E \rangle \in \mathbf{DL}(\mathbf{Ho}^*)$  by 2.9 and 2.13 below. Many other examples can be derived from the preceding, since  $\mathbf{DL}(\mathbf{Ho}^*)$  is closed under the operation  $\vee$  (with any number of summands) and under  $\wedge$ ; the proof for  $\vee$  uses the equality  $\langle X \rangle \vee (\langle X \rangle \wedge \langle Y \rangle) = \langle X \rangle$ . With the operations  $\vee$  and  $\wedge$ ,  $\mathbf{DL}(\mathbf{Ho}^*)$  is clearly a distributive lattice with  $0, 1$  as defined in the next paragraph.

We refer the reader to [Dwinger] or [Grätzer] for an exposition of distributive lattice theory, but for convenience we recall that a class  $L$  with binary operations  $\vee, \wedge$  is a *distributive lattice with  $0, 1$*  if:

- (i)  $x \wedge x = x$  and  $x \vee x = x$  for  $x \in L$ .
- (ii)  $x \wedge y = y \wedge x$  and  $x \vee y = y \vee x$  for  $x, y \in L$ .
- (iii)  $x \wedge (y \wedge z) = (x \wedge y) \wedge z$  and  $x \vee (y \vee z) = (x \vee y) \vee z$  for  $x, y, z \in L$ .
- (iv)  $x \wedge (x \vee y) = x$  and  $x \vee (x \wedge y) = x$  for  $x, y \in L$ .
- (v)  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$  and  $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$  for  $x, y, z \in L$ .
- (vi) There exist elements  $0, 1 \in L$  such that  $0 \vee x = x$  and  $1 \wedge x = x$  for all  $x \in L$ .

(Clearly,  $0$  and  $1$  are unique.) Now let  $L$  be a distributive lattice with  $0, 1$ . For  $x, y \in L$  one writes  $x \leq y$  if the equivalent conditions  $x \wedge y = x$  and  $x \vee y = y$  are satisfied. Then  $\leq$  is a partial order relation on  $L$ , and  $x \vee y$  (resp.  $x \wedge y$ ) is the l.u.b. (resp. g.l.b.) of  $x, y \in L$ , cf. [Dwinger, p. 44] or [Grätzer, p. 6]. We also recall that  $L$  is called a *Boolean algebra* if for each  $x \in L$  there exists  $y \in L$  with  $x \wedge y = 0$  and  $x \vee y = 1$ .

For  $\langle X \rangle, \langle Y \rangle \in \mathbf{DL}(\mathbf{Ho}^*)$  we conclude that  $\langle X \rangle \wedge \langle Y \rangle$  is the g.l.b. of  $\langle X \rangle$  and  $\langle Y \rangle$ , where  $\mathbf{DL}(\mathbf{Ho}^*)$  has the partial ordering inherited from  $\mathbf{A}(\mathbf{Ho}^*)$ . Of course, we previously observed that  $\langle X \rangle \vee \langle Y \rangle$  is the l.u.b. of  $\langle X \rangle$  and  $\langle Y \rangle$ . Thus the algebraic structure of  $\mathbf{DL}(\mathbf{Ho}^*)$  is contained in its partial ordering.

We call  $\langle Y \rangle \in \mathbf{A}(\mathbf{Ho}^*)$  the *complement* of  $\langle X \rangle \in \mathbf{A}(\mathbf{Ho}^*)$  if  $\langle X \rangle \wedge \langle Y \rangle = \langle 0 \rangle$  and  $\langle X \rangle \vee \langle Y \rangle = \langle S \rangle$ . Note that if  $\langle Y_1 \rangle$  is also the complement of  $\langle X \rangle$ , then  $\langle Y \rangle = \langle Y_1 \rangle$

because

$$\langle Y \rangle = \langle Y \rangle \wedge (\langle X \rangle \vee \langle Y_1 \rangle) = \langle Y \rangle \wedge \langle Y_1 \rangle = (\langle X \rangle \vee \langle Y \rangle) \wedge \langle Y_1 \rangle = \langle Y_1 \rangle.$$

If  $\langle X \rangle$  has a complement  $\langle Y \rangle$ , then  $\langle X \rangle \in \mathbf{DL}(\mathbf{Ho}^*)$  because  $\langle X \rangle = \langle X \rangle \wedge (\langle X \rangle \vee \langle Y \rangle) = \langle X \rangle \wedge \langle X \rangle$ , but the members of  $\mathbf{DL}(\mathbf{Ho}^*)$  need not have complements.

LEMMA 2.7.  $\langle H \rangle \in \mathbf{DL}(\mathbf{Ho}^*)$ , but  $\langle H \rangle$  does not have a complement.

*Proof.*  $\langle H \rangle \in \mathbf{DL}(\mathbf{Ho}^*)$  since  $H$  is a ring spectrum. Suppose  $\langle H \rangle$  has a complement  $\langle L \rangle$ . Let  $\langle cX \rangle$  be as in 2.5, and recall that  $\langle cX \rangle \neq \langle 0 \rangle = \langle H \rangle \wedge \langle cX \rangle$  and  $\langle cX \rangle \leq \langle H \rangle$ . Thus

$$\langle cX \rangle = (\langle H \rangle \vee \langle L \rangle) \wedge \langle cX \rangle = \langle L \rangle \wedge \langle cX \rangle \leq \langle L \rangle \wedge \langle H \rangle = \langle 0 \rangle$$

and this contradicts  $\langle cX \rangle \neq \langle 0 \rangle$ . Therefore  $\langle H \rangle$  cannot have a complement.

We now introduce

## 2.8 The Boolean algebra of spectra $\mathbf{BA}(\mathbf{Ho}^*)$

Let  $\mathbf{BA}(\mathbf{Ho}^*)$  consist of all  $\langle X \rangle \in \mathbf{A}(\mathbf{Ho}^*)$  such that  $\langle X \rangle$  has a complement (written  $\langle X \rangle^c$ ), and note that  $\mathbf{BA}(\mathbf{Ho}^*) \subset \mathbf{DL}(\mathbf{Ho}^*)$ . If  $E$  is a Moore spectrum or a (possibly infinite) wedge of finite CW spectra, then  $\langle E \rangle \in \mathbf{BA}(\mathbf{Ho}^*)$  by 2.9 and 2.13 below. Many other members of  $\mathbf{BA}(\mathbf{Ho}^*)$  can be derived from the preceding, since  $\mathbf{BA}(\mathbf{Ho}^*)$  is clearly closed under  $(\ )^c$  and the binary operations  $\vee, \wedge$ ; indeed, for  $\langle X \rangle, \langle Y \rangle \in \mathbf{BA}(\mathbf{Ho}^*)$

$$\langle X \rangle^{cc} = \langle X \rangle$$

$$(\langle X \rangle \vee \langle Y \rangle)^c = \langle X \rangle^c \wedge \langle Y \rangle^c$$

$$(\langle X \rangle \wedge \langle Y \rangle)^c = \langle X \rangle^c \vee \langle Y \rangle^c.$$

With these operations,  $\mathbf{BA}(\mathbf{Ho}^*)$  is clearly a Boolean algebra.

As promised, we now prove

PROPOSITION 2.9. *If  $E \in \mathbf{Ho}^*$  is a (possibly infinite) wedge of finite CW-spectra, then  $\langle E \rangle \in \mathbf{BA}(\mathbf{Ho}^*)$ . Moreover,  $\langle E \rangle = \langle {}^E S \rangle$  and  $\langle E \rangle^c = \langle S^E \rangle$ .*

*Proof.* Assume  $E = \vee_{\alpha} B_{\alpha}$  where each  $B_{\alpha}$  is a finite CW spectrum. A spectrum  $Y \in \mathbf{Ho}^*$  is  $[E, ]$ -trivial iff  $(DB_{\alpha}) \wedge Y \simeq 0 \in \mathbf{Ho}^*$  for all  $\alpha$ , where  $DB_{\alpha}$  is the

Spanier-Whitehead dual of  $B_{\alpha}$ . Thus if  $Y$  is  $[E, ]_*$ -trivial, then so is  $X \wedge Y$  for all  $X \in \mathbf{Ho}^*$ . In particular,  ${}^E S \wedge S^E$  is  $[E, ]_*$ -trivial as well as  $[E, ]_*$ -colocal (by 1.4), and thus  ${}^E S \wedge S^E \simeq 0 \in \mathbf{Ho}^*$ . Using the cofibering  ${}^E S \rightarrow S \rightarrow S^E$  of (1.7), we conclude that  $\langle {}^E S \rangle \vee \langle S^E \rangle = \langle S \rangle$ , so  $\langle {}^E S \rangle \in \mathbf{BA}(\mathbf{Ho}^*)$  with  $\langle {}^E S \rangle^c = \langle S^E \rangle$ . It remains to show  $\langle E \rangle = \langle {}^E S \rangle$ . Applying (1.8) to the cofibre sequence

$$X \wedge {}^E S \rightarrow X \wedge S \rightarrow X \wedge S^E,$$

we find that  $X \wedge {}^E S \simeq {}^E X$  and  $X \wedge S^E \simeq X^E$  for all  $X \in \mathbf{Ho}^*$ . Since  $E$  is  $[E, ]_*$ -colocal, this implies  $E \wedge {}^E S \simeq E$ , and thus  $\langle E \rangle \leq \langle {}^E S \rangle$ . Since  ${}^E S$  is  $[E, ]_*$ -colocal, we know  $\langle {}^E S \rangle \leq \langle E \rangle$ , and therefore  $\langle E \rangle = \langle {}^E S \rangle$ .

We remark that the above spectra  ${}^E S$  and  $S^E$  satisfy the strong idempotency conditions  ${}^E S \wedge {}^E S \simeq {}^E S$  and  $S^E \wedge S^E \simeq S^E$ . Indeed  $S^E$  is a commutative ring spectrum whose multiplication map  $S^E \wedge S^E \rightarrow S^E$  is an equivalence. We next observe

**PROPOSITION 2.10.** *If  $E \in \mathbf{Ho}^*$  is a finite CW spectrum, then  $\langle E \rangle = \langle DE \rangle$ . Consequently, for any  $G \in \mathbf{Ho}^*$ ,  $G_*(E) = 0 \Leftrightarrow G^*(E) = 0$ .*

*Proof.* Since  $[E, S^E]_* = 0$ , we have  $(DE) \wedge S^E \simeq 0$  and thus  $(DE) \wedge {}^E S \simeq DE$ . Since  $\langle {}^E S \rangle = \langle E \rangle$ , this implies  $\langle DE \rangle \wedge \langle E \rangle = \langle DE \rangle$ . Dually one shows  $\langle E \rangle \wedge \langle DE \rangle = \langle E \rangle$ , and therefore  $\langle DE \rangle = \langle E \rangle$ . The last statement is deduced using  $G^*(E) = G_*(DE)$ .

We next prove “triangle (in)equalities” for cofibre sequences. Call a map  $f: A \rightarrow X \in \mathbf{Ho}^*$  smash nilpotent if the  $m$ -fold smash product

$$f \wedge \cdots \wedge f: A \wedge \cdots \wedge A \rightarrow X \wedge \cdots \wedge X \in \mathbf{Ho}^*$$

is the 0 map for some  $m \geq 1$ . Note that the smash nilpotent maps form a subgroup of  $[A, X]$ , and a composite  $fg$  is smash nilpotent if either  $f$  or  $g$  is. Moreover, if  $i \neq 0$  then each map  $S^i \rightarrow S^0 \in \mathbf{Ho}^*$  is a smash nilpotent by [Nishida].

**PROPOSITION 2.11.** *If  $A \xrightarrow{f} X \rightarrow B$  is a cofibre sequence in  $\mathbf{Ho}^*$ , then:*

- (i)  $\langle A \rangle \leq \langle X \rangle \vee \langle B \rangle$ ,  $\langle X \rangle \leq \langle A \rangle \vee \langle B \rangle$ , and  $\langle B \rangle \leq \langle A \rangle \vee \langle X \rangle$ .
- (ii) If  $A, X \in \mathbf{DL}(\mathbf{Ho}^*)$  and  $f$  is smash nilpotent, then  $\langle B \rangle = \langle A \rangle \vee \langle X \rangle$ .

*Proof.* Part (i) is obvious. For (ii) we assume  $f \wedge \cdots \wedge f = 0$  in  $\mathbf{Ho}^*$  and form a cofibre sequence

$$A \wedge \cdots \wedge A \xrightarrow{f \wedge \cdots \wedge f} X \wedge \cdots \wedge X \rightarrow C \in \mathbf{Ho}^*$$

Then

$$\langle C \rangle = \langle A \wedge \cdots \wedge A \rangle \vee \langle X \wedge \cdots \wedge X \rangle = \langle A \rangle \vee \langle X \rangle$$

and also  $\langle C \rangle \leq \langle B \rangle$  because  $C$  is  $[B, ]_*$ -colocal (i.e.  $C \in \text{Class-}B$ ). Thus  $\langle A \rangle \vee \langle X \rangle \leq \langle B \rangle$  and the opposite inequality is given by (i).

**COROLLARY 2.12.** *If  $n \neq 1$  and  $\alpha : S^{n-1} \rightarrow S^0$  in  $\mathbf{Ho}^*$ , then  $\langle S^0 \cup_\alpha e^n \rangle = \langle S \rangle$ .*

Because of Nishida’s theorem, this follows from 2.11; or instead of using 2.11, we could have used the easy result that if  $\sum^m A \xrightarrow{f} A \rightarrow B$  is a cofibre sequence in  $\mathbf{Ho}^*$  with  $f$  nilpotent in  $[A, A]_*$ , then  $\langle B \rangle = \langle A \rangle$ . By combining 2.12 with R. Wood’s result  $K \simeq (S^0 \cup_\eta e^2) \wedge KO$ , we recover the result  $\langle K \rangle = \langle KO \rangle$ , cf. [Meier], [Ravenel].

Of course, 2.12 fails when  $n = 1$ , and we now consider  $\langle SG \rangle$  where  $G$  is an abelian group and  $SG \in \mathbf{Ho}$  is a Moore spectrum of type  $(G, 0)$  (i.e.  $\pi_i SG = 0$  for  $i < 0$ ,  $H_0 SG = G$ , and  $H_i SG = 0$  for  $i > 0$ ). There is a short exact sequence

$$0 \rightarrow G \otimes \pi_n X \rightarrow \pi_n (SG \wedge X) \rightarrow \text{Tor}(G, \pi_{n-1} X) \rightarrow 0.$$

Thus

$$\langle S \rangle = \langle SQ \rangle \vee \bigvee_{p \text{ prime}} \langle SZ/p \rangle$$

$$\langle SQ \rangle \wedge \langle SZ/p \rangle = \langle 0 \rangle = \langle SZ/p \rangle \wedge \langle SZ/q \rangle \text{ for primes } p \neq q.$$

It follows that  $\mathbf{BA}(\mathbf{Ho}^*)$  has a sub-Boolean algebra  $\mathbf{MBA}(\mathbf{Ho}^*)$  whose members are the wedges of subsets of

$$I = \{ \langle SQ \rangle, \langle SZ/2 \rangle, \langle SZ/3 \rangle, \langle SZ/5 \rangle, \dots \}.$$

Moreover, there is an obvious Boolean algebra isomorphism between  $\mathbf{MBA}(\mathbf{Ho}^*)$  and the power set  $P(I)$ . Note that for any set  $J$  of primes

$$\langle SZ_{(J)} \rangle = \langle SQ \rangle \vee \bigvee_{p \in J} \langle SZ/p \rangle$$

where  $Z_{(J)}$  is the localization of  $Z$  at  $J$ . More generally,

**PROPOSITION 2.13.** *For each abelian group  $G$ ,  $\langle SG \rangle \in \mathbf{MBA}(\mathbf{Ho}^*)$ .*

*Proof.* Let  $C$  be the class of all abelian groups  $A$  such that  $G \otimes A = 0 = \text{Tor}(G, A)$ , and note that  $(SG)_* X = 0$  if  $\pi_i X \in C$  for all  $i$ . The result now follows easily from [Bousfield 1,2.3] since  $C$  is a "special" class.

We conclude by noting that  $\mathbf{BA}(\mathbf{Ho}^*)$  contains many elements outside  $\mathbf{MBA}(\mathbf{Ho}^*)$ . For  $p$  prime let

$$A(p): \Sigma^m SZ/p \rightarrow SZ/p \in \mathbf{Ho}^*$$

be the  $K_*$ -equivalence of [Adams 1, §12] where  $m = 2p - 2$  for  $p$  odd and  $m = 8$  for  $p = 2$ . It is easy to check that the cofibre of  $A(p)$  represents an element of  $\mathbf{BA}(\mathbf{Ho}^*)$  outside  $\mathbf{MBA}(\mathbf{Ho}^*)$ . In [Bousfield 3] we will show that  $\langle K \rangle = \langle E \rangle^c$  where  $E$  is the wedge of the cofibres of all the  $A(p)$ .

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