EQUIVARIANT STEENROD OPERATIONS

P. BHATTACHARYA, A. WAUGH, M. ZENG, AND F. ZOU

Abstract. In this paper, we introduce the concept of R-Eulerian sequences for any $\mathcal{N}_\infty$-ring spectrum R with finite orientation order. We show that each R-Eulerian sequence corresponds to a stable R-cohomology operation. Moreover, we show that the collection of R-Eulerian sequences admits an additive and a multiplicative structure that are linear over a coefficient ring. It remains an open question whether operations coming from Eulerian sequences generate the algebra of all stable R-cohomology operations. Finally, we apply our theory to equivariant ordinary cohomology with coefficients in finite fields to produce genuine equivariant lifts of the classical Steenrod operations for all finite groups.

Contents

1. Introduction

In the 1950s, stable homotopy theory underwent a period of rapid development, driven by profound applications in geometry [SW51, Tho53, Tho54]. Key breakthroughs included Steenrod and Whitehead's work on the vector fields on spheres problem [SW51, Ste62], the calculation of cobordism rings [Tho53, Tho54, Mil60], the resolution of the Hopf invariant one problem [Ada60], among many others. These results relied crucially on Steenrod operations, which Norman Steenrod introduced in 1947 [Ste47]. The usefulness of Steenrod operations extends to differential topology via Wu's reformulation of Stiefel-Whitney classes [Wu50], and in homotopy theory through the Adams spectral sequence [Ada58]. Over the years, the applications of Steenrod operations have only expanded, establishing them as one of the most formidable tools in current homotopy theory research.

Equivariant homotopy theory is an extension of classical homotopy theory that is sensitive to symmetries. In 1967, Bredon introduced an equivariant cohomology theory that refines Borel cohomology by incorporating fixed-point data for all subgroups [Bre67]. The coefficients for Bredon cohomology are provided by Mackey functors, which Dress introduced [Dre71]. In 1981, Lewis-May-McClure [LMM81] extended the indexing system of a G-equivariant Bredon cohomology from integers to the real representation ring, usually denoted by $\mathrm{RO}(\mathrm{G})$, for any compact Lie group G. Equivariant stable homotopy theory was developed in the 1980s to address problems in equivariant geometry. However, despite a few striking applications [tD70, HK01, KW08, HHR16, HLSX22], its geometric utility has remained limited and sporadic. This constraint is largely due to the fact that equivariant Steenrod operations are not known beyond the group of order 2 [HK01, Voe03].

The $\mathrm{RO}(\mathrm{G})$-graded Bredon cohomology with coefficients in $\underline{\mathbb{F}}_p$ (the constant Tambara functor at the field of order p) is represented by a genuine G-equivariant $\mathbb{E}_\infty^{\mathrm{G}}$-ring spectrum $\mathrm{H}\underline{\mathbb{F}}_p$. The **G-equivariant mod p Steenrod algebra** is defined as the stable homotopy class of G-equivariant self-maps of $\mathrm{H}\underline{\mathbb{F}}_p$:

$$\mathcal{A}^{\star}_{\mathrm{G},p} := [\mathrm{H}\underline{\mathbb{F}}_p, \mathrm{H}\underline{\mathbb{F}}_p]^{\mathrm{G}}_{-\star}.$$

This Steenrod algebra is a module over its **coefficient ring**, the $\mathrm{RO}(\mathrm{G})$-graded cohomology of a point:

$$\mathbb{M}_p^{\mathrm{G}} := \mathrm{H}^{\star}_{\mathrm{G}}(\mathrm{pt}_+; \underline{\mathbb{F}}_p) \cong \pi^{\mathrm{G}}_{-\star}\mathrm{H}\underline{\mathbb{F}}_p.$$

The calculation of this coefficient ring is a notoriously difficult problem. After Stong's calculation of $\mathrm{RO}(\mathrm{C}_2)$-graded cohomology of a point (see [Lew88, §2]), progress was stalled for many years. However, a series of recent breakthroughs have revitalized the field: In 2017, Holler and Kriz [HK17] calculated the coefficient ring for $\mathrm{G} = \mathrm{C}_2^{\times n}$, soon after the third author identified the coefficient ring for $\mathrm{G} = \mathrm{C}_{p^2}$ [Zen], G. Yan [Yanb] for $\mathrm{G} = \mathrm{C}_{2^n}$, and Kriz-Lu [KL20] and G. Yan [Yana] for the dihedral groups.

In 2001, Hu and Kriz, in their seminal work [HK01], determined the structure of C_2-equivariant dual Steenrod algebra. This was followed in 2003 by Voevodsky [Voe03], who determined the $\mathbb{R}$-motivic Steenrod algebra, whose Betti realization provided a complete description of the C_2-equivariant Steenrod algebra. The case $\mathrm{G} = \mathrm{C}_2$ (as well as the nonequivariant case) is often considered special, as its Steenrod algebra is free over its coefficient ring, a property that is crucial for its determination. In fact, recent work by Wilson-Sarkar [SW22] as well as Hu-Kriz-Somberg-Zou [HKSZ] determines the C_p-equivariant mod p dual Steenrod algebra as a module over the coefficient ring and shows that it is not free. This non-freeness may explain why the methods in [Ste47, HK01, Voe03] did not adapt to identify the equivariant Steenrod algebra for other groups (also see [HKSZ]).

In this paper, we introduce a general *theoretical framework* to construct G-equivariant Steenrod operations for G-equivariant cohomology theories. This framework requires the following assumptions:

- G is a finite group.
- The cohomology theory is represented by a homotopy $\mathcal{O}$-ring R (as in (6.8)), for some $\mathcal{N}_\infty$ G-operad $\mathcal{O}$ (in the sense of [BH15]).

- The G-equivariant vector bundle of (3.2) admits an R-orientation (in the sense of [BZ24]).

Building on these conditions, we introduce the concept of a **V-stable R-Eulerian sequence** (see (4.12)), where V is a finite orthogonal G-representation. Our main result is the following theorem:

Main Theorem 1 **((4.18)).** *For every* V*-stable* R*-Eulerian sequence* χ*, there exists an* R*-cohomology operation*

$$\mathfrak{S}^{\chi} : \mathrm{R}^{\star}(-) \longrightarrow \mathrm{R}^{\star+\|\chi\|}(-),$$

of degree $\|\chi\|$ *(see (4.12)), where* $\star \in \mathrm{RO}(\mathrm{G}, \mathrm{V})$ *(see (1.7)), which commutes with the* V*-suspension isomorphism*

$$\sigma_{\mathrm{V}} : \mathrm{R}^{\star}(-) \xrightarrow{\cong} \mathrm{R}^{\star+\mathrm{V}}(\Sigma^{\mathrm{V}}(-)), \tag{1.1}$$

i.e., $\sigma_{\mathrm{V}}(\mathfrak{S}^{\chi}(x)) = \mathfrak{S}^{\chi}(\sigma_{\mathrm{V}}(x))$ *for any* R*-cohomology class* x*.*

Definition 1.2. For a G-spectrum R, we say an R-cohomology operation is **genuine stable** if it commutes with σ_{ρ}, where ρ is the real regular representation of G.

In the nonequivariant case, when G is the trivial group and R is $\mathrm{H}\mathbb{F}_2$—the ordinary cohomology with $\mathbb{F}_2$-coefficients, the vector bundle (3.2) can be chosen to be the tautological line bundle over $\mathrm{B}\Sigma_2 \simeq \mathbb{RP}^{\infty}$. We recall the graded module structure on the mod 2 homology of the classifying space:

$$\mathrm{H}_*((\mathrm{B}\Sigma_2)_+; \mathbb{F}_2) \cong \mathbb{F}_2\{\mathbf{b}_0, \mathbf{b}_1, \dots\},$$

where $\mathbf{b}_i$ are generators is degree i. As shown in (4.3), the $\mathrm{H}\mathbb{F}_2$-Eulerian sequences

$$\beta[k] = (\overbrace{0, \dots, 0}^{k}, \mathbf{b}_0, \mathbf{b}_1, \dots)$$

generate the classical k-th Steenrod squaring operation Sq^k by Main Theorem 1. We also show that our framework recovers all classical odd primary Steenrod operations as well as C_2-equivariant Steenrod operations, as detailed in (4.3) and (4.5). In Section 7, we demonstrate the strength of our theory by constructing new G-equivariant Steenrod operations for all finite group G:

Main Theorem 2. *Suppose* G *is a finite group. Then for every* $k \in \mathbb{N}$ *there exist genuine stable cohomology operations:*

$$\mathrm{Sq}_{\lambda}^{k\rho_{\mathrm{G}}} : \mathrm{H}_{\mathrm{G}}^{\star}(-; \underline{\mathbb{F}_2}) \longrightarrow \mathrm{H}_{\mathrm{G}}^{\star+k\rho_{\mathrm{G}}}(-; \underline{\mathbb{F}_2})$$

$$\mathrm{Sq}^{k\rho_{\mathrm{G}}+1} : \mathrm{H}_{\mathrm{G}}^{\star}(-; \underline{\mathbb{F}_2}) \longrightarrow \mathrm{H}_{\mathrm{G}}^{\star+k\rho_{\mathrm{G}}+1}(-; \underline{\mathbb{F}_2})$$

where $\lambda \in \mathrm{Irr}_1(\mathrm{G})$*—the isomorphism classes of* 1*-dimensional orthogonal* G*-representation.*

When p *is an odd prime, there exist genuine stable cohomology operations:*

$$\mathrm{P}_{\lambda}^{2\epsilon k\rho_{\mathrm{G}}} : \mathrm{H}_{\mathrm{G}}^{\star}(-; \underline{\mathbb{F}_p}) \longrightarrow \mathrm{H}_{\mathrm{G}}^{\star+2\epsilon k\rho_{\mathrm{G}}}(-; \underline{\mathbb{F}_p})$$

$$\mathrm{P}^{2\epsilon k\rho_{\mathrm{G}}+1} : \mathrm{H}_{\mathrm{G}}^{\star}(-; \underline{\mathbb{F}_p}) \longrightarrow \mathrm{H}_{\mathrm{G}}^{\star+2\epsilon k\rho_{\mathrm{G}}+1}(-; \underline{\mathbb{F}_p})$$

where $k \in \mathbb{N}$, $\lambda \in \widetilde{\mathrm{Irr}}_1(\mathrm{G})$*—the isomorphism classes of complex* 1*-dimensional orthogonal* G*-representation whose character factors through* $\mathrm{C}_p \subset \mathrm{S}^1 \subset \mathbb{C}^{\times}$*, and*

$$\epsilon = \begin{cases} (p-1)/2 & \textit{if } |\mathrm{G}| \textit{ is even} \\ (p-1) & \textit{if } |\mathrm{G}| \textit{ is odd.} \end{cases}$$

Remark 1.3. When G is the trivial subgroup e in Main Theorem 2, then the operations $\mathrm{Sq}_1^{k\rho_{\mathsf{e}}}$, $\mathrm{Sq}^{k\rho_{\mathsf{e}}+1}$, $\mathrm{P}_1^{2\epsilon k\rho_{\mathsf{e}}}$, $\mathrm{P}^{2\epsilon\rho_{\mathsf{e}}+1}$ are the classical Steenrod operations Sq^k, Sq^{k+1}, P^k, $\beta\mathrm{P}^k$ in the notation of [Ste62], respectively.

An R-Eulerian sequence χ is a sequence of homology classes. By restricting the action to a subgroup $\mathrm{K} \subset \mathrm{G}$, we define its restriction $\iota_{\mathrm{K}}(\chi)$, which is an ι_{K}R-Eulerian sequence (see (4.25)). The stable ι_{K}R-cohomology operation $\mathfrak{S}^{\iota_{\mathrm{K}}(\chi)}$ is then the restriction of the $\mathfrak{S}^{\chi}$ (see (4.29)). From this we notice that the underlying nonequivariant operations of the equivariant operations from Main Theorem 2 are precisely the classical Steenrod operations:

Main Theorem 3. *Suppose* K *is a subgroup of a finite group* G*, and let* $x \in \mathrm{H}^{\star}_{\mathrm{G}}(\mathrm{X}; \underline{\mathbb{F}}_p)$ *denote an arbitrary cohomology class for a* G*-space (or* G*-spectrum)* X*. Then:*

(1) *When* $p = 2$
   (a) $\iota_{\mathrm{K}*}(\mathrm{Sq}_{\lambda}^{k\rho_{\mathrm{G}}}(x)) = \mathrm{Sq}_{\iota_{\mathrm{K}}\lambda}^{|\mathrm{G}/\mathrm{K}|k\rho_{\mathrm{K}}}(\iota_{\mathrm{K}*}(x))$.
   (b) $\iota_{\mathrm{K}*}(\mathrm{Sq}^{k\rho_{\mathrm{G}}+1}(x)) = \mathrm{Sq}^{|\mathrm{G}/\mathrm{K}|k\rho_{\mathrm{K}}+1}(\iota_{\mathrm{K}*}(x))$.

(2) *When* p *is odd*
   (a) $\iota_{\mathrm{K}*}(\mathrm{P}_{\lambda}^{2\epsilon k\rho_{\mathrm{G}}}(x)) = \mathrm{P}_{\iota_{\mathrm{K}}\lambda}^{2\epsilon|\mathrm{G}/\mathrm{K}|k\rho_{\mathrm{K}}}(\iota_{\mathrm{K}*}(x))$.
   (b) $\iota_{\mathrm{K}*}(\mathrm{P}^{2\epsilon k\rho_{\mathrm{G}}+1}(x)) = \mathrm{P}^{2\epsilon|\mathrm{G}/\mathrm{K}|k\rho_{\mathrm{K}}+1}(\iota_{\mathrm{K}*}(x))$.

Notation 1.4. Given a subgroup K of G, we let N(K) denote the normalizer subgroup of K, and W(K) = N(K)/K denote the Weyl group of K.

In (4.27), we define geometric fixed-points $\varphi^{\mathrm{K}}(\chi)$ of an R-Eulerian sequences χ. In (4.29), we show that the geometric fixed-point of $\mathfrak{S}^{\chi}$ on a given cohomology class is equal to $\mathfrak{S}^{\varphi^{\mathrm{K}}}(\chi)$ on the geometric fixed-point of that class. While this result is satisfying, it does not quite compare the operations introduced in Main Theorem 2 across geometric fixed-point functors. This is because K-geometric fixed-point of $\mathrm{H}\underline{\mathbb{F}}_p \in \mathcal{S}\mathrm{p}^{\mathrm{G}}$, denote it by $\Phi^{\mathrm{K}}(\mathrm{H}\underline{\mathbb{F}}_p)$, is *not* equivalent to $\mathrm{H}\underline{\mathbb{F}}_p \in \mathcal{S}\mathrm{p}^{\mathrm{W(K)}}$. However, $\mathrm{H}\underline{\mathbb{F}}_p$ is a split summand of $\Phi^{\mathrm{K}}(\mathrm{H}\underline{\mathbb{F}}_p)$ as an $\mathbb{E}_{\infty}^{\mathrm{W(K)}}$-ring spectrum. This leads us to consider a modified K-geometric fixed-point functor:

$$\widetilde{\varphi}^{\mathrm{K}} : \mathrm{H}^{\mathrm{G}}_{\star}(-; \underline{\mathbb{F}}_p) \longrightarrow \mathrm{H}^{\mathrm{W(K)}}_{\star}(-; \underline{\mathbb{F}}_p).$$

In (4.30), we define the modified geometric K-fixed-point $\widetilde{\varphi}^{\mathrm{K}}(\chi)$ of an $\mathrm{H}\underline{\mathbb{F}}_p$-Eulerian sequence χ. We summarize the relation between $\mathfrak{S}^{\chi}$ and $\mathfrak{S}^{\widetilde{\varphi}^{\mathrm{K}}(\chi)}$ in (4.33). From this result we conclude:

Main Theorem 4. *Let* K *be a subgroup of a finite group* G*. For a* G*-space (or* G*-spectrum)* X*, let* $x \in \mathrm{H}^{\star}_{\mathrm{G}}(\mathrm{X}; \underline{\mathbb{F}}_p)$ *be a cohomology class. Then:*

(1) $\widetilde{\varphi}^{\mathrm{K}}\left(\mathrm{Sq}_{\lambda}^{k\rho_{\mathrm{G}}}(x)\right)=\mathrm{Sq}_{\lambda^{\mathrm{K}}}^{k|\mathrm{G}/\mathrm{N}(\mathrm{K})|\rho_{\mathrm{W}(\mathrm{K})}}\left(\widetilde{\varphi}^{\mathrm{K}}(x)\right)$ *when* $p=2$

(2) $\widetilde{\varphi}^{\mathrm{K}}\left(\mathrm{P}_{\lambda}^{2\epsilon k\rho_{\mathrm{G}}}(x)\right)=\mathrm{P}_{\lambda^{\mathrm{K}}}^{2\epsilon k|\mathrm{G}/\mathrm{N}(\mathrm{K})|\rho_{\mathrm{W}(\mathrm{K})}}\left(\widetilde{\varphi}^{\mathrm{K}}(x)\right)$ *when* p *is odd*

where it is assumed that $\mathrm{Sq}_{\lambda^{\mathrm{K}}}^{k\rho_{\mathrm{W}(\mathrm{K})}}$ *and* $\mathrm{P}_{\lambda^{\mathrm{K}}}^{2\epsilon k\rho_{\mathrm{W}(\mathrm{K})}}$ *are trivial operations when* $\lambda^{\mathrm{K}}=\mathbf{0}$.

The operations in Main Theorem 2 are derived from $\mathrm{H}\underline{\mathbb{F}}_p$-Eulerian sequences in the homology of $\mathrm{B}_{\mathrm{G}}\Sigma_p$. Calculation of $\mathrm{H}_\star^{\mathrm{G}}(\mathrm{B}_{\mathrm{G}}\Sigma_p;\underline{\mathbb{F}}_p)$ is an extremely difficult problem and is largely unsolved for groups larger than C_2. The technical part of this paper identifies infinite families of homology classes in $\mathrm{H}_\star^{\mathrm{G}}(\mathrm{B}_{\mathrm{G}}\Sigma_p;\underline{\mathbb{F}}_p)$ that are specifically designed to form $\mathrm{H}\underline{\mathbb{F}}_p$-Eulerian sequences. These homology classes can be tracked along restrictions and geometric fixed-points which leads to the results in Main Theorem 3 and Main Theorem 4. When $\mathrm{G}=\mathrm{C}_2$, the homology groups $\mathrm{H}_\star^{\mathrm{C}_2}(\mathrm{B}_{\mathrm{C}_2}\Sigma_2;\underline{\mathbb{F}}_2)$ are fully known [HK01] (also see (3.12) and (4.5)) and we show that our list of $\mathrm{H}\underline{\mathbb{F}}_2$-Eulerian sequences in $\mathrm{H}_\star^{\mathrm{C}_2}(\mathrm{B}_{\mathrm{C}_2}\Sigma_2;\underline{\mathbb{F}}_2)$ is complete. However, recent unpublished calculations of $\mathrm{H}_\star^{\mathrm{C}_4}(\mathrm{B}_{\mathrm{C}_4}\Sigma_2;\underline{\mathbb{F}}_2)$ [Geo] reveal that our list of $\mathrm{H}\underline{\mathbb{F}}_2$-Eulerian sequences for $\mathrm{G}=\mathrm{C}_4$ is far from complete. Consequently, we do not expect the cohomology operations of Main Theorem 2 to generate the full set of G-equivariant Steenrod operations for groups larger than C_2.

The structural properties of classical and C_2-equivariant Steenrod operations—namely, the Cartan formula, the Adem relations, and the total squaring operation—make them a potent tool. Given the difficult nature of calculating $\mathrm{H}_\star^{\mathrm{G}}(\mathrm{B}_{\mathrm{G}}\Sigma_p;\underline{\mathbb{F}}_p)$, our strategy is to pursue an abstract formulation of the Cartan formula and Adem relations solely through the Eulerian sequence framework. This will ensure that any new G-equivariant Eulerian sequence that is discovered will provide direct insight into the structural properties of the G-equivariant Steenrod algebra.

In (5.8), we establish a generalized Cartan formula for Eulerian sequences that remains applicable even without a Künneth isomorphism. Furthermore, we develop the framework for defining Adem relations purely in terms of Eulerian sequences. Our (4.12) provides a very general definition for Eulerian sequences, which, among other applications, allows us to define Eulerian sequences in $\mathrm{H}_\star^{\mathrm{G}}(\mathrm{B}_{\mathrm{G}}\Sigma_n;\underline{\mathbb{F}}_p)$ for all $n\in\mathbb{N}$. We refer to these as ρ_{G}-stable $\mathrm{H}\underline{\mathbb{F}}_p$-Eulerian sequences of weight n, and let $\mathcal{E}_{\mathrm{G},p}^{(n)}$ denote their collection. We then define a strictly associative product

$$\odot:\mathcal{E}_{\mathrm{G},p}^{(n)}\times\mathcal{E}_{\mathrm{G},p}^{(m)}\longrightarrow\mathcal{E}_{\mathrm{G},p}^{(nm)}$$

and show that $\mathfrak{S}^{\chi_1\odot\chi_2}=\mathfrak{S}^{\chi_1}\circ\mathfrak{S}^{\chi_2}$ (see (6.54)). This product thus realizes the composition of genuine stable $\mathrm{H}\underline{\mathbb{F}}_p$-cohomology operations. The Adem relations arise from the fact that the map

$$\Sigma_n\times\Sigma_n\longrightarrow\Sigma_n\wr\Sigma_n\longrightarrow\Sigma_{n^2}$$

and its composition with the twist map on $\Sigma_n\times\Sigma_n$ are conjugates. We are currently investigating if we can use this fact to describe Adem relations abstractly in terms of Eulerian sequences avoiding explicit calculations of $\mathrm{H}_\star^{\mathrm{G}}(\mathrm{B}_{\mathrm{G}}\Sigma_n;\underline{\mathbb{F}}_p)$.

Our theory, which is sensitive to $\mathcal{N}_\infty$-ring structures, applies to a wide range of equivariant and nonequivariant cohomology theories. Nonequivariantly, it is applicable to any cohomology theory represented by an $\mathbb{E}_\infty$-ring spectrum with finite orientation order (as defined in [BC22]). This includes complex oriented theories such as $\mathrm{H}\mathbb{Z}$, ku and Morava E-theory, along with real K-theory, topological modular forms, Johnson-Wilson theories and EO-theories. While we do not know if every stable cohomology operation for an $\mathcal{N}_\infty$-ring R can be obtained from an R-Eulerian sequence using Main Theorem 1, this is the case for $\mathrm{H}\underline{\mathbb{F}}_p$ when $\mathrm{G} = \mathsf{e}$ and $\mathrm{G} = \mathrm{C}_2$. We therefore conjecture:

Conjecture 1.5. *The collection of genuine stable* ρ_{G}*-stable cohomology operations*

$$\mathbf{S}_{\mathrm{G},p} := \{\mathfrak{S}^{\chi} : \chi \in \bigsqcup_i \mathcal{E}^{(p^i)}_{\mathrm{G},p}\}$$

generate $\mathcal{A}^{\star}_{\mathrm{G},p}$ *for all finite group* G *at all prime* p.

One can formulate a stronger version of this conjecture. First, note that the classical Steenrod algebra $\mathcal{A}^*_p$ and the C_2-equivariant Steenrod algebra $\mathcal{A}^{\star}_{\mathrm{C}_2,p}$ are multiplicatively generated over the coefficient ring by the stable cohomology operations arising from $\mathcal{E}^{(p)}_{\mathrm{G},p}$. This motivates the following question for future investigation:

Question 1.6. Does there exist an $n \in \mathbb{N}$ such that

$$\mathbf{S}_{\mathrm{G},p}\langle n\rangle := \{\mathfrak{S}^{\chi} : \chi \in \bigsqcup_{i=1}^{n} \mathcal{E}^{(p^i)}_{\mathrm{G},p}\}$$

generate the algebra $\mathcal{A}^{\star}_{\mathrm{G},p}$ for all finite group G?

Notation 1.7. Throughout this paper:

- G is a finite group,
- ρ_{G} is the regular representation of G (we drop the subscript when the underlying group is clear from the context),
- τ_n and $\widetilde{\tau}_n$ denote the permutation and the standard representations of Σ_n respectively,
- $\mathcal{U}_{\mathrm{G}}$ denotes the complete G-universe,
- V denotes an orthogonal G-representation which contains the trivial subrepresentation $\mathbb{R}$,
- $\mathcal{U}_{\mathrm{G},\mathrm{V}}$ denotes the sub G-universe generated by the G-representation V,
- $\mathcal{S}\mathrm{p}_{\mathrm{G}}$ be a category of orthogonal G spectra in the universe $\mathcal{U}_{\mathrm{G}}$,
- $\iota_{\mathrm{K}} : \mathcal{S}\mathrm{p}_{\mathrm{G}} \longrightarrow \mathcal{S}\mathrm{p}_{\mathrm{K}}$ will denote the restriction functor for the subgroup K,
- $\Phi^{\mathrm{K}} : \mathcal{S}\mathrm{p}_{\mathrm{G}} \longrightarrow \mathcal{S}\mathrm{p}_{\mathrm{W(K)}}$ denote the geometric fixed point functor for the subgroup K, where W(K) denotes the Weyl group of K in G,
- $\mathrm{RO}(\mathrm{G},\mathrm{V})$ denote the subring of $\mathrm{RO}(\mathrm{G})$ generated by $\mathcal{U}_{\mathrm{G},\mathrm{V}}$.

Convention. Throughout this paper, all G-spaces will be assumed to have a basepoint. We denote the reduced Bredon homology and cohomology with coefficients in the Mackey functor $\underline{\mathrm{A}}$ by

$$\mathrm{H}^{\mathrm{G}}_{\star}(-;\underline{\mathrm{A}}) \text{ and } \mathrm{H}_{\mathrm{G}}^{\star}(-;\underline{\mathrm{A}}),$$

respectively. Unreduced versions will be indicated by adding a disjoint basepoint.

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Organization of the paper In Section 2, we introduce equivariant power operations. In Section 3, we use orientations of certain equivariant bundles to define shifted power operations.

In Section 4, we introduce the theory of Eulerian sequences and prove Main Theorem 1. We also introduce restrictions, geometric fixed-points , and modified geometric fixed-points of Eulerian sequences. We discuss Cartan formula for Eulerian sequences in Section 5.

In Section 6, we introduce the notion of V-shifted homotopy $\mathcal{N}_\infty$-rings and use it define composition of Eulerian sequences.

In Section 7, we identify new $\mathrm{H}\underline{\mathbb{F}}_p$-Eulerian sequences and prove Main Theorem 2, Main Theorem 3, and Main Theorem 4.

2. Power operations associated to $\mathcal{N}_\infty$ ring spectra

Classical Steenrod operations are constructed from power operations, which are defined using the $\mathbb{E}_\infty$-structure of $\mathrm{H}\mathbb{F}_2$. In equivariant homotopy theory, $\mathcal{N}_\infty$-operads generalize their nonequivariant $\mathbb{E}_\infty$-operad counterparts. First studied in [BH15], these operads differ from their nonequivariant counterparts in that there can be multiple distinct homotopy classes for a given finite group G (see [Rub21], [GW18], [BP21]).

In this section, we first generalize the extended power construction (see (2.3)) to equivariant settings. We then use the multiplicative $\mathcal{N}_\infty$-structure of a ring spectrum R to define equivariant generalizations of the classical power operations (see (2.12)). We also study the interaction of equivariant power operations with restriction, fixed-points, geometric fixed-points, and modified geometric fixed-points. First, we recall some standard definitions from equivariant homotopy theory.

Definition 2.1. A collection of subgroups $\mathscr{F}$ of a group Γ is called a **family** if it is closed with respect to subgroups up to conjugation. We will call a family $\mathscr{F}$ of $\Gamma = \mathrm{G} \times \Pi$ a **G-closed family** if it contains all subgroups of the form $\mathrm{K} \times \{\mathsf{e}\}$.

Notation 2.2. For every family $\mathscr{F}$ of Γ, let $\mathrm{E}\mathscr{F}$ denote the universal Γ-space satisfying

$$(\mathrm{E}\mathscr{F})^{\mathrm{H}} \simeq \begin{cases} * & \mathrm{H} \in \mathscr{F} \\ \emptyset & \text{otherwise.} \end{cases}$$

When $\mathscr{F}$ consists of all subgroups $\mathrm{G} \times \Pi$ such that its intersection with $1 \times \Pi$ is the trivial group, then we use $\mathrm{E}_{\mathrm{G}}\Pi$ to denote $\mathrm{E}\mathscr{F}$.

2.1. **Extended powers and power operations**

Definition 2.3. Let X be a G-space or a G-spectrum, and let T be a finite Π-set. For a $(\mathrm{G}\times\Pi)$-family $\mathscr{F}$, the $(\mathscr{F},\mathrm{T})$**-th extended power of** X is defined as the G-space (or G-spectrum) given by the formula:

$$\mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(\mathrm{X}) := (\mathrm{E}\mathscr{F})_+ \wedge_{\Pi} (\mathrm{X}^{\wedge \mathrm{T}})$$

where the G-action is the diagonal action.

Remark 2.4. When X is a G-spectrum, the object $\mathrm{X}^{\wedge\mathrm{T}}$ is regarded as a $(\mathrm{G}\times\Pi)$-spectrum. The defining feature is that its underlying G-spectrum is genuine, while its Π-spectrum structure is naive. Specifically, the universe of $\mathrm{X}^{\wedge\mathrm{T}}$ is generated by finite-dimensional orthogonal representations of the form $\alpha\otimes\epsilon$, where α is a G-representation and ϵ is a trivial representation of Π.

Since the diagonal map

$$\Delta : \mathrm{E}\mathscr{F} \longrightarrow \mathrm{E}(\mathscr{F}\times\mathscr{F}) \longrightarrow \mathrm{E}\mathscr{F}\times\mathrm{E}\mathscr{F}$$

is $\mathrm{G}\times\Pi$-equivariant, we get the following lemma.

Lemma 2.5. *Given a* Π*-set* T*, there exists a* G*-equivariant natural map*

$$\partial^{\mathscr{F}}_{\mathrm{T}} : \mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(\mathrm{X}\wedge\mathrm{Y}) \longrightarrow \mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(\mathrm{X})\wedge\mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(\mathrm{Y}) \tag{2.6}$$

for any pair of G*-spaces (or* G*- spectra)* X *and* Y *that satisfies the external associativity condition. This means that the diagram*

$$\begin{tikzcd}
\mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(\mathrm{X}\wedge\mathrm{Y}\wedge\mathrm{Z}) \arrow[r] \arrow[d] & \mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(\mathrm{X}\wedge\mathrm{Y})\wedge\mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(\mathrm{Z}) \arrow[d] \\
\mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(\mathrm{X}\wedge\mathrm{Y})\wedge\mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(\mathrm{Z}) \arrow[r] & \mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(\mathrm{X})\wedge\mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(\mathrm{Y})\wedge\mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(\mathrm{Z})
\end{tikzcd} \tag{2.7}$$

commutes for any triplet $(\mathrm{X},\mathrm{Y},\mathrm{Z})$.

Proof. The map $\partial^{\mathscr{F}}_{\mathrm{T}}$, which is defined as the composite map

$$\begin{tikzcd}
\mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(\mathrm{X}\wedge\mathrm{Y}) := \mathrm{E}\mathscr{F}_+ \wedge_{\Pi} (\mathrm{X}\wedge\mathrm{Y})^{\wedge\mathrm{T}} \arrow[d, "\Delta\wedge_{\Pi}\mathrm{Id}"] \\
(\mathrm{E}\mathscr{F}\times\mathrm{E}\mathscr{F})_+ \wedge_{\Pi} (\mathrm{X}\wedge\mathrm{Y})^{\wedge\mathrm{T}} \arrow[d] \\
(\mathrm{E}\mathscr{F}_+\wedge_{\Pi}\mathrm{X}^{\wedge\mathrm{T}})\wedge(\mathrm{E}\mathscr{F}_+\wedge_{\Pi}\mathrm{Y}^{\wedge\mathrm{T}}) =: \mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(\mathrm{X})\wedge\mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(\mathrm{Y}),
\end{tikzcd}$$

satisfies (2.7) as the diagonal map Δ satisfies an external associativity condition. □

Notation 2.8. Suppose $\mathcal{O}$ is an $\mathcal{N}_\infty$-operad. Let $\mathscr{F}_n(\mathcal{O})$ denote the G-closed family of $\mathrm{G}\times\Sigma_n$ such that $\mathrm{E}\mathscr{F}_n(\mathcal{O})$ is equivalent to $\mathcal{O}(n)$, the n-th space of $\mathcal{O}$.

The collection $\{\mathscr{F}_n(\mathcal{O})\}_{n\in\mathbb{N}}$ must satisfy certain compatibility criteria, which gives rise to the operadic structure maps of $\mathcal{O}$:

$$\mu : \mathcal{O}(k)\times(\mathcal{O}(n_1)\times\cdots\times\mathcal{O}(n_k)) \longrightarrow \mathcal{O}(n_1+\cdots+n_k), \tag{2.9}$$

where $\mathcal{O}(i)$ denotes the i-th space of $\mathcal{O}$. This compatibility is encoded using a symmetric monoidal coefficient system $\underline{\mathcal{C}}(\mathcal{O})$, called the **indexing system** of $\mathcal{O}$.

Notation 2.10. Let n denote the set $\{1, 2, \ldots, n\}$ on which Σ_n acts by permutation. For an $\mathcal{N}_\infty$ G-operad $\mathcal{O}$, we will use the abbreviation $\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{X}) := \mathrm{D}^{\mathscr{F}_n(\mathcal{O})}_{\mathsf{n}}(\mathrm{X})$.

Let $\mathrm{R} \in \mathcal{S}\mathrm{p}_{\mathrm{G}}$ be a spectrum. We call R an $\mathcal{N}_\infty$-ring if it is an algebra over some $\mathcal{N}_\infty$G-operad $\mathcal{O}$. By definition, an $\mathcal{O}$-algebra $\mathrm{R} \in \mathcal{S}\mathrm{p}_{\mathrm{G}}$ is equipped with a compatible family of G-equivariant structure maps:

$$\theta^{\mathrm{R}}_n : \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{R}) \longrightarrow \mathrm{R}$$

for each $n \in \mathbb{N}$.

Given a group homomorphism $\kappa : \Pi \longrightarrow \Sigma_n$, one obtains a $\mathrm{G} \times \Pi$-family $(1 \times \kappa)^*(\mathscr{F}_n(\mathcal{O}))$ by pulling back the $\mathrm{G} \times \Sigma_n$-family $\mathscr{F}_n(\mathcal{O})$ along the map $1 \times \kappa$. For any sub-family $\mathscr{F}$ of $(1 \times \kappa)^*(\mathscr{F}_n(\mathcal{O}))$, we define a composite map

$$\theta^{\mathrm{R}}_{\mathscr{F},\kappa} : \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{R}) \longrightarrow \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{R}) \xrightarrow{\theta^{\mathrm{R}}_n} \mathrm{R}, \tag{2.11}$$

whose initial map is induced by the sequence of maps

$$\mathrm{E}\mathscr{F} \longrightarrow \mathrm{E}((1 \times \kappa)^*\mathscr{F}_n(\mathcal{O})) \longrightarrow \mathrm{E}\mathscr{F}_n(\mathcal{O})$$

combined with the Π-equivariant map $\mathrm{R}^{\wedge \kappa^*\mathsf{n}} \longrightarrow \mathrm{R}^{\wedge \mathsf{n}}$.

Definition 2.12. Let $\mathcal{O}$ be an $\mathcal{N}_\infty$ G-operad, $\kappa : \Pi \to \Sigma_n$ denote a group homomorphism and $\mathscr{F}$ be a sub-family of the $\mathrm{G} \times \Pi$-family $(1 \times \kappa)^*(\mathscr{F}_n(\mathcal{O}))$. Then the **$\mathscr{F}$-th power operation** of an $\mathcal{O}$-ring R is the natural map

$$\mathcal{P}^{\mathscr{F}} : \mathrm{R}^0(-) \longrightarrow \mathrm{R}^0(\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(-))$$

which sends $x : \mathrm{X} \to \mathrm{R}$ to the composite

$$\mathcal{P}^{\mathscr{F}}(x) : \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{X}) \xrightarrow{\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(x)} \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{R}) \xrightarrow{\theta^{\mathrm{R}}_{\mathscr{F},\kappa}} \mathrm{R} \tag{2.13}$$

in $\mathsf{Ho}(\mathcal{S}\mathrm{p}_{\mathrm{G}})$.

2.2. Restrictions and fixed points of power operations

Suppose $\mathscr{F}$ is a G-closed family of $\mathrm{G} \times \Pi$ and K a subgroup of G. The **restriction** of $\mathscr{F}$ to K, defined as

$$\iota_{\mathrm{K}}\mathscr{F} := \{\mathrm{F} \cap (\mathrm{K} \times \Pi) : \mathrm{F} \in \mathscr{F}\}$$

is an K-closed family of $\mathrm{K} \times \Pi$. The K-**fixed point family** of $\mathscr{F}$ is given by

$$\mathscr{F}^{\mathrm{K}} := \{(\mathrm{F} \cap (\mathrm{N}(\mathrm{K}) \times \Pi))/(\mathrm{K} \times \{1\}) : \mathrm{K} \times \{1\} \subset \mathrm{F} \in \mathscr{F}\}$$

and is a W(K)-closed family. These constructions lead to the following equivalences:

- $\mathrm{E}(\iota_{\mathrm{K}}\mathscr{F}) \simeq \iota_{\mathrm{K}}(\mathrm{E}\mathscr{F})$ as $(\mathrm{K} \times \Pi)$-spaces.
- $(\mathrm{E}\mathscr{F})^{\mathrm{K}} \simeq \mathrm{E}(\mathscr{F}^{\mathrm{K}})$ as $\mathrm{W}(\mathrm{K}) \times \Pi$-spaces.

Therefore, for an $\mathcal{N}_\infty$ G-operad $\mathcal{O}$, the collections

- $\iota_K\mathcal{O} := \{\iota_K\mathcal{O}(n)\}_{n\in\mathbb{N}}$
- $\mathcal{O}^K := \{\mathcal{O}(n)^K\}_{n\in\mathbb{N}}$

form an $\mathcal{N}_\infty$ K-operad and an $\mathcal{N}_\infty$ W(K)-operad, respectively.

Suppose R is an $\mathcal{O}$-ring in $\mathcal{S}\mathrm{p}_G$. Then its restriction $\iota_K R$ is an $\iota_K\mathcal{O}$-ring in $\mathcal{S}\mathrm{p}_K$ with structure maps

$$\theta_n^{\iota_K R} : \mathsf{D}_{\mathsf{n}}^{\iota_K\mathcal{O}}(\iota_K R) \cong \iota_K \mathsf{D}_{\mathsf{n}}^{\mathcal{O}}(R) \xrightarrow{\iota_K(\theta_n^R)} \iota_K R,$$

and its K-fixed points R^K is an $\mathcal{O}^K$-ring in $\mathcal{S}\mathrm{p}_{W(K)}$ with structure maps

$$\theta_n^{R^K} : \mathsf{D}_{\mathsf{n}}^{\mathcal{O}^K}(R^K) \xrightarrow{\lambda} \mathsf{D}_{\mathsf{n}}^{\mathcal{O}}(R)^K \xrightarrow{(\theta_n^R)^K} R^K,$$

where $n \in \mathbb{N}$ and λ is the map defined in (2.15).

Remark 2.14. For any space with an action of $K \times \Sigma_n$ its Σ_n-orbits of K-fixed points is a subspace of K-fixed points of Σ_n-orbits. Thus, there is a natural map of the form

$$\lambda : \mathsf{D}_{\mathsf{n}}^{\mathscr{F}^K}([-]^K) \longrightarrow [\mathsf{D}_{\mathsf{n}}^{\mathscr{F}}(-)]^K \tag{2.15}$$

in $\mathcal{T}\mathrm{op}_*^{W(K)}$. This natural map extends to $\mathcal{S}\mathrm{p}_G$ because the K-fixed points functor is lax monoidal.

Notation 2.16. For $x \in R_G^0(X) \cong [X, R]^G$, let

$$\iota_{K*}(x) : \iota_K X \longrightarrow \iota_K R \quad \in R_K^0(\iota_K X) = [\iota_K X, \iota_K R]^K$$

denote the restriction of x to the subgroup K, and

$$x^K : X^K \longrightarrow R^K \quad \in (R^K)^0_{W(K)}(X^K)$$

is the map induced by x on K-fixed points.

Assumption 1. Let

- $\mathcal{O}$ be an $\mathcal{N}_\infty$-operad,
- R an $\mathcal{O}$-ring,
- $\kappa : \Pi \to \Sigma_n$ a group homomorphism, and
- $\mathscr{F}$ a sub $(G \times \Pi)$-family of $(1 \times \kappa)^*\mathscr{F}_n(\mathcal{O})$.

Lemma 2.17. *Suppose* R, κ, $\mathscr{F}$ *are as in Assumption 1 and* K *be a subgroup of* G. *Then for any* $X \in \mathcal{S}\mathrm{p}_G$ *and* $x \in R_G^0(X)$

(1) $\iota_{K*}(\mathcal{P}^{\mathscr{F}}(x)) = \mathcal{P}^{\iota_K\mathscr{F}}(\iota_{K*}(x))$,

(2) $\lambda^*(\mathcal{P}^{\mathscr{F}}(x)^K) = \mathcal{P}^{\mathscr{F}^K}(x^K)$,

where λ *is the natural map of* (2.14).

Proof. The claim *(1)* follows from the fact that there is a natural equivalence

$$\mathsf{D}_{\kappa^*\mathsf{n}}^{\iota_K\mathscr{F}}(\iota_K(-)) \xrightarrow{\simeq} \iota_K\left(\mathsf{D}_{\kappa^*\mathsf{n}}^{\mathscr{F}}(-)\right).$$

Whereas, *(2)* follows from the naturality of λ (see (2.14)) in that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{D}^{\mathscr{F}^{\mathrm{K}}}_{\kappa^*\mathsf{n}}(\mathrm{X}^{\mathrm{K}}) \arrow[r, "\mathrm{D}^{\mathscr{F}^{\mathrm{K}}}_{\kappa^*\mathsf{n}}(x^{\mathrm{K}})"] \arrow[d, "\lambda"'] & \mathrm{D}^{\mathscr{F}^{\mathrm{K}}}_{\kappa^*\mathsf{n}}(\mathrm{R}^{\mathrm{K}}) \arrow[d, "\lambda"'] \arrow[dr, "\theta^{\mathrm{R}^{\mathrm{K}}}_{\mathscr{F}^{\mathrm{K}},\kappa}"] & \\
\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{X})^{\mathrm{K}} \arrow[r, "\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(x)^{\mathrm{K}}"'] & \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{R})^{\mathrm{K}} \arrow[r, "(\theta^{\mathrm{R}}_{\mathscr{F},\kappa})^{\mathrm{K}}"'] & \mathrm{R}^{\mathrm{K}}
\end{tikzcd} \tag{2.18}$$

in which the composition of red arrows represents $\lambda^*(\mathcal{P}^{\mathscr{F}}(x)^{\mathrm{K}})$ and the composition of blue arrows represents $\mathcal{P}^{\mathscr{F}^{\mathrm{K}}}(x^{\mathrm{K}})$. □

2.3. **Power operations and geometric fixed-points.**

In equivariant homotopy theory, the K-fixed points functor does not commute with $\Sigma^\infty_{\mathrm{G}} : \mathcal{T}\mathrm{op}^{\mathrm{G}}_* \to \mathcal{S}\mathrm{p}_{\mathrm{G}}$ and the error term is explained by the tom Dieck splitting. One of the summands of $(\Sigma^\infty_{\mathrm{G}}\mathrm{X})^{\mathrm{K}}$ is defined using the K-geometric fixed point functor

$$\Phi^{\mathrm{K}} : \mathcal{S}\mathrm{p}_{\mathrm{G}} \longrightarrow \mathcal{S}\mathrm{p}_{\mathrm{W(K)}},$$

which is given by the formula (also see (2.19))

$$\Phi^{\mathrm{K}}(\mathrm{R}) := \left(\widetilde{\mathrm{E}\mathscr{P}_{\mathrm{K}}} \wedge \iota_{\mathrm{N(K)}}\mathrm{R}\right)^{\mathrm{K}},$$

where $\mathscr{P}_{\mathrm{K}}$ is the N(K)-family consisting of those subgroups which do not contain K. The functor Φ^{K} is a symmetric monoidal functor and commutes with $\Sigma^\infty_{\mathrm{G}}$, i.e.,

$$\Phi^{\mathrm{K}}(\Sigma^\infty_{\mathrm{G}}\mathrm{X}) \simeq \Sigma^\infty_{\mathrm{W(K)}}(\mathrm{X}^{\mathrm{K}})$$

for any G-space X.

Notation 2.19. Let $\widetilde{\mathrm{E}\mathscr{F}}$ denote the cofiber of the G-equivariant map $\mathrm{E}\mathscr{F}_+ \longrightarrow \mathrm{S}^0$.

Notation 2.20. For $x \in \mathrm{R}^0_{\mathrm{G}}(\mathrm{X}) \cong [\mathrm{X},\mathrm{R}]^{\mathrm{G}}$, let

$$\varphi^{\mathrm{K}}(x) : \Phi^{\mathrm{K}}(\mathrm{X}) \longrightarrow \Phi^{\mathrm{K}}(\mathrm{R}) \quad \in [\Phi^{\mathrm{K}}(\mathrm{X}), \Phi^{\mathrm{K}}(\mathrm{R})]^{\mathrm{W(K)}}$$

denote the map induced by x on the geometric fixed points with respect to K.

Using the connecting map $\mathrm{S}^0 \longrightarrow \widetilde{\mathrm{E}\mathscr{P}_{\mathrm{K}}}$ one gets a lax monoidal natural transformation (see [BH15, Appendix B])

$$\eta_{\mathrm{K}} : (-)^{\mathrm{K}} \longrightarrow \Phi^{\mathrm{K}}(-) \tag{2.21}$$

from the K-fixed point functor to the K-geometric fixed point functor. Consequently, for any family $\mathscr{F}$ of $\mathrm{G} \times \Pi$ and Π-set T, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{D}^{\mathscr{F}^{\mathrm{K}}}_{\mathrm{T}}((-)^{\mathrm{K}}) \arrow[r, "\lambda"] \arrow[d, "\mathrm{D}^{\mathscr{F}^{\mathrm{K}}}_{\mathsf{n}}(\eta_{\mathrm{K}})"'] & (\mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(-))^{\mathrm{K}} \arrow[d, "\eta_{\mathrm{K}}"] \\
\mathrm{D}^{\mathscr{F}^{\mathrm{K}}}_{\mathrm{T}}(\Phi^{\mathrm{K}}(-)) \arrow[r, "\widehat{\lambda}"'] & \Phi^{\mathrm{K}}(\mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(-)),
\end{tikzcd} \tag{2.22}$$

in $\mathrm{Ho}(\mathcal{S}\mathrm{p}_{\mathrm{W(K)}})$, where the map $\widehat{\lambda}$ is the map defined using λ and the fact that $\widetilde{\mathrm{E}\mathscr{P}_{\mathrm{K}}} \wedge \widetilde{\mathrm{E}\mathscr{P}_{\mathrm{K}}}$ is equivalent to $\widetilde{\mathrm{E}\mathscr{P}_{\mathrm{K}}}$.

When R is an $\mathcal{O}$-ring in $\mathcal{S}\mathrm{p}_{\mathrm{G}}$ then $\Phi^{\mathrm{K}}(\mathrm{R})$ is an $\mathcal{O}^{\mathrm{K}}$-ring with structure maps

$$\theta_n^{\Phi^{\mathrm{K}}(\mathrm{R})} : \mathrm{D}_{\mathsf{n}}^{\mathcal{O}^{\mathrm{K}}}(\Phi^{\mathrm{K}}(\mathrm{R})) \xrightarrow{\widehat{\lambda}} \Phi^{\mathrm{K}}(\mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathrm{R})) \xrightarrow{\varphi^{\mathrm{K}}(\theta_n^{\mathrm{R}})} \Phi^{\mathrm{K}}(\mathrm{R}),$$

and η_{K} of (2.21) is a map of $\mathcal{O}^{\mathrm{K}}$-rings. Thus, we get the following result.

Lemma 2.23. *Suppose* R, κ, $\mathscr{F}$ *are as in Assumption 1 and* K *be a subgroup of* G*. Then, for* $\mathrm{X} \in \mathcal{S}\mathrm{p}_{\mathrm{G}}$ *and* $x \in \mathrm{R}^0_{\mathrm{G}}(\mathrm{X})$

(1) $\widehat{\lambda}^* \left(\varphi^{\mathrm{K}}\left(\mathcal{P}^{\mathscr{F}}(x)\right)\right) = \mathcal{P}^{\mathscr{F}^{\mathrm{K}}}\left(\varphi^{\mathrm{K}}(x)\right)$

(2) $\mathrm{D}_{\mathsf{n}}^{\mathscr{F}^{\mathrm{K}}}(\eta_{\mathrm{K}})_* \left(\mathcal{P}^{\mathscr{F}^{\mathrm{K}}}(x^{\mathrm{K}})\right) = \mathcal{P}^{\mathscr{F}^{\mathrm{K}}}(\eta_{\mathrm{K}*}(x^{\mathrm{K}})) = \mathcal{P}^{\mathscr{F}^{\mathrm{K}}}(\varphi^{\mathrm{K}}(x)).$

Proof. It is easy to check the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{R}^0_{\mathrm{G}}(\mathrm{X}) \arrow[rr, "\mathcal{P}^{\mathscr{F}}"] \arrow[d, "(\ )^{\mathrm{K}}"] \arrow[dd, bend right=60, "\varphi^{\mathrm{K}}(\ )"'] & & \mathrm{R}^0_{\mathrm{G}}(\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{X})) \arrow[d, "(\ )^{\mathrm{K}}"] \arrow[dd, bend left=60, "\varphi^{\mathrm{K}}(\ )"] \\
(\mathrm{R}^{\mathrm{K}})^0_{\mathrm{W(K)}}(\mathrm{X}^{\mathrm{K}}) \arrow[r, "\mathcal{P}^{\mathscr{F}^{\mathrm{K}}}"] \arrow[d, "\eta_{\mathrm{K}*}"] & (\mathrm{R}^{\mathrm{K}})^0_{\mathrm{W(K)}}(\mathrm{D}^{\mathscr{F}^{\mathrm{K}}}_{\kappa^*\mathsf{n}}(\mathrm{X}^{\mathrm{K}})) \arrow[r, "\lambda^*"] \arrow[d, "\mathrm{D}_{\mathsf{n}}^{\mathscr{F}^{\mathrm{K}}}(\eta_{\mathrm{K}})_*"'] & (\mathrm{R}^{\mathrm{K}})^0_{\mathrm{W(K)}}(\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{X})^{\mathrm{K}}) \arrow[d, "\eta_{\mathrm{K}*}"] \\
(\Phi^{\mathrm{K}}\mathrm{R})^0_{\mathrm{W(K)}}(\Phi^{\mathrm{K}}(\mathrm{X})) \arrow[r, "\mathcal{P}^{\mathscr{F}^{\mathrm{K}}}"'] & (\Phi^{\mathrm{K}}\mathrm{R})^0_{\mathrm{W(K)}}(\mathrm{D}^{\mathscr{F}^{\mathrm{K}}}_{\kappa^*\mathsf{n}}(\Phi^{\mathrm{K}}\mathrm{X})) \arrow[r, "\widehat{\lambda}^*"'] & (\Phi^{\mathrm{K}}\mathrm{R})^0_{\mathrm{W(K)}}(\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{X})^{\mathrm{K}}).
\end{tikzcd}$$

Then *(1)* follows from the commutativity of the outer square and *(2)* follows from the commutativity of the lower left square. □

Notation 2.24. We call a G-operad $\mathcal{O}$ an $\mathbb{E}^{\mathrm{G}}_\infty$-operad if its n-th space is equivalent to $\mathrm{E}_{\mathrm{G}}\Sigma_n$, the total space of the universal principal G-equivariant Σ_n-bundle. By definition

$$\mathrm{E}_{\mathrm{G}}\Sigma_n := \mathrm{E}\mathcal{A}\ell\ell_n,$$

where $\mathcal{A}\ell\ell_n$ is the G-family consisting of all subgroups of Γ of $\mathrm{G} \times \Sigma_n$ whose intersection with Σ_n is trivial.

In this paper, we consider the special case when $\mathrm{R} = \mathrm{H}\underline{\mathbb{F}}_p$, which is an $\mathbb{E}^{\mathrm{G}}_\infty$-ring in $\mathcal{S}\mathrm{p}_{\mathrm{G}}$. For a subgroup $\mathrm{K} \subset \mathrm{G}$, the K-fixed points of $\mathrm{H}\underline{\mathbb{F}}_p$ is $\mathrm{H}\underline{\mathbb{F}}_p$ at the Weyl group W(K), however, the K-geometric fixed points need not be Eilenberg MacLane[1]. Nevertheless, there exists an $\mathbb{E}^{\mathrm{W(K)}}_\infty$-ring map

$$\pi : \Phi^{\mathrm{K}}(\mathrm{H}\underline{\mathbb{F}}_p) \longrightarrow \mathrm{H}\underline{\mathbb{F}}_p,$$

as the zeroth Postnikov tower is a lax monoidal functor. Consequently, the power operations (as in (2.12)) commute with the "modified" K-geometric fixed point

---

[1] When $\mathrm{G} = \mathrm{C}_2$ then $\Phi^{\mathrm{C}_2}(\mathrm{H}\underline{\mathbb{F}}_2) \simeq \bigvee_{n\in\mathbb{N}} \Sigma^n \mathrm{H}\underline{\mathbb{F}}_2$ (see [HK01]).

functor

$$\widetilde{\varphi}^{\mathrm{K}} : \mathrm{H}^{\star}_{\mathrm{G}}(-;\underline{\mathbb{F}}_2) \longrightarrow \mathrm{H}^{\star}_{\mathrm{W(K)}}(\Phi^{\mathrm{K}}(-);\underline{\mathbb{F}}_2) \tag{2.25}$$

which sends a class x to $\widetilde{\varphi}^{\mathrm{K}}(x) := \pi^* \varphi^{\mathrm{K}}(x)$:

Lemma 2.26. *Suppose* R, κ, $\mathscr{F}$ *are as in* Assumption 1 *and* K *be a subgroup of* G*. Then*

$$\widehat{\lambda}^* \left(\mathcal{P}^{\mathscr{F}^{\mathrm{K}}}(\widetilde{\varphi}^{\mathrm{K}}(x)) \right) = \widetilde{\varphi}^{\mathrm{K}} \left(\mathcal{P}^{\mathscr{F}}(x) \right)$$

for any $x \in \mathrm{H}^0_{\mathrm{G}}(\mathrm{X};\underline{\mathbb{F}}_p)$*, where* $\mathrm{X} \in \mathcal{S}\mathrm{p}_{\mathrm{G}}$*.*

3. Equivariant orientations and shifted power operations

In nonequivariant stable homotopy theory, shifted power operations are a feature of the $\mathbb{H}^d_\infty$-ring structures introduced in [BMMS86]. The main purpose of this section is twofold: first, to generalize this concept to the equivariant setting, and second, to extend (2.12) to define power operations on classes in nonzero degrees. As demonstrated in [BMMS86, VII], an $\mathbb{H}^d_\infty$-structure is equivalent to existence of a certain compatible family of orientations. To develop the equivariant analog of these results, we consider the following equivariant bundles:

Notation 3.1. Given an $\mathcal{N}_\infty$ G-operad $\mathcal{O}$, a group homomorphism $\kappa : \Pi \to \Sigma_n$ and a sub-family $\mathscr{F}$ of $(1 \times \kappa)^* \mathscr{F}_n(\mathcal{O})$, define the G-equivariant vector bundle:

$$\gamma^{\mathscr{F}}_{\mathrm{V}} := \begin{tikzcd} (\mathrm{E}\mathscr{F}) \times_{\Pi} (\mathrm{V} \otimes \kappa^* \tau_n) \arrow[d] \\ \mathrm{B}\mathscr{F} := (\mathrm{E}\mathscr{F}) \times_{\Pi} \mathbf{0} \end{tikzcd} \tag{3.2}$$

where V is a finite dimensional real G-representation and τ_n is the permutation representation of Σ_n, i.e., the orthogonal Σ_n-representation generated by the set $\mathsf{n} = \{1, \dots, n\}$.

Remark 3.3. The $(\mathscr{F}, \kappa^*\mathsf{n})$-th extended power (as in (2.3)) of the representation sphere $\mathrm{S}^{k\mathrm{V}}$ is G-equivariantly homeomorphic to the Thom space of the bundle

$$\gamma^{\mathscr{F}}_{k\mathrm{V}} \cong \underbrace{\gamma^{\mathscr{F}}_{\mathrm{V}} \oplus \cdots \oplus \gamma^{\mathscr{F}}_{\mathrm{V}}}_{k\text{-fold}},$$

the k-fold direct sum of $\gamma^{\mathscr{F}}_{\mathrm{V}}$. In other words,

$$\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{k\mathrm{V}}) \cong \mathsf{Th}(\gamma^{\mathscr{F}}_{k\mathrm{V}})$$

for all $k \in \mathbb{N}$.

Remark 3.4. In this section, we examine the relationship between the shifted power operations across restriction and geometric fixed-point functors. Our comparison result, (3.8), hinges on the fact that the restriction and the geometric fixed-point functors on the category of G-spectra are strictly monoidal. Consequently, our arguments do not compare these shifted power operations across categorical fixed-point functors, as they are not strictly monoidal functors.

Assumption 2. Suppose R, κ, $\mathscr{F}$ are as in Assumption 1 such that $\gamma^{\mathscr{F}}_{\mathrm{V}}$ is R-orientable.

If $\gamma^{\mathscr{F}}_{k\mathrm{V}}$ is R-orientable in the sense of [BZ24, Definition 2.26], then an R-Thom class exists:

$$\mathbf{u}_{k\mathrm{V}} \in \mathrm{R}^{nk\mathrm{V}}(\mathsf{Th}(\gamma^{\mathscr{F}}_{k\mathrm{V}})). \tag{3.5}$$

We utilize this class to extend (2.12) in the following manner.

Definition 3.6. Under Assumption 2, define the **$\mathscr{F}$-th power operation** of R as the natural map

$$\mathcal{P}^{\mathscr{F}}_{\mathrm{V}} : \mathrm{R}^{k\mathrm{V}}(-) \longrightarrow \mathrm{R}^{nk\mathrm{V}}(\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(-))$$

which sends $x : \mathrm{X} \to \Sigma^{k\mathrm{V}}\mathrm{R}$ to the composite
(3.7)

$$\begin{tikzcd}
\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{X}) \arrow[r, "\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(x)"] & \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\Sigma^{k\mathrm{V}}\mathrm{R}) \arrow[d, "\partial_{\kappa^*\mathsf{n}}"] & & \\
& \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{k\mathrm{V}}) \wedge \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{R}) \arrow[r, phantom, "\cong"] & \mathsf{Th}(\gamma^{\mathscr{F}}_{k\mathrm{V}}) \wedge \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{R}) \arrow[d, "\mathbf{u}_{k\mathrm{V}} \wedge \theta^{\mathrm{R}}_{\mathscr{F},\kappa}"] & \\
& & \Sigma^{nk\mathrm{V}}\mathrm{R} \wedge \mathrm{R} \arrow[r, "\mu_{\mathrm{R}}"'] & \Sigma^{nk\mathrm{V}}\mathrm{R}
\end{tikzcd}$$

in $\mathsf{Ho}(\mathcal{S}\mathrm{p}_{\mathrm{G}})$ for all $k \in \mathbb{N}$.

Theorem 3.8. *The $\mathscr{F}$-th power operation of (3.6) satisfies*

(1) $\mathcal{P}^{\iota_{\mathrm{K}}\mathscr{F}}_{\iota_{\mathrm{K}}\mathrm{V}}(\iota_{\mathrm{K}}(x)) = \iota_{\mathrm{K}}(\mathcal{P}^{\mathscr{F}}_{\mathrm{V}}(x))$

(2) $\mathcal{P}^{\mathscr{F}^{\mathrm{K}}}_{\mathrm{V}^{\mathrm{K}}}(\varphi^{\mathrm{K}}(x)) = \widehat{\lambda}^*(\varphi^{\mathrm{K}}(\mathcal{P}^{\mathscr{F}}_{\mathrm{V}}(x)))$

for any subgroup $\mathrm{K} \subset \mathrm{G}$.

Proof. Since the restriction of a R-Thom class of $\gamma^{\mathscr{F}}_{\mathrm{V}}$ is an $\iota_{\mathrm{K}}(\mathrm{R})$-Thom class of $\gamma^{\iota_{\mathrm{K}}\mathscr{F}}_{\iota_{\mathrm{K}}\mathrm{V}}$

$$\mathbf{u}_{\iota_{\mathrm{K}}\mathrm{V}} = \iota_{\mathrm{K}*}(\mathbf{u}_{\mathrm{V}}),$$

statement *(1)* follows simply from applying the restriction functor.

To prove statement *(2)* we first observe that the geometric K-fixed points of an R-Thom class of $\gamma^{\mathscr{F}}_{\mathrm{V}}$ composed with

$$\begin{tikzcd}
\widehat{\lambda} : \Sigma^{\infty}_{\mathrm{W(K)}}\mathrm{D}^{\mathscr{F}^{\mathrm{K}}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{k\mathrm{V}^{\mathrm{K}}}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \Phi^{\mathrm{K}}\left(\Sigma^{\infty}_{\mathrm{G}}\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{k\mathrm{V}})\right) \arrow[d, phantom, sloped, "\cong"] \\
\Sigma^{\infty}_{\mathrm{W(K)}}\,\mathsf{Th}(\gamma^{\mathscr{F}^{\mathrm{K}}}_{\mathrm{V}^{\mathrm{K}}}) & \Phi^{\mathrm{K}}(\Sigma^{\infty}_{\mathrm{G}}\,\mathsf{Th}(\gamma^{\mathscr{F}}_{\mathrm{V}}))
\end{tikzcd}$$

is an $\Phi^{\mathrm{K}}(\mathrm{R})$-Thom class of $\gamma^{\mathscr{F}^{\mathrm{K}}}_{\mathrm{V}^{\mathrm{K}}}$. Then we have a homotopy commutative diagram

$$\begin{tikzcd}
\mathrm{D}^{\mathscr{F}^{\mathrm{K}}}_{\kappa^*\mathsf{n}}(\Phi^{\mathrm{K}}(\mathrm{X})) \arrow[r, "\widehat{\lambda}"] \arrow[d, "\mathrm{D}^{\mathscr{F}^{\mathrm{K}}}_{\kappa^*\mathsf{n}}(\varphi^{\mathrm{K}}(x))"'] & \Phi^{\mathrm{K}}(\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{X})) \arrow[d, "\varphi^{\mathrm{K}}(\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(x))"] \\
\mathrm{D}^{\mathscr{F}^{\mathrm{K}}}_{\kappa^*\mathsf{n}}(\Phi^{\mathrm{K}}(\Sigma^{k\mathrm{V}}\mathrm{R})) \arrow[r, "\widehat{\lambda}"] \arrow[d, "\partial^{\mathscr{F}^{\mathrm{K}}}_{\kappa^*\mathsf{n}}"'] & \Phi^{\mathrm{K}}(\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\Sigma^{k\mathrm{V}}\mathrm{R})) \arrow[d, "\varphi^{\mathrm{K}}(\partial^{\mathscr{F}}_{\kappa^*\mathsf{n}})"] \\
\mathrm{D}^{\mathscr{F}^{\mathrm{K}}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{k\mathrm{V}^{\mathrm{K}}}) \wedge \mathrm{D}^{\mathscr{F}^{\mathrm{K}}}_{\kappa^*\mathsf{n}}(\Phi^{\mathrm{K}}(\mathrm{R})) \arrow[r, "\widehat{\lambda}\wedge\widehat{\lambda}"] \arrow[d, phantom, sloped, "\cong"] & \Phi^{\mathrm{K}}(\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{k\mathrm{V}})) \wedge \Phi^{\mathrm{K}}(\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{R})) \arrow[d, phantom, sloped, "\cong"] \\
\mathsf{Th}(\gamma^{\mathscr{F}^{\mathrm{K}}}_{\mathrm{V}^{\mathrm{K}}}) \wedge \mathrm{D}^{\mathscr{F}^{\mathrm{K}}}_{\kappa^*\mathsf{n}}(\Phi^{\mathrm{K}}(\mathrm{R})) \arrow[d, "\widehat{\lambda}^*\varphi^{\mathrm{K}}(\mathbf{u}_{\mathrm{V}})\wedge\theta^{\Phi^{\mathrm{K}}(\mathrm{R})}_n"'] & \Phi^{\mathrm{K}}(\mathsf{Th}(\gamma^{\mathscr{F}}_{\mathrm{V}})) \wedge \Phi^{\mathrm{K}}(\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{R})) \arrow[d, "\varphi^{\mathrm{K}}(\mathbf{u}_{\mathrm{V}})\wedge\varphi^{\mathrm{K}}(\theta^{\mathrm{R}}_n)"] \\
\Sigma^{k\mathrm{V}^{\mathrm{K}}}\Phi^{\mathrm{K}}(\mathrm{R}) \wedge \Phi^{\mathrm{K}}(\mathrm{R}) \arrow[r, equal] \arrow[d, "\mu_{\Phi^{\mathrm{K}}(\mathrm{R})}"'] & \Sigma^{k\mathrm{V}^{\mathrm{K}}}\Phi^{\mathrm{K}}(\mathrm{R}) \wedge \Phi^{\mathrm{K}}(\mathrm{R}) \arrow[d, "\mu_{\Phi^{\mathrm{K}}(\mathrm{R})}"] \\
\Sigma^{k\mathrm{V}^{\mathrm{K}}}\Phi^{\mathrm{K}}(\mathrm{R}) \arrow[r, equal] & \Sigma^{k\mathrm{V}^{\mathrm{K}}}\Phi^{\mathrm{K}}(\mathrm{R})
\end{tikzcd}$$

using the naturality of $\widehat{\lambda}$, $\partial^{\mathscr{F}^{\mathrm{K}}}_{\kappa^*\mathsf{n}}$ and $\partial^{\mathscr{F}}_{\kappa^*\mathsf{n}}$ (defined in (2.6)), as well as the strong symmetric monoidal property of Φ^{K}. Now observe that the composition of the blue arrows and the red arrows represent the left hand side and the right hand side of *(2)* respectively. Hence, the result. □

Consider a G-space X as a $\mathrm{G}\times\Pi$-space with the trivial action of Π. Then the diagonal

$$\Delta : \mathrm{X} \longrightarrow \mathrm{X}^{\wedge\mathrm{T}}$$

is $(\mathrm{G}\times\Pi)$-equivariant for any finite Π-set T, and it induces the following map:

$$\delta : \mathrm{B}\mathscr{F} \times \mathrm{X} \simeq \mathrm{E}\mathscr{F}_+ \wedge_{\Pi} \mathrm{X} \xrightarrow{1_{\mathrm{E}\mathscr{F}_+}\wedge_{\Pi}(\Delta)} \mathrm{E}\mathscr{F}_+ \wedge_{\Pi} \mathrm{X}^{\wedge\mathrm{T}} \simeq \mathrm{D}^{\mathscr{F}}_{\mathrm{T}}(\mathrm{X}). \tag{3.9}$$

(specifically, under the conditions of Assumption 2), we use the induced map δ to obtain a class

$$\delta^*\mathcal{P}^{\mathscr{F}}_{k\mathrm{V}}(x) \in \mathrm{R}^{nk\mathrm{V}}_{\mathrm{G}}(\mathrm{B}\mathscr{F} \times \mathrm{X})_+$$

which is crucial to the construction of Steenrod operations.

3.1. **Known examples of equivariant Steenrod operations**

Steenrod operations have been constructed nonequivariantly for $\mathrm{H}\mathbb{F}_p$-cohomology for all primes [Ste62], and for $\mathrm{H}\underline{\mathbb{F}}_2$-cohomology when $\mathrm{G} = \mathrm{C}_2$ [HK01, Voe03].

We begin by discussing the case $p = 2$. In the following discussion, we simplify our notation by letting

$$\mathcal{P} : \mathrm{H}^{k\rho_{\mathrm{G}}}_{\mathrm{G}}(-;\underline{\mathbb{F}}_2) \longrightarrow \mathrm{H}^{2k\rho_{\mathrm{G}}}_{\mathrm{G}}(\mathrm{D}^{\mathcal{A}\ell\ell_2}_2(-);\underline{\mathbb{F}}_2)$$

denote the $\mathcal{A}\ell\ell_2$-th power operation for all $k \in \mathbb{N}$.

When G is trivial, the coefficient ring is the field $\mathbb{F}_2$, and we therefore have a Künneth isomorphism. Consequently, for any $\mathrm{X} \in \mathcal{T}\mathrm{op}$

$$\mathrm{H}^*((\mathrm{B}\Sigma_2 \times \mathrm{X})_+;\mathbb{F}_2) \cong \mathrm{H}^*(\mathrm{X})[\![\mathbf{e}_1]\!], \tag{3.10}$$

where $\mathbf{e}_1 \in \mathrm{H}^1(\mathrm{B}\Sigma_2; \mathbb{F}_2)$ is the $\mathrm{H}\mathbb{F}_2$-Euler class of the tautological line bundle. Under the identification (3.10), we have the formula

$$\delta^*\mathcal{P}(x) = \sum_{i=0}^{k} \mathrm{Sq}^i(x)\mathbf{e}_1^{k-i} \tag{3.11}$$

which defines the classical Steenrod operations.

When $\mathrm{G} = \mathrm{C}_2$, the coefficient ring $\mathbb{M}_2^{\mathrm{C}_2} := \pi_\star^{\mathrm{C}_2}(\mathrm{H}\underline{\mathbb{F}}_2)$ is not a field. Therefore, a Künneth isomorphism should not be expected to hold in general. However, Hu and Kriz [HK01] showed that

$$\mathrm{H}^\star_{\mathrm{C}_2}((\mathrm{B}_{\mathrm{G}}\Sigma_2)_+; \underline{\mathbb{F}}_2) \cong \mathbb{M}_2^{\mathrm{C}_2}[\![\mathbf{y}, \mathbf{e}_\rho]\!]/(\mathbf{y}^2 = a\mathbf{y} + u\mathbf{e}_\rho), \tag{3.12}$$

where $\mathbf{e}_\rho$ is the $\mathrm{H}\underline{\mathbb{F}}_2$-Euler class of a ρ-dimensional C_2-equivariant vector bundle $\bar{\gamma}$ as in [BGL22, pg 17] (also see (4.8)), and a and u are specific elements in the coefficient ring. Importantly, $\mathrm{H}^\star_{\mathrm{C}_2}(\mathrm{B}_{\mathrm{G}}\Sigma_2; \underline{\mathbb{F}}_2)$ is free as over $\mathbb{M}_2^{\mathrm{C}_2}$, and there is a Künneth isomorphism. Under this isomorphism, we have the formula:

$$\delta^*\mathcal{P}(x) = \sum_{i=0}^{k} \mathrm{Sq}^{2i}(x)\mathbf{e}_\rho^{k-i} + \sum_{i=0}^{k-1} \mathrm{Sq}^{2i+1}(x)\mathbf{y}\mathbf{e}_\rho^{k-i-1} \tag{3.13}$$

for any x in degree $k\rho$. This formula defines the C_2-equivariant Steenrod operations [BGL22, §3].

Remark 3.14. The above definition of C_2-equivariant Steenrod operations may not generalize to an arbitrary group G because the Künneth map

$$\mathfrak{K} : \mathrm{R}^\star_{\mathrm{G}}(\mathrm{B}\mathscr{F}_+) \otimes_{\pi_\star^{\mathrm{G}}\mathrm{R}} \mathrm{R}^\star_{\mathrm{G}}(\mathrm{X}_+) \longrightarrow \mathrm{R}^\star_{\mathrm{G}}(\mathrm{B}\mathscr{F}_+ \wedge \mathrm{X}_+) \tag{3.15}$$

is not always an isomorphism[2]. In Section 4, we will introduce the theory of Eulerian sequence to circumvent this failure of the Künneth map to be an isomorphism.

In the classical case, i.e., when G is the trivial group, Epstein and Steenrod (see [Ste62, Chapter VII]) use the inclusion of $\Pi = \mathrm{C}_p$

$$\kappa : \mathrm{C}_p \hookrightarrow \Sigma_p \tag{3.16}$$

to identify odd primary Steenrod operations. In this case, they consider the power operation $\mathcal{P}_{k(p-1)}^{\kappa^*\mathcal{A}\ell\ell_p}$, which will be denote by

$$\mathcal{P} : \mathrm{H}_{\mathrm{G}}^{k(p-1)}(-; \underline{\mathbb{F}}_p) \longrightarrow \mathrm{H}_{\mathrm{G}}^{pk(p-1)}(\mathrm{D}_{\kappa^*\mathsf{p}}^{\kappa^*\mathcal{A}\ell\ell_p}(-); \underline{\mathbb{F}}_p)$$

for all $k \in \mathbb{N}$ to alleviate notation. Note that

$$\mathrm{H}^*((\mathrm{BC}_p)_+; \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\mathbf{y})[\![\mathbf{u}]\!],$$

where $|\mathbf{y}| = 1$ and $|\mathbf{u}| = 2$. Then they utilize the Künneth isomorphism to note that

(3.17)

$$\nu(k(p-1))\,\delta^*\mathcal{P}(x) = \sum_{i=0}^{k}(-1)^i\mathrm{P}^i(x)\mathbf{u}^{(k-i)(p-1)} + \sum_{i=0}^{k-1}(-1)^i\beta\mathrm{P}^i(x)\mathbf{y}\mathbf{u}^{(k-i)(p-1)-1}$$

---

[2] In the unpublished work [Geo], Nick Georgakopoulus showed that the $\mathrm{RO}(\mathrm{C}_4)$-graded cohomology of $\mathrm{B}_{\mathrm{C}_4}\Sigma_2$ is not free over its coefficient ring.

where $x \in \mathrm{H}^{k(p-1)}(\mathrm{X}_+;\mathbb{F}_p)$, β is the Bockstein homomorphism, and

$$\nu(q) = \left(\left(\frac{p-1}{2}\right)!\right)^q (-1)^{(p-1)(q^2+q)/4}.$$

The above equation can be used to define mod p Steenrod operations.

Remark 3.18. If x is a cohomology class in a degree which is not a multiple of $p-1$ then one can use the suspension isomorphism

$$\sigma : \mathrm{H}^*(\mathrm{X}_+;\mathbb{F}_p) \cong \mathrm{H}^{*+1}(\Sigma\mathrm{X}_+;\mathbb{F}_p) \tag{3.19}$$

to define the mod p Steenrod operations on x

$$\beta^\epsilon\mathrm{P}^i(x) = \sigma^{-n}(\beta^\epsilon\mathrm{P}^i(\sigma^n(x))),$$

where n is a positive integer such that $n + |x|$ is a multiple of $p-1$. This is well-defined as Epstein and Steenrod have shown that the mod p Steenrod operations are stable, i.e., the operations P^i and β commutes with the suspension isomorphism.

Remark 3.20. It is possible to define odd primary Steenrod operations using the power operation

$$\mathcal{P} := \mathcal{P}^{\mathcal{A}\ell\ell_p}_{k(p-1)} : \mathrm{H}^{k(p-1)}_{\mathrm{G}}(-;\underline{\mathbb{F}}_2) \longrightarrow \mathrm{H}^{pk(p-1)}_{\mathrm{G}}(\mathrm{D}^{\mathcal{A}\ell\ell_p}_{\mathsf{p}}(-);\underline{\mathbb{F}}_2)$$

avoiding the group homomorphism κ of (3.16). To see this we first note that the map

$$\mathrm{B}\kappa^* : \mathrm{H}^*((\mathrm{B}\Sigma_p)_+;\mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\mathbf{v})[\![\mathbf{e}]\!] \longrightarrow \mathrm{H}^*((\mathrm{BC}_p)_+;\mathbb{F}_p)$$

sends $\mathbf{v} \mapsto \mathbf{y}\mathbf{u}^{p-2}$ and $\mathbf{e} \mapsto \mathbf{u}^{p-1}$, and therefore,

$$\nu(k(p-1))\,\delta^*\mathcal{P}(x) = \sum_{i=0}^{k}(-1)^i\mathrm{P}^i(x)\mathbf{e}^{(k-i)} + \sum_{i=0}^{k-1}(-1)^i\beta\mathrm{P}^i(x)\mathbf{v}\mathbf{e}^{k-i-1} \tag{3.21}$$

for any x in degree $k(p-1)$.

4. Eulerian sequences and stable cohomology operations

In this section, we introduce the concept of Eulerian sequences. These sequences are designed to resolve the problem posed in (3.14)—the definition of Steenrod operations in the absence of a Künneth isomorphism. We then prove the fundamental result of this paper, (4.18), which establishes that a stable cohomology operation exists corresponding to every such sequence.

To motivate this definition, we first observe that the known Steenrod operations (described in Section 3.1) can be defined using the slant product (see (4.3) and (4.5) below), thereby avoiding the need for a Künneth isomorphism

Notation 4.1. For any $\mathrm{R} \in \mathcal{S}\mathrm{p}_{\mathrm{G}}$, define the RO(G)**-graded slant product** as the pairing

$$(\ )|_{(\ )} : \mathrm{R}^{\mathrm{W}}_{\mathrm{G}}(\mathrm{B}\mathscr{F}_+ \wedge \mathrm{X}) \otimes \mathrm{R}^{\mathrm{G}}_{\mathrm{V}}(\mathrm{B}\mathscr{F}_+) \longrightarrow \mathrm{R}^{\mathrm{V}-\mathrm{W}}_{\mathrm{G}}(\mathrm{X}), \tag{4.2}$$

where $x|_{\mathsf{b}}$ is the homotopy class of the composite

$$x|_{\mathsf{b}} : \mathrm{S}^{\mathrm{V}} \wedge \mathrm{X} \xrightarrow{\mathsf{b}\wedge 1_{\mathrm{X}}} \mathrm{R} \wedge \mathrm{B}\mathscr{F}_+ \wedge \mathrm{X} \xrightarrow{1_{\mathrm{R}}\wedge x} \mathrm{R} \wedge \Sigma^{\mathrm{W}}\mathrm{R} \xrightarrow{\Sigma^{\mathrm{W}}\mu_{\mathrm{R}}} \Sigma^{\mathrm{W}}\mathrm{R}$$

for any $x \in \mathrm{R}_{\mathrm{G}}^{\mathrm{W}}(\mathrm{B}\mathscr{F}_{+} \wedge \mathrm{X})$ and $\mathsf{b} \in \mathrm{R}_{\mathrm{V}}^{\mathrm{G}}(\mathrm{B}\mathscr{F}_{+})$.

Remark 4.3. In the classical case, we note that the Steenrod squaring operations (as in (3.11)) can be equivalently defined as

$$\mathrm{Sq}^{i}(x) := \delta^{*}\mathcal{P}(x)|_{\mathbf{b}_{k-i}} \tag{4.4}$$

whenever $|x| = k$, where $\mathbf{b}_{k-i} \in \mathrm{H}_{k-i}((\mathrm{B}\Sigma_2)_{+};\mathbb{F}_2)$ is the class dual to $\mathbf{e}_1^{k-i} \in \mathrm{H}^{k-i}((\mathrm{B}\Sigma_2)_{+};\mathbb{F}_2)$. Likewise, at an odd prime p, one may also define P^i of (3.17) using the slant product:

$$\mathrm{P}^{i}(x) := (-1)^{i}\nu(k(p-1))\, \delta^{*}\mathcal{P}(x)|_{\mathbf{b}_{k-i}}$$
$$\beta\mathrm{P}^{i}(x) := (-1)^{i}\nu(k(p-1))\, \delta^{*}\mathcal{P}(x)|_{\mathbf{c}_{k-i}}$$

whenever $|x| = k(p-1)$, where $\mathbf{b}_{k-i}, \mathbf{c}_{k-i} \in \mathrm{H}_{*}((\mathrm{B}\Sigma_p)_{+};\mathbb{F}_p)$ are elements which satisfy

$$\langle \mathbf{u}^{(k-i)(p-1)}, \mathbf{b}_{k-i}\rangle = 1 = \langle \mathbf{y}\mathbf{u}^{(k-i)(p-1)-1}, \mathbf{c}_{k-i}\rangle$$

where $\langle\ ,\ \rangle$ is the Kronecker product [Ada95, III.9].

Remark 4.5. Note that $\mathrm{H}\underline{\mathbb{F}}_2$-homology and $\mathrm{H}\underline{\mathbb{F}}_2$-cohomology of $\mathrm{B}_{\mathrm{C}_2}\Sigma_2$ are $\mathbb{M}_2^{\mathrm{C}_2}$-linear dual to each other, as they are free over the coefficient ring $\mathbb{M}_2^{\mathrm{C}_2}$. Let

$$\mathbf{b}_{k\rho}, \mathbf{c}_{k\rho+\sigma} \in \mathrm{H}_{\star}^{\mathrm{C}_2}((\mathrm{B}_{\mathrm{C}_2}\Sigma_2)_{+};\underline{\mathbb{F}}_2)$$

denote the elements $\mathbb{M}_2^{\mathrm{C}_2}$-linear dual to $\mathbf{e}_{\rho}^{k}$ and $\mathbf{y}\mathbf{e}_{\rho}^{k}$ (described in (3.12)), respectively. Then the C_2-equivariant Steenrod operations of (3.13) can be defined as

$$\mathrm{Sq}^{2i}(x) := \delta^{*}\mathcal{P}(x)|_{\mathbf{b}_{(k-i)\rho}} \text{ and } \mathrm{Sq}^{2i+1}(x) := \delta^{*}\mathcal{P}(x)|_{\mathbf{c}_{(k-i)\rho+\sigma}}$$

whenever $|x| = k\rho$.

Notation 4.6. For a ring spectrum $\mathrm{R} \in \mathcal{S}\mathrm{p}_{\mathrm{G}}$, the RO(G)**-graded cap product**

$$- \frown - : \mathrm{R}_{\mathrm{W}}^{\mathrm{G}}(\mathrm{X}) \times \mathrm{R}_{\mathrm{G}}^{\mathrm{V}}(\mathrm{X}) \longrightarrow \mathrm{R}_{\mathrm{W}-\mathrm{V}}^{\mathrm{G}}(\mathrm{X})$$

is defined for any G-space X by sending the pair $b \in \mathrm{R}_{\mathrm{W}}^{\mathrm{G}}(\mathrm{X})$ and $e \in \mathrm{R}_{\mathrm{G}}^{\mathrm{V}}(\mathrm{X})$ to the composite

$$\mathrm{S}^{\mathrm{W}} \xrightarrow{b} \mathrm{X} \wedge \mathrm{R} \xrightarrow{\Delta \wedge 1_{\mathrm{R}}} \mathrm{X} \wedge \mathrm{X} \wedge \mathrm{R} \xrightarrow{1_{\mathrm{X}} \wedge e \wedge 1_{\mathrm{R}}} \mathrm{X} \wedge \Sigma^{\mathrm{V}}\mathrm{R} \wedge \mathrm{R} \xrightarrow{1_{\mathrm{X}} \wedge \mu_{\mathrm{R}}} \Sigma^{\mathrm{V}}\mathrm{X} \wedge \mathrm{R},$$

where Δ is the diagonal map of X.

Note, in (4.4), the definition of the i-th Steenrod operation depends on the degree of the class x. Thus one may ask why the Steenrod squaring operations are stable, i.e.,

$$\sigma(\delta^{*}\mathcal{P}(x)|_{\mathbf{b}_{k-i}}) = \sigma(\mathrm{Sq}^{i}(x)) = \mathrm{Sq}^{i}(\sigma(x)) = \delta^{*}\mathcal{P}(\sigma(x))|_{\mathbf{b}_{k+1-i}}$$

where σ is the suspension isomorphism of (3.19). We observe that this is a consequence of the relation

$$\mathbf{b}_{k+1} \frown \mathbf{e}_1 = \mathbf{b}_k,$$

where '$\frown$' denotes the cap product and $\mathbf{e}_1$ is the Euler class of the tautological line bundle over $\mathrm{B}\Sigma_2 \simeq \mathbb{RP}^{\infty}$. This leads us to the notion of Eulerian sequences.

4.1. **The theory of Eulerian sequences**

A G-equivariant Eulerian sequence is defined using an Euler class of certain G-equivariant vector bundles. To describe these bundles consider τ_n—the orthogonal permutation representation of Σ_n (see (3.1)), and let $\widetilde{\tau}_n$ denote the orthogonal complement of the 1-dimensional trivial subrepresentation of τ_n (spanned by the sum of all elements in n).

For any finite orthogonal G-representation V, let $\widetilde{\gamma}^{\mathscr{F}}_{\mathrm{V}}$ denote the $(n-1)\mathrm{V}$-dimensional G-equivariant vector bundle

$$\widetilde{\gamma}^{\mathscr{F}}_{\mathrm{V}} := \begin{tikzcd} (\mathrm{E}\mathscr{F}) \times_{\Pi} (\mathrm{V} \otimes \kappa^* \widetilde{\tau}_n) \arrow[d] \\ \mathrm{B}\mathscr{F}. \end{tikzcd} \tag{4.7}$$

Note that $\widetilde{\gamma}^{\mathscr{F}}_{\mathrm{V}} \oplus \epsilon_{\mathrm{V}} \cong \gamma^{\mathscr{F}}_{\mathrm{V}}$, and consequently, $\Sigma^{\mathrm{V}}\, \mathsf{Th}(\widetilde{\gamma}^{\mathscr{F}}_{\mathrm{V}}) \simeq \mathsf{Th}(\gamma^{\mathscr{F}}_{\mathrm{V}})$.

Remark 4.8. Recall the G-closed family $\mathcal{A}\ell\ell_n$ of $\mathrm{G} \times \Sigma_n$ from (2.24). When G is the trivial group, then $\widetilde{\gamma}^{\mathcal{A}\ell\ell_2}_{\rho}$ is the tautological line bundle over $\mathrm{B}\Sigma_2$. When $\mathrm{G} = \mathrm{C}_2$, $\widetilde{\gamma}^{\mathcal{A}\ell\ell_2}_{\rho}$ is the ρ-dimensional bundle $\overline{\gamma}$ described in [BGL22], which is used in the construction of the C_2-equivariant Steenrod operations.

Notation 4.9. Let $\mathrm{R} \in \mathcal{S}\mathrm{p}_{\mathrm{G}}$ be a ring spectrum. If $\widetilde{\gamma}^{\mathscr{F}}_{\rho}$ admits and is equipped with an R-orientation (in the sense of [BZ24]), then we let

$$\widetilde{\mathbf{u}}_{\mathrm{V}} \in \mathrm{R}^{(n-1)\mathrm{V}}_{\mathrm{G}}(\mathsf{Th}(\widetilde{\gamma}^{\mathscr{F}}_{\mathrm{V}}))$$

denote the corresponding R-Thom class. Let

$$\zeta : \mathrm{B}\mathscr{F}_+ \longrightarrow \mathsf{Th}(\widetilde{\gamma}^{\mathscr{F}}_{\mathrm{V}})$$

denote the zero section of the G-equivariant vector bundle $\widetilde{\gamma}^{\mathscr{F}}_{\mathrm{V}}$. When $\widetilde{\gamma}^{\mathscr{F}}_{\mathrm{V}}$ is R-orientable then its R-**Euler class**

$$\widetilde{\mathbf{e}}_{\mathrm{V}} := \zeta^*(\widetilde{\mathbf{u}}_{\mathrm{V}}) \in \mathrm{R}^{(n-1)\mathrm{V}}_{\mathrm{G}}(\mathrm{B}\mathscr{F}_+) \tag{4.10}$$

is the pullback of its R-Thom class along the zero section.

Remark 4.11. If $\widetilde{\gamma}^{\mathscr{F}}_{\rho}$ is R-orientable, then $\gamma^{\mathscr{F}}_{\mathrm{V}} \cong \widetilde{\gamma}^{\mathscr{F}}_{\mathrm{V}} \oplus \epsilon_{\mathrm{V}}$ is also R-orientable, and we set

$$\mathbf{u}_{\mathrm{V}} := \sigma_{\mathrm{V}}(\widetilde{\mathbf{u}}_{\mathrm{V}})$$

as the R-Thom class of $\gamma^{\mathscr{F}}_{\mathrm{V}}$ in (3.5).

Definition 4.12. Suppose $\mathrm{R}, \kappa, \mathcal{O}, \mathscr{F}$ be as in Assumption 2. Then a **V-stable** R-**Eulerian sequence** is a sequence

$$\chi = (\mathsf{x}_0, \mathsf{x}_1, \dots)$$

such that

- $|\mathsf{x}_i| \in \mathrm{RO}(\mathrm{G}, \mathrm{V})$ is non-virtual for all $i \in \mathbb{N}$,
- $\mathsf{x}_{i+1} \frown \widetilde{\mathbf{e}}_{\mathrm{V}} = \mathsf{x}_i$,
- $\mathsf{x}_0 \frown \widetilde{\mathbf{e}}_{\mathrm{V}} = 0$,

where $\widetilde{\mathbf{e}}_{\mathrm{V}}$ is an R-Euler class of $\widetilde{\gamma}_{\mathrm{V}}^{\mathscr{F}}$. Since $\widetilde{\mathbf{e}}_{\mathrm{V}}$ has degree $(n-1)\mathrm{V}$

$$\|\chi\| := t(n-1)\mathrm{V} - |\mathsf{x}_t| \in \mathrm{RO}(\mathrm{G},\mathrm{V})$$

is independent of t. We call $\|\chi\|$ the **degree of** χ, and n the **weight** of χ.

Remark 4.13. The requirement that $|\mathsf{x}_i|$ is required to define composition of Eulerian sequences (see (6.48)).

Notation 4.14. For a G-representation V and $\mathrm{R} \in \mathcal{S}\mathrm{p}_{\mathrm{G}}$, let

$$\sigma_{\mathrm{V}} : \mathrm{R}_{\mathrm{G}}^{\star}(-) \xrightarrow{\cong} \mathrm{R}_{\mathrm{G}}^{\star+\mathrm{V}}(\Sigma^{\mathrm{V}}(-))$$

denote the V-th suspension isomorphism. We will simply use σ when V is the 1-dimensional trivial representation.

Definition 4.15. Given a V-stable R-Eulerian sequence $\chi := (\mathsf{x}_0, \mathsf{x}_1, \dots)$ of weight n, define the RO(G, V)-graded **χ-th Steenrod operation**

$$\mathfrak{S}^{\chi} : \mathrm{R}_{\mathrm{G}}^{\star}(-) \longrightarrow \mathrm{R}_{\mathrm{G}}^{\star+\|\chi\|}(-)$$

as follows:

- If $x \in \mathrm{R}_{\mathrm{G}}^{t\mathrm{V}}(\mathrm{X})$ where X is a G-space , $t \in \mathbb{N}$, then let

$$\mathfrak{S}^{\chi}(x) := \delta^{*}\mathcal{P}_{t\mathrm{V}}^{\mathscr{F}}(x)|_{\mathsf{x}_t}. \tag{4.16}$$

- If $x \in \mathrm{R}_{\mathrm{G}}^{\mathrm{W}}(\mathrm{X})$ for some $\mathrm{W} \underset{\text{finite}}{\subset} \mathcal{U}_{\mathrm{G},\mathrm{V}}$, then choose W′ such that $\mathrm{W} \oplus \mathrm{W}' = t\widetilde{\mathrm{V}}$ for some $t \in \mathbb{N}$ and define

$$\mathfrak{S}^{\chi}(x) := \sigma_{\mathrm{W}'}^{-1}\mathfrak{S}^{\chi}(\sigma_{\mathrm{W}'}(x)). \tag{4.17}$$

Our next result, (4.18), demonstrates that the operation $\mathfrak{S}^{\chi}$ commutes with the V-th suspension σ_{V}. This simultaneously implies two key points:

- the equation (4.17) is well-defined, i.e., independent of the choice of W′,
- the operation $\mathfrak{S}^{\chi}$ is stable in the universe $\mathcal{U}_{\mathrm{G},\mathrm{V}}$, and therefore extends to classes of degrees contained in RO(G, V).

Theorem 4.18. *Let* X *be a* G*-space and let* $\mathrm{R}, \kappa, \mathcal{O}, \mathscr{F}$ *be as in Assumption 2. Given a* V*-stable* R*-Eulerian sequence*

$$\chi = (\mathsf{x}_0, \mathsf{x}_1, \dots),$$

the χ*-th Steenrod operation defined in* (4.16) *satisfies the naturality condition with respect to suspension:*

$$\mathfrak{S}^{\chi}(\sigma_{\mathrm{V}}(x)) = \sigma_{\mathrm{V}}(\mathfrak{S}^{\chi}(x))$$

for any $x \in \mathrm{R}_{\mathrm{G}}^{t\mathrm{V}}(\mathrm{X})$ *for any* $t \in \mathbb{N}$.

Proof. Consider the diagram (4.20) in which the map

(1) is the composition

$$\begin{tikzcd}
\mathrm{S}^{|\mathsf{x}_{t+1}|} \wedge \Sigma^{\mathrm{V}}\mathrm{X} \arrow[r, "\sigma_{(n-1)\mathrm{V}}(\mathsf{x}_t \wedge 1_{\Sigma\mathrm{V}})"] & \Sigma^{(n-1)\mathrm{V}}\mathrm{B}\mathscr{F}_{+} \wedge \mathrm{R} \wedge \Sigma\mathrm{X} \arrow[d, phantom, "\cong" rotate=90] \\
 & \Sigma^{n\mathrm{V}}\mathrm{B}\mathscr{F}_{+} \wedge \mathrm{X} \wedge \mathrm{R},
\end{tikzcd}$$

(2) is the composition

$$\mathrm{S}^{|\mathsf{x}_{t+1}|} \xrightarrow{\mathsf{x}_{t+1} \wedge 1_{\Sigma^{\mathrm{V}}\mathrm{X}}} \mathrm{B}\mathscr{F}_+ \wedge \mathrm{R} \wedge \Sigma^{\mathrm{V}}\mathrm{X} \quad \cong \quad \mathrm{B}\mathscr{F}_+ \wedge \Sigma^{\mathrm{V}}\mathrm{X} \wedge \mathrm{R},$$

(3) is induced by the diagonal map $\Delta : \mathrm{B}\mathscr{F} \longrightarrow \mathrm{B}\mathscr{F} \times \mathrm{B}\mathscr{F}$,

(4) is $1_{\mathrm{B}\mathscr{F}_+} \wedge \widetilde{\mathbf{e}}_{\mathrm{V}} \wedge 1_{\Sigma^{\mathrm{V}}\mathrm{X} \wedge \mathrm{R}}$,

(5) is induced by the multiplication of R,

(6) is the natural map δ of (3.9),

(7) is the composition

$$\begin{tikzcd}
\mathrm{B}\mathcal{F}_+ \wedge \mathrm{B}\mathcal{F}_+ \wedge \Sigma^{\mathrm{V}}\mathrm{X} \wedge \mathrm{R} \arrow[d, phantom, "\cong" rotate=90] & \\
(\mathrm{B}\mathcal{F}_+ \wedge \mathrm{S}^{\mathrm{V}}) \wedge (\mathrm{B}\mathcal{F}_+ \wedge \mathrm{X}) \wedge \mathrm{R} \arrow[r, "\delta\wedge\delta\wedge 1_{\mathrm{R}}"] & \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{\mathrm{V}}) \wedge \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{X}) \wedge \mathrm{R},
\end{tikzcd}$$

(8) is the composition

$$\begin{tikzcd}
\mathrm{B}\mathscr{F}_+ \wedge \Sigma^{(n-1)\mathrm{V}}\mathrm{R} \wedge \Sigma^{\mathrm{V}}\mathrm{X} \wedge \mathrm{R} \arrow[d, phantom, "\cong" rotate=90] & \\
\Sigma^{n\mathrm{V}}\mathrm{R} \wedge (\mathrm{B}\mathscr{F}_+ \wedge \mathrm{X}) \wedge \mathrm{R} \arrow[r, "\sigma_{n\mathrm{V}}(1_{\mathrm{R}}\wedge\delta\wedge 1_{\mathrm{R}})"] & \Sigma^{n\mathrm{V}}\mathrm{R} \wedge \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{X}) \wedge \mathrm{R},
\end{tikzcd}$$

(9) is $\sigma_{n\mathrm{V}}(\delta \wedge 1_{\mathrm{R}})$,

(10) is $\partial^{\mathscr{F}}_{\kappa^*\mathsf{n}} \wedge 1_{\mathrm{R}}$,

(11) is $\sigma_{\mathrm{V}}(\widetilde{\mathbf{u}}_{\mathrm{V}}) \wedge 1_{\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{X})} \wedge 1_{\mathrm{R}}$,

(12) is induced by the multiplication of R,

(13) is $\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\sigma_{\mathrm{V}}(x)) \wedge 1_{\mathrm{R}}$,

(14) is $1_{\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{\mathrm{V}})} \wedge \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(x) \wedge 1_{\mathrm{R}}$,

(15) is $\sigma_{n\mathrm{V}}(1_{\mathrm{R}} \wedge \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(x) \wedge 1_{\mathrm{R}})$,

(16) is $\sigma_{n\mathrm{V}}(\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(x) \wedge 1_{\mathrm{R}})$,

(17) is $\partial^{\mathscr{F}}_{\kappa^*\mathsf{n}} \wedge 1_{\mathrm{R}}$,

(18) is $\sigma_{\mathrm{V}}(\widetilde{\mathbf{u}}_{\mathrm{V}}) \wedge 1_{\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\Sigma^{t\mathrm{V}}\mathrm{R})} \wedge 1_{\mathrm{R}}$,

(19) is induced by the multiplication of R,

(20) is $\partial^{\mathscr{F}}_{\kappa^*\mathsf{n}} \wedge 1_{\mathrm{R}}$,

(21) is $1_{\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{\mathrm{V}})} \wedge \partial^{\mathscr{F}}_{\kappa^*\mathsf{n}} \wedge 1_{\mathrm{R}}$,

(22) is $\sigma_{n\mathrm{V}}(1_{\mathrm{R}} \wedge \partial^{\mathscr{F}}_{\kappa^*\mathsf{n}} \wedge 1_{\mathrm{R}})$,

(23) is $\sigma_{n\mathrm{V}}(\partial^{\mathscr{F}}_{\kappa^*\mathsf{n}} \wedge 1_{\mathrm{R}})$,

(24) is $\partial^{\mathscr{F}}_{\kappa^*\mathsf{n}} \wedge 1_{\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}} \wedge 1_{\mathrm{R}}$,

(25) is $\sigma_{\mathrm{V}}(\widetilde{\mathbf{u}}_{\mathrm{V}}) \wedge 1_{\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{t\mathrm{V}})} \wedge 1_{\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{R})} \wedge 1_{\mathrm{R}}$,

(26) is induced by the multiplication of R,

(27) is $\sigma_{(t+1)\mathrm{V}}(\widetilde{\mathbf{u}}_{(t+1)\mathrm{V}}) \wedge \theta^{\mathrm{R}}_{\mathscr{F},\kappa} \wedge 1_{\mathrm{R}}$,

(28) is $\sigma_{\mathrm{V}}(\widetilde{\mathbf{u}}_{\mathrm{V}}) \wedge \sigma_{t\mathrm{V}}(\widetilde{\mathbf{u}}_{t\mathrm{V}}) \wedge \theta^{\mathrm{R}}_{\mathscr{F},\kappa} \wedge 1_{\mathrm{R}}$,

(29) is $\sigma_{n\mathrm{V}}(1_{\mathrm{R}} \wedge \sigma_{t\mathrm{V}}(\widetilde{\mathbf{u}}_{t\mathrm{V}}) \wedge \theta^{\mathrm{R}}_{\mathscr{F},\kappa} \wedge 1_{\mathrm{R}})$,

(30) is $\sigma_{n\mathrm{V}}(\sigma_{t\mathrm{V}}(\widetilde{\mathbf{u}}_{t\mathrm{V}}) \wedge \theta^{\mathrm{R}}_{\mathscr{F},\kappa} \wedge 1_{\mathrm{R}})$,

(31) is induced by the multiplication of R,

(32) is induced by the multiplication of R.

We will now argue that (4.20) commutes up to homotopy by showing that squares **(S1)** through **(S16)** commutes up to homotopy.

The square **(S1)** commutes because $\mathsf{x}_{t+1} \frown \widetilde{\mathbf{e}}_{\mathrm{V}} = \mathsf{x}_t$. The square **(S2)** commutes as the definition of the map $\partial^{\mathscr{F}}_{\kappa^*\mathsf{n}}$ involves the diagonal map of $\mathrm{B}\mathscr{F}$. The square **(S3)** commutes because the map

$$\delta : \mathrm{B}\mathscr{F}_+ \wedge \mathrm{S}^{\mathrm{V}} \longrightarrow \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{\mathrm{V}}) \simeq \Sigma^{\mathrm{V}}\, \mathsf{Th}(\widetilde{\gamma}_{\mathrm{V}})$$

is equivalent to $\sigma_{\mathrm{V}}(\zeta)$, where ζ is the zero section of $\widetilde{\gamma}_{\mathrm{V}}$, and the fact that $\widetilde{\mathbf{e}}_{\mathrm{V}} = \zeta^*(\widetilde{\mathbf{u}}_{\mathrm{V}})$.

The squares **(S5)**, **(S8)**, **(9)**, **(10)** commutes from the naturality of $\partial^{\mathscr{F}}_{\kappa^*\mathsf{n}}$. The square **(S11)** commutes because $\widetilde{\mathbf{u}}_{(t+1)\mathrm{V}} = \omega^*(\widetilde{\mathbf{u}}_{\mathrm{V}} \times \widetilde{\mathbf{u}}_{t\mathrm{V}})$ (see (5.1)) as

(4.19)

$$\begin{tikzcd}
\mathsf{Th}(\widetilde{\gamma}^{\mathscr{F}}_{(t+1)\mathrm{V}}) \arrow[rr, "\widetilde{\mathbf{u}}_{(t+1)\mathrm{V}}"] \arrow[d, "\omega"'] & & \Sigma^{(t+1)(n-1)\mathrm{V}}\mathrm{R} \arrow[d, equal] \\
\mathsf{Th}(\widetilde{\gamma}^{\mathscr{F}}_{\mathrm{V}}) \wedge \mathsf{Th}(\widetilde{\gamma}^{\mathscr{F}}_{t\mathrm{V}}) \arrow[r, "\widetilde{\mathbf{u}}_{\mathrm{V}} \wedge \widetilde{\mathbf{u}}_{t\mathrm{V}}"] \arrow[rr, bend right, "\widetilde{\mathbf{u}}_{\mathrm{V}} \times \widetilde{\mathbf{u}}_{t\mathrm{V}}"'] & \Sigma^{(n-1)\mathrm{V}}\mathrm{R} \wedge \Sigma^{t(n-1)\mathrm{V}}\mathrm{R} \arrow[r, "\mu_{\mathrm{R}}"] & \Sigma^{(t+1)(n-1)\mathrm{V}}\mathrm{R},
\end{tikzcd}$$

where ω is the natural map from the Thom space of the Whitney sum to Thom space of the product of two vector bundles. The remaining squares, i.e. **(S4)**, **(S4)**, **(S6)**, **(S7)**, **(S12)**, **(S13)**, **(S14)**, **(S15)**, **(S16)** commutes trivially.

(4.20)

$$\begin{tikzcd}
\mathrm{S}^{|x_{t+1}|} \wedge \Sigma^{\mathrm{V}}\mathrm{X} \arrow[d, "(2)"'] \arrow[rrrd, bend left, "(1)"] \arrow[rrrd, phantom, "\textbf{(S1)}"] & & & \\
\mathrm{B}\mathscr{F}_+ \wedge \Sigma^{\mathrm{V}}\mathrm{X} \wedge \mathrm{R} \arrow[r, "(3)"] \arrow[d, "(6)"'] \arrow[dr, phantom, "\textbf{(S2)}"] & \mathrm{B}\mathscr{F}_+ \wedge \mathrm{B}\mathscr{F}_+ \wedge \Sigma^{\mathrm{V}}\mathrm{X} \wedge \mathrm{R} \arrow[r, "(4)"] \arrow[d, "(7)"'] \arrow[dr, phantom, "\textbf{(S3)}"] & \mathrm{B}\mathscr{F}_+ \wedge \Sigma^{(n-1)\mathrm{V}}\mathrm{R} \wedge \Sigma^{\mathrm{V}}\mathrm{X} \wedge \mathrm{R} \arrow[r, "(5)"] \arrow[d, "(8)"'] \arrow[dr, phantom, "\textbf{(S4)}"] & \Sigma^{n\mathrm{V}}\mathrm{B}\mathscr{F}_+ \wedge \mathrm{X} \wedge \mathrm{R} \arrow[d, "(9)"] \\
\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\Sigma^{\mathrm{V}}\mathrm{X}) \wedge \mathrm{R} \arrow[r, "(10)"'] \arrow[d, "(13)"'] \arrow[dr, phantom, "\textbf{(S5)}"] & \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{\mathrm{V}}) \wedge \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{X}) \wedge \mathrm{R} \arrow[r, "(11)"'] \arrow[d, "(14)"'] \arrow[dr, phantom, "\textbf{(S6)}"] & \Sigma^{n\mathrm{V}}\mathrm{R} \wedge \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{X}) \wedge \mathrm{R} \arrow[r, "(12)"'] \arrow[d, "(15)"'] \arrow[dr, phantom, "\textbf{(S7)}"] & \Sigma^{n\mathrm{V}}\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{X}) \wedge \mathrm{R} \arrow[d, "(16)"] \\
\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\Sigma^{(t+1)\mathrm{V}}\mathrm{R}) \wedge \mathrm{R} \arrow[r, "(17)"'] \arrow[d, "(20)"'] \arrow[dr, phantom, "\textbf{(S8)}"] & \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{\mathrm{V}}) \wedge \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\Sigma^{t\mathrm{V}}\mathrm{R}) \wedge \mathrm{R} \arrow[r, "(18)"'] \arrow[d, "(21)"'] \arrow[dr, phantom, "\textbf{(S9)}"] & \Sigma^{n\mathrm{V}}\mathrm{R} \wedge \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\Sigma^{t\mathrm{V}}\mathrm{R}) \wedge \mathrm{R} \arrow[r, "(19)"'] \arrow[d, "(22)"'] \arrow[dr, phantom, "\textbf{(S10)}"] & \Sigma^{n\mathrm{V}}\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\Sigma^{t\mathrm{V}}\mathrm{R}) \wedge \mathrm{R} \arrow[d, "(23)"] \\
\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{(t+1)\mathrm{V}}) \wedge \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{R}) \wedge \mathrm{R} \arrow[r, "(24)"'] \arrow[d, "(27)"'] \arrow[dr, phantom, "\textbf{(S11)}"] & \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{\mathrm{V}}) \wedge \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{t\mathrm{V}}) \wedge \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{R}) \wedge \mathrm{R} \arrow[r, "(25)"'] \arrow[d, "(28)"'] \arrow[dr, phantom, "\textbf{(S12)}"] & \Sigma^{n\mathrm{V}}\mathrm{R} \wedge \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{t\mathrm{V}}) \wedge \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{R}) \wedge \mathrm{R} \arrow[r, "(26)"'] \arrow[d, "(29)"'] \arrow[dr, phantom, "\textbf{(S13)}"] & \Sigma^{n\mathrm{V}}\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{S}^{t\mathrm{V}}) \wedge \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{R}) \wedge \mathrm{R} \arrow[d, "(30)"] \\
\Sigma^{n(t+1)\mathrm{V}}\mathrm{R} \wedge \mathrm{R} \wedge \mathrm{R} \arrow[d, "(34)"'] \arrow[dr, phantom, "\textbf{(S14)}"] & \Sigma^{n\mathrm{V}}\mathrm{R} \wedge \Sigma^{nt\mathrm{V}}\mathrm{R} \wedge \mathrm{R} \wedge \mathrm{R} \arrow[l, "(31)"] \arrow[r, equal] \arrow[d, "(34)"'] \arrow[dr, phantom, "\textbf{(S15)}"] & \Sigma^{n\mathrm{V}}\mathrm{R} \wedge \Sigma^{nt\mathrm{V}}\mathrm{R} \wedge \mathrm{R} \wedge \mathrm{R} \arrow[r, "(32)"'] \arrow[d, "(35)"'] \arrow[dr, phantom, "\textbf{(S16)}"] & \Sigma^{n\mathrm{V}}(\Sigma^{nt\mathrm{V}}\mathrm{R} \wedge \mathrm{R} \wedge \mathrm{R}) \arrow[d, "(36)"] \\
\Sigma^{n(t+1)\mathrm{V}}\mathrm{R} \arrow[r, equal] & \Sigma^{n(t+1)\mathrm{V}}\mathrm{R} \arrow[r, equal] & \Sigma^{n(t+1)\mathrm{V}}\mathrm{R} \arrow[r, equal] & \Sigma^{n(t+1)\mathrm{V}}\mathrm{R}
\end{tikzcd}$$

Therefore, the class represented by the composition of blue arrows in (4.20)

$$[(34)\circ(27)\circ(20)\circ(13)\circ(6)\circ(2)] = \mathfrak{S}^{\chi}(\sigma_{\mathrm{V}}(x)) \in \mathrm{R}_{\mathrm{G}}^{t\mathrm{V}+\|\chi\|}(\mathrm{X})$$

must equal the class represented by the composition of red arrows

$$[(36)\circ(30)\circ(23)\circ(16)\circ(9)\circ(1)] = \sigma_{\mathrm{V}}(\mathfrak{S}^{\chi}(x)) \in \mathrm{R}_{\mathrm{G}}^{t\mathrm{V}+\|\chi\|}(\mathrm{X}),$$

and hence, the result. □

Remark 4.21. Observe that if χ is a V-stable R-Eulerian sequence then its k**-th shift**

$$\chi[k] = (\overbrace{0,\dots,0}^{k\text{-fold}}, \mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_2, \dots)$$

is also a V-stable R-Eulerian sequence. Note that

$$\|\chi[k]\| = \|\chi\| + k(n-1)\mathrm{V}$$

if χ has weight n.

Example 4.22. In the classical case $\mathrm{H}_*((\mathrm{B}\Sigma_2)_+;\mathbb{F}_2) = \mathbb{F}_2\{\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \dots\}$, where $\mathbf{b}_i$ is the element dual to $\mathbf{e}_1^i$ as in (4.3). Then

$$\beta = (\mathbf{b}_0, \mathbf{b}_1, \dots)$$

and its shifts accounts for all $\mathrm{H}\mathbb{F}_2$-stable Eulerian sequences of weight 2. It follows form (4.3) that $\mathfrak{S}^{\beta[k]} = \mathrm{Sq}^k$ for all $k \in \mathbb{N}$.

Example 4.23. When $\mathrm{G} = \mathrm{C}_2$, let

$$\mathbf{b}_{i\rho}, \mathbf{c}_{i\rho+\sigma} \in \mathrm{H}_{\star}^{\mathrm{C}_2}((\mathrm{B}_{\mathrm{C}_2}\Sigma_2)_+;\underline{\mathbb{F}}_2)$$

be the generators discussed in (4.5). Then

$$\beta = (\mathbf{b}_0, \mathbf{b}_\rho, \mathbf{b}_{2\rho}, \dots)$$
$$\zeta = (0, \mathbf{c}_\sigma, \mathbf{c}_{\rho+\sigma}, \dots)$$

and their shifts are ρ-stable Eulerian sequence of weight 2 such that

$$\mathfrak{S}^{\beta[k]} = \mathrm{Sq}^{2k} \quad \text{and} \quad \mathfrak{S}^{\zeta[k]} = \mathrm{Sq}^{2k+1}$$

for all $k \in \mathbb{N}$.

Remark 4.24. Given an V-stable R-Eulerian sequence $\chi = (\mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_2, \dots)$ of weight n, we get a kV-stable R-Eulerian sequence

$$\mathrm{t}_k(\chi) = (\mathrm{x}_0, \mathrm{x}_k, \mathrm{x}_{2k}, \dots)$$

of weight n. This is because

- $\widetilde{\mathbf{e}}_{k\mathrm{V}} = \underbrace{\widetilde{\mathbf{e}}_{\mathrm{V}} \smile \cdots \smile \widetilde{\mathbf{e}}_{\mathrm{V}}}_{k}$ is an R-Euler class of $\widetilde{\gamma}_{k\mathrm{V}}^{\mathscr{F}} = (\widetilde{\gamma}_{\mathrm{V}}^{\mathscr{F}})^{\oplus k}$, and,
- $\mathrm{x}_{(i+1)k} \frown \widetilde{\mathbf{e}}_{k\mathrm{V}} = \mathrm{x}_{ik}$.

It follows immediately from (4.18) that $\mathfrak{S}^{\chi} = \mathfrak{S}^{\mathrm{t}_k\chi}$.

4.2. **Restrictions and geometric fixed points of Eulerian sequences**

Suppose $\mathrm{K} \subset \mathrm{G}$. The cap product (as in (4.6)) is well-behaved with restrictions and fixed points:

- $\iota_{\mathrm{K}*}(x \frown e) = \iota_{\mathrm{K}*}(x) \frown \iota_{\mathrm{K}*}(e)$
- $\varphi^{\mathrm{K}}(x \frown e) = \varphi^{\mathrm{K}}(x) \frown \varphi^{\mathrm{K}}(e)$.

Definition 4.25. Suppose $\chi = (\mathsf{x}_0, \mathsf{x}_1, \mathsf{x}_2, \dots)$, where $\mathsf{x}_i \in \mathrm{R}^{\mathrm{G}}_{\star}(\mathrm{B}\mathscr{F}_+)$, is a V-stable R-Eulerian sequence. Then $\iota_{\mathrm{K}*}\mathsf{x}_i \in \mathrm{R}^{\mathrm{G}}_{\star}(\mathrm{B}\iota_{\mathrm{K}}\mathscr{F}_+)$, and

$$\iota_{\mathrm{K}}(\chi) := (\iota_{\mathrm{K}*}\mathsf{x}_0, \iota_{\mathrm{K}*}\mathsf{x}_1, \iota_{\mathrm{K}*}\mathsf{x}_2, \dots)$$

is a ι_{K}V-stable ι_{K}R-Eulerian sequence, which we will call the **restriction of χ to the subgroup** K.

Defining the geometric fixed points of an Eulerian sequence is somewhat subtle as the natural map

$$\widehat{\lambda} : \mathrm{B}(\mathscr{F}^{\mathrm{K}})_+ \longrightarrow (\mathrm{B}\mathscr{F})^{\mathrm{K}}_+$$

is not generally an equivalence. However, $\mathrm{B}(\mathscr{F}^{\mathrm{K}})$ is a path component of $\mathrm{B}\mathscr{F}^{\mathrm{K}}$ [LM86, Theorem 10] (also see [BZ24, Theorem 2.13]). Therefore, there exists a natural collapse map

$$\mathsf{c} : (\mathrm{B}\mathscr{F})^{\mathrm{K}}_+ \longrightarrow \mathrm{B}(\mathscr{F}^{\mathrm{K}})_+ \tag{4.26}$$

which is identity on $\mathrm{B}(\mathscr{F}^{\mathrm{K}})$ and maps the elements from other components to the disjoint basepoint. Moreover, an argument similar to [BZ24, Lemma 2.18] reveals that the pullback of the the K-fixed point of the bundle $\widetilde{\gamma}^{\mathscr{F}}_{\mathrm{V}}$ along $\widehat{\lambda}$ is isomorphic to $\widetilde{\gamma}^{\mathscr{F}^{\mathrm{K}}}_{\mathrm{V}^{\mathrm{K}}}$:

$$\widehat{\lambda}^* \left((\widetilde{\gamma}^{\mathscr{F}}_{\mathrm{V}})^{\mathrm{K}}\right) \cong \widetilde{\gamma}^{\mathscr{F}^{\mathrm{K}}}_{\mathrm{V}^{\mathrm{K}}}$$

Consequently, $\widehat{\lambda}^*(\widetilde{\mathbf{e}}_{\mathrm{V}})$ (where $\widetilde{\mathbf{e}}_{\mathrm{V}}$ as defined in (4.10)) is an $\Phi^{\mathrm{K}}(\mathrm{R})$-Euler class of $\widetilde{\gamma}^{\mathscr{F}^{\mathrm{K}}}_{\mathrm{V}}$. This motivates the following definition:

Definition 4.27. Suppose $\chi = (\mathsf{x}_0, \mathsf{x}_1, \mathsf{x}_2, \dots)$, where $\mathsf{x}_i \in \mathrm{R}^{\mathrm{G}}_{\star}(\mathrm{B}\mathscr{F}_+)$, is a V-stable R-Eulerian sequence. Then $\mathsf{c}_*\varphi^{\mathrm{K}}(\mathsf{x}_i) \in \mathrm{R}^{\mathrm{W(K)}}_{\star}(\mathrm{B}\mathscr{F}^{\mathrm{K}}_+)$ and

$$\varphi^{\mathrm{K}}(\chi) := (\mathsf{c}_*\varphi^{\mathrm{K}}(\mathsf{x}_0), \mathsf{c}_*\varphi^{\mathrm{K}}(\mathsf{x}_1), \mathsf{c}_*\varphi^{\mathrm{K}}(\mathsf{x}_2), \dots)$$

is a V^{K}-stable $\Phi^{\mathrm{K}}(\mathrm{R})$-Eulerian sequence, which we will call the **K-geometric fixed points of χ**.

The fact that $\varphi^{\mathrm{K}}(\chi)$ in (4.27) is a V^{K}-stable $\Phi^{\mathrm{K}}(\mathrm{R})$-Eulerian sequence follows from:

Lemma 4.28. *Suppose $x \in \mathrm{R}^{\mathrm{G}}_{\star}(\mathrm{B}\mathscr{F}_+)$ and $e \in \mathrm{R}^{\star}_{\mathrm{G}}(\mathrm{B}\mathscr{F}_+)$, then*

$$\mathsf{c}_*\varphi^{\mathrm{K}}(x) \frown \widehat{\lambda}^*(e) = \mathsf{c}_*\varphi^{\mathrm{K}}(x \frown e).$$

Proof. The result follows from the commutative diagram:

$$\begin{tikzcd}
S^{V^K} \arrow[d, "c_*\varphi^K(x)"'] \arrow[dr, bend left, "\varphi^K(x)"] & \\
B(\mathscr{F}^K)_+ \wedge \Phi^K(R) \arrow[r, "\widehat{\lambda}\wedge 1"] \arrow[d, "\Delta\wedge 1"'] & (B\mathscr{F}^K)_+ \wedge \Phi^K(R) \arrow[d, "\Delta\wedge 1"] \\
B(\mathscr{F}^K)_+ \wedge B(\mathscr{F}^K)_+ \wedge \Phi^K(R) \arrow[r, "\widehat{\lambda}\wedge\widehat{\lambda}\wedge 1"] \arrow[d, "1\wedge\widehat{\lambda}^*(e)\wedge 1"'] & (B\mathscr{F}^K)_+ \wedge (B\mathscr{F}^K)_+ \wedge \Phi^K(R) \arrow[d, "1\wedge e\wedge 1"] \\
B(\mathscr{F}^K)_+ \wedge \Sigma^{W^K}\Phi^K(R) \wedge \Phi^K(R) \arrow[r, "\widehat{\lambda}\wedge 1\wedge 1"'] \arrow[dr, bend right, "1\wedge\mu_R"'] & (B\mathscr{F}^K)_+ \wedge \Phi^K(\Sigma^W R) \wedge \Phi^K(R) \arrow[d, "c\wedge\mu_R"] \\
 & B(\mathscr{F}^K)_+ \wedge \Sigma^{W^K}\Phi^K(R)
\end{tikzcd}$$

where the composition of the red arrows is the left hand side and the composition of the blue arrows is the right hand side. □

Theorem 4.29. *Suppose* $\chi = (x_0, x_1, \dots)$ *in an* V*-stable* R*-Eulerian sequence. Then*

(1) $\iota_{K*}\mathfrak{S}^{\chi}(-) = \mathfrak{S}^{\iota_K(\chi)}(\iota_{K*}(-))$

(2) $\varphi^K(\mathfrak{S}^{\chi}(-)) = \mathfrak{S}^{\varphi^K(\chi)}(\varphi^K(-))$

for all RO(G, V)*-graded cohomology classes.*

Proof. By (4.18), it is enough to show that *(1)* and *(2)* hold for any R-cohomology class $x \in R^{tV}_G(X)$, where $t \in \mathbb{N}$.

From (3.8)*(1)* and the definition of slant product (see (4.1))

$$\begin{aligned}
\iota_{K*}\mathfrak{S}^{\chi}(x) &= \iota_{K*}(\delta^*\mathcal{P}^{\mathscr{F}}_{tV}(x)|_{x_t}) \\
&= \delta^*\iota_{K*}(\mathcal{P}^{\mathscr{F}}_{tV}(x))|_{\iota_{K*}(x_t)} \\
&= \delta^*\mathcal{P}^{\iota_K\mathscr{F}}_{t\iota_{K*}V}(\iota_K(x))|_{\iota_{K*}(x_t)} \\
&= \mathfrak{S}^{\iota_K(\chi)}(\iota_{K*}(x)).
\end{aligned}$$

To prove *(2)*, we first observe that there is a commutative diagram

$$\begin{tikzcd}
B(\mathscr{F}^K)_+ \wedge \Phi^K(X) \arrow[r, "\delta"] \arrow[d, "\widehat{\lambda}\wedge 1"'] & D^{\mathscr{F}^K}_{\kappa^*n}(\Phi^K(X)) \arrow[d, "\widehat{\lambda}"] \\
\Phi^K(B\mathscr{F}_+ \wedge X) \cong (B\mathscr{F})^K_+ \wedge \Phi^K(X) \arrow[r, "\Phi^K(\delta)"'] & \Phi^K(D^{\mathscr{F}}_{\kappa^*n}(X)).
\end{tikzcd}$$

Combining this with (3.8)*(2)* and the fact that $\mathsf{c} \circ \widehat{\lambda}$ is identity, we get

$$\begin{aligned} \mathfrak{S}^{\varphi^{\mathrm{K}}(\chi)}(\varphi^{\mathrm{K}}(x)) &= \delta^* \mathcal{P}^{\mathscr{F}^{\mathrm{K}}}_{t\mathrm{V}^{\mathrm{K}}}(\varphi^{\mathrm{K}}(x))|_{\mathsf{c}_*\varphi^{\mathrm{K}}(\mathsf{x}_t)} \\ &= \delta^* \widehat{\lambda}^* \varphi^{\mathrm{K}}(\mathcal{P}^{\mathscr{F}}_{t\mathrm{V}}(x))|_{\mathsf{c}_*\varphi^{\mathrm{K}}(\mathsf{x}_t)} \\ &= (\widehat{\lambda} \wedge 1)^* \Phi^{\mathrm{K}}(\delta)^* \varphi^{\mathrm{K}}(\mathcal{P}^{\mathscr{F}}_{t\mathrm{V}}(x))|_{\mathsf{c}_*\varphi^{\mathrm{K}}(\mathsf{x}_t)} \\ &= \varphi^{\mathrm{K}}\left(\delta^* \mathcal{P}^{\mathscr{F}}_{t\mathrm{V}}(x)|_{\mathsf{x}_t}\right) \\ &= \varphi^{\mathrm{K}}(\mathfrak{S}^{\chi}(\mathsf{x}_t)) \end{aligned}$$

as desired. □

4.3. Modified geometric fixed-point of $\mathrm{H}\underline{\mathbb{F}}_p$-Eulerian Sequence

In Section 7, we demonstrate that $\widetilde{\gamma}^{\mathcal{A}\ell\ell_n}_{\rho_{\mathrm{G}}}$ (as defined in (3.1)) is $\mathrm{H}\underline{\mathbb{F}}_p$-orientable when $|\mathrm{G}|$ is even. Furthermore, we show that $2\widetilde{\gamma}^{\mathcal{A}\ell\ell_n}_{\rho_{\mathrm{G}}}$ is always $\mathrm{H}\underline{\mathbb{F}}_p$-orientable for all finite groups G. We then proceed to identify the ρ_{G}-stable $\mathrm{H}\underline{\mathbb{F}}_p$-Eulerian sequence. The primary purpose of this subsection is to lay the foundation for analyzing the modified geometric fixed-points of the corresponding cohomology operations.

Recall that the modified geometric fixed-point map of (2.25) is the composition:

$$\widetilde{\varphi}^{\mathrm{K}}(-) := \pi_* \varphi^{\mathrm{K}}(-)$$

where

$$\pi : \Phi^{\mathrm{K}}(\mathrm{H}\underline{\mathbb{F}}_p) \longrightarrow \mathrm{H}\underline{\mathbb{F}}_p,$$

is the $\mathbb{E}_\infty^{\mathrm{W}(\mathrm{K})}$-ring map from the K-geometric fixed-point of the G-Eilenberg MacLane spectrum $\mathrm{H}\underline{\mathbb{F}}_p$ to the W(K)-Eilenberg MacLane spectrum $\mathrm{H}\underline{\mathbb{F}}_p$, induced by the zeroth Postnikov approximation.

Definition 4.30. For a $k\rho_{\mathrm{G}}$-stable $\mathrm{H}\underline{\mathbb{F}}_p$-Eulerian sequence $\chi = (\mathsf{x}_0, \mathsf{x}_1, \mathsf{x}_2, \dots)$ such that $\mathsf{x}_i \in \mathrm{H}^{\mathrm{G}}_\star(\mathrm{B}_{\mathrm{G}}\Sigma_n; \underline{\mathbb{F}}_p)$, we define its modified geometric fixed-point as

$$\widetilde{\varphi}^{\mathrm{K}}(\chi) := \pi_*\varphi^{\mathrm{K}}(\chi) = (\pi_*\mathsf{c}_*\varphi^{\mathrm{K}}(\mathsf{x}_0), \pi_*\mathsf{c}_*\varphi^{\mathrm{K}}(\mathsf{x}_1), \pi_*\mathsf{c}_*\varphi^{\mathrm{K}}(\mathsf{x}_2), \dots). \tag{4.31}$$

Remark 4.32. Since c_* and π_* commute, we may rewrite (4.31)

$$\widetilde{\varphi}^{\mathrm{K}}(\chi) = (\mathsf{c}_*\widetilde{\varphi}^{\mathrm{K}}(\mathsf{x}_0), \mathsf{c}_*\widetilde{\varphi}^{\mathrm{K}}(\mathsf{x}_1), \mathsf{c}_*\widetilde{\varphi}^{\mathrm{K}}(\mathsf{x}_2), \dots).$$

Note $\widetilde{\varphi}^{\mathrm{K}}$ sends the $\mathrm{H}\underline{\mathbb{F}}_p$-Euler class of $k\widetilde{\gamma}^{\mathcal{A}\ell\ell_n}_{\rho_{\mathrm{G}}}$, denote it by $\widetilde{\mathbf{e}}_{\mathrm{G},p}$, to $\widetilde{\mathbf{e}}_{\mathrm{W}(\mathrm{K}),p}$, the $\mathrm{H}\underline{\mathbb{F}}_p$-Euler class of $k\widetilde{\gamma}^{\mathcal{A}\ell\ell_n}_{\rho_{\mathrm{W}(\mathrm{K})}}$. This along with the fact that

$$\widetilde{\varphi}^{\mathrm{K}}(x \frown e) = \widetilde{\varphi}^{\mathrm{K}}(x) \frown \widetilde{\varphi}^{\mathrm{K}}(e)$$

implies that $\widetilde{\varphi}^{\mathrm{K}}(\chi)$ is a $k\rho_{\mathrm{W}(\mathrm{K})}$-stable $\mathrm{H}\underline{\mathbb{F}}_p$-Eulerian sequence.

Combining (4.29) with the fact that π is an $\mathbb{E}_\infty^{\mathrm{W}(\mathrm{K})}$-ring map, we conclude:

Theorem 4.33. *Any $k\rho_{\mathrm{G}}$-stable $\mathrm{H}\underline{\mathbb{F}}_p$-Eulerian sequence $\chi = (\mathsf{x}_1, \mathsf{x}_2, \dots)$ satisfies:*

$$\widetilde{\varphi}^{\mathrm{K}}(\mathfrak{S}^{\chi}(-)) = \mathfrak{S}^{\widetilde{\varphi}^{\mathrm{K}}(\chi)}(\widetilde{\varphi}(-)) \tag{4.34}$$

for any subgroup K *of* G.

Proof. Applying π_* to (4.29) *(2)*, we get

$$\begin{aligned}
\widetilde{\varphi}^{\mathrm{K}}(\mathfrak{S}^{\mathsf{X}}(-)) &= \pi_*\left(\varphi^{\mathrm{K}}(\mathfrak{S}^{\mathsf{X}}(-))\right)\\
&= \pi_*\left(\mathfrak{S}^{\varphi^{\mathrm{K}}(\mathsf{X})}(\varphi^{\mathrm{K}}(-))\right)\\
&= \pi_*\left(\delta^*\mathcal{P}^{\mathscr{F}^{\mathrm{K}}}_{t\mathrm{V}^{\mathrm{K}}}(\varphi^{\mathrm{K}}(-))|_{\mathsf{c}_*\varphi^{\mathrm{K}}(\mathsf{x}_t)}\right)\\
&= \delta^*\mathcal{P}^{\mathscr{F}^{\mathrm{K}}}_{t\mathrm{V}^{\mathrm{K}}}(\pi_*\varphi^{\mathrm{K}}(-))|_{\pi_*(\mathsf{c}_*\varphi^{\mathrm{K}}(\mathsf{x}_t))}\\
&= \delta^*\mathcal{P}^{\mathscr{F}^{\mathrm{K}}}_{t\mathrm{V}^{\mathrm{K}}}(\widetilde{\varphi}^{\mathrm{K}}(-))|_{\mathsf{c}_*\widetilde{\varphi}^{\mathrm{K}}(\mathsf{x}_t)}\\
&= \mathfrak{S}^{\widetilde{\varphi}^{\mathrm{K}}(\mathsf{X})}(\widetilde{\varphi}(-))
\end{aligned}$$

as desired. ☐

5. Generalized Cartan formula

The classical Cartan formula encodes the relationship between Steenrod operations and the external product. The standard formulation, however, relies on the Künneth isomorphism, which does not hold for most equivariant cohomology theories satisfying Assumption 2. In this section, we overcome this challenge by introducing the diagonal of an Eulerian sequence and formulating a generalization of the Cartan formula (see (5.8)).

Recall that any multiplicative R-cohomology theory admits a natural external product pairing:

$$(-)\times(-):\mathrm{R}^{\star}_{\mathrm{G}}(\mathrm{X})\times\mathrm{R}^{\star}_{\mathrm{G}}(\mathrm{Y})\longrightarrow\mathrm{R}^{\star}_{\mathrm{G}}(\mathrm{X}\wedge\mathrm{Y})$$

This pairing sends elements $x\in\mathrm{R}^{\star}_{\mathrm{G}}(\mathrm{X})$ and $y\in\mathrm{R}^{\star}_{\mathrm{G}}(\mathrm{Y})$ to the composite map

$$x\times y:\mathrm{X}\wedge\mathrm{Y}\xrightarrow{x\wedge y}\Sigma^{|x|}\mathrm{R}\wedge\Sigma^{|y|}\mathrm{R}\xrightarrow{\mu_{\mathrm{R}}}\Sigma^{|x|+|y|}\mathrm{R}. \tag{5.1}$$

The first step toward establishing the Cartan formula is to examine the following relationship between power operations and the external product:

Theorem 5.2. *Suppose* $\mathrm{R},\mathsf{\kappa},\mathscr{F}$ *satisfy the condition of Assumption 2. For elements* $x\in\mathrm{R}^{i_1\mathrm{V}}_{\mathrm{G}}(\mathrm{X})$ *and* $y\in\mathrm{R}^{i_2\mathrm{V}}_{\mathrm{G}}(\mathrm{Y})$ *with* $i_1,i_2\in\mathbb{N}$ *the formula*

$$\mathcal{P}^{\mathscr{F}}_{i\mathrm{V}}(x\times y)=(\partial^{\mathscr{F}}_{\mathsf{\kappa}^*\mathsf{n}})^*\left(\mathcal{P}^{\mathscr{F}}_{i_1\mathrm{V}}(x)\times\mathcal{P}^{\mathscr{F}}_{i_2\mathrm{V}}(y)\right), \tag{5.3}$$

where $i=i_1+i_2$*, relates power operations with external products.*

Proof. For simplicity, let τ^{R}_k denote the composite

$$\tau^{\mathrm{R}}_i:\mathrm{D}^{\mathscr{F}}_{\mathsf{\kappa}^*\mathsf{n}}(\Sigma^{i\mathrm{V}}\mathrm{R})\xrightarrow{\partial^{\mathscr{F}}_{\mathsf{\kappa}^*\mathsf{n}}}\mathrm{D}^{\mathscr{F}}_{\mathsf{\kappa}^*\mathsf{n}}(\mathrm{S}^{i\mathrm{V}})\wedge\mathrm{D}^{\mathscr{F}}_{\mathsf{\kappa}^*\mathsf{n}}(\mathrm{R})\xrightarrow{\mathbf{u}_{i\mathrm{V}}\wedge\theta^{\mathrm{R}}_n}\Sigma^{in\mathrm{V}}\mathrm{R}\wedge\mathrm{R}\xrightarrow{\mu_{\mathrm{R}}}\Sigma^{in\mathrm{V}}\mathrm{R},$$

where $\mathbf{u}_{i\mathrm{V}}$ be an R-Thom class of $\gamma^{\mathscr{F}}_{i\mathrm{V}}$ (as in (3.5)). These R-Thom classes are compatible as i varies, in the sense that the diagram

$$\begin{tikzcd}
\mathsf{Th}(\gamma^{\mathscr{F}}_{i\mathrm{V}}) \arrow[r, "\partial_{\mathsf{\kappa}^*\mathsf{n}}"] \arrow[rrd, bend right, "\mathbf{u}_{i\mathrm{V}}"'] & \mathsf{Th}(\gamma^{\mathscr{F}}_{i_1\mathrm{V}})\wedge\mathsf{Th}(\gamma^{\mathscr{F}}_{i_2\mathrm{V}}) \arrow[r, "\mathbf{u}_{i_1\mathrm{V}}\wedge\mathbf{u}_{i_2\mathrm{V}}"] & \Sigma^{i_1\mathrm{V}}\mathrm{R}\wedge\Sigma^{i_2\mathrm{V}}\mathrm{R} \arrow[d, "\mu_{\mathrm{R}}"] \\
 & & \Sigma^{i\mathrm{V}}\mathrm{R}
\end{tikzcd} \tag{5.4}$$

commutes whenever $i = i_1 + i_2$. Consequently, we have a commutative diagram

$$\begin{tikzcd}
D^{\mathscr{F}}_{\kappa^*\mathsf{n}}(X \wedge Y) \arrow[r, "\partial_{\kappa^*\mathsf{n}}"] \arrow[d, "D^{\mathscr{F}}_{\kappa^*\mathsf{n}}(x \wedge y)"'] & D^{\mathscr{F}}_{\kappa^*\mathsf{n}}(X) \wedge D^{\mathscr{F}}_{\kappa^*\mathsf{n}}(Y) \arrow[d, "D^{\mathscr{F}}_{\kappa^*\mathsf{n}}(x) \wedge D^{\mathscr{F}}_{\kappa^*\mathsf{n}}(y)"] \\
D^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\Sigma^{i_1 V}R \wedge \Sigma^{i_2 V}R) \arrow[r, "\partial_{\kappa^*\mathsf{n}}"'] \arrow[dd, "D^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mu_R)"'] & D^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\Sigma^{i_1 V}R) \wedge D^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\Sigma^{i_2 V}R) \ , \arrow[d, "\tau_{i_1} \wedge \tau_{i_2}"] \\
 & \Sigma^{i_1 nV}R \wedge \Sigma^{i_2 nV}R \arrow[d, "\mu_R"] \\
D^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\Sigma^{iV}R) \arrow[r, "\tau^R_i"'] & \Sigma^{inV}R
\end{tikzcd}$$

where the top square commutes because of naturality $\partial^{\mathscr{F}}_{\kappa^*\mathsf{n}}$ and bottom square commutes because of (5.4). In the above diagram, the composition of the blue arrows is the left hand side and composition of the red arrows is the right hand side of (5.3), hence the result. ☐

Now notice, there is a commutative diagram
(5.5)

$$\begin{tikzcd}
B\mathscr{F}_+ \wedge X_1 \wedge \ldots \wedge X_k \arrow[r, "\Delta_k \wedge 1_{X_1 \wedge \ldots \wedge X_k}"] \arrow[dd, "\delta"'] & \overbrace{B\mathscr{F}_+ \wedge \ldots \wedge B\mathscr{F}_+}^{k\text{-fold}} \wedge X_1 \wedge \ldots \wedge X_k \arrow[d, phantom, "\cong" rotate=90] \arrow[dd, bend left=60, "\delta^{(k)}"] \\
 & B\mathscr{F}_+ \wedge X_1 \wedge \ldots \wedge B\mathscr{F}_+ \wedge X_k \arrow[d, "\delta \wedge \ldots \wedge \delta"] \\
D^{\mathscr{F}}_{\kappa^*\mathsf{n}}(X_1 \wedge \ldots \wedge X_k) \arrow[r, "\partial^{\mathscr{F}}_{\kappa^*\mathsf{n}}"'] & D^{\mathscr{F}}_{\kappa^*\mathsf{n}}(X_1) \wedge \ldots \wedge D^{\mathscr{F}}_{\kappa^*\mathsf{n}}(X_k),
\end{tikzcd}$$

where

$$\Delta_k : B\mathscr{F} \longrightarrow \overbrace{B\mathscr{F} \times \cdots \times B\mathscr{F}}^{k\text{-fold}}$$

is the k-fold diagonal map.

Since the pullback of $\widetilde{\gamma}^{\mathscr{F}}_V \times \cdots \times \widetilde{\gamma}^{\mathscr{F}}_V$ along Δ_k is $k\widetilde{\gamma}^{\mathscr{F}}_V$, we make the following definition.

Definition 5.6. For a V-stable R-Eulerian sequence $\chi = (\mathsf{x}_0, \mathsf{x}_1, \mathsf{x}_2, \ldots)$, the sequence

$$\Delta_k(\chi) := (\Delta_{k*}\mathsf{x}_0, \Delta_{k*}\mathsf{x}_k, \Delta_{k*}\mathsf{x}_{2k}, \ldots)$$

is defined as the **diagonal of** χ.

This sequence exhibits Eulerian like properties with respect to the R-Euler class $\widetilde{\mathbf{e}}^{\times k}_V$ of the bundle $(\widetilde{\gamma}^{\mathscr{F}}_V)^{\times k}$. Indeed, the naturality of cap product implies:

$$\begin{aligned}
(\Delta_{k*}\mathsf{x}_{ik}) \frown \widetilde{\mathbf{e}}^{\times k}_V &= \Delta_{k*}\left(\mathsf{x}_{ik} \frown \Delta^*_k(\widetilde{\mathbf{e}}^{\times k}_V)\right) \\
&= \Delta_{k*}\left(\mathsf{x}_{ik} \frown \widetilde{\mathbf{e}}_{kV}\right) \\
&= \Delta_{k*}\mathsf{x}_{(i-1)k}.
\end{aligned}$$

However, it is not an Eulerian sequence according to (4.12), as $(\widetilde{\gamma}_{\mathrm{V}}^{\mathscr{F}})^{\times k}$ is not of the form (3.2). This is because the $\Sigma_n \times \cdots \times \Sigma_n$-representation $\widetilde{\tau}_n \times \cdots \times \widetilde{\tau}_n$ is not a pullback of $\widetilde{\tau}_d$ for any $d \in \mathbb{N}$.

We call a sequence $\widehat{\chi} = (\widehat{\mathsf{x}}_0, \widehat{\mathsf{x}}_1, \dots)$ a **pseudo R-Eulerian sequence** if:

- $\widehat{\mathsf{x}}_i \in \mathrm{R}^{\mathrm{G}}_{\star}(\overbrace{\mathrm{B}\mathscr{F}_+ \wedge \ldots \wedge \mathrm{B}\mathscr{F}_+}^{k\text{-fold}})$
- $\widehat{\mathsf{x}}_{i+1} \frown \widetilde{\mathbf{e}}_{\mathrm{V}}^{\times k} = \widehat{\mathsf{x}}_i$
- $\widehat{\mathsf{x}}_0 \frown \widetilde{\mathbf{e}}_{\mathrm{V}}^{\times k} = 0.$

Given a pseudo R-Eulerian sequence $\widehat{\chi} = (\widehat{\mathsf{x}}_0, \widehat{\mathsf{x}}_1, \dots)$, define the cohomology operation

$$\mathfrak{S}^{\widehat{\chi}} : \mathrm{R}_{\mathrm{G}}^{t_1\mathrm{V}}(\mathrm{X}_1) \times \cdots \times \mathrm{R}_{\mathrm{G}}^{t_k\mathrm{V}}(\mathrm{X}_k) \longrightarrow \mathrm{R}_{\mathrm{G}}^{t\mathrm{V}}(\mathrm{X}_1 \wedge \ldots \wedge \mathrm{X}_k).$$

by setting

$$\mathfrak{S}^{\widehat{\chi}}(x_1, \dots, x_n) = (\delta^{(k)})^* \left(\mathcal{P}^{\mathscr{F}}_{t_1\mathrm{V}}(x_1) \times \cdots \times \mathcal{P}^{\mathscr{F}}_{t_1\mathrm{V}}(x_k)\right) |_{\widehat{\mathsf{x}}_t}, \tag{5.7}$$

where $t = t_1 + \cdots + t_k,$. A diagram chase similar to (4.20) shows that above operation is stable, i.e.,

$$\mathfrak{S}^{\widehat{\chi}}\left(\sigma_{\mathrm{V}}(x_1), \dots, \sigma_{\mathrm{V}}(x_k)\right) = \sigma_{k\mathrm{V}}\left(\mathfrak{S}^{\widehat{\chi}}(x_1, \dots, x_k)\right)$$

and extends to $\mathrm{RO}(\mathrm{G}, \mathrm{V})$-graded cohomology classes.

For any R-Eulerian sequence $\chi = (\mathsf{x}_0, \mathsf{x}_1, \dots)$, $\Delta_k(\chi)$ is a pseudo R-Eulerian sequence and the corresponding stable R-cohmology operation satisfies the following relation:

Theorem 5.8 (**Generalized Cartan formula**). *For any* R*-Eulerian sequence* χ

$$\mathfrak{S}^{\chi}(x_1 \times \cdots \times x_k) = \mathfrak{S}^{\Delta_k(\chi)}(x_1, \dots, x_k),$$

where x_j*, for each* $1 \leq j \leq k$*, is an* $\mathrm{RO}(\mathrm{G}, \mathrm{V})$*-cohomology class.*

Proof. It is enough to establish the result when $x_i \in \mathrm{R}_{\mathrm{G}}^{t_i\mathrm{V}}(\mathrm{B}\mathscr{F}_+)$ for some $t_i \in \mathbb{N}$ for each $j \in \{1,\dots,k\}$. In the diagram

$$\begin{tikzcd}
\mathrm{S}^{|\mathsf{x}_{ik}|} \wedge \left(\bigwedge_{j=1}^k \mathrm{X}_j\right) \arrow[d, "\mathsf{x}_{\mathsf{i}k}"'] \arrow[dr, bend left, "\Delta_{k*}\mathsf{x}_{\mathsf{i}k}"] \arrow[dr, phantom, "\textbf{(I)}"] & \\
\mathrm{B}\mathscr{F}_+ \wedge \left(\bigwedge_{j=1}^k \mathrm{X}_j\right) \wedge \mathrm{R} \arrow[r, "\Delta_k \wedge 1"] \arrow[d, "\delta\wedge 1_{\mathrm{R}}"'] \arrow[dr, phantom, "\textbf{(II)}"] & \mathrm{B}\mathscr{F}_+^{\times k} \wedge \left(\bigwedge_{j=1}^k \mathrm{X}_j\right) \wedge \mathrm{R} \arrow[d, "\delta^{(k)}\wedge 1_{\mathrm{R}}"] \\
\mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}} \left(\bigwedge_{j=1}^k \mathrm{X}_j\right) \wedge \mathrm{R} \arrow[r, "\partial^{\mathscr{F}}_{\kappa^*\mathsf{n}}\wedge 1_{\mathrm{R}}"'] \arrow[d, "\mathcal{P}^{\mathscr{F}}_{t\mathrm{V}}(\mathsf{x}_1\times\cdots\times\mathsf{x}_k)\wedge 1_{\mathrm{R}}"'] & \left(\bigwedge_{j=1}^k \mathrm{D}^{\mathscr{F}}_{\kappa^*\mathsf{n}}(\mathrm{X}_j)\right) \wedge \mathrm{R} \arrow[d, "\left(\bigwedge \mathcal{P}^{\mathscr{F}}_{t_i\mathrm{V}}(x_j)\right)\wedge 1_{\mathrm{R}}"] \\
\Sigma^{t\mathrm{V}}\mathrm{R}\wedge\mathrm{R} \arrow[r, phantom, "\textbf{(III)}"] \arrow[dr, bend right, "\mu_{\mathrm{R}}"'] & \left(\bigwedge_{j=1}^k \Sigma^{t_j\mathrm{V}}\mathrm{R}\right)\wedge\mathrm{R} \arrow[d, "\mu_{\mathrm{R}}"] \\
& \Sigma^{t\mathrm{V}}\mathrm{R},
\end{tikzcd}$$

it is easy to see **(I)** and **(II)** commutes, and **(III)** commutes because of (5.2). Since the composition of blue arrows is the left hand side and the composition of the red arrows is the right hand side, the result follows. ☐

In the following remark, we show that (5.8) recovers the standard Cartan formula in the classical setting where the Künneth isomorphism holds.

Remark 5.9. Recall that $\mathrm{Sq}^k = \mathfrak{S}^{\chi[k]}$ where $\chi[k] = (\overbrace{0,\dots,0}^{k\text{-fold}}, \mathbf{b}_0, \mathbf{b}_1, \dots)$ and that

$$\Delta_2(\mathbf{b}_l) = \mathbf{b}_l \otimes \mathbf{b}_0 + \mathbf{b}_{l-1}\otimes\mathbf{b}_1 + \cdots + \mathbf{b}_0\otimes\mathbf{b}_l.$$

Suppose, $x \in \mathrm{H}^i(\mathrm{X};\mathbb{F}_2)$ and $y \in \mathrm{H}^j(\mathrm{Y};\mathbb{F}_2)$ such that $i+j$ is an even number greater than k. Set $l = i+j-k$. By (5.8)

$$\begin{aligned}
\mathrm{Sq}^k(x\otimes y) &= (\delta\wedge\delta)^*(\mathcal{P}_i(x)\otimes\mathcal{P}_j(y))|_{\Delta_2(\mathbf{b}_l)} \\
&= (\delta^*(\mathcal{P}_i(x))\otimes\delta^*(\mathcal{P}_j(x)))\,|_{\sum_{l'+l''=l}\mathbf{b}_{l'}\otimes\mathbf{b}_{l''}} \\
&= \sum_{l'+l''=l} \delta^*(\mathcal{P}_i(x))|_{\mathbf{b}_{l'}} \otimes \delta^*(\mathcal{P}_j(y))|_{\mathbf{b}_{l''}} \\
&= \sum_{l'+l''=l} \mathrm{Sq}^{i-l'}(x)\otimes\mathrm{Sq}^{j-l''}(y)
\end{aligned}$$

which is equivalent to the classical Cartan formula after implementing the unstable conditions.

6. Homotopy $\mathcal{N}_\infty$ rings and the composition law

The main goal of this section is to establish a product law for R-Eulerian sequences in a way that emulates the composition of their corresponding stable cohomology operations. We note that this product law does not require the strict commutativity among the structure maps of an $\mathcal{N}_\infty$-ring R; instead we need these diagrams to commute up to a homotopy. However, we require certain compatibility among the R-Euler classes of (4.10) across all $n \in \mathbb{N}$. In the classical case, this requirement is precisely the $\mathbb{H}^d_\infty$-ring structures introduced in [BMMS86]. We generalize this to equivariant settings to introduce shifted homotopy $\mathcal{N}_\infty$-rings before establishing the composition law. We also identify the shift degree of the homotopy $\mathbb{E}^{\mathrm{G}}_\infty$-ring structures for Eilenberg MacLane spectra.

Suppose $\mathcal{O}$ is an $\mathcal{N}_\infty$ G-operad. Let $\mathscr{F}_n$ denote the G-closed family of $(\mathrm{G} \times \Sigma_n)$ (see (2.1)) such that the n-th space $\mathcal{O}(n)$ is equivalent to the universal space $\mathrm{E}\mathscr{F}_n$. Since $\mathscr{F}_n$ is G-closed, the G-fixed point space of $\mathcal{O}(n)$ is contractible. Consequently, there is a contractible choice of G-equivariant maps

$$\iota_n : * \hookrightarrow \mathcal{O}(n). \tag{6.1}$$

This gives rise to a diagram of G-equivariant maps

$$\begin{tikzcd}
* \times (* \times \cdots \times *) \arrow[r, "\iota_n \times (\iota_{i_1} \times \cdots \times \iota_{i_k})"] \arrow[d, equal] & \mathcal{O}(n) \times \mathcal{O}(i_1) \times \cdots \times \mathcal{O}(i_n) \arrow[d] \\
* \arrow[r, "\iota_{i_1+\cdots+i_n}"'] & \mathcal{O}(i_1 + \cdots + i_n)
\end{tikzcd} \tag{6.2}$$

which commutes up to a G-equivariant homotopy for all $n, i_1, \ldots, i_n \in \mathbb{N}$.

Remark 6.3. When $n = 1$, we choose ι_1 such that it maps $*$ to the distinguished element $1 \in \mathcal{O}(1)$. Since, $\mathrm{G} \times \Sigma_1$ has exactly one G-closed family, ι_1 of (6.1) is an equivalence.

By combining ι_n of (6.1) with the structure map (2.9) of the operad $\mathcal{O}$, we obtain a map

$$\mathcal{O}(i_1) \times \cdots \times \mathcal{O}(i_n) \longrightarrow \mathcal{O}(i_1 + \cdots + i_n).$$

This gives rise to a natural G-equivariant map

$$\alpha_{i_1,\ldots,i_k} : \mathrm{D}^{\mathcal{O}}_{\mathrm{i}_1}(\mathrm{X}) \wedge \ldots \wedge \mathrm{D}^{\mathcal{O}}_{\mathrm{i}_n}(\mathrm{X}) \longrightarrow \mathrm{D}^{\mathcal{O}}_{i_1+\cdots+i_n}(\mathrm{X})$$

which satisfies the following property:

Lemma 6.4. *For all $i, j, k \in \mathbb{N}$,*

$$\alpha_{i+j,k} \circ (\alpha_{i,j} \wedge 1) \simeq \alpha_{i,j,k} \simeq \alpha_{i,j+k} \circ (1 \wedge \alpha_{j,k}).$$

Proof. To abbreviate notations, let $\mathcal{O}(i_1, \dots, i_n) := \mathcal{O}(i_1) \times \cdots \times \mathcal{O}(i_n)$. Consider the commutative diagram:

$$\begin{tikzcd}
* \times (* \times *) \times \mathcal{O}(i,j,k) \arrow[r, equal] \arrow[d] & * \times \mathcal{O}(i,j,k) \arrow[r, equal] \arrow[d] & * \times (* \times *) \times \mathcal{O}(i,j,k) \arrow[d] \\
\mathcal{O}(2) \times \mathcal{O}(2,1) \times \mathcal{O}(i,j,k) \arrow[r] \arrow[d] & \mathcal{O}(3) \times \mathcal{O}(i,j,k) \arrow[d] & \mathcal{O}(2) \times \mathcal{O}(1,2) \times \mathcal{O}(i,j,k) \arrow[l] \arrow[d] \\
\mathcal{O}(2) \times \mathcal{O}(i+j,k) \arrow[r] & \mathcal{O}(i+j+k) & \mathcal{O}(2) \times \mathcal{O}(i+j,k) \arrow[l]
\end{tikzcd}$$

In this diagram, the lower squares commute by the usual compatibility of the evident structure maps of $\mathcal{O}$. The upper squares commute up to homotopy due to (6.2). The result follows immediately from the homotopy commutativity of the entire diagram above. ☐

If R is an $\mathcal{O}$-ring, one can readily verify the existence of the following diagrams, which commute in the homotopy category $\mathrm{Ho}(\mathcal{S}\mathrm{p}_\mathrm{G})$ for all $i, j \in \mathbb{N}$:

$$\begin{tikzcd}
\mathrm{D}^{\mathcal{O}}_{\mathsf{i}}(\mathrm{R}) \wedge \mathrm{D}^{\mathcal{O}}_{\mathsf{j}}(\mathrm{R}) \arrow[rr, "\alpha_{\mathsf{i},\mathsf{j}}"] \arrow[d] & & \mathrm{D}^{\mathcal{O}}_{\mathsf{i}+\mathsf{j}}(\mathrm{R}) \arrow[d] \\
\mathrm{R} \wedge \mathrm{R} \arrow[r] & \mathrm{D}^{\mathcal{O}}_{2}(\mathrm{R}) \arrow[r] & \mathrm{R}
\end{tikzcd} \tag{6.5}$$

$$\begin{tikzcd}
\mathrm{D}^{\mathcal{O}}_{\mathsf{i}}\mathrm{D}^{\mathcal{O}}_{\mathsf{j}}(\mathrm{R}) \arrow[r, "\beta_{\mathsf{i},\mathsf{j}}"] \arrow[d, "\mathrm{D}^{\mathcal{O}}_{\mathsf{i}}(\theta^{\mathrm{R}}_{\mathsf{j}})"'] & \mathrm{D}^{\mathcal{O}}_{\mathsf{ij}}(\mathrm{R}) \arrow[d, "\theta^{\mathrm{R}}_{\mathsf{ij}}"] \\
\mathrm{D}^{\mathcal{O}}_{\mathsf{i}}(\mathrm{R}) \arrow[r, "\theta^{\mathrm{R}}_{\mathsf{i}}"] & \mathrm{R}
\end{tikzcd} \tag{6.6}$$

where the map $\beta_{i,j}$ is induced by the inclusion map $\Sigma_j \wr \Sigma_i := \Sigma_j^{\times i} \rtimes \Sigma_i \hookrightarrow \Sigma_{ij}$.

Remark 6.7. As a consequence of (6.4), the diagram of (6.5) implies the homotopy commutativity of the general diagram

$$\begin{tikzcd}
\mathrm{D}^{\mathcal{O}}_{\mathsf{i}_1}(\mathrm{R}) \wedge \dots \wedge \mathrm{D}^{\mathcal{O}}_{\mathsf{i}_n}(\mathrm{R}) \arrow[rr, "\alpha_{\mathsf{i}_1,\dots,\mathsf{i}_n}"] \arrow[d, "\theta^{\mathrm{R}}_{\mathsf{i}_1} \wedge \dots \wedge \theta^{\mathrm{R}}_{\mathsf{i}_n}"'] & & \mathrm{D}^{\mathcal{O}}_{\mathsf{i}_1+\cdots+\mathsf{i}_\mathsf{n}}(\mathrm{R}) \arrow[d, "\theta^{\mathrm{R}}_{\mathsf{i}_1+\cdots+\mathsf{i}_\mathsf{n}}"] \\
\mathrm{R}^{\wedge n} \arrow[r, "\alpha_{1,\dots,1}"'] & \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{R}) \arrow[r, "\theta^{\mathrm{R}}_{n}"'] & \mathrm{R}.
\end{tikzcd}$$

The details are left to the reader to verify.

However, the commutativity of diagrams (6.5) and (6.6) does not guarantee an $\mathcal{O}$-ring structure on R. For example, in the nonequivariant case [Noe14] gave an example of an $\mathbb{H}_\infty$ rings which is not $\mathbb{E}_\infty$.

Definition 6.8. Let $\mathcal{O}$ be an $\mathcal{N}_\infty$-operad. An $\mathrm{R} \in \mathcal{S}\mathrm{p}_\mathrm{G}$ is a **homotopy $\mathcal{O}$-ring** (or simply an $\mathcal{O}^\mathsf{h}$-ring) if for all $n \in \mathbb{N}$, there exists a map

$$\theta^{\mathrm{R}}_n : \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{R}) \longrightarrow \mathrm{R}$$

such that

(1) $\theta_1^{R} = 1_{R}$, and

(2) the diagrams in (6.5) and (6.6) commutes for all $i, j \in \mathbb{N}$

in $\mathrm{Ho}(\mathcal{S}\mathrm{p}_{G})$.

Remark 6.9. When G is the trivial group and $\mathcal{O}$ is an $\mathbb{E}_\infty$-operad then (6.8) recovers the classical definition of $\mathbb{H}_\infty$ ring spectra [BMMS86, I.3.1].

Remark 6.10. It is easy to check that the unit map $e = \theta_0^{R} : \mathbb{S} \to R$ along with the composition

$$\mu^{R} : R \wedge R \xrightarrow{\alpha_{1,1}} D_2^{\mathcal{O}}(R) \xrightarrow{\theta_2^{R}} R$$

gives R the structure of a homotopy commutative G-ring, i.e. the diagrams

$$\begin{tikzcd}
\mathbb{S} \wedge R \arrow[r, "e\wedge 1"] \arrow[dr, equal] & R \wedge R \arrow[d, "\mu^{R}"] & R \wedge \mathbb{S} \arrow[l, "1\wedge e"'] \arrow[dl, equal] \\
 & R &
\end{tikzcd}
\qquad
\begin{tikzcd}
R \wedge R \wedge R \arrow[r, "1\wedge\mu^{R}"] \arrow[d, "\mu^{R}\wedge 1"'] & R \wedge R \arrow[d, "\mu^{R}"] \\
R \wedge R \arrow[r, "\mu^{R}"'] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \wedge R \arrow[dr, "\mu^{R}"] \arrow[dd, "\tau"] & \\
 & R \\
R \wedge R \arrow[ur, "\mu^{R}"'] &
\end{tikzcd}$$

in which τ is the shuffle map, commutes up to G-equivariant homotopies.

6.1. Shifted $\mathcal{N}_\infty$-rings

The definition of $\mathscr{F}$-th power operation (as in (2.12)) applies to any $\mathcal{O}^{\mathsf{h}}$-ring R. Consequently, the results of Section 2.2 and Section 2.3 generalize directly to $\mathcal{O}^{\mathsf{h}}$-ring spectra. To extend the notion of shifted power operations (as in Section 3) to a broader class of equivariant ring spectra, we introduce the equivariant analog of $\mathbb{H}_\infty^{\mathsf{d}}$-rings [BMMS86, I.4.1]:

Definition 6.11. Suppose $\mathcal{O}$ is an $\mathcal{N}_\infty$ G-operad and V is a finite G-representation. We say that R is a **V-shifted homotopy $\mathcal{O}$-ring** (or simply an $\mathcal{O}^{\mathsf{h}}_{[V]}$-ring) if there are maps

$$\theta_{i,t}^{R} : D_i^{\mathcal{O}}(\Sigma^{tV} R) \longrightarrow \Sigma^{itV} R \tag{6.12}$$

for each $i, t \in \mathbb{N}$, such that $\theta_{1,t}^{R}$ is equivalent to the identity map $1_{\Sigma^{tV}R}$, and the diagrams

$$\begin{tikzcd}
D_i^{\mathcal{O}}(\Sigma^{tV}R) \wedge D_j^{\mathcal{O}}(\Sigma^{tV}R) \arrow[r, "\alpha_{i,j}"] \arrow[d, "\theta_{i,t}^{R}\wedge\theta_{j,t}^{R}"'] & D_{i+j}^{\mathcal{O}}(\Sigma^{tV}R) \arrow[d, "\theta_{i+j,t}^{R}"] \\
\Sigma^{itV}R \wedge \Sigma^{jtV}R \arrow[r, "\mu"'] & \Sigma^{(i+j)t}R
\end{tikzcd} \tag{6.13}$$

$$\begin{tikzcd}
D_i^{\mathcal{O}}\left(D_j^{\mathcal{O}}(\Sigma^{tV}R)\right) \arrow[r, "\beta_{i,j}"] \arrow[d, "D_i^{\mathcal{O}}(\theta_{j,t}^{R})"'] & D_{ij}^{\mathcal{O}}(\Sigma^{tV}R) \arrow[d, "\theta_{ij,t}^{R}"] \\
D_i^{\mathcal{O}}(\Sigma^{jtV}R) \arrow[r, "\theta_{i,jt}^{R}"'] & \Sigma^{ijtV}R
\end{tikzcd} \tag{6.14}$$

(6.15)
$$\begin{tikzcd}
\mathrm{D}_{\mathsf{i}}^{\mathcal{O}}(\Sigma^{s\mathrm{V}}\mathrm{R}\wedge\Sigma^{t\mathrm{V}}\mathrm{R}) \arrow[r, "\mathrm{D}_{\mathsf{i}}^{\mathcal{O}}(\mu)"] \arrow[d] & \mathrm{D}_{\mathsf{i}}^{\mathcal{O}}(\Sigma^{(s+t)\mathrm{V}}\mathrm{R}) \arrow[dd, "\theta^{\mathrm{R}}_{\mathsf{i},s+t}"] \\
\mathrm{D}_{\mathsf{i}}^{\mathcal{O}}(\Sigma^{s\mathrm{V}}\mathrm{R})\wedge\mathrm{D}_{\mathsf{i}}^{\mathcal{O}}(\Sigma^{t\mathrm{V}}\mathrm{R}) \arrow[d, "\theta^{\mathrm{R}}_{\mathsf{i},s}\wedge\theta^{\mathrm{R}}_{\mathsf{i},t}"'] & \\
\Sigma^{is\mathrm{V}}\mathrm{R}\wedge\Sigma^{it\mathrm{V}}\mathrm{R} \arrow[r, "\mu^{\mathrm{R}}"'] & \Sigma^{i(s+t)\mathrm{V}}\mathrm{R}
\end{tikzcd}$$

commute for all $i, j, s, t \in \mathbb{N}$ in $\mathrm{Ho}(\mathcal{S}\mathrm{p}_{\mathrm{G}})$.

Remark 6.16. When $\mathcal{O}$ is a nonequivariant $\mathbb{E}_\infty$-operad and $\mathrm{V} = \mathbb{R}^{\mathsf{d}}$ then (6.11) recovers the classical definition of $\mathbb{H}^{\mathsf{d}}_\infty$-rings [BMMS86, I.4.1].

Remark 6.17. When $\mathsf{i} = 0$, we have $\mathrm{D}_{\mathsf{i}}^{\mathcal{O}}(\Sigma^{t\mathrm{V}}\mathrm{R}) \simeq \mathbb{S}$ and the map $\theta^{\mathrm{R}}_{\mathsf{i},\mathsf{t}}$ of (6.12) is the same map for all $t \in \mathbb{N}$. This map serves as the unit map of R which we will denoted by θ^{R}_0.

Remark 6.18. The diagrams (6.13) and (6.14) restricted to $t = 0$ implies that any $\mathcal{O}^{\mathsf{h}}_{[\mathrm{V}]}$-ring spectrum is automatically an $\mathcal{O}^{\mathsf{h}}$-ring spectrum.

Notation 6.19. For brevity, we set $\gamma^{(i)}_{\mathrm{V}} \coloneqq \gamma^{\mathscr{F}_i(\mathcal{O})}_{\mathrm{V}}$, $\widetilde{\gamma}^{(i)}_{\mathrm{V}} \coloneqq \widetilde{\gamma}^{\mathscr{F}_i(\mathcal{O})}_{\mathrm{V}}$ and $\partial_{\mathsf{i}} = \partial_{\mathsf{i}}^{\mathscr{F}_i(\mathcal{O})}$ with $\mathscr{F}_i(\mathcal{O})$ as given in (2.8).

Using the unit map of R, we get a map

$$\mathbf{u}^{(i)}_{t\mathrm{V}} : \mathsf{Th}(\gamma^{(i)}_{t\mathrm{V}}) \simeq \mathrm{D}_{\mathsf{i}}^{\mathcal{O}}(\mathrm{S}^{t\mathrm{V}}) \longrightarrow \mathrm{D}_{\mathsf{i}}^{\mathcal{O}}(\Sigma^{t\mathrm{V}}\mathrm{R}) \longrightarrow \Sigma^{it\mathrm{V}}\mathrm{R} \tag{6.20}$$

which serves as an R-Thom class of the bundle $\gamma^{\mathscr{F}_i}_{t\mathrm{V}}$. Since $\gamma^{(i)}_{t\mathrm{V}} \cong \widetilde{\gamma}^{(i)}_{t\mathrm{V}} \oplus \epsilon_{t\mathrm{V}}$, we also get an R-Thom class

$$\widetilde{\mathbf{u}}^{(i)}_{t\mathrm{V}} \in \mathrm{R}^{(\mathsf{i}-1)t\mathrm{V}}\,\mathsf{Th}(\widetilde{\gamma}^{(i)}_{t\mathrm{V}}).$$

We arrange these Thom classes so that

$$\mathbf{u}^{(i)}_{t\mathrm{V}} = \sigma_{t\mathrm{V}}(\widetilde{\mathbf{u}}^{(i)}_{t\mathrm{V}}) \tag{6.21}$$

for all $i \geq 2$ and $t \in \mathbb{N}$.

The diagrams in (6.13), (6.14) and (6.15) forces certain relationships between these R-Thom classes defined in (6.20). These relations are nothing but the equivariant analogs of those in [BMMS86, VII].

Proposition 6.22. *The family* $\{\mathbf{u}^{(i)}_{t\mathrm{V}} : i, \mathsf{t} \in \mathbb{N}\}$ *of* R*-Thom classes satisfy*

$$\mathbf{u}^{(i)}_{(s+t)\mathrm{V}} = \partial_{\mathsf{i}}^*(\mathbf{u}^{(i)}_{s\mathrm{V}} \times \mathbf{u}^{(i)}_{t\mathrm{V}}), \tag{6.23}$$

where $\partial_{\mathsf{i}}^{\mathscr{F}_i}$ *is the natural map defined in* (2.6).

Proof. The result follows from the diagram

$$\begin{tikzcd}
\mathrm{D}_{\mathsf{i}}^{\mathcal{O}}(\mathrm{S}^{(s+t)\mathrm{V}}) \arrow[r, "(\mathbf{A})"] \arrow[d, "\partial_{\mathsf{i}}"'] & \mathrm{D}_{\mathsf{i}}^{\mathcal{O}}(\Sigma^{s\mathrm{V}}\mathrm{R}\wedge\Sigma^{t\mathrm{V}}\mathrm{R}) \arrow[r, "\mathrm{D}_{\mathsf{i}}^{\mathcal{O}}(\mu)"] \arrow[d, "\partial_{\mathsf{i}}"'] & \mathrm{D}_{\mathsf{i}}^{\mathcal{O}}(\Sigma^{(s+t)\mathrm{V}}\mathrm{R}) \arrow[d, "\theta^{\mathrm{R}}_{\mathsf{i},s+t}"] \\
\mathrm{D}_{\mathsf{i}}^{\mathcal{O}}(\Sigma^{s\mathrm{V}})\wedge\mathrm{D}_{\mathsf{i}}^{\mathcal{O}}(\Sigma^{t\mathrm{V}}) \arrow[r, "(\mathbf{B})"'] & \mathrm{D}_{\mathsf{i}}^{\mathcal{O}}(\Sigma^{s\mathrm{V}}\mathrm{R})\wedge\mathrm{D}_{\mathsf{i}}^{\mathcal{O}}(\Sigma^{t\mathrm{V}}\mathrm{R}) \arrow[r, "\mu^{\mathrm{R}}(\theta^{\mathrm{R}}_{\mathsf{i},s}\wedge\theta^{\mathrm{R}}_{\mathsf{i},t})"'] & \Sigma^{(s+t)i\mathrm{V}}\mathrm{R}
\end{tikzcd}$$

where $(\mathbf{A}) = \mathrm{D}^{\mathcal{O}}_{\mathsf{i}}(e \wedge e)$ and $(\mathbf{B}) = \mathrm{D}^{\mathcal{O}}_{\mathsf{i}}(e) \wedge \mathrm{D}^{\mathcal{O}}_{\mathsf{i}}(e)$, which commutes due to (6.15) and the naturality of $\partial^{\mathscr{F}_i}_{\mathsf{i}}$. ☐

Corollary 6.24. *The family* $\{\widetilde{\mathbf{u}}^{(i)}_{t\mathrm{V}} : i, t \in \mathbb{N}\}$ *of* R-*Thom classes satisfy*

$$\widetilde{\mathbf{u}}^{(i)}_{(s+t)\mathrm{V}} = \partial^{*}_{\mathsf{i}}(\widetilde{\mathbf{u}}^{(i)}_{s\mathrm{V}} \times \widetilde{\mathbf{u}}^{(i)}_{t\mathrm{V}}),$$

where $\partial^{\mathscr{F}_i}_{\mathsf{i}}$ *is the natural map defined in* (2.6).

Likewise, using the diagram (6.13) and naturality of $\alpha_{i,j}$ we get:

Proposition 6.25. *The family* $\{\mathbf{u}^{(i)}_{t\mathrm{V}} : i, t \in \mathbb{N}\}$ *of* R-*Thom classes satisfy*

$$\alpha^{*}_{i,j}\mathbf{u}^{(i+j)}_{t\mathrm{V}} = \mathbf{u}^{(i)}_{t\mathrm{V}} \times \mathbf{u}^{(j)}_{t\mathrm{V}}, \tag{6.26}$$

for all $i, j, \mathsf{t} \in \mathbb{N}$.

Corollary 6.27. *The family* $\{\widetilde{\mathbf{u}}^{(i)}_{t\mathrm{V}} : i, \mathsf{t} \in \mathbb{N}\}$ *of* R-*Thom classes satisfy*

$$\alpha^{*}_{i,j}\widetilde{\mathbf{u}}^{(i+j)}_{t\mathrm{V}} = \sigma_{t\mathrm{V}}(\widetilde{\mathbf{u}}^{(i)}_{t\mathrm{V}} \times \widetilde{\mathbf{u}}^{(j)}_{t\mathrm{V}}),$$

for all $i, j, t \in \mathbb{N}$.

Now we discuss the consequence of (6.14) which will be crucial in defining composition of Eulerian sequences. Using unit map of R and (6.14), we get the diagram

$$\begin{tikzcd}
\mathrm{D}^{\mathcal{O}}_{\mathsf{i}}\left(\mathrm{D}^{\mathcal{O}}_{\mathsf{j}}(\mathrm{S}^{t\mathrm{V}})\right) \arrow[rr, "\beta_{\mathsf{i,j}}"] \arrow[d, "\mathrm{D}^{\mathcal{O}}_{\mathsf{i}}(\mathbf{u}^{(j)}_{t\mathrm{V}})"'] & & \mathrm{D}^{\mathcal{O}}_{\mathsf{ij}}(\mathrm{S}^{t\mathrm{V}}) \arrow[dd, "\mathbf{u}^{(ij)}_{t\mathrm{V}}"] \\
\mathrm{D}^{\mathcal{O}}_{\mathsf{i}}(\Sigma^{jt\mathrm{V}}\mathrm{R}) \arrow[d, "\partial_{\mathsf{i}}"'] \arrow[rrd, bend left, "\theta^{\mathrm{R}}_{\mathsf{i,jt}}"] & & \\
\mathrm{D}^{\mathcal{O}}_{\mathsf{i}}(\mathrm{S}^{jt\mathrm{V}}) \wedge \mathrm{D}^{\mathcal{O}}_{\mathsf{i}}(\mathrm{R}) \arrow[r, "\mathbf{u}^{(i)}_{jt\mathrm{V}} \wedge \theta^{\mathrm{R}}_{i}"'] & \Sigma^{ijt\mathrm{V}}\mathrm{R} \wedge \mathrm{R} \arrow[r, "\mu^{\mathrm{R}}"'] & \Sigma^{ijt\mathrm{V}}\mathrm{R}
\end{tikzcd}$$

which implies:

Proposition 6.28. *The system* $\{\widetilde{\mathbf{u}}^{(i)}_{\mathsf{t}\mathrm{V}} : i, \mathsf{t} \in \mathbb{N}\}$ *of* R-*Thom classes satisfy*

$$\mu^{\mathrm{R}}\left(\mathbf{u}^{(i)}_{jt\mathrm{V}} \wedge \theta^{\mathrm{R}}_{i}\right) \circ \partial_{\mathsf{i}} \circ \mathrm{D}^{\mathcal{O}}_{\mathsf{i}}(\mathbf{u}^{(j)}_{\mathsf{t}\mathrm{V}}) = \mathbf{u}^{(ij)}_{\mathsf{t}\mathrm{V}} \circ \beta_{i,j} \tag{6.29}$$

for all $i, j, t \in \mathbb{N}$.

Corollary 6.30. *The family* $\{\widetilde{\mathbf{u}}^{(i)}_{t\mathrm{V}} : i, \mathsf{t} \in \mathbb{N}\}$ *of* R-*Thom classes satisfy*

$$\mu^{\mathrm{R}}\left(\sigma_{jt\mathrm{V}}(\widetilde{\mathbf{u}}^{(i)}_{jt\mathrm{V}}) \wedge \theta^{\mathrm{R}}_{i}\right) \circ \partial_{\mathsf{i}} \circ \mathrm{D}^{\mathcal{O}}_{\mathsf{i}}(\sigma_{t\mathrm{V}}(\widetilde{\mathbf{u}}^{(j)}_{t\mathrm{V}})) = \sigma_{t\mathrm{V}}(\widetilde{\mathbf{u}}^{(ij)}_{t\mathrm{V}}) \circ \beta_{i,j}$$

for all $i, j, t \in \mathbb{N}$.

Conversely:

Proposition 6.31. *If* R *is an* $\mathcal{O}^{\mathsf{h}}$*-ring spectrum such that there exists a family of* R-*Thom classes* $\{\mathbf{u}^{(i)}_{\mathrm{V}} : i \in \mathbb{N}\}$ *satisfying* (6.26) *and* (6.29)*, then* R *is an* $\mathcal{O}^{\mathsf{h}}_{[\mathrm{V}]}$*-ring spectrum.*

Remark 6.32. Note that $\mathbf{u}_{t\mathrm{V}}^{(i)}$, for $t \geq 2$ can be defined inductively from $\mathbf{u}_{\mathrm{V}}^{(i)}$ using the formula in (6.23). This will satisfy both (6.26) and (6.23) as long as the initial family $\{\mathbf{u}_{\mathrm{V}}^{(i)} : i \in \mathbb{N}\}$ does. This consistency allows us to omit the R-Thom classes for $t \geq 2$ in the family of R-Thom classes considered in (6.31).

The proof of (6.31) is identical to the arguments provided in [BMMS86, VII.5,VII.6] (in particular see [BMMS86, VII, Proposition 6.2]) which covers the nonequivariant case. Hence, we leave the details to the readers. Also note:

Lemma 6.33. *Suppose* R *is an* $\mathcal{O}^{\mathsf{h}}_{[\mathrm{V}]}$*-ring and suppose have a map of* $\mathcal{O}^{\mathsf{h}}$*-rings*

$$f : \mathrm{R} \longrightarrow \mathrm{S},$$

then S *is also an* $\mathcal{O}^{\mathsf{h}}_{[\mathrm{V}]}$*-ring.*

Proof. By assumption, we have a family $\{\mathbf{u}_{\mathrm{V}}^{(i)} : i \in \mathbb{N}\}$ of R-Thom classes satisfying (6.26) and (6.29). Since f is a homotopy $\mathcal{O}$-ring map, the family

$$\{f_*\mathbf{u}_{\mathrm{V}}^{(i)} : i \in \mathbb{N}\}$$

of T-Thom classes also satisfies (6.26) and (6.29). Therefore, S is an $\mathcal{O}^{\mathsf{h}}_{[\mathrm{V}]}$-ring. ☐

6.2. **Norms and geometric fixed points of V-shifted $\mathbb{E}_\infty^{\mathrm{G}}$-rings**

In [BMMS86], the authors demonstrate that $\mathrm{H}\mathbb{F}_p$ is an $\mathbb{H}_\infty^d$-ring, where $d = 1$ when $p = 2$ and $d = 2$ when p is odd. The primary goal of this subsection is to leverage this result by employing the norm (introduced in [HHR16]) and geometric fixed-point functors to identify V for which $\mathrm{H}\underline{\mathbb{F}}_p$ admits a V-shifted homotopy $\mathbb{E}_\infty^{\mathrm{G}}$-ring structure.

Suppose K is a subgroup of G. Then for a pointed K-space X its norm is nothing but the G-space $\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}(\mathrm{X}) := \mathrm{Map}_{\mathrm{H}}(\mathrm{G}_+, \mathrm{X})$. In [HHR16], the authors introduced a spectrum level norm

$$\mathrm{N}_{\mathrm{K}}^{\mathrm{G}} : \mathcal{S}\mathrm{p}_{\mathrm{K}} \longrightarrow \mathcal{S}\mathrm{p}_{\mathrm{G}},$$

which is

- — strong symmetric monoidal,
- — commutes with sifted colimits,
- — and $\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}(\Sigma_{\mathrm{K}}^\infty \mathrm{X}) \simeq \Sigma_{\mathrm{G}}^\infty(\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}(\mathrm{X}))$.

In particular, $\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}(\Sigma^{\mathrm{V}}\mathbb{S}_{\mathrm{K}}) \simeq \Sigma^{\mathrm{Ind}\,\mathrm{V}}\mathbb{S}_{\mathrm{G}}$, where $\mathrm{Ind}\,\mathrm{V} := \mathbb{R}[\mathrm{G}] \otimes_{\mathbb{R}[\mathrm{H}]} \mathrm{V}$ for any orthogonal K-representation V.

The smash product serves as the coproduct in the category of $\mathbb{E}_\infty^{\mathrm{G}}$ rings. The norm $\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}$ is constructed inductively from the smash product. Since $\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}$ is strong symmetric monoidal, it lifts to a functor from the category of (homotopy) $\mathbb{E}_\infty^{\mathrm{K}}$-rings to the homotopy category of (homotopy) $\mathbb{E}_\infty^{\mathrm{G}}$-rings. We will now show that:

Proposition 6.34. *If* R *is a* V*-shifted homotopy* $\mathbb{E}_\infty^{\mathrm{K}}$*-ring spectrum, then* $\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}\mathrm{R}$ *is an* Ind V*-shifted homotopy* $\mathbb{E}_\infty^{\mathrm{G}}$*-ring spectrum.*

Notation 6.35. In this subsection, we simplify the notation of (3.1) by writing $\gamma_{\mathrm{V}}^{(i)}$ as an abbreviation for $\gamma_{\mathrm{V}}^{\mathcal{A}\ell\ell_i}$, where $\mathcal{A}\ell\ell_i$ denotes the $\mathrm{G} \times \Sigma_i$ family defined in (2.24). The notation implicitly indicates that $\gamma_{\mathrm{V}}^{(i)}$ is a G-equivariant bundle, a property that follows from V being a G-representation.

Suppose R is a V-shifted homotopy $\mathbb{E}_{\infty}^{\mathrm{K}}$-ring for some orthogonal K-representation V. Then there exist a R-Thom class $\mathsf{u}_{\mathrm{V}}^{(i)}$ of the bundle $\gamma_{\mathrm{V}}^{(i)}$ such that the collection $\{\mathbf{u}_{\mathrm{V}}^{(i)} : i \in \mathbb{N}\}$ satisfies (6.26) and (6.29). It is easy to see

$$\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}\,\mathsf{Th}(\gamma_{\mathrm{V}}^{(i)}) \cong \mathsf{Th}(\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}(\gamma_{\mathrm{V}}^{(i)})).$$

An R-Thom class, restricted to (G-orbit of) a point of the base space is a unit in the homotopy group of R. Since norms are strong symmetric monoidal they preserve units. Thus:

Lemma 6.36. *The class* $\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}\mathsf{u}_{\mathrm{V}}^{(i)}$ *is an* $\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}\mathrm{R}$*-Thom class for* $\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}(\gamma_{\mathrm{V}}^{(i)}))$.

The non-basepointed space-level restriction-coinduction adjunction yields a natural unit map

$$\eta : \mathrm{X} \longrightarrow \mathrm{Map}_{\mathrm{K}}(\mathrm{G}, \mathrm{X})$$

for any G-space X. Because G is a finite group, we have a G-equivariant homeomorphism

$$\mathrm{Map}_{\mathrm{K}}(\mathrm{G}, \mathrm{X})_+ \cong \mathrm{N}_{\mathrm{K}}^{\mathrm{G}}(\mathrm{X}_+).$$

Using this homeomorphism, we identify $\eta^*\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}(\gamma_{\mathrm{V}}^{(i)}) \cong \gamma_{\mathrm{Ind\,V}}^{(i)}$. Thus,

$$\mathbf{u}_{\mathrm{Ind\,V}}^{(i)} := \eta^*\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}\mathbf{u}_{\mathrm{V}}^{(i)}$$

is an $\mathrm{N}_{\mathsf{e}}^{\mathrm{G}}\mathrm{R}$-Thom class for $\gamma_{\mathrm{Ind\,V}}^{(i)}$.

Proof of (6.34). Since $\{\mathbf{u}_{\mathrm{V}}^{(i)} : i \in \mathbb{N}\}$ satisfy the conditions of (6.31), the family

$$\{\mathbf{u}_{\mathrm{Ind\,V}}^{(i)} : i \in \mathbb{N}\}$$

also satisfies those conditions (see (6.37) below). This immediately implies the result. □

Lemma 6.37. *The family* $\{\mathsf{u}_{\mathrm{Ind\,V}}^{(i)} : i \in \mathbb{N}\}$ *of* $\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}(\mathrm{R})$ *satisfies* (6.26) *and* (6.29).

Proof. Consider the diagram:

$$\begin{tikzcd}
\mathrm{D}_{\mathsf{i}}^{\mathbb{E}_\infty^{\mathrm{G}}}(\mathrm{S}^{n\,\mathrm{Ind\,V}}) \wedge \mathrm{D}_{\mathsf{j}}^{\mathbb{E}_\infty^{\mathrm{G}}}(\mathrm{S}^{n\,\mathrm{Ind\,V}}) \arrow[r, "\alpha_{\mathsf{i},\mathsf{j}}"] \arrow[d] \arrow[dd, bend right=60, "\mathsf{u}_{\mathrm{Ind\,V}}^{(i)} \wedge \mathsf{u}_{\mathrm{Ind\,V}}^{(j)}"'] & \mathrm{D}_{\mathsf{i+j}}^{\mathbb{E}_\infty^{\mathrm{G}}}(\mathrm{S}^{n\,\mathrm{Ind\,V}}) \arrow[d] \arrow[dd, bend left=60, "\mathsf{u}_{\mathrm{Ind\,V}}^{(i)}"] \\
\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}(\mathrm{D}_{\mathsf{i}}^{\mathbb{E}_\infty^{\mathrm{K}}}\mathrm{S}^{n\mathrm{V}}) \wedge \mathrm{N}_{\mathrm{K}}^{\mathrm{G}}(\mathrm{D}_{\mathsf{j}}^{\mathbb{E}_\infty^{\mathrm{K}}}\mathrm{S}^{n\mathrm{V}}) \arrow[r, "\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}(\alpha_{\mathsf{i},\mathsf{j}})"] \arrow[d, "\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}\mathsf{u}_{\mathrm{V}}^{(i)} \wedge \mathrm{N}_{\mathrm{K}}^{\mathrm{G}}\mathsf{u}_{\mathrm{V}}^{(j)}"'] & \mathrm{N}_{\mathrm{K}}^{\mathrm{G}}\mathrm{D}_{\mathsf{i+j}}^{\mathbb{E}_\infty^{\mathrm{K}}}\mathrm{S}^{n\mathrm{V}} \arrow[d, "\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}\mathsf{u}_{\mathrm{V}}^{(i+j)}"] \\
\Sigma^{in\,\mathrm{Ind\,V}}\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}\mathrm{R} \wedge \Sigma^{jn\,\mathrm{Ind\,V}}\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}\mathrm{R} \arrow[r, "\mathrm{N}_{\mathrm{G}}^{\mathrm{G}}(\mu_{\mathrm{R}})"] & \Sigma^{(i+j)n\,\mathrm{Ind\,V}}\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}\mathrm{R}
\end{tikzcd}$$

where the top vertical arrows are induced from the unit map η for $\mathrm{E_G}\Sigma_{(-)}$. Thus, the top square commutes due to the naturality of η. The bottom square commutes because it is the norm of (6.26) for the family $\{\mathsf{u}_{\mathrm{V}}^{(i)} : i \in \mathbb{N}\}$. Consequently, the entire diagram commutes which shows that the family $\{\mathsf{u}_{\mathrm{Ind\,V}}^{(i)} : i \in \mathbb{N}\}$ also satisfies the condition specified in (6.26).

Next, we consider the diagram:

$$\begin{tikzcd}
 & \mathrm{D}_{\mathsf{i}}^{\mathbb{E}_\infty^{\mathrm{G}}} \mathrm{D}_{\mathsf{j}}^{\mathbb{E}_\infty^{\mathrm{G}}} \mathrm{S}^{n\,\mathrm{Ind\,V}} \arrow[r, "\beta_{\mathsf{i,j}}"] \arrow[d] \arrow[ld, bend right, "\mathrm{D}_{\mathsf{i}}\mathsf{u}_{\mathrm{Ind\,V}}^{(j)}"'] & \mathrm{D}_{\mathsf{ij}}^{\mathbb{E}_\infty^{\mathrm{G}}} \mathrm{S}^{n\,\mathrm{Ind\,V}} \arrow[d] \arrow[dd, bend left=60, "\mathsf{u}_{\mathrm{Ind\,V}}^{(ij)}"] \\
\mathrm{D}_{\mathsf{i}}^{\mathbb{E}_\infty^{\mathrm{G}}} \Sigma^{jn\,\mathrm{Ind\,V}} \mathrm{N}_{\mathrm{K}}^{\mathrm{G}}\mathrm{R} \arrow[rd, bend right] & \mathrm{N}_{\mathrm{K}}^{\mathrm{G}} \mathrm{D}_{\mathsf{i}}^{\mathbb{E}_\infty^{\mathrm{K}}} \mathrm{D}_{\mathsf{j}}^{\mathbb{E}_\infty^{\mathrm{K}}} \mathrm{S}^{n\mathrm{V}} \arrow[r, "\beta_{\mathsf{i,j}}"] \arrow[d, "\mathrm{D}_{\mathsf{i}}\mathsf{u}_{\mathrm{V}}^{(j)}"] & \mathrm{N}_{\mathrm{K}}^{\mathrm{G}} \mathrm{D}_{\mathsf{ij}}^{\mathrm{E}_\infty^{\mathrm{K}}} \mathrm{S}^{n\mathrm{V}} \arrow[d, "\mathsf{u}_{\mathrm{V}}^{(ij)}"] \\
 & \mathrm{N}_{\mathrm{K}}^{\mathrm{G}} \mathrm{D}_{\mathsf{i}}^{\mathbb{E}_\infty^{\mathrm{K}}} \Sigma^{jn}\mathrm{R} \arrow[r, "\mathrm{N}_{\mathrm{K}}^{\mathrm{G}}(\theta_{\mathsf{i,j}}^{\mathrm{R}})"] & \Sigma^{ijn\,\mathrm{Ind\,V}} \mathrm{N}_{\mathrm{K}}^{\mathrm{G}}\mathrm{R}
\end{tikzcd}$$

where the unlabelled arrows in the top square are induced from the unit map η for $\mathrm{E}_{\mathrm{G}}\Sigma_{(-)}$. It is straightforward to check that this diagram commutes, and therefore the family $\{\mathsf{u}_{\mathrm{Ind\,V}}^{(i)} : i \in \mathbb{N}\}$ satisfies (6.29). ☐

Rephrasing the result [BMMS86, Proposition I.4.5] in our language we conclude that $\mathrm{H}\mathbb{F}_p$ is an $\mathbb{R}^d$-shifted homotopy $\mathbb{E}_\infty$-ring, where $d = 1$ when $p = 2$ and $d = 2$ when p is odd. Using the counit of the norm-restriction adjunction, we have an $\mathbb{E}_\infty^{\mathrm{G}}$-ring map

$$\mathrm{N}_{\mathsf{e}}^{\mathrm{G}}\mathrm{H}\mathbb{F}_p \longrightarrow \mathrm{H}\underline{\mathbb{F}}_p.$$

Therefore, from (6.33) we get:

Theorem 6.38. *The spectrum* $\mathrm{H}\underline{\mathbb{F}}_p$ *admits a functorial* $d\rho$*-shifted homotopy* $\mathbb{E}_\infty^{\mathrm{G}}$*-ring structure, where* $d = 1$ *when* $p = 2$ *and* $d = 2$ *when* p *is odd.*

Proposition 6.39. *If* R *is a* V*-shifted homotopy* $\mathbb{E}_\infty^{\mathrm{G}}$ *ring, then* $\Phi^{\mathrm{K}}\mathrm{R}$ *is a* V^{K}*-shifted homotopy* $\mathbb{E}_\infty^{\mathrm{K}}$*-ring.*

Proof. Let $\{\mathsf{u}_{\mathrm{V}}^{(i)} : i \in \mathbb{N}\}$ denote the family R-Thom classes satisfying (6.26) and (6.29) that determine the V-shifted homotopy ring structure of R. Define $\mathsf{u}_{\mathrm{V}^{\mathrm{K}}}^{(i)}$ to be the composite

$$\mathsf{u}_{\mathrm{V}^{\mathrm{K}}}^{(i)} :\ \mathrm{D}_{\mathsf{i}}^{\mathbb{E}_\infty^{\mathrm{K}}}(\mathrm{S}^{\mathrm{V}^{\mathrm{K}}}) \xrightarrow{\widehat{\lambda}} \Phi^{\mathrm{K}}\mathrm{D}_{\mathsf{i}}^{\mathbb{E}_\infty^{\mathrm{G}}}(\mathrm{S}^{\mathrm{V}}) \xrightarrow{\varphi^{\mathrm{K}}(\mathsf{u}_{\mathrm{V}}^{(i)})} \Phi^{\mathrm{K}}\Sigma^{\mathsf{i}\mathrm{V}}\mathrm{R} \quad \cong \quad \Sigma^{\mathsf{i}\mathrm{V}^{\mathrm{K}}}\Phi^{\mathrm{K}}\mathrm{R}\ ,$$

where $\widehat{\lambda}$ is the natural map defined in (2.22). It is easy to see $\mathsf{u}_{\mathrm{V}^{\mathrm{K}}}^{(i)}$ is a $\Phi^{\mathrm{K}}\mathrm{R}$-Thom class of $\gamma_{\mathrm{V}^{\mathrm{K}}}^{(i)}$. A straightforward diagram chase shows that the family $\{\mathsf{u}_{\mathrm{V}^{\mathrm{K}}}^{(i)} : i \in \mathbb{N}\}$ also satisfies (6.26) and (6.29). Hence, the result. ☐

Theorem 6.40. *When* p *is odd,* $\mathrm{H}\underline{\mathbb{F}}_p$ *does not admit a* ρ*-shifted homotopy* $\mathbb{E}_\infty^{\mathrm{G}}$*-ring structure.*

Proof. Assume, for the sake of contradiction, $\mathrm{H}\underline{\mathbb{F}}_p$ admits such a structure. Then, by (6.39), its geometric fixed points $\Phi^{\mathrm{G}}\mathrm{H}\underline{\mathbb{F}}_p$ would admit an $\mathbb{R}$-shifted homotopy $\mathbb{E}_\infty$-ring structure. Since the zeroth Postnikov approximation provides an $\mathbb{E}_\infty$-ring map

$$\Phi^{\mathrm{G}}\mathrm{H}\underline{\mathbb{F}}_p \longrightarrow \mathrm{H}\mathbb{F}_p$$

which implies that $\mathrm{H}\mathbb{F}_p$ itself admits an $\mathbb{R}$-shifted homotopy $\mathbb{E}_\infty$-ring structure (by (6.33)). This, however, contradicts [BMMS86, Proposition I.4.5]. □

6.3. **Composition of Eulerian sequences**

Let $\mathcal{E}^{(n)}_{\mathrm{R,V}}$ denote the collection of V-stable R-Eulerian sequence of weight n for a V-shifted homotopy $\mathcal{O}$-ring R, where $\mathcal{O}$ is an $\mathcal{N}_\infty$-operad. Our goal is to define a strictly associative pairing

$$\odot : \mathcal{E}^{(n)}_{\mathrm{R,V}} \times \mathcal{E}^{(m)}_{\mathrm{R,V}} \longrightarrow \mathcal{E}^{(nm)}_{\mathrm{R,V}}$$

and prove (6.54).

Since R is an $\mathcal{O}^{\mathsf{h}}_{[\mathrm{V}]}$-ring, $\widetilde{\gamma}^{(i)}_{\mathrm{V}}$ admits an R-Thom class $\widetilde{\mathbf{u}}^{(i)}_{\mathrm{V}}$ for all $i \in \mathbb{N}$ such that the family $\{\mathbf{u}^{(i)}_{\mathrm{V}} : i \in \mathbb{N}\}$ satisfies the conditions of (6.26) and (6.29). Let $\widetilde{\mathbf{e}}^{(i)}_{\mathrm{V}}$ denote the corresponding Euler class.

Notation 6.41. We introduce a slight abuse of notation for a non-basepointed G-space X, defining its $\mathscr{F}$-th extended power as:

$$\mathrm{D}^{\mathscr{F}}_{\mathsf{n}}(\mathrm{X}) := \mathrm{E}\mathscr{F} \times_{\Sigma_n} \mathrm{X}^{\times \mathsf{n}}$$

This notation is consistent with the basepointed case, as there is a canonical homeomorphism $\mathrm{D}^{\mathscr{F}}_{\mathsf{n}}(\mathrm{X}_+) \cong \mathrm{D}^{\mathscr{F}}_{\mathsf{n}}(\mathrm{X})_+$.

We now consider the standard inclusion of the wreath product

$$i : \Sigma_m \wr \Sigma_n \cong \Sigma_m^{\times n} \rtimes \Sigma_n \hookrightarrow \Sigma_{mn}.$$

The pullback of τ_{mn} along this inclusion is isomorphic to the representation

$$\tau_m^{\times \mathsf{n}} := \mathrm{Map}(\mathsf{n}, \tau_m),$$

where Σ_n acts on $\mathsf{n} = \{1, \ldots, n\}$ by permutation. The action of the element $((s_1, \ldots, s_n), \sigma^{-1}) \in \Sigma_m \wr \Sigma_n$ on $\tau_m^{\times \mathsf{n}}$ is given by the formula

$$((s_1, \ldots, s_n), \sigma) \cdot (v_1, \ldots, v_n) = (s_{\sigma(1)} \cdot v_{\sigma(1)}, \ldots, s_{\sigma(n)} \cdot v_{\sigma(n)}).$$

Thus, the pullback of $\widetilde{\tau}_{mn}$ along i decomposes into a direct sum of the form

$$i^* \widetilde{\tau}_{mn} \cong \widehat{\tau}_n \oplus \widetilde{\tau}_m^{\times \mathsf{n}},$$

where $\widehat{\tau}_n$ is the pullback of $\widetilde{\tau}_n$ along the quotient map $\Sigma_m \wr \Sigma_n \twoheadrightarrow \Sigma_n$.

Notation 6.42. Let $\mathscr{B}_n$ denote the G-space $\mathrm{B}\mathscr{F}_n(\mathcal{O}) = \mathcal{O}(n) \times_{\Sigma_n} *$.

The map i also results in the map

$$\iota : \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_m) \cong \mathcal{O}(n) \times_{\Sigma_n} (\mathscr{B}_m)^{\times n} \longrightarrow \mathscr{B}_{mn}. \tag{6.43}$$

The pullback of $\gamma^{(mn)}_{\mathrm{V}}$ along ι is the bundle

$$\begin{tikzcd}
(\mathcal{O}(n) \times (\mathcal{O}(m)^{\times n}) \times_{\Sigma_m \wr \Sigma_n} ((\mathrm{V} \otimes (\tau_m^{\times \mathsf{n}})) \arrow[d] \\
\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_m).
\end{tikzcd}$$

The above bundle is isomorphic to $\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\gamma_{\mathrm{V}}^{(m)})$, which is obtained by applying $\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(-)$ to both the total space and the base space of $\gamma_{\mathrm{V}}^{(m)}$. Likewise,

$$\iota^*\widetilde{\gamma}_{\mathrm{V}}^{(mn)} \cong \begin{tikzcd} (\mathcal{O}(n) \times (\mathcal{O}(m)^{\times n}) \times_{\Sigma_m \wr \Sigma_n} ((\mathrm{V} \otimes (\widehat{\tau}_n \oplus \widetilde{\tau}_m^{\times \mathsf{n}})) \arrow[d] \\ \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_m). \end{tikzcd}$$

is isomorphic to the direct sum $q^*\widetilde{\gamma}_{\mathrm{V}}^{(n)} \oplus \mathrm{D}_{\mathsf{n}}(\widetilde{\gamma}_{\mathrm{V}}^{(m)})$, where

$$q : \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_m) \longrightarrow \mathscr{B}_{\mathsf{n}}$$

is map induced by the quotient $\Sigma_m \wr \Sigma_n \twoheadrightarrow \Sigma_n$. This enables us to express the R-Euler class of $\iota^*\widetilde{\gamma}_{\mathrm{V}}^{(mn)}$ in terms of the R-Euler class of $\widetilde{\gamma}_{\mathrm{V}}^{(m)}$ and $\widetilde{\gamma}_{\mathrm{V}}^{(n)}$.

Now fix two R-Eulerian sequence

$$\chi^{(1)} = (\mathsf{x}_0, \mathsf{x}_1, \mathsf{x}_2, \dots) \in \mathcal{E}_{\mathrm{R},\mathrm{V}}^{(n)} \quad \text{and} \quad \chi^{(2)} = (\mathsf{y}_0, \mathsf{y}_1, \mathsf{y}_2, \dots) \in \mathcal{E}_{\mathrm{R},\mathrm{V}}^{(m)}$$

of weight m and n respectively.

Notation 6.44. For convenience, we introduce the notations:

- $\mathbf{e}_x := \widetilde{\mathbf{e}}_{\mathrm{V}}^{(n)} \in \mathrm{R}^{(n-1)\mathrm{V}}((\mathscr{B}_n)_+)$.
- $\mathbf{e}_y := \widetilde{\mathbf{e}}_{\mathrm{V}}^{(m)} \in \mathrm{R}^{(m-1)\mathrm{V}}((\mathscr{B}_m)_+)$.

Let

$$\mathbf{e}_{xy} \in \mathrm{R}^{(mn-1)\mathrm{V}}(\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_m)_+)$$

denote the class represented by the following composite map:

$$\begin{tikzcd} \mathbf{e}_{xy} : \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_m)_+ \arrow[r, "\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathbf{e}_y)"] & \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\Sigma^{(m-1)\mathrm{V}}\mathrm{R}) \arrow[d, "\partial_{\mathsf{n}}"] & \\ & \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{S}^{(m-1)\mathrm{V}}) \wedge \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{R}) \arrow[d, "1\wedge\theta^{\mathrm{R}}_n"] & \\ & \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{S}^{(m-1)\mathrm{V}}) \wedge \mathrm{R} \arrow[d, phantom, "\wr\ \mathrm{T}_{m-1}"] & \\ & \Sigma^{(m-1)n\mathrm{V}}\mathscr{B}_{n+} \wedge \mathrm{R} \arrow[r, "\mathbf{e}_x"] & \Sigma^{(nm-1)\mathrm{V}}\mathrm{R} \end{tikzcd}$$

Here T_k is the R-Thom isomorphism

$$\mathrm{T}_k : \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{S}^{k\mathrm{V}}) \wedge \mathrm{R} \simeq \Sigma^{kn\mathrm{V}}\mathscr{B}_{n+} \wedge \mathrm{R} \tag{6.45}$$

defined using the R-Thom class $\mathbf{u}_{k\mathrm{V}}^{(n)}$ (via the Thom diagonal) and the identification $\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{S}^{k\mathrm{V}}) \cong \mathsf{Th}(k\gamma_{\mathrm{V}}^{(mn)})$. Note that the R-Euler class of the bundle $\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\gamma_{\mathrm{V}}^{(m)})$ and $q^*\widetilde{\gamma}_{\mathrm{V}}^{(n)}$ are given by the following composites, respectively:

$$\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_m)_+ \xrightarrow{\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathbf{e}_y)} \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\Sigma^{(m-1)\mathrm{V}}\mathrm{R}) \xrightarrow{\theta^{\mathrm{R}}_{n,m-1}} \Sigma^{n(m-1)\mathrm{V}}\mathrm{R}$$

$$\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_m)_+ \xrightarrow{q} \mathscr{B}_{n+} \xrightarrow{\mathbf{e}_x} \Sigma^{(n-1)\mathrm{V}}\mathrm{R}$$

The class $\mathbf{e}_{xy}$ is their cup product. This is implicit in the diagram in the following lemma:

Lemma 6.46. *The class* $\mathbf{e}_{xy}$ *is the pullback of the* R*-Euler class* $\widetilde{\mathbf{e}}^{(mn)}_{\mathrm{V}}$ *of the bundle* $\widetilde{\gamma}^{(mn)}_{\mathrm{V}}$ *along the map* ι *define in* (6.43)*.*

Proof. Let $\widetilde{\zeta}^{(k)}_{t\mathrm{V}}$ and $\zeta^{(k)}_{t\mathrm{V}}$ denote the zero sections of $t\widetilde{\gamma}^{(k)}_{\mathrm{V}}$ and $t\gamma^{(k)}_{\mathrm{V}}$, respectively. Using the identification $\mathrm{D}^{\mathcal{O}}_{\mathsf{k}}(\mathrm{S}^{t\mathrm{V}}) \cong \mathsf{Th}(t\gamma^{(k)}_{\mathrm{V}})$, we construct the following diagram:

$$\begin{tikzcd}
\Sigma^{\mathrm{V}}\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_{m+}) \arrow[rr, "\iota"] \arrow[d, "\sigma_{\mathrm{V}}\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\widetilde{\zeta}^{(m)}_{\mathrm{V}})"'] & & \Sigma^{\mathrm{V}}\mathscr{B}_{mn} \arrow[d, "\sigma_{\mathrm{V}}\left(\widetilde{\zeta}^{(mn)}_{\mathrm{V}}\right)"'] \\
\Sigma^{\mathrm{V}}\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathsf{Th}(\widetilde{\gamma}^{(m)}_{\mathrm{V}})) \arrow[r, "\Delta_1"] \arrow[d, "\sigma_{\mathrm{V}}\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}\left(\widetilde{\mathbf{u}}^{(m)}_{\mathrm{V}}\right)"'] & \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{D}^{\mathcal{O}}_{\mathsf{m}}(\mathrm{S}^{\mathrm{V}})) \arrow[r, "\beta_{n,m}"] \arrow[d, "\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}\left(\mathbf{u}^{(m)}_{\mathrm{V}}\right)"] & \mathrm{D}^{\mathcal{O}}_{\mathsf{mn}}(\mathrm{S}^{\mathrm{V}}) \arrow[ddd, "\mathbf{u}^{(i)}_{\mathrm{V}}"] \\
\Sigma^{\mathrm{V}}\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\Sigma^{(m-1)\mathrm{V}}\mathrm{R}) \arrow[r, "\Delta_2"] \arrow[d, "\Theta_1"'] & \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\Sigma^{m\mathrm{V}}\mathrm{R}) \arrow[d, "\Theta_2"] & \\
\Sigma^{\mathrm{V}}\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{S}^{(m-1)\mathrm{V}}) \wedge \mathrm{R} \arrow[r, "\Delta_3"] \arrow[d, "\mathrm{D}"'] & \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{S}^{m\mathrm{V}}) \wedge \mathrm{R} \arrow[d, "\partial_{\mathsf{n}}"] \arrow[dr, bend left, "\mathrm{U}_m"] & \\
\Sigma^{\mathrm{V}}\mathscr{B}_n \wedge \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{S}^{(m-1)\mathrm{V}}) \wedge \mathrm{R} \arrow[r, "\sigma_{\mathrm{V}}(\widetilde{\zeta}^{(n)}_{\mathrm{V}})\wedge 1"'] & \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{S}^{\mathrm{V}}) \wedge \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{S}^{(\mathsf{m}-1)\mathrm{V}}) \wedge \mathrm{R} \arrow[r, "\mathrm{U}_{1,m-1}"'] & \Sigma^{mn}\mathrm{R}
\end{tikzcd}$$

The constituent maps are defined as follows: D is induced by the Thom diagonal map for $\mathsf{Th}((m-1)\gamma^{(n)}_{\mathrm{V}})$, Θ_i uses ∂_{n} and $\theta^{\mathrm{R}}_{\mathsf{n}} : \mathrm{D}_{\mathsf{n}}(\mathrm{R}) \to \mathrm{R}$, U_m uses the R-Thom class $\mathbf{u}^{(n)}_{m\mathrm{V}}$, $\mathrm{U}_{1,m-1}$ uses $\mathbf{u}^n_{\mathrm{V}} \times \mathbf{u}^n_{(m-1)\mathrm{V}}$, and Δ_i are all induced by the diagonal map

$$\Delta_{\mathrm{V}} : \mathrm{S}^{\mathrm{V}} \longrightarrow \mathrm{S}^{\mathrm{V}} \wedge \cdots \wedge \mathrm{S}^{\mathrm{V}} \cong \mathrm{S}^{n\mathrm{V}}.$$

The naturality of these diagonal maps, along with (4.19), and (6.28), shows that the diagram commutes.

From, (6.21), and the property that the zero section composed with the Thom class yields the Euler class, we conclude that compositions of blue arrows and red arrows are $\mathbf{e}_{xy}$ and $\iota_*\widetilde{\mathbf{e}}^{(mn)}_{\mathrm{V}}$, respectively. This establishes the result. ☐

As a necessary precursor to defining the product $\chi^{(1)} \odot \chi^{(2)}$, we first define the class

$$\mathsf{x}_{mk} \circ \mathsf{y}_k \in \mathrm{R}^{\mathrm{G}}_{|\mathsf{x}_{km}|+|\mathsf{y}_k|}(\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_m)_+)$$

as the following composition:
(6.47)

$$\begin{tikzcd}
S^{|x_{mk}|+|y_k|} \arrow[r, "\sigma_{|y_k|}(x_{mk})"] & \Sigma^{|y_k|}\mathscr{B}_{n+} \wedge R \arrow[d, "\Delta'_k \wedge 1_R"] & \\
& D^{\mathcal{O}}_{n}(S^{|y_k|}) \wedge R \arrow[d, "D^{\mathcal{O}}_{n}(y_k)\wedge 1_R"] & \\
& D^{\mathcal{O}}_{n}(\mathscr{B}_{m+} \wedge R) \wedge R \arrow[d, "\partial_n \wedge 1"'] & \\
& D^{\mathcal{O}}_{n}(\mathscr{B}_{m+}) \wedge D^{\mathcal{O}}_{n}(R) \wedge R \arrow[r, "1\wedge\mu^R(\theta^R_n \wedge 1)"'] & D^{\mathcal{O}}_{n}(\mathscr{B}_{m+}) \wedge R
\end{tikzcd}$$

Remark 6.48. Here, we make use of the fact that $|y_k| = kV+|y_0|$ is an isomorphism class of a non-virtual G-representation. This condition allows for the existence of a diagonal map $S^{|y_k|} \longrightarrow S^{n|y_k|}$ which induces Δ'_k in the diagram above. Note that the map Δ'_k is nothing but a suspension of the zero section of the bundle $\widetilde{\gamma}^{(n)}_{|y_k|}$.

Lemma 6.49. *For all $k \in \mathbb{N}$, we have:*

$$(x_{m(k+1)} \circ y_{k+1}) \frown \mathbf{e}_{xy} = x_{mk} \circ y_k \tag{6.50}$$

Proof. We will prove this result by showing that the diagram in (6.51) commutes. In this diagram, the map labeled

(1) is induced by $\Delta : \mathscr{B}_n \longrightarrow \mathscr{B}_n \times \mathscr{B}_n$;

(2) is the composite map:

$$\begin{tikzcd}
\Sigma^{|y_{k+1}|}\mathscr{B}_{n+} \wedge \mathscr{B}_{n+} \wedge R \arrow[d, equal] \\
\Sigma^{|y_k|}\mathscr{B}_{n+} \wedge \Sigma^{(n-1)V}\mathscr{B}_{n+} \wedge R \arrow[d, "\Delta'_k \wedge e_y^{m-1} \wedge 1_R"] \\
\Sigma^{(n-1)V}D^{\mathcal{O}}_{n}(S^{y_k}) \wedge \Sigma^{(m-1)(n-1)V}\mathscr{B}_{n+} \wedge R \wedge R \arrow[d, "1\wedge\mu"] \\
\Sigma^{(n-1)V}D^{\mathcal{O}}_{n}(S^{y_k}) \wedge \Sigma^{(m-1)(n-1)V}\mathscr{B}_{n+} \wedge R \arrow[d, phantom, "\cong"] \\
\Sigma^{m(n-1)V}D^{\mathcal{O}}_{n}(S^{y_k}) \wedge \mathscr{B}_{n+} \wedge R
\end{tikzcd}$$

(3) is $\sigma_{(mn-1)V}(\Delta'_k) \wedge 1_R$;

(4) is induced by T_{m-1} in (6.45);

(5) is induced by $\mu(e_y \wedge 1) : \mathscr{B}_{n+} \wedge R \to \Sigma^{(n-1)V}R \wedge R \to \Sigma^{(n-1)V}R$;

(6) is induced by the composition:

$$\begin{tikzcd}
\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_{n+}\wedge \mathrm{R}) \arrow[d, "\partial^{\mathrm{R}}_{\mathsf{n}}\wedge 1_{\mathrm{R}}"'] \\
\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_{n+})\wedge \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{R}) \arrow[d, "\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\Delta)\wedge\theta^{\mathrm{R}}_{\mathsf{n}}"'] \\
\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_{n+}\wedge \mathscr{B}_{n+})\wedge \mathrm{R} \arrow[d, "\partial_{\mathsf{n}}\wedge 1_{\mathrm{R}}"'] \\
\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_{n+})\wedge \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_{n+})\wedge \mathrm{R} \arrow[d, "1\wedge\zeta^{(n)}_{(m-1)\mathrm{V}}\wedge 1_{\mathrm{R}}"'] \\
\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_{n+})\wedge \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{S}^{(m-1)\mathrm{V}})\wedge \mathrm{R};
\end{tikzcd}$$

(7) is the composition of the last two maps in (6.47);

(8) is the composition:

$$\begin{tikzcd}
\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_{n+})\wedge \mathrm{R} \arrow[d, "\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\Delta)"'] \\
\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_{n+}\wedge \mathscr{B}_{n+})\wedge \mathrm{R} \arrow[d, "\partial_{\mathsf{n}}"'] \\
\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_{n+})\wedge \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_{n+})\wedge \mathrm{R} \arrow[d, "1\wedge\zeta^{(n)}_{(m-1)\mathrm{V}}\wedge 1_{\mathrm{R}}"'] \\
\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathscr{B}_{n+})\wedge \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathrm{S}^{(m-1)\mathrm{V}})\wedge \mathrm{R}
\end{tikzcd}$$

Now observe that:

(S1) commutes because $\mathsf{y}_{m(k+1)} \frown \mathsf{e}_y^m = \mathsf{y}_{mk}$ (Eulerian of χ_2);

(S2) commutes by inspection;

(S3) commutes by the relation between Euler class and Thom class;

(S4) commutes because $\mathsf{e}_y \smile \mathsf{e}_y^{m-1} = \mathsf{e}_y^m$;

(S5) commutes because $\mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathsf{x}_k \frown \mathsf{e}_x) = \mathrm{D}^{\mathcal{O}}_{\mathsf{n}}(\mathsf{x}_{k-1})$ (Eulerian of χ_1);

(**S6**) – (**S10**) commute by naturality. The composition of the blue arrows and the red arrows yields the two sides of (6.50), proving the result.

(6.51)

$$\begin{tikzcd}
\mathrm{S}^{|\mathsf{x}_{k+1}m|+|\mathsf{y}_{(k+1)}|} \arrow[rrr, "="] \arrow[d, "\mathsf{x}_{m(k+1)}"'] \arrow[drrr, phantom, "\mathbf{(S1)}"] & & & \Sigma^{(mn-1)\mathrm{V}}\mathrm{S}^{|\mathsf{x}_{mk}|+|\mathsf{y}_k|} \arrow[d, "\sigma_{(mn-1)\mathrm{V}}(\mathsf{x}_{mk})"] \\
\Sigma^{|y_{k+1}|}\mathscr{B}_{n+} \wedge \mathrm{R} \arrow[r, "(1)"] \arrow[d, "\Delta'_{k+1}"'] \arrow[dr, phantom, "\mathbf{(S2)}"] & \Sigma^{|y_{k+1}|}\mathscr{B}_{n+} \wedge \mathscr{B}n+ \wedge \mathrm{R} \arrow[r, "1\wedge \mathsf{e}_{\mathsf{y}}^m\wedge 1"] \arrow[d, "(2)"'] \arrow[dr, "(3)"] \arrow[dr, phantom, "\mathbf{(S3)}"'] & \Sigma^{|y_{k+1}|}\mathscr{B}_{n+} \wedge \Sigma^{m(n-1)\mathrm{V}}\mathrm{R} \wedge \mathrm{R} \arrow[r, "1\wedge\mu"] \arrow[dr, phantom, "\mathbf{(S4)}"] & \Sigma^{(mn-1)\mathrm{V}}\Sigma^{|\mathsf{y}_k|}\mathscr{B}_{n+} \wedge \mathrm{R} \arrow[d, "\sigma_{(mn-1)\mathrm{V}}(\Delta'_k)\wedge 1_{\mathrm{R}}"] \\
\mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathrm{S}^{|y_{k+1}|}) \wedge \mathrm{R} \arrow[r, "\partial_{\mathsf{n}}\wedge 1_{\mathrm{R}}"'] \arrow[d, "\mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathsf{y}_{k+1})\wedge 1"'] \arrow[dr, phantom, "\mathbf{(S5)}"] & \mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathrm{S}^{|y_k|}) \wedge \mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathrm{S}^{(m-1)\mathrm{V}}) \wedge \mathrm{R} \arrow[r, "(4)"'] \arrow[d, "\mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathsf{y}_k)\wedge 1"'] \arrow[dr, phantom, "\mathbf{(S6)}"] & \mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathrm{S}^{|y_k|}) \wedge \Sigma^{n(m-1)\mathrm{V}}\mathscr{B}_{n+} \wedge \mathrm{R} \arrow[r, "(5)"'] \arrow[d, "\mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathsf{y}_k)\wedge 1"'] \arrow[dr, phantom, "\mathbf{(S7)}"] & \Sigma^{(mn-1)\mathrm{V}}\mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathrm{S}^{|\mathsf{y}_k|}) \wedge \mathrm{R} \arrow[d, "\mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathsf{y}_k)\wedge 1"] \\
\mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathscr{B}_{m+} \wedge \mathrm{R}) \wedge \mathrm{R} \arrow[r, "(6)"'] \arrow[d, "(7)"'] \arrow[dr, phantom, "\mathbf{(S8)}"] & \mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathscr{B}_{m+} \wedge \mathrm{R}) \wedge \mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathrm{S}^{(m-1)\mathrm{V}}) \wedge \mathrm{R} \arrow[r, "(4)"'] \arrow[d, "(7)"'] \arrow[dr, phantom, "\mathbf{(S9)}"] & \mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathscr{B}_{m+} \wedge \mathrm{R}) \wedge \Sigma^{n(m-1)\mathrm{V}}\mathscr{B}_{n+} \wedge \mathrm{R} \arrow[r, "(5)"'] \arrow[d, "(7)"'] \arrow[dr, phantom, "\mathbf{(S10)}"] & \Sigma^{(mn-1)\mathrm{V}}\mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathscr{B}_{m+} \wedge \mathrm{R}) \wedge \mathrm{R} \arrow[d, "(7)"] \\
\mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathscr{B}_{m+}) \wedge \mathrm{R} \arrow[r, "(8)"'] & \mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathscr{B}_{m+}) \wedge \mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathrm{S}^{(m-1)\mathrm{V}}) \wedge \mathrm{R} \arrow[r, "(4)"'] & \mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathscr{B}_{m+}) \wedge \Sigma^{n(m-1)\mathrm{V}}\mathscr{B}_{n+} \wedge \mathrm{R} \arrow[r, "(5)"'] & \Sigma^{(mn-1)\mathrm{V}}\mathrm{D}_{\mathsf{n}}^{\mathcal{O}}(\mathscr{B}_{m+}) \wedge \mathrm{R}
\end{tikzcd}$$

□

Definition 6.52 (**Product Law**)**.** Suppose V is an orthogonal G-representation that contains a trivial subrepresentation. We define a **product operation**

$$\odot : \mathcal{E}^{(n)}_{\mathrm{R,V}} \times \mathcal{E}^{(m)}_{\mathrm{R,V}} \longrightarrow \mathcal{E}^{(nm)}_{\mathrm{R,V}} \tag{6.53}$$

for $n, m \geq 2$ as follows: Given V-stable R-Eulerian sequences $\chi^{(1)} = (\mathsf{x}_1, \mathsf{x}_2, \dots) \in \mathcal{E}^{(n)}_{\mathrm{R,V}}$ and $\chi^{(2)} = (\mathsf{y}_1, \mathsf{y}_2, \dots) \in \mathcal{E}^{(m)}_{\mathrm{R,V}}$ define

$$\chi^{(1)} \odot \chi^{(2)} = (\mathsf{z}_0, \mathsf{z}_1, \dots) := (\iota_*(\mathsf{x}_0 \circ \mathsf{y}_0), \iota_*(\mathsf{x}_m \circ \mathsf{y}_1), \iota_*(\mathsf{x}_{2m} \circ \mathsf{y}_2), \cdots).$$

From (6.46) and (6.49), we conclude that $\chi^{(1)} \odot \chi^{(2)}$ is a V-stable R-Eulerian sequence of weight nm.

Theorem 6.54. *Let $\chi^{(1)}$ and $\chi^{(2)}$ be* V*-stable* R*-Eulerian sequences of weight m and n, respectively. Then, $\chi^{(1)} \odot \chi^{(2)}$ is an Eulerian sequence of weight mn such that*

$$\mathfrak{S}^{\chi^{(1)} \odot \chi^{(2)}} = \mathfrak{S}^{\chi^{(1)}} \circ \mathfrak{S}^{\chi^{(2)}}.$$

Proof. The result follows from a direct inspection using (4.15) and (6.47). ☐

Remark 6.55. We may extend the definition of V-stable R-Eulerian sequence to weight 1 using $\widetilde{\tau}_1 = 0$ in (4.7), thereby setting $\widetilde{\gamma}^{(1)}_{\mathrm{V}}$ as the 0-dimensional bundle over $\mathscr{B}_1 \simeq *$. Then the R-Euler class of $\widetilde{\gamma}^{(1)}_{\mathrm{V}}$ is the identity element $1 \in \pi^{\mathrm{G}}_0(\mathrm{R}) \cong \mathrm{R}^0(\mathscr{B}_{1+})$. Thus, for any $a \in \pi^{\mathrm{G}}_{\mathrm{V}}(\mathrm{R})$, where V is a non-virtual,

$$\mathbf{a} = (a, a, \dots)$$

is an R-Eulerian sequence of weight 1. We extend the product $\odot$ in (6.52) by allowing n and m to equal 1. Under such pairing the weight 1 Eulerian sequence

$$\mathbf{1} = (1, 1, \dots)$$

satisfies the identity condition namely $\mathbf{1} \odot \chi = \chi = \chi \odot \mathbf{1}$.

Remark 6.56. It is easy to see that the point-wise sum turns $\mathcal{E}^{(n)}_{\mathrm{R,V}}$ into an Abelian group for each n and the product law $\odot$ gives the collection

$$\mathcal{E}_{\mathrm{R,V}} = \bigsqcup_{n \geq 1} \mathcal{E}^{(n)}_{\mathrm{R,V}}$$

a structure of a ring. In fact, it is a bimodule over the ring $\pi^{\mathrm{G}}_{\star}(\mathrm{R})$, where $\star$ varies over non-virtual orthogonal G-representations. Further, by (6.54) we get a ring homomorphism

$$\mathfrak{S}^{(-)} : \mathcal{E}_{\mathrm{R,V}} \longrightarrow [\mathrm{R}, \mathrm{R}]^{\mathrm{G}}_{-\star}.$$

Thus it is natural to ask if the image of $\mathfrak{S}^{(-)}$ generate the collection of all stable R-cohomology operation in the universe generated by V. We expect this to be the case when $\mathrm{R} = \mathrm{H}\underline{\mathbb{F}}_p$ and $\mathrm{V} = \rho_{\mathrm{G}}$ as outlined in (1.5).

7. New equivariant cohomology operations

In this section, we construct genuine stable $\mathrm{H}\underline{\mathbb{F}}_p$-cohomology operations by identifying ρ_{G}-stable Eulerian sequences in $\mathrm{H}^{\mathrm{G}}_{\star}\left((\mathrm{B}_{\mathrm{G}}\Sigma_p)_+;\underline{\mathbb{F}}_p\right)$ for all finite G and primes p.

To find $\mathrm{H}\underline{\mathbb{F}}_p$-Eulerian sequences, first we need to fix an $\mathrm{H}\underline{\mathbb{F}}_p$-Euler class of the n-fold sum of the G-vector bundle (see (4.24))

$$\widetilde{\gamma}_{\mathrm{G},p} := \begin{tikzcd} \mathrm{E}_{\mathrm{G}}\Sigma_p \times_{\Sigma_p} (\rho_{\mathrm{G}} \otimes \widetilde{\tau}_p) \arrow[d] \\ \mathrm{B}_{\mathrm{G}}\Sigma_p \end{tikzcd}$$

for some nonzero $n \in \mathbb{N}$.

Remark 7.1. The G-bundle $\widetilde{\gamma}_{\mathrm{G},p}$ is nothing but $\widetilde{\gamma}^{\mathcal{A}\ell\ell_p}_{\rho_{\mathrm{G}}}$ according to (3.1).

Notation 7.2. When underlying group G is clear from the context, we will simply use $\widetilde{\gamma}_p$ to denote $\widetilde{\gamma}_{\mathrm{G},p}$.

By [BZ24, Theorem 1.12], $\widetilde{\gamma}_{\mathrm{G},p}$ is a homogeneous bundle, therefore, we use the equivariant first Steifel Whitney class (as in [BZ24, Definition 3.2]) to detect orientation. When $p = 2$, any homogeneous bundle is $\mathrm{H}\underline{\mathbb{F}}_2$-orientable as $\mathbb{F}_2^{\times}$ is the trivial group. In particular, $\widetilde{\gamma}_{\mathrm{G},2}$ is $\mathrm{H}\underline{\mathbb{F}}_2$-orientable.

Proposition 7.3. *When $p > 2$, $\widetilde{\gamma}_{\mathrm{G},p}$ is $\mathrm{H}\underline{\mathbb{Z}}$-orientable if and only if $|\mathrm{G}|$ is even.*

Proof. In this proof, we subscribe to (7.2) and write $\widetilde{\gamma}_p$ as $\widetilde{\gamma}_{\mathrm{G},p}$. The obstruction to $\widetilde{\gamma}_{\mathrm{G},p}$ being $\mathrm{H}\underline{\mathbb{Z}}$-orientable is the first equivariant Stiefel-Whitney class

$$w_1^{\mathrm{H}\underline{\mathbb{Z}}}(\widetilde{\gamma}_{\mathrm{G},p}) \in \mathrm{H}^1(\mathrm{B}_{\mathrm{G}}\Sigma_p;\underline{\mathbb{F}}_2)$$

according to [BZ24, Theorem 1.15]. Since the restriction of $\widetilde{\gamma}_{\mathrm{G},p}$ to the trivial group is $|\mathrm{G}|\widetilde{\gamma}_{\mathsf{e},p}$, we conclude

$$\mathsf{res}^{\mathrm{G}}_{\mathsf{e}}(w_1^{\mathrm{H}\underline{\mathbb{Z}}}(\widetilde{\gamma}_{\mathrm{G},p})) = |\mathrm{G}|\, w_1^{\mathrm{H}\mathbb{Z}}(\widetilde{\gamma}_{\mathsf{e},p}),$$

which is nonzero when $|\mathrm{G}|$ is odd.

Conversely, suppose $|\mathrm{G}|$ is even and fix $\mathrm{K} \subset \mathrm{G}$ of order 2. Using [BZ24, Theorem 3.14] and the fact that $\widetilde{\gamma}_{\mathrm{G},p}$ is induced up from $\widetilde{\gamma}_{\mathsf{e},p}$ (as in [BZ24, Notation 3.13]), we conclude

$$w_1^{\mathrm{H}\underline{\mathbb{Z}}}(\widetilde{\gamma}_{\mathrm{G},p}) = \mathsf{tr}^{\mathrm{G}}_{\mathsf{e}}\, w_1^{\mathrm{H}\mathbb{Z}}(\widetilde{\gamma}_{\mathsf{e},p}) = \mathsf{tr}^{\mathrm{G}}_{\mathrm{K}}\, \mathsf{tr}^{\mathrm{K}}_{\mathsf{e}}\, w_1^{\mathrm{H}\mathbb{Z}}(\widetilde{\gamma}_{\mathsf{e},p}).$$

It is a standard fact that $w_1^{\mathrm{H}\mathbb{Z}}(\widetilde{\gamma}_{\mathsf{e},p})$ is nonzero and can be written as $\mathsf{sgn}^*(\iota)$, where

$$\mathsf{sgn}^* : \mathbb{Z}/2\langle\iota\rangle \cong \mathrm{H}^1(\mathrm{B}\Sigma_{2+};\mathbb{F}_2) \xrightarrow{\cong} \mathrm{H}^1(\mathrm{B}\Sigma_{p+};\mathbb{F}_2)$$

is the isomorphism induced by the sign homomorphism $\mathsf{sgn} : \Sigma_p \to \Sigma_2$ and ι is the generator as indicated. Thus,

$$\begin{aligned} \mathsf{tr}^{\mathrm{K}}_{\mathsf{e}}\, w_1^{\mathrm{H}\mathbb{Z}}(\widetilde{\gamma}_{\mathsf{e},p}) &= \mathsf{tr}^{\mathrm{K}}_{\mathsf{e}}\, \mathsf{sgn}^*(\iota) \\ &= \mathsf{sgn}^*\, \mathsf{tr}^{\mathrm{K}}_{\mathsf{e}}(\iota) \\ &= 0 \end{aligned}$$

as $\mathrm{H}^1_{\mathrm{K}}(\mathrm{B}_{\mathrm{K}}\Sigma_{2+};\underline{\mathbb{F}}_2) = 0$ by [HK01]. □

Remark 7.4. The proposition above, combined with the fact that the ring $\mathbb{Z} \to \mathbb{F}_p$ induce an injection on units

$$\mathbb{Z}^\times \hookrightarrow \mathbb{F}_p^\times$$

when p is odd, shows that $\widetilde{\gamma}_{G,p}$ is not $H\underline{\mathbb{F}}_p$-orientable when p and $|G|$ are odd. However, the 2-fold sum of $\widetilde{\gamma}_{G,p}$ is $H\underline{\mathbb{F}}_p$-orientable for all G by [BZ24, Theorem 1.19]). Thus, when p and $|G|$ are odd, we will work with of $2\widetilde{\gamma}_{G,p}$.

Notation 7.5. For the rest of the section we fix a finite group G and a prime p, and let

- $\epsilon = [1 + (|G| \mod 2)](p-1)/2$, i.e.,
$$\epsilon = \begin{cases} (p-1)/2 & \text{if } |G| \text{ is even} \\ (p-1) & \text{if } |G| \text{ is odd,} \end{cases}$$
- $\widetilde{\mathbf{u}}_{G,2}$ denote an $H\underline{\mathbb{F}}_2$-Thom class of $\widetilde{\gamma}_{G,2}$,
- $\widetilde{\mathbf{u}}_{G,p}$ denote an $H\underline{\mathbb{F}}_p$-Thom class of $(1 + |G| \mod 2)\widetilde{\gamma}_{G,p}$ when $p \neq 2$,
- $\widetilde{\mathbf{e}}_{G,p}$ denote the $H\underline{\mathbb{F}}_p$-Euler class corresponding to $\widetilde{\mathbf{u}}_{G,p}$ at all prime p.

Remark 7.6. Note that $\widetilde{\mathbf{e}}_G \in H_G^{\epsilon(p-1)\rho_G}(B_G\Sigma_p; \underline{\mathbb{F}}_p)$ is a nonzero class as its image under the restriction map to trivial group is nonzero.

At the prime 2, we filter $B_G\Sigma_2$ as

$$* \hookrightarrow \mathbb{P}(\rho) \hookrightarrow \mathbb{P}(2\rho) \hookrightarrow \mathbb{P}(3\rho) \hookrightarrow \cdots \hookrightarrow B_G\Sigma_2, \tag{7.7}$$

where

$$\mathbb{P}(-) := S(- \otimes \widetilde{\tau}_2) \times_{\Sigma_2} *$$

denotes the G-equivariant projective space as a functor of G-representations. This results in an Atiyah-Hirzebruch like spectral sequence

$$E^1_{*,V} := \bigoplus_{n\in\mathbb{N}}^{\infty} H_V^G(\mathbb{P}((n+1)\rho)/\mathbb{P}(n\rho); \underline{\mathbb{F}}_2) \Longrightarrow H_V^G(B_G\Sigma_{2+}; \underline{\mathbb{F}}_2) \tag{7.8}$$

calculating the homology of $H_V^G(B_G\Sigma_{2+}; \underline{\mathbb{F}}_2)$.

Notation 7.9. Fix an injection $\kappa : C_p \hookrightarrow \Sigma_p$ and let

$$\kappa : B_GC_p \hookrightarrow B_G\Sigma_p \tag{7.10}$$

denote the map induced by κ on the classifying spaces. Note that the map κ, up to homotopy, is independent of the choice of the injection κ. This is because all injections from C_p to Σ_p are conjugates of each other.

When p is odd, we focus on identifying $H\underline{\mathbb{F}}_p$-Eulerian sequence in $H_\star^G(B_GC_p; \underline{\mathbb{F}}_p)$ defined using the Euler class $\kappa^*\widetilde{\mathbf{e}}_{G,p}$ of the bundle $\epsilon\kappa^*\widetilde{\gamma}_{G,p}$. This is because if $\chi = (x_0, x_1, \dots)$ is a $H\underline{\mathbb{F}}_p$-Eulerian sequence in $H_\star^G(B_GC_p; \underline{\mathbb{F}}_p)$ then

$$\kappa_*\chi = (\kappa_*x_0, \kappa_*x_1, \dots)$$

is a $H\underline{\mathbb{F}}_p$-Eulerian sequence in $H_\star^G(B_G\Sigma_p; \underline{\mathbb{F}}_p)$ as

$$\kappa_*(x_{i+1}) \frown \widetilde{\mathbf{e}}_{G,p} = \kappa_*(x_{i+1} \frown \kappa^*\widetilde{\mathbf{e}}_{G,p}) = \kappa_*(x_i).$$

Notation 7.11. Let $\rho_{G,\mathbb{C}}$ denote the complex regular representation of G. We will simply use $\rho_{\mathbb{C}}$ to denote $\rho_{G,\mathbb{C}}$ when the group G is clear from the context.

Similar to (7.7), B_GC_p admits a filtration
(7.12)

$$* \hookrightarrow \mathbb{L}_p(\epsilon\rho_{\mathbb{C}}) \hookrightarrow \mathbb{L}_p(2\epsilon\rho_{\mathbb{C}}) \hookrightarrow \mathbb{L}_p(3\epsilon\rho_{\mathbb{C}}) \hookrightarrow \cdots \hookrightarrow B_GC_p,$$

where

$$\mathbb{L}_p(-) := S(- \otimes_{\mathbb{C}} \mathsf{r}_{2\pi/p}) \times_{C_p} *$$

denotes the G-equivariant lens space as a functor of complex G-representations defined using $\mathsf{r}_{2\pi/p}$, the rotation by $2\pi/p$ representation of C_p. This yields an Atiyah-Hirzebruch like spectral sequence

$$\text{(7.13)} \qquad E^1_{*,V} := \bigoplus_{n\in\mathbb{N}}^{\infty} H^G_V(\mathbb{L}_p(\epsilon(n+1)\rho_{\mathbb{C}})/\mathbb{L}_p(\epsilon n\rho_{\mathbb{C}}); \underline{\mathbb{F}}_p) \Longrightarrow H^G_V(B_GC_{p+}; \underline{\mathbb{F}}_p)$$

calculating the homology groups of B_GC_p.

To identify classes in first pages of (7.8) and (7.13) we pause briefly to discuss the equivariant analogs of some of the classical results of Atiyah [Ati61] on projective spaces.

7.1. Some equivariant analogs of Atiyah's result

In 1961, Atiyah proved that the quotient $\mathbb{RP}^{n+k}/\mathbb{RP}^n$ is the Thom complex of the n-fold sum of the tautological bundle over $\mathbb{RP}^k$ [Ati61, Proposition 4.3]. The equivariant analog of this result takes the following form.

Proposition 7.14. *There exist a* G*-equivariant homeomorphism*

$$\mathsf{Th}\left(n\widetilde{\gamma}^{(k)}_{G,2}\right) \cong \mathbb{P}((n+k)\rho)/\mathbb{P}(n\rho)$$

for all $n, k \in \mathbb{N}$, *where* $\widetilde{\gamma}^{(k)}_{G,2}$ *is the pullback of* $\widetilde{\gamma}_{G,2}$ *along* $\mathbb{P}(k\rho) \hookrightarrow B_G\Sigma_2$.

Proof. We first note that $\widetilde{\gamma}^{(k)}_{G,2}$ can be explicitly described as

$$\widetilde{\gamma}^{(k)}_{G,2} := \begin{tikzcd} S(k\rho \otimes \widetilde{\tau}_2) \times_{\Sigma_2} (\rho \otimes \widetilde{\tau}_2) \arrow[d] \\ \mathbb{P}(k\rho), \end{tikzcd}$$

where S(V) is the space of subspace of unit length vectors in V. Let

$$D(V) := \{v \in V : \|v\| \leq 1\}$$

denote the subspace of V consisting of vectors of length less than or equal to 1. Note than when V is an orthogonal representation then both S(V) and D(V) inherits the action of the ambient group. Thus, $S(k\rho \otimes \widetilde{\tau}_2)$ and $D(k\rho \otimes \widetilde{\tau}_2)$ are $G \times \Sigma_2$-spaces.

Now notice that the $G \times \Sigma_2$-equivariant map

$$f : S(k\rho \otimes \widetilde{\tau}_2) \times D(n\rho \otimes \widetilde{\tau}_2) \to S((n+k)\rho \otimes \widetilde{\tau}_2)$$

defined by the formula $f(v,w) = \left(w, v\sqrt{1-\|w\|^2}\right)$, restricts to an equivariant homeomorphism

$$\mathrm{S}(k\rho\otimes\widetilde{\tau}_2)\times(\mathrm{D}(n\rho\otimes\widetilde{\tau}_2)-\mathrm{S}(n\rho\otimes\widetilde{\tau}_2))\to\mathrm{S}((n+k)\rho\otimes\widetilde{\tau}_2)-\mathrm{S}(n\rho\otimes\widetilde{\tau}_2).$$

Taking Σ_2-orbits, we get a G-equivariant homeomorphism

$$\mathrm{D}\left(n\widetilde{\gamma}_2^{(k)}\right)-\mathrm{S}\left(n\widetilde{\gamma}_2^{(k)}\right)\to\mathbb{P}((n+k)\rho)-\mathbb{P}(n\rho),$$

where D(−) and S(−) on the left hand side are the unit disk bundle and the unit sphere bundle functors. By one-point compactifying the map above, we get the desired homeomorphism. ☐

Thus, we have the following $\mathrm{H}\underline{\mathbb{F}}_2$-Thom isomorphism.

Corollary 7.15. *For any finite group* G

$$\mathrm{H}^{\mathrm{G}}_{\mathrm{V}+n\rho}(\mathbb{P}(n+1)\rho)/\mathbb{P}(n\rho);\underline{\mathbb{F}}_2)\cong\mathrm{H}^{\mathrm{G}}_{\mathrm{V}}(\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2).$$

An analogous result can be obtained for equivariant lens spaces using the bundle

$$\widetilde{\omega}^{(k)}_{\mathrm{G},p} := \begin{tikzcd} \mathrm{S}(k\rho_{\mathbb{C}}\otimes_{\mathbb{C}}\mathsf{r}_{2\pi/p})\times_{\mathrm{C}_p}(\rho_{\mathbb{C}}\otimes_{\mathbb{C}}\mathsf{r}_{2\pi/p}) \arrow[d] \\ \mathbb{L}_p(k\rho), \end{tikzcd} \tag{7.16}$$

where $\rho_{\mathbb{C}}$ is the complex regular representation of G.

Remark 7.17. For a complex $\mathrm{G}\times\mathrm{C}_p$-representation V, let $\mathsf{u}(\mathrm{V})$ denote its underlying real representation. Then

$$\mathsf{u}(\rho_{\mathbb{C}}\otimes_{\mathbb{C}}\mathsf{r}_{2\pi/p})\cong\mathsf{u}\left((\rho\otimes\mathbb{C})\otimes_{\mathbb{C}}\mathsf{r}_{2\pi/p}\right)\cong\mathsf{u}\left(\rho\otimes(\mathbb{C}\otimes_{\mathbb{C}}\mathsf{r}_{2\pi/p})\right)\cong\rho\otimes\mathsf{u}(\mathsf{r}_{2\pi/p}).$$

Thus $\widetilde{\omega}^{(k)}_{\mathrm{G},p}$ is a bundle of rank 2ρ in the sense of [BZ24].

Proposition 7.18. *There exist a* G*-equivariant homeomorphism*

$$\mathsf{Th}\left(n\,\widetilde{\omega}^{(k)}_{\mathrm{G},p}\right)\cong\mathbb{L}_p((n+k)\rho_{\mathbb{C}})/\mathbb{L}_p(n\rho_{\mathbb{C}})$$

for all $n,k\in\mathbb{N}$, *where* $\widetilde{\omega}^{(k)}_{\mathrm{G},p}$ *is the pullback of* $\widetilde{\omega}_{\mathrm{G},p}$ *as in* (7.16).

Proof. Notice that the $\mathrm{G}\times\mathrm{C}_p$-equivariant map

$$f:\mathrm{S}(k\rho_{\mathbb{C}}\otimes_{\mathbb{C}}\mathsf{r}_{2\pi/p})\times\mathrm{D}(n\rho_{\mathbb{C}}\otimes_{\mathbb{C}}\mathsf{r}_{2\pi/p})\longrightarrow\mathrm{S}((n+k)\rho_{\mathbb{C}}\otimes_{\mathbb{C}}\mathsf{r}_{2\pi/p})$$

defined by the formula $f(v,w) = \left(w, v\sqrt{1-\|w\|^2}\right)$, restricts to an equivariant homeomorphism

$$\begin{tikzcd} \mathrm{S}(k\rho_{\mathbb{C}}\otimes_{\mathbb{C}}\mathsf{r}_{2\pi/p})\times\left(\mathrm{D}(n\rho_{\mathbb{C}}\otimes_{\mathbb{C}}\mathsf{r}_{2\pi/p}\right)-\mathrm{S}(n\rho_{\mathbb{C}}\otimes_{\mathbb{C}}\mathsf{r}_{2\pi/p})) \arrow[d] \\ \mathrm{S}((n+k)\rho_{\mathbb{C}}\otimes_{\mathbb{C}}\mathsf{r}_{2\pi/p})-\mathrm{S}(n\rho_{\mathbb{C}}\otimes_{\mathbb{C}}\mathsf{r}_{2\pi/p}). \end{tikzcd}$$

Taking C_p-orbits, we get a G-equivariant homeomorphism

$$\mathrm{D}\left(n\widetilde{\omega}_2^{(k)}\right)-\mathrm{S}\left(n\widetilde{\omega}_2^{(k)}\right)\longrightarrow\mathbb{L}((n+k)\rho_{\mathbb{C}})-\mathbb{L}(n\rho_{\mathbb{C}}).$$

By one-point compactifying the map above, we get the desired homeomorphism. ☐

Since $2\widetilde{\omega}^{(k)}_{G,p}$ is $H\underline{\mathbb{F}}_p$-orientable by (7.4), we conclude:

Corollary 7.19. *For any finite group* G

$$H^{G}_{V+4n\rho}\left(\mathbb{L}_p((2n+k)\rho_{\mathbb{C}})/\mathbb{L}_p(2n\rho_{\mathbb{C}});\underline{\mathbb{F}}_p\right) \cong H^{G}_{V}(\mathbb{L}_p(k\rho_{\mathbb{C}})_+;\underline{\mathbb{F}}_p).$$

Using (7.15), we may rewrite the spectral sequence (7.8) as

$$E^1_{*,V} := \bigoplus_{k\in\mathbb{N}}^{\infty} H^{G}_{V}(\Sigma^{k\rho}\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2) \Longrightarrow H^{G}_{V}(B_G\Sigma_{2+};\underline{\mathbb{F}}_2). \tag{7.20}$$

Likewise, we use (7.19) to rewrite (7.13) as

$$E^1_{*,V} := \bigoplus_{k\in\mathbb{N}}^{\infty} H^{G}_{V}(\Sigma^{2\epsilon k\rho}\mathbb{L}_p(\epsilon\rho_{\mathbb{C}})_+;\underline{\mathbb{F}}_p) \Longrightarrow H^{G}_{V}(B_G C_{p+};\underline{\mathbb{F}}_p) \tag{7.21}$$

when p is an odd prime.

Atiyah showed that [Ati61, Lemma 4.5] the sum of the tangent bundle $T\mathbb{RP}^{n-1}$ of $\mathbb{RP}^{n-1} := \mathbb{P}(\mathbb{R}^n)$ and a 1-dimensional trivial bundle is isomorphic to the n-fold Whitney sum $n\widetilde{\gamma}^{(n)}_2$ of the tautological line bundle. Using this and Atiyah duality [Ati61, Proposition 3.2], one can reproduce Poincaré duality

$$\begin{aligned} H^k((\mathbb{RP}^{n-1})_+;\mathbb{F}_2) &\cong H_{-k}(\mathsf{Th}(-T\mathbb{RP}^{n-1});\mathbb{F}_2) \\ &\cong H_{-k}(\mathsf{Th}(-n\widetilde{\gamma}^{(n)}_2 + \epsilon_1);\mathbb{F}_2) \\ &\cong H_{-k}(\Sigma^{1-n}(\mathbb{RP}^{n-1})_+;\mathbb{F}_2) \\ &\cong H_{n-1-k}((\mathbb{RP}^{n-1})_+;\mathbb{F}_2), \end{aligned}$$

where the third isomorphism is the evident $H\mathbb{F}_2$-Thom isomorphism.

We would now like to establish an equivariant analog of [Ati61, Lemma 4.5] to establish the equivariant Poincaré duality results for $\mathbb{P}(n\rho)$ and $\mathbb{L}_p(n\rho)$.

Lemma 7.22. *The following are isomorphisms of* G*-equivariant vector bundles:*

(1) $T\mathbb{P}(k\rho) \oplus \epsilon_1 \cong k\widetilde{\gamma}^{(k)}_2$.

(2) $T\mathbb{L}_p(k\rho) \oplus \epsilon_1 \cong k\widetilde{\omega}^{(k)}_p$ *when p is odd.*

Proof. Note that the normal bundle of $\mathbb{S}(k\rho \otimes \widetilde{\tau}_2) \hookrightarrow k\rho \otimes \widetilde{\tau}_2$ is isomorphic to ϵ_1, the trivial 1-dimensional real $(G \times \Sigma_2)$- vector bundle. Therefore, we get $(G \times \Sigma_2)$-equivariant isomorphism

$$TS(k\rho \otimes \widetilde{\tau}_2) \oplus \epsilon_1 \cong S(k\rho \otimes \widetilde{\tau}_2) \times (k\rho \otimes \widetilde{\tau}_2)$$

of $(G \times \Sigma_2)$-vector bundle. By taking Σ_2-orbit, we get *(1)*.

Similarly, the homeomorphism of *(2)* is the map on the C_p-orbits of the $(G \times C_p)$-isomorphism

$$TS(k\rho_{\mathbb{C}} \otimes_{\mathbb{C}} \mathsf{r}_{2\pi/p}) \oplus \epsilon_1 \cong S(k\rho_{\mathbb{C}} \otimes_{\mathbb{C}} \mathsf{r}_{2\pi/p}) \times (k\rho_{\mathbb{C}} \otimes_{\mathbb{C}} \mathsf{r}_{2\pi/p}),$$

where ϵ_1 is the trivial 1 dimensional real $(G \times C_p)$-representation. ☐

Lemma 7.23 (Poincaré duality)**.** *Let* G *be a finite group. Then there are isomorphisms of Abelian groups*

(1) $\mathrm{H}^{\star}_{\mathrm{G}}(\mathbb{P}(k\rho)_+;\underline{\mathbb{F}}_2) \cong \mathrm{H}^{\mathrm{G}}_{k\rho-1-\star}(\mathbb{P}(k\rho)_+;\underline{\mathbb{F}}_2)$

(2) $\mathrm{H}^{\mathrm{G}}_{\star}(\mathbb{L}_p(\epsilon k\rho_{\mathbb{C}})_+;\underline{\mathbb{F}}_p) \cong \mathrm{H}^{2\epsilon k\rho-1-\star}_{\mathrm{G}}(\mathbb{L}_p(\epsilon k\rho_{\mathbb{C}})_+;\underline{\mathbb{F}}_p)$

for all $k \in \mathbb{N}$*.*

Proof. Since the tangent bundle of $\mathbb{P}(k\rho)$ is homogeneous [BZ24, Lemma 2.21], we may combine equivariant Atiyah duality [May96, XV1§8] with (7.22) to obtain

$$\begin{aligned}
\mathrm{H}^{\star}_{\mathrm{G}}(\mathbb{P}(k\rho)_+;\underline{\mathbb{F}}_2) &\cong \mathrm{H}^{\mathrm{G}}_{-\star}(\mathsf{Th}(-\mathrm{T}\mathbb{P}(k\rho));\underline{\mathbb{F}}_2)\\
&\cong \mathrm{H}^{\mathrm{G}}_{-\star}(\mathsf{Th}(-k\widetilde{\gamma}^{(k)}_{\mathrm{G},2}+\epsilon_1);\underline{\mathbb{F}}_2)\\
&\cong \mathrm{H}^{\mathrm{G}}_{k\rho-1-\star}(\mathbb{P}(k\rho)_+);\underline{\mathbb{F}}_2).
\end{aligned}$$

Likewise, when p is odd, we have

$$\begin{aligned}
\mathrm{H}^{\star}_{\mathrm{G}}(\mathbb{L}_p(\epsilon k\rho_{\mathbb{C}})_+;\underline{\mathbb{F}}_p) &\cong \mathrm{H}^{\mathrm{G}}_{-\star}(\mathsf{Th}(-\mathrm{T}\mathbb{L}_p(\epsilon k\rho_{\mathbb{C}}));\underline{\mathbb{F}}_p)\\
&\cong \mathrm{H}^{\mathrm{G}}_{-\star}(\mathsf{Th}(-\epsilon k\widetilde{\omega}^{(k)}_{\mathrm{G},p}+\epsilon_1);\underline{\mathbb{F}}_p)\\
&\cong \mathrm{H}^{\mathrm{G}}_{2\epsilon k\rho-1-\star}(\mathbb{L}_p(\epsilon k\rho_{\mathbb{C}})_+);\underline{\mathbb{F}}_p),
\end{aligned}$$

where the last isomorphism is an $\mathrm{H}\underline{\mathbb{F}}_p$-Thom isomorphism which holds because ϵ is an even number when $|\mathrm{G}|$ and p are odd (see (7.3) and (7.4)). ☐

7.2. Identifying $\mathrm{H}\underline{\mathbb{F}}_p$-Eulerian sequences

In this subsection, we begin by analyzing the path components of $\mathbb{P}(k\rho)^{\mathrm{G}}$ and $\mathbb{L}_p(k\rho_{\mathbb{C}})^{\mathrm{G}}$. This will lead us to a calculation of $\mathrm{H}^{\mathrm{G}}_0(\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2)$ as well as $\mathrm{H}^{\mathrm{G}}_0(\mathbb{L}(\epsilon\rho_{\mathbb{C}})_+;\underline{\mathbb{F}}_p)$, and consequently, new elements in the E_1-page of (7.20) and (7.21). We then show that these elements are nonzero permanent cycles resulting in identification of $\mathrm{H}\underline{\mathbb{F}}_p$-Eulerian sequences.

Notation 7.24. For a finite real or complex G-representation V and an irreducible G-representation λ, we let V_λ be the subrepresentation of V such that $\mathrm{V}/\mathrm{V}_\lambda$ does not contain an irreducible sub-representation isomorphic to λ.

The fixed-points of $\mathbb{P}(k\rho_{\mathrm{G}})$ and $\mathbb{L}_p(k\rho_{\mathrm{G}})$ have been studied and described explicitly in [KL24]. We simply state their conclusion.

Lemma 7.25. *Let* $\mathrm{Irr}_1(\mathrm{G})$ *is the set of isomorphism classes of* 1*-dimensional irreducible representations. Then*

$$\mathbb{P}(\mathrm{V})^{\mathrm{G}} = \coprod_{\lambda\in\mathrm{Irr}_1(\mathrm{G})} \mathbb{P}(\mathrm{V}_\lambda).$$

For $p > 2$*,*

$$\mathbb{L}_p(\mathrm{V})^{\mathrm{G}} = \coprod_{[\lambda]\in\widetilde{\mathrm{Irr}_1}(\mathrm{G})} \mathbb{L}_p(\mathrm{V}_\lambda),$$

where $\widetilde{\mathrm{Irr}_1}(\mathrm{G})$ *is the set of isomorphism classes of complex* 1*-dimensional irreducible representations whose character factors through* C_p*.*

For each $\lambda \in \mathrm{Irr}_1(\mathrm{G})$, we consider a based map

$$b_\lambda : \mathrm{S}^0 \xrightarrow{b'_\lambda} \mathbb{P}(\rho_\lambda)_+ \xhookrightarrow{i} \mathbb{P}(\rho)_+,$$

where b'_λ sends the non-basepoint of S^0 to a point in $\mathbb{P}(\omega_\lambda)$ and i is the inclusion along the G-fixed points. By (7.25), such a map is unique up to a contractible choice and determines a unique class in $[b_\lambda] \in \pi_0^{\mathrm{G}}(\mathbb{P}(\rho)_+)$.

Remark 7.26. The $\mathrm{H}\underline{\mathbb{F}}_2$-Hurewicz image of $[b_\lambda] \in \pi_0^{\mathrm{G}}(\mathbb{P}(\rho)_+)$, denote it by

$$\mathsf{b}_\lambda \in \mathrm{H}_0^{\mathrm{G}}(\mathbb{P}(\rho)_+; \underline{\mathbb{F}}_2),$$

is nonzero because its underlying nonequivariant Hurewicz image is nonzero.

Remark 7.27. Moreover, $\mathsf{b}_\lambda = \mathsf{b}_{\lambda'}$ if and only if $\lambda = \lambda'$ in $\mathrm{Irr}_1(\mathrm{G})$. This is because images of b_λ and $\mathsf{b}_{\lambda'}$ under modified geometric fixed-point functor

$$\widetilde{\varphi}^{\mathrm{G}} : \mathrm{H}_0^{\mathrm{G}}(\mathbb{P}(\rho)_+; \underline{\mathbb{F}}_2) \longrightarrow \textstyle\bigoplus_{\lambda \in \mathrm{Irr}_1(\mathrm{G})} \mathrm{H}_0(\mathbb{P}(\lambda)_+; \mathbb{F}_2)$$

are different classes when $\lambda \neq \lambda'$.

The classes b_λ leads to a nonzero element

$$\overline{\mathsf{b}}_{k\rho,\lambda} \in \mathrm{E}^1_{k,k\rho}$$

for each $k \in \mathbb{N}$ in the E_1-page of (7.20).

Remark 7.28 (When p is odd)**.** In this case, for each $\lambda \in \widetilde{\mathrm{Irr}_1}(\mathrm{G})$, we consider a basepoint-preserving map:

$$b_\lambda : \mathrm{S}^0 \xhookrightarrow{b'_\lambda} \mathbb{L}_p((\epsilon\rho_{\mathbb{C}})_\lambda)_+ \xrightarrow{\ i\ } \mathbb{L}_p(\epsilon\rho_{\mathbb{C}})_+.$$

Here, b'_λ sends the non-basepoint of S^0 to a point in $\mathbb{L}_p((\epsilon\rho_{\mathbb{C}})_\lambda)$. The Hurewicz image, $\mathsf{b}_\lambda \in \mathrm{H}_0^{\mathrm{G}}(\mathbb{L}_p((\epsilon\rho_{\mathbb{C}})_\lambda)_+; \underline{\mathbb{F}}_p)$, is a nonzero class because its restriction is also nonzero. This results in the element

$$\overline{\mathsf{b}}_{2\epsilon k\rho,\lambda} \in \mathrm{E}^1_{k,2\epsilon k\rho}$$

in the (7.21) for each $k \in \mathbb{N}$ and $\lambda \in \widetilde{\mathrm{Irr}_1}(\mathrm{G})$.

Our next goal is to prove (7.33), where we show that $\overline{\mathsf{b}}_{k\rho,\lambda}$ in (7.20) are nonzero permanent cycles. We will need the following two lemmas.

Lemma 7.29. *Suppose r is an integer. Then*

$$\mathrm{H}^{\mathrm{G}}_{r\rho-1}(\mathbb{P}(\rho)_+; \underline{\mathbb{F}}_2) \cong \begin{cases} \mathbb{F}_2 & \textit{when } r = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover, the restriction map

$$\iota_{e*} : \mathrm{H}^{\mathrm{G}}_{\rho-1}(\mathbb{P}(\rho)_+; \underline{\mathbb{F}}_2) \longrightarrow \mathrm{H}_{|\mathrm{G}|-1}(\mathbb{RP}^{|\mathrm{G}|-1}; \mathbb{F}_2) \tag{7.30}$$

is an isomorphism.

Proof. When $r \leq 0$, then

$$\begin{aligned} \mathrm{H}^{\mathrm{G}}_{r\rho-1}(\mathbb{P}(\rho)_+; \underline{\mathbb{F}}_2) &\cong \mathrm{H}^{\mathrm{G}}_{r-1}(\Sigma^{-r(\rho-1)}\mathbb{P}(\rho)_+; \underline{\mathbb{F}}_2) \\ &\cong 0 \end{aligned}$$

as negative Bredon homology groups of any G-space is trivial.

When $r > 1$, we use (7.23) (Poincaré duality) to conclude that

$$\begin{aligned} \mathrm{H}^{\mathrm{G}}_{r\rho-1}(\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2) &\cong \mathrm{H}_{\mathrm{G}}^{(1-r)\rho}(\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2) \\ &\cong \mathrm{H}_{\mathrm{G}}^{(1-r)}\left(\Sigma^{(r-1)(\rho-1)}\mathbb{P}(\rho)_+;\mathbb{F}_2\right) \\ &\cong 0 \end{aligned}$$

as negative Bredon cohomology of G-spaces are trivial.

When $r = 1$, we use Poincaré duality and the fact that the action of G on $\mathrm{H}\underline{\mathbb{F}}_p$ is trivial to relate the $(\rho - 1)$-th Bredon homology group

$$\begin{aligned} \mathrm{H}^{\mathrm{G}}_{\rho-1}(\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2) &\cong \mathrm{H}_{\mathrm{G}}^{0}\left(\Sigma^{(r-1)(\rho-1)}\mathbb{P}(\rho)_+;\mathbb{F}_2\right) \\ &\cong \mathrm{H}^{0}\left(\mathrm{Orb}_{\mathrm{G}}(\Sigma^{(r-1)(\rho-1)}\mathbb{P}(\rho))_+;\mathbb{F}_2\right) \end{aligned}$$

with the zeroth ordinary cohomology of the space of G-orbits of $\Sigma^{(r-1)(\rho-1)}\mathbb{P}(\rho)$. Since the underlying space of $\Sigma^{(r-1)(\rho-1)}\mathbb{P}(\rho)$ is path connected, its G-orbits also form a path connected space, and therefore, $\mathrm{H}^{\mathrm{G}}_{\rho-1}(\mathbb{P}(\rho);\underline{\mathbb{F}}_2) \cong \mathbb{F}_2$.

Note that the nonzero element $\mathsf{c} \in \mathrm{H}^{\mathrm{G}}_{\rho-1}(\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2)$ is Poincaré dual to

$$1 \in \mathrm{H}^{0}_{\mathrm{G}}(\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2)$$

whose restriction is $1 \in \mathrm{H}^{0}_{\mathrm{G}}(\mathbb{RP}^{|\mathrm{G}|-1}_+;\mathbb{F}_2)$. Since Poincaré duality isomorphism (PD) commutes with restriction

$$\begin{tikzcd} \mathrm{H}^{\mathrm{G}}_{\rho-1}(\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2) \arrow[r, "\iota_{e*}"] \arrow[d, "\mathrm{PD}"'] & \mathrm{H}_{|\mathrm{G}|-1}(\mathbb{RP}^{|\mathrm{G}|-1};\underline{\mathbb{F}}_2) \arrow[d, "\mathrm{PD}"] \\ \mathrm{H}^{0}_{\mathrm{G}}(\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2) \arrow[r, "\cong", "\iota_{e*}"'] & \mathrm{H}^{0}(\mathbb{RP}^{|\mathrm{G}|-1};\underline{\mathbb{F}}_2), \end{tikzcd}$$

it follows that (7.30) is also an isomorphism. □

Remark 7.31. The element $\mathsf{c} \in \mathrm{H}^{\mathrm{G}}_{\rho-1}(\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2)$ in the proof of the Lemma above maps to $\mathbf{b}_{|\mathrm{G}|-1} \in \mathrm{H}^{\mathrm{G}}_{|\mathrm{G}|-1}(\mathbb{RP}^{|\mathrm{G}|-1};\mathbb{F}_2)$ (in the notation of (4.22)) under restriction.

Lemma 7.32. *Suppose r is an integer. Then*

$$\mathbb{F}_2\{\mathsf{b}_\lambda : \lambda \in \mathrm{Irr}_1(\mathrm{G})\} \subset \mathrm{H}^{\mathrm{G}}_{0}(\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2)$$

and $\mathrm{H}^{\mathrm{G}}_{r\rho}(\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2) = 0$ for $r \neq 0$.

Proof. An argument similar to (7.29) establishes $\mathrm{H}^{\mathrm{G}}_{r\rho}(\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2) \cong 0$ for all integers $r \neq 0$. From (7.27), we conclude that

$$\mathbb{F}_2\{\mathsf{b}_\lambda : \lambda \in \mathrm{Irr}_1(\mathrm{G})\} \subset \mathrm{H}^{\mathrm{G}}_{0}(\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2).$$

The above inclusion may not be an isomorphism. This is a consequence of (7.57) and (7.60). □

Proposition 7.33. *The element $\overline{\mathsf{b}}_{k\rho,\lambda}$ in* (7.20) *is a nonzero permanent cycle for all $k \in \mathbb{N}$.*

Proof. By (7.29)

$$d_r(\overline{\mathsf{b}}_{k\rho,\lambda}) \in \mathrm{E}^{r}_{k-r,k\rho-1} \cong \mathrm{H}^{\mathrm{G}}_{k\rho-1}(\Sigma^{(k-r)\rho}\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2)$$

is trivial for $r \geq 2$. Thus, to establish $\mathsf{b}_{k\rho,\lambda}$ as a permanent cycle we must show that it is a d_1-cycle.

By (7.29), there is a unique $\mathbb{F}_2$-generator

$$\bar{\mathsf{c}}_{k\rho-1} \in \mathrm{E}^1_{k-1,k\rho-1} \tag{7.34}$$

which can be a potential target of a d_1-differential on $\bar{\mathsf{b}}_{k\rho,\lambda}$. In other words,

$$d_1(\bar{\mathsf{b}}_{k\rho,\lambda}) = t\bar{\mathsf{c}}_{k\rho-1}$$

where $t \in \{0,1\}$. We will now show that t must equal 0.

In contrary, suppose this differential is nontrivial, i.e. $t = 1$, it would imply a nontrivial differential in the spectral sequence

$$\mathrm{E}^1_{*,\mathrm{V}} := \bigoplus_{n=0}^{k+r} \mathrm{H}^{\mathrm{G}}_{\mathrm{V}}(\Sigma^{n\rho}\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2) \Longrightarrow \mathrm{H}^{\mathrm{G}}_{\mathrm{V}}(\mathbb{P}((k+r)\rho)_+;\underline{\mathbb{F}}_2) \tag{7.35}$$

for all $k \geq 1$, and consequently,

$$\mathrm{H}_{\mathrm{G}}^{k\rho}(\mathbb{P}((k+r)\rho)_+;\underline{\mathbb{F}}_2) \cong \mathrm{H}^{\mathrm{G}}_{k\rho-1}(\mathbb{P}((k+r)\rho)_+;\underline{\mathbb{F}}_2) \cong 0$$

by Poincare duality (see (7.23)). It would follow that $\mathrm{H}\underline{\mathbb{F}}_2$-Euler class of $k\tilde{\gamma}^{(k+r)}_{\mathrm{G},2}$ and hence its restrictions, must be zero. However, this is a contradiction as the restriction of $k\tilde{\gamma}^{(k+r)}_{\mathrm{G},2}$

$$\iota_{\mathsf{e}}\left(k\tilde{\gamma}^{(k+r)}_{\mathrm{G},2}\right) = k|\mathrm{G}|\tilde{\gamma}^{(k+r)}_{\mathsf{e},2}$$

is the $k|\mathrm{G}|$-fold sum of the tautological bundle over $\mathbb{RP}^{(k+r)|\mathrm{G}|-1}$, whose $\mathrm{H}\mathbb{F}_2$-Euler class is nontrivial.

To show that $\bar{\mathsf{b}}_{k\rho,\lambda}$ is nonzero in the E_∞-page, we consider the map of spectral sequences

$$\begin{tikzcd} \bigoplus\limits_{n\in\mathbb{N}} \mathrm{H}^{\mathrm{G}}_{\star}(\Sigma^{n\rho}\mathbb{P}(\rho)_+;\underline{\mathbb{F}}_2) \arrow[r, Rightarrow] \arrow[d, "\iota_{\mathsf{e}*}"] & \mathrm{H}^{\mathrm{G}}_{\star}(\mathrm{B}_{\mathrm{G}}\Sigma_{2+};\underline{\mathbb{F}}_2) \arrow[d, "\iota_{\mathsf{e}*}"] \\ \bigoplus\limits_{n\in\mathbb{N}} \mathrm{H}_*(\Sigma^{n\rho}\mathbb{RP}^{|\mathrm{G}|-1}_+;\underline{\mathbb{F}}_2) \arrow[r, Rightarrow] & \mathrm{H}_*(\mathbb{RP}^\infty_+;\mathbb{F}_2) \end{tikzcd} \tag{7.36}$$

induced by the restriction to trivial group. Note that the bottom spectral sequence collapses at the E_1-page. Since, the image of $\bar{\mathsf{b}}_{n\rho,\lambda}$ under restriction is a nonzero permanent cycle (follows from (7.26)), it follows that $\bar{\mathsf{b}}_{n\rho,\lambda}$ cannot be a target of a differential, thus nonzero in the E_∞-page. ☐

Proposition 7.37. *The elements $\bar{\mathsf{c}}_{k\rho-1}$ defined in* (7.34) *is nonzero permanent cycle for all $k \geq 0$ in the spectral sequence* (7.20)*.*

Proof. If $\bar{\mathsf{c}}_{k\rho-1}$ supports or is a target of a differential in (7.20) then the same will hold in the spectral sequence (7.35) for some $r \gg 0$. Then, by Poincaré duality, i.e. (7.23),

$$\mathrm{H}_{\mathrm{G}}^{r\rho}(\mathbb{P}((k+r)\rho)_+;\underline{\mathbb{F}}_2) \cong \mathrm{H}^{\mathrm{G}}_{k\rho-1}(\mathbb{P}((k+r)\rho)_+;\underline{\mathbb{F}}_2) \cong 0$$

which contradicts the fact that $\mathrm{H}\underline{\mathbb{F}}_2$-Euler class of $k\tilde{\gamma}^{(k+r)}_{\mathrm{G},2}$ is nonzero. ☐

By (7.29) and (7.32), the nonzero permanent cycles $\overline{\mathsf{b}}_{k\rho,\lambda}$ and $\overline{\mathsf{c}}_{k\rho-1}$ determine unique elements in $\mathrm{H}^{\mathrm{G}}_{\star}(\mathrm{B}_{\mathrm{G}}\Sigma_2;\underline{\mathbb{F}}_2)$ with no indeterminacies.

Notation 7.38. When G is nontrivial, let $\mathbf{b}_{k\rho,\lambda} \in \mathrm{H}^{\mathrm{G}}_{k\rho}((\mathrm{B}_{\mathrm{G}}\Sigma_2)_+;\underline{\mathbb{F}}_2)$ denote the element detected by $\overline{\mathsf{b}}_{k\rho,\lambda}$ in (7.20). Likewise, we let $\mathbf{c}_{k\rho-1} \in \mathrm{H}^{\mathrm{G}}_{k\rho-1}(\mathrm{B}_{\mathrm{G}}\Sigma_2)_+;\underline{\mathbb{F}}_2)$ denote the element detected by $\overline{\mathsf{c}}_{k\rho-1}$ of (7.34). Using these elements, we define the sequences

$$\beta_{\lambda,(2)} := (\mathbf{b}_{0,\lambda}, \mathbf{b}_{\rho,\lambda}, \mathbf{b}_{2\rho,\lambda}, \dots)$$
$$\zeta_{(2)} := (0, \mathbf{c}_{\rho-1}, \mathbf{c}_{2\rho-1}, \dots)$$

which are candidates for ρ-stable $\mathrm{H}\underline{\mathbb{F}}_2$-Eulerian sequences.

Theorem 7.39. *The sequence $\zeta_{(2)}$ is a ρ-stable $\mathrm{H}\underline{\mathbb{F}}_2$-Eulerian sequence.*

Proof. We must verify that $\zeta_{(2)}$ satisfies the condition of (4.12). Firstly note,

$$\mathbf{c}_{\rho} \frown \widetilde{\mathbf{e}}_{\mathrm{G},2} = 0$$

as $\mathrm{H}^{\mathrm{G}}_{-1}(\mathrm{B}_{\mathrm{G}}\Sigma_2;\underline{\mathbb{F}}_2) \cong 0$. Now notice that

$$\iota_{\mathsf{e}*}(\widetilde{\mathbf{e}}_{\mathrm{G},2}) = \widetilde{\mathbf{e}}^{|\mathrm{G}|}_{\mathsf{e},2}$$

as the underlying nonequivariant bundle of $\widetilde{\gamma}_{\mathrm{G},2}$ is the $|\mathrm{G}|$-fold sum of the tautological line bundle $\widetilde{\gamma}_{\mathsf{e},2}$ over $\mathrm{B}\Sigma_2$. It follows from the arguments in the proof of (7.33) that

$$\mathrm{H}^{\mathrm{G}}_{k\rho-1}(\mathrm{B}_{\mathrm{G}}\Sigma_2;\underline{\mathbb{F}}_2) \cong \mathbb{F}_2$$

generated by $\mathbf{c}_{k\rho-1}$. Further, the diagram in (7.36) implies that the restriction map

$$\iota_{\mathsf{e}*} : \mathrm{H}^{\mathrm{G}}_{k\rho-1}(\mathrm{B}_{\mathrm{G}}\Sigma_2;\underline{\mathbb{F}}_2) \longrightarrow \mathrm{H}_{k|\mathrm{G}|-1}(\mathrm{B}\Sigma_2;\mathbb{F}_2)$$

is an isomorphism for all $k \geq 1$ sending $\mathbf{c}_{k\rho-1}$ to $\mathbf{b}_{k|\mathrm{G}|-1}$ following the notations of (4.22). Since, $\mathbf{b}_{(k+1)|\mathrm{G}|-1} \frown \widetilde{\mathbf{e}}^{|\mathrm{G}|}_{\mathsf{e},2} = \mathbf{b}_{k\rho-1}$, we conclude that

$$\mathbf{c}_{(k+1)\rho-1} \frown \widetilde{\mathbf{e}}_{\mathrm{G},2} = \mathbf{c}_{k\rho-1}$$

for all $k \geq 1$. Thus, $\zeta_{(2)}$ is a ρ-stable $\mathrm{H}\underline{\mathbb{F}}_2$-Eulerian sequences. ☐

Notation 7.40. Let $\mathrm{Sq}^{k\rho_{\mathrm{G}}+1}$ to denote the genuine stable cohomology operation $\mathfrak{S}^{\zeta_{(2)}[k]}$ corresponding to the k-shift of $\zeta_{(2)}$.

Theorem 7.41. *The genuine stable cohomology operation $\mathrm{Sq}^{k\rho+1}_{\mathrm{G}}$ is a nonzero element in $[\mathrm{H}\underline{\mathbb{F}}_2, \mathrm{H}\underline{\mathbb{F}}_2]^{k\rho+1}_{\mathrm{G}}$ as its restriction*

$$\iota_{\mathsf{e}*}(\mathrm{Sq}^{k\rho_{\mathrm{G}}+1}) = \mathrm{Sq}^{k|\mathrm{G}|+1}$$

is the classical $(k|\mathrm{G}|+1)$-th classical mod 2 Steenrod squaring operation.

Proof. Since $\iota_{\mathsf{e}*}(\mathbf{c}_{k\rho-1}) = \mathbf{b}_{k|\mathrm{G}|-1}$ (see proof of (7.39)), we get

$$\iota_{\mathsf{e}*}(\zeta_{(2)}[k]) = \mathsf{t}_{|\mathrm{G}|}(\beta_1[k|\mathrm{G}|+1])$$

and the result follows from (4.22) and (4.24). ☐

In fact, we can make a much more general statement. First notice that, for all subgroups $\mathrm{K} \subset \mathrm{G}$, all the restriction maps involved in

$$\iota_{\mathsf{e}*} : \mathrm{H}^{\mathrm{G}}_{k\rho_{\mathrm{G}}-1}(\mathrm{B}_{\mathrm{G}}\Sigma_2;\underline{\mathbb{F}}_2) \xrightarrow{\iota_{\mathrm{K}*}} \mathrm{H}^{\mathrm{K}}_{k|\mathrm{G}/\mathrm{K}|\rho_{\mathrm{K}}-1}(\mathrm{B}_{\mathrm{K}}\Sigma_2;\underline{\mathbb{F}}_2) \xrightarrow{\iota^{\mathrm{K}}_{\mathsf{e}*}} \mathrm{H}_{k|\mathrm{G}|-1}(\mathrm{B}\Sigma_2;\mathbb{F}_2)$$

are isomorphisms. Consequently, $\iota_{\mathrm{K}*}(\mathbf{c}_{k\rho_{\mathrm{G}}} - 1) = \mathbf{c}_{k|\mathrm{G}/\mathrm{K}|\rho_{\mathrm{K}}-1}$ and we get:

Theorem 7.42. *Suppose* $K \subset G$. *Then*

$$\iota_{K*}(\mathrm{Sq}^{k\rho_G+1}(x)) = \mathrm{Sq}^{k|G/K|\rho_K+1}(\iota_{K*}(x))$$

for any $H\underline{\mathbb{F}}_2$*-cohomology class* x.

Remark 7.43. We leave the identification of the G-geometric fixed point of ζ (as defined in (4.27)) for the future, as our methods do not identify the elements

$$\mathsf{c}_*\widetilde{\varphi}^K(\mathbf{c}_{k\rho-1}) \in H^{k-1}(B_K\Sigma_2;\mathbb{F}_2),$$

where c is the collapse map of (4.26) and $\widetilde{\varphi}^K$ is as in modified geometric fixed-point functor of (2.25), in general.

Remark 7.44. Whether the sequence $\beta_{\lambda,(2)}$ satisfy the Eulerian criteria of (4.12) is the subject of Section 7.3.

The odd primary analog of (7.29) is the following lemma:

Lemma 7.45. *Let* G *be any finite group and* r *be an integer. Then*

$$H^G_{2r\epsilon\rho-1}(\mathbb{L}_p(\epsilon\rho_{\mathbb{C}})_+;\underline{\mathbb{F}}_p) \cong \begin{cases} \mathbb{F}_p & \textit{when } r=1 \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover, the restriction map

$$\iota_{e*} : H^G_{2\epsilon\rho-1}(\mathbb{L}_p(\epsilon\rho_{\mathbb{C}})_+;\underline{\mathbb{F}}_p) \longrightarrow H_{2\epsilon|G|-1}(\mathbb{L}_p(\mathbb{C}^{\epsilon|G|})_+;\underline{\mathbb{F}}_2) \tag{7.46}$$

is an isomorphism.

Proof. The proof is identical to that of (7.29), so we leave it to the readers. ☐

From (7.45), we conclude $E^1_{k,2k\epsilon\rho-1} \cong \mathbb{F}_p$ in (7.21).

Lemma 7.47. *Any nonzero element in* $E^1_{k,2k\epsilon\rho-1} \cong \mathbb{F}_p$ *in the spectral sequence* (7.21) *is a nonzero permanent cycle.*

Proof. We consider a truncated version (7.21) along with its restriction to trivial group

$$\begin{tikzcd}
\displaystyle\bigoplus_{n=0}^{n=k+r} H^G_\star(\Sigma^{2n\epsilon\rho}\mathbb{L}_p(\epsilon\rho_{\mathbb{C}})_+;\underline{\mathbb{F}}_p) \arrow[r, Rightarrow] \arrow[d, "\iota_{e*}"] & H^G_\star(\mathbb{L}_p((k+r)\epsilon\rho_{\mathbb{C}})_+;\underline{\mathbb{F}}_p) \arrow[d, "\iota_{e*}"] \\
\displaystyle\bigoplus_{n=0}^{n=k+r} H_*(\Sigma^{k\rho}\mathbb{L}_p(\mathbb{C}^{\epsilon|G|})_+;\underline{\mathbb{F}}_p) \arrow[r, Rightarrow] & H_*(\mathbb{L}_p(\mathbb{C}^{(k+r)\epsilon|G|})_+;\mathbb{F}_p)
\end{tikzcd} \tag{7.48}$$

where the bottom spectral sequence collapses at the E_1-page. Thus, a nonzero element $\bar{\mathsf{c}} \in E^1_{k,2k\epsilon\rho-1}$ cannot be the target of a differential.

Suppose $\bar{\mathsf{c}}$ supports a differential, then this differential will appear in the top spectral sequence of (7.48). This, along with (7.45) and (7.23) (Poincaré duality), would imply

$$H_G^{2k\epsilon\rho}(\mathbb{L}_p(\epsilon(k+r)\rho_{\mathbb{C}})_+;\underline{\mathbb{F}}_p) \cong 0$$

and hence, the $H\underline{\mathbb{F}}_p$-Euler class of $\epsilon k\widetilde{\omega}^{\epsilon(k+r)}_{G,p}$ must be zero. However, this is a contradiction as its restriction is the $H\mathbb{F}_p$-Euler class of

$$\iota_e(\epsilon k\widetilde{\omega}^{(\epsilon(k+r))}_{G,p}) = \epsilon k|G|\widetilde{\omega}^{(\epsilon(k+r)|G|)}_{e,p},$$

which is nonzero. □

The proof above reveals that not only $\mathrm{H}^{\mathrm{G}}_{2\epsilon k\uprho-1}\left((\mathrm{B_G C}_p)_+;\underline{\mathbb{F}}_2\right) \cong \mathbb{F}_p$, but also that the restriction map

$$\iota_{\mathsf{e}*} : \mathrm{H}^{\mathrm{G}}_{2\epsilon k\uprho-1}\left((\mathrm{B_G C}_p)_+;\underline{\mathbb{F}}_p\right) \longrightarrow \mathrm{H}_{2\epsilon k|\mathrm{G}|-1}((\mathrm{BC}_p)_+;\mathbb{F}_p) \tag{7.49}$$

is an isomorphism. The mod p cohomology of BC_p is isomorphic to

$$\mathrm{H}^*((\mathrm{BC}_p)_+;\mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\mathbf{y})[\![\mathbf{u}]\!],$$

where $|\mathbf{a}| = 1$ and $\mathbf{b}$ can be chosen to be $\mathrm{H}\mathbb{F}_p$-Euler class of the real 2-dimensional bundle

$$\mathfrak{R}_k := \begin{tikzcd} \mathrm{EC}_p \times_{\mathrm{C}_p} \left(\mathsf{r}_{2\pi k/p}\right) \arrow[d] \\ \mathrm{BC}_p \end{tikzcd}$$

for any $k \in \{1, \dots, p-1\}$ (these Euler classes differ up to a multiple of unit). An element dual to $\mathbf{y}\mathbf{u}^{\epsilon k|\mathrm{G}|-1}$ generates $\mathrm{H}_{2\epsilon k|\mathrm{G}|-1}((\mathrm{BC}_p)_+;\mathbb{F}_p)$.

Recall κ and $\upkappa$ from (7.9). The pullback $\widetilde{\uptau}_p$ along κ is isomorphic to

$$\kappa^*\widetilde{\uptau}_p \cong \bigoplus_{k=1}^{(p-1)/2} \mathsf{r}_{\frac{2k\pi}{p}}$$

and therefore,

$$\upkappa^*\widetilde{\upgamma}_{\mathsf{e},p} \cong \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_{(p-1)/2}.$$

Consequently, $\mathrm{H}\mathbb{F}_p$-Euler class of $(1 + |\mathrm{G}| \mod 2)\upkappa^*\widetilde{\upgamma}_{\mathsf{e},p}$ is nonzero, in fact it is a unit multiple of $\mathbf{u}^\epsilon$. Since,

- the underlying nonequivariant bundle of $\upkappa^*\widetilde{\upgamma}_{\mathrm{G},p}$ is $|\mathrm{G}|$-fold sum of $\upkappa^*\widetilde{\upgamma}_{\mathsf{e},p}$,
- capping with $\mathbf{u}^\epsilon$

  $$- \frown \mathbf{u}^\epsilon : \mathrm{H}_{2\epsilon(k+1)-1}\left((\mathrm{BC}_p)_+;\mathbb{F}_p\right) \longrightarrow \mathrm{H}_{2\epsilon k-1}\left((\mathrm{BC}_p)_+;\mathbb{F}_p\right)$$

  is an isomorphism,
- the restricted class $\iota_{\mathsf{e}*}(\widetilde{\mathbf{e}}_{\mathrm{G},p})$ is a unit multiple of $\mathbf{u}^{\epsilon|\mathrm{G}|}$, and
- the restriction map (7.49) is an isomorphism,

we conclude that:

Lemma 7.50. *For any finite group* G*, the map*

$$- \frown \upkappa^*\widetilde{\mathbf{e}}_{\mathrm{G},p} : \mathrm{H}^{\mathrm{G}}_{2\epsilon(k+1)\uprho-1}\left((\mathrm{B_G C}_p)_+;\underline{\mathbb{F}}_2\right) \longrightarrow \mathrm{H}^{\mathrm{G}}_{2\epsilon k\uprho-1}\left((\mathrm{B_G C}_p)_+;\underline{\mathbb{F}}_2\right)$$

is an isomorphism.

Notation 7.51. Give a finite group G and a prime p, choose a generator $\mathbf{c}_{\uprho-1} \in \mathrm{H}^{\mathrm{G}}_{2\epsilon\uprho-1}((\mathrm{B_G C}_p)_+;\underline{\mathbb{F}}_p)$. Using (7.50), we define the elements

$$\mathbf{c}_{k\uprho-1} \in \mathrm{H}^{\mathrm{G}}_{2\epsilon k\uprho-1}((\mathrm{B_G C}_p)_+;\underline{\mathbb{F}}_p)$$

so that $\mathbf{c}_{(k+1)\uprho-1} \frown \upkappa^*\tilde{\mathbf{e}}_{\mathrm{G},p} = \mathbf{c}_{k\uprho-1}$.

Now define the sequence

$$\zeta_{(p)} := (0, \mathbf{c}_{\rho-1}, \mathbf{c}_{2\rho-1}, \dots)$$

which is a ρ-stable $\mathrm{H}\underline{\mathbb{F}}_p$-Eulerian sequence by construction. Invoking (4.18), we let $\mathrm{P}^{k\rho_{\mathrm{G}}+1}$ denote the genuine stable $\mathrm{H}\underline{\mathbb{F}}_p$-cohomology operation induced by $\zeta_{(p)}[k]$. Then:

Theorem 7.52. *The genuine stable cohomology operation* $\mathrm{P}^{2\epsilon k\rho_{\mathrm{G}}+1}$ *is a nonzero element in* $[\mathrm{H}\underline{\mathbb{F}}_p, \mathrm{H}\underline{\mathbb{F}}_p]_{\mathrm{G}}^{2\epsilon k\rho+1}$ *as its restriction is the classical odd primary classical Steenrod operation*

$$\iota_{\mathrm{e}*}\mathrm{P}^{2\epsilon k\rho_{\mathrm{G}}+1} = \beta\mathrm{P}^{2k|\mathrm{G}|}$$

in the notation of [Ste62].

Proof. This follows from the fact that $\iota_{\mathrm{e}*}(\mathbf{c}_{k\rho-1})$ is the element dual to $\mathbf{yu}^{\epsilon k|\mathrm{G}|-1}$ under Kronecker product, (3.17) and (4.3). ☐

An argument identical to that of (7.42), shows that:

Theorem 7.53. *Suppose* $\mathrm{K} \subset \mathrm{G}$. *Then*

$$\iota_{\mathrm{K}*}(\mathrm{P}^{2\epsilon k\rho_{\mathrm{G}}+1}(x)) = \mathrm{P}^{2\epsilon|\mathrm{G}/\mathrm{K}|k\rho_{\mathrm{K}}+1}(\iota_{\mathrm{K}*}(x))$$

for any $\mathrm{H}\underline{\mathbb{F}}_p$*-cohomology class* x.

Another important consequence of (7.45) is that the odd primary classes $\overline{\mathsf{b}}_{2\epsilon k\rho,\lambda}$ from (7.28) are immune to all differentials:

Corollary 7.54. *The class* $\overline{\mathsf{b}}_{2\epsilon k\rho,\lambda}$ *of* (7.21) *is a nonzero permanent cycle for all* $k \in \mathbb{N}$.

Proof. By (7.29), $\overline{\mathsf{b}}_{2\epsilon k\rho,\lambda}$ can only possibly support a d_1-differential. However, (7.45) shows that the potential targets of any such permanent d_1-differential are already nonzero permanent cycles. Consequently, $\overline{\mathsf{b}}_{2\epsilon k\rho,\lambda}$ must also be permanent cycles.

Furthermore, by construction, as detailed in (7.28)), the restriction of $\overline{\mathsf{b}}_{2\epsilon k\rho,\lambda}$ in (7.48) are nonzero permanent cycles. Therefore, $\overline{\mathsf{b}}_{2\epsilon k\rho,\lambda}$ cannot be the target of a differential. Hence, the result. ☐

Arguing exactly the same way as (7.32), we get:

Lemma 7.55. *Suppose* r *is an integer. Then*

$$\mathbb{F}_2\{\mathsf{b}_\lambda : \lambda \in \widetilde{\mathrm{Irr}_1}(\mathrm{G})\} \subset \mathrm{H}_0^{\mathrm{G}}(\mathbb{L}_p(\epsilon\rho_{\mathbb{C}})_+; \underline{\mathbb{F}}_2),$$

and $\mathrm{H}_{r\rho}^{\mathrm{G}}(\mathbb{L}_p(\epsilon\rho_{\mathbb{C}})_+; \underline{\mathbb{F}}_2) = 0$ *when* $r \neq 0$.

Notation 7.56. As a consequence of (7.55), $\overline{\mathsf{b}}_{2\epsilon k\rho,\lambda}$ detects a unique class, denote it by $\mathsf{b}_{2\epsilon k\rho,\lambda}$, in $\mathrm{H}_{2\epsilon k\rho}^{\mathrm{G}}((\mathrm{B}_{\mathrm{G}}\mathrm{C}_p)_+; \underline{\mathbb{F}}_p)$.

Next, we will show that the elements in $\mathrm{H}_{\mathrm{G}}^{2\epsilon k\rho}((\mathrm{B}_{\mathrm{G}}\mathrm{C}_p)_+; \underline{\mathbb{F}}_p)$ detected by $\mathsf{b}_{2\epsilon k\rho,\lambda}$ can be arranged to give rise to $\mathrm{H}\underline{\mathbb{F}}_p$-Eulerian sequences which we will denote by $\upbeta_{\lambda,(p)}$.

7.3. The Eulerian sequence $\beta_{\lambda,(p)}$.

We now establish the main technical result for this subsection:

Lemma 7.57. *Suppose* X *path-connected space with an action of* G. *Then the kernel of the map*

$$\mathrm{H}_0^{\mathrm{G}}(\mathrm{X}_+;\underline{\mathbb{F}}_p) \longrightarrow \mathrm{H}_0^{\mathrm{G}}(\widetilde{\mathrm{EG}}\wedge \mathrm{X}_+;\underline{\mathbb{F}}_p) \tag{7.58}$$

is the image of the $\mathsf{tr}_{\mathsf{e}}^{\mathrm{G}}:\mathrm{H}_0(\iota_{\mathsf{e}}\mathrm{X}_+;\underline{\mathbb{F}}_p)\longrightarrow \mathrm{H}_0^{\mathrm{G}}(\mathrm{X}_+;\underline{\mathbb{F}}_p)$. *Moreover, if the transfer is nontrivial, then*

$$0 \longrightarrow \mathrm{H}_0(\iota_{\mathsf{e}}\mathrm{X}_+) \xrightarrow{\mathsf{tr}_{\mathsf{e}}^{\mathrm{G}}} \mathrm{H}_0^{\mathrm{G}}(\mathrm{X}_+) \longrightarrow \mathrm{H}_0^{\mathrm{G}}(\widetilde{\mathrm{EG}}\wedge \mathrm{X}_+) \longrightarrow 0. \tag{7.59}$$

is a short exact sequence of $\mathbb{F}_p$*-vector spaces.*

Proof. By running the long exact sequence associated to the cofiber sequence $\mathrm{EG}_+ \to \mathrm{S}^0 \to \widetilde{\mathrm{EG}}$, and using the fact that negative Bredon homology of any space is trivial, we notice that the map (7.58) is a surjection whose kernel is the image of map

$$\mathrm{H}_0^{\mathrm{G}}(\mathrm{EG}_+\wedge \mathrm{X}_+;\underline{\mathbb{F}}_p) \longrightarrow \mathrm{H}_0^{\mathrm{G}}(\mathrm{X}_+;\underline{\mathbb{F}}_p).$$

Since $\mathrm{G}_+ \to \mathrm{EG}_+$ is the zeroth G-equivariant skeletal approximation, the map

$$\mathrm{H}_0^{\mathrm{G}}(\mathrm{G}_+\wedge \mathrm{X}_+;\underline{\mathbb{F}}_p) \longrightarrow \mathrm{H}_0^{\mathrm{G}}(\mathrm{EG}_+\wedge \mathrm{X}_+;\underline{\mathbb{F}}_p)$$

is an isomorphism. Moreover the composite $\mathrm{G}_+\wedge \mathrm{X}_+ \to \mathrm{EG}_+\wedge \mathrm{X}_+ \to \mathrm{S}^0\wedge \mathrm{X}_+$ induces the transfer map:

$$\mathsf{tr}_{\mathsf{e}}^{\mathrm{G}}:\mathrm{H}_0(\iota_{\mathsf{e}}\mathrm{X}_+;\underline{\mathbb{F}}_p)\cong \mathrm{H}_0^{\mathrm{G}}(\mathrm{G}_+\wedge \mathrm{X}_+;\underline{\mathbb{F}}_p) \longrightarrow \mathrm{H}_0^{\mathrm{G}}(\mathrm{X}_+;\underline{\mathbb{F}}_p).$$

The claim regarding the short exact sequence holds because $\mathrm{H}_0^{\mathsf{e}}(\iota_{\mathsf{e}}\mathrm{X}_+;\underline{\mathbb{F}}_p)$ is isomorphic to $\mathbb{F}_p$, a consequence of $\iota_{\mathsf{e}}\mathrm{X}$ being path connected. Therefore, if the transfer is nontrivial, it is necessarily an injection. □

Although we do not compute the transfer in (7.59) explicitly, it is not hard to show that there are many cases where it is nontrivial. For example:

Example 7.60. The map $\mathrm{X}_+ \to \mathrm{S}^0$ induces a map of Mackey functors in homology. The transfer $\mathsf{tr}_{\mathsf{e}}^{\mathrm{G}}$ in $\underline{\mathrm{H}}_0^{\mathrm{G}}(\mathrm{S}^0;\underline{\mathbb{F}}_2)$ is the transfer in the constant Mackey functor $\underline{\mathbb{F}}_2$, i.e., multiplication by $|\mathrm{G}|$. This is nonzero when $|\mathrm{G}|$ is odd. Thus, the inequalities of (7.32) and (7.55) are often strict.

Remark 7.61. In the case when the transfer is nontrivial, the short exact sequence (7.59) splits naturally. This is because

$$\mathrm{H}_0^{\mathrm{G}}(\mathrm{X}_+^{\mathrm{G}};\underline{\mathbb{F}}_p) \longrightarrow \mathrm{H}_0^{\mathrm{G}}(\mathrm{X}_+;\underline{\mathbb{F}}_p) \longrightarrow \mathrm{H}_0^{\mathrm{G}}(\widetilde{\mathrm{EG}}\wedge \mathrm{X}_+;\underline{\mathbb{F}}_p)$$

is an isomorphism. To prove this, we first note that $\mathrm{H}_0^{\mathrm{G}}(\mathrm{X}_+^{\mathrm{G}};\underline{\mathbb{F}}_p)\cong \mathrm{H}_0(\mathrm{X}^{\mathrm{G}};\mathbb{F}_p)$, which is free $\mathbb{F}_p$-vector space generated by G-connected components of X. We then compute the right-hand side as follows:

$$\begin{aligned} \mathrm{H}_0^{\mathrm{G}}(\widetilde{\mathrm{EG}} \wedge \mathrm{X}_+; \underline{\mathbb{F}}_p) &\cong [\mathrm{S}^0, \mathrm{H}\underline{\mathbb{F}}_p \wedge \widetilde{\mathrm{EG}} \wedge \mathrm{X}_+]^{\mathrm{G}} \\ &\cong [\mathrm{S}^0, \Phi^{\mathrm{G}}\mathrm{H}\underline{\mathbb{F}}_p \wedge \mathrm{X}_+^{\mathrm{G}}] \\ &\cong (\Phi^{\mathrm{G}}\mathrm{H}\underline{\mathbb{F}}_p)_0(\mathrm{X}_+^{\mathrm{G}}) \\ &\cong (\mathrm{H}\underline{\mathbb{F}}_p)_0(\mathrm{X}_+^{\mathrm{G}}), \end{aligned}$$

where the last isomorphism folows from the fact that $\Phi^{\mathrm{G}}\mathrm{H}\underline{\mathbb{F}}_p$ splits as a wedge of $\mathrm{H}\mathbb{F}_p$ and a 1-connected $\mathrm{H}\mathbb{F}_p$-module.

Now, we focus on the specific case when $\mathrm{X} = \mathbb{P}(\rho)$. From (7.60). Let

$$\mathsf{t} = \mathsf{tr}_{\mathsf{e}}^{\mathrm{G}}(\mathbf{b}_0) \in \mathrm{H}_0^{\mathrm{G}}(\mathbb{P}(\rho)_+; \underline{\mathbb{F}}_2),$$

where $\mathbf{b}_0$ is the generator of $\mathrm{H}_0(\mathbb{RP}_+^{|\mathrm{G}|-1}; \mathbb{F}_2)$. Then:

Proposition 7.62. *The class* $\mathsf{tr}_{\mathsf{e}}^{\mathrm{G}}(\mathbf{b}_{k|\mathrm{G}|}) \in \mathrm{H}_{k\rho}^{\mathrm{G}}((\mathrm{B_G}\Sigma_2)_+; \underline{\mathbb{F}}_2)$ *is nonzero iff* $\mathsf{t} = 0$, *for all* $k \in \mathbb{N}$.

Proof. This follows immediately from the study of the transfer map between the spectral sequences of (7.36). ☐

Next, we show that:

Theorem 7.63. *For all* $k \geq 0$, *the map*

$$- \frown \widetilde{\mathbf{e}}_{\mathrm{G},2} : \mathrm{H}_{(k+1)\rho}^{\mathrm{G}}((\mathrm{B_G}\Sigma_2)_+; \underline{\mathbb{F}}_2) \longrightarrow \mathrm{H}_{k\rho}^{\mathrm{G}}((\mathrm{B_G}\Sigma_2)_+; \underline{\mathbb{F}}_2)$$

is an isomorphism.

Proof. It follows from the theory of equivariant vector bundle (see [BZ24, Lemma 2.18]) that G-fixed point $\widetilde{\gamma}_{\mathrm{G},2}$ is

$$(\widetilde{\gamma}_{\mathrm{G},2})^{\mathrm{G}} \cong \coprod_{\lambda \in \mathrm{Irr}_1(\mathrm{G})} \widetilde{\gamma}_{\mathsf{e},2}.$$

Thus, the modified G-geometric functor sends the $\mathrm{H}\underline{\mathbb{F}}_2$-Euler class $\widetilde{\mathbf{e}}_{\mathrm{G},2}$ to

$$\widetilde{\varphi}^{\mathrm{G}}(\widetilde{\mathbf{e}}_{\mathrm{G},2}) = \sum_{\lambda} \widetilde{\mathbf{e}}_{\lambda},$$

where $\widetilde{\mathbf{e}}_{\lambda}$ is an Euler class of $\widetilde{\gamma}_{\mathsf{e},2}$ over the component corresponding to λ. Easy to see that

$$- \frown \widetilde{\varphi}^{\mathrm{G}}(\widetilde{\mathbf{e}}_{\mathrm{G},2}) : \bigoplus_{\lambda} \mathrm{H}_{k+1}((\mathrm{B}\Sigma_2)_+; \mathbb{F}_2) \longrightarrow \bigoplus_{\lambda} \mathrm{H}_{k}((\mathrm{B}\Sigma_2)_+; \mathbb{F}_2)$$

is an isomorphism for all $k \geq 0$. Since $\widetilde{\varphi}^{\mathrm{G}}$ satisfies the general formula

$$\widetilde{\varphi}^{\mathrm{G}}(b \frown e) = \widetilde{\varphi}^{\mathrm{G}}(b) \frown \widetilde{\varphi}^{\mathrm{G}}(e),$$

we have a commutative diagram

$$\begin{tikzcd} \mathrm{H}_{(k+1)\rho}^{\mathrm{G}}((\mathrm{B_G}\Sigma_2)_+; \underline{\mathbb{F}}_2) \arrow[r, "\widetilde{\varphi}^{\mathrm{G}}"] \arrow[d, "- \frown \widetilde{\mathbf{e}}_{\mathrm{G},2}"'] & \mathrm{H}_{(k+1)}((\mathrm{B}\Sigma_2); \mathbb{F}) \arrow[d, "- \frown \widetilde{\varphi}^{\mathrm{G}}(\widetilde{\mathbf{e}}_{\mathrm{G},2})"] \\ \mathrm{H}_{k\rho}^{\mathrm{G}}((\mathrm{B_G}\Sigma_2)_+; \underline{\mathbb{F}}_2) \arrow[r, "\widetilde{\varphi}^{\mathrm{G}}"] & \mathrm{H}_{k}((\mathrm{B}\Sigma_2); \mathbb{F}), \end{tikzcd}$$

where the right vertical arrow is an isomorphism for all $k \geq 0$. When $\mathsf{t} = 0$, (7.57) and (7.61) implies that the horizontal arrows in the diagram above are also isomorphisms. Consequently, the left vertical arrow, which is capping with $\widetilde{\mathbf{e}}_{\mathrm{G},2}$ is an isomorphism for all $k \geq 0$, as desired.

When $\mathsf{t} \neq 0$, we consider the diagram

$$\begin{tikzcd}
\mathrm{H}_{(k+1)|\mathrm{G}|}((\mathrm{B}\Sigma_2)_+; \mathbb{F}_2) \arrow[r, hook, "\mathsf{tr}_{\mathsf{e}}^{\mathrm{G}}"] \arrow[d, "- \frown \widetilde{\mathbf{e}}_{\mathrm{e},2}^{|\mathrm{G}|}"'] & \mathrm{H}^{\mathrm{G}}_{(k+1)\rho}((\mathrm{B}_{\mathrm{G}}\Sigma_2)_+; \underline{\mathbb{F}}_2) \arrow[r, two heads, "\widetilde{\varphi}^{\mathrm{G}}"] \arrow[d, "- \frown \widetilde{\mathbf{e}}_{\mathrm{G},2}"'] & \bigoplus_\lambda \mathrm{H}_{(k+1)}((\mathrm{B}\Sigma_2); \mathbb{F}) \arrow[d, "- \frown \widetilde{\varphi}^{\mathrm{G}}(\widetilde{\mathbf{e}}_{\mathrm{G},2})"] \\
\mathrm{H}_{k|\mathrm{G}|}((\mathrm{B}\Sigma_2)_+; \mathbb{F}_2) \arrow[r, hook, "\mathsf{tr}_{\mathsf{e}}^{\mathrm{G}}"'] & \mathrm{H}^{\mathrm{G}}_{k\rho}((\mathrm{B}_{\mathrm{G}}\Sigma_2)_+; \underline{\mathbb{F}}_2) \arrow[r, two heads, "\widetilde{\varphi}^{\mathrm{G}}"'] & \bigoplus_\lambda \mathrm{H}_{k}((\mathrm{B}\Sigma_2); \mathbb{F})
\end{tikzcd}$$

where the rows are short exact sequences (follows from (7.57) and (7.62)) and the left square commutes because of (7.65). Since the left and the right vertical arrows are isomorphism, it follows that the middle vertical arrow is also an isomorphism as desired. ☐

A nearly identical argument yields the odd primary analog of the above result:

Theorem 7.64. *For all $k \geq 0$, the map*

$$- \frown \kappa^* \widetilde{\mathbf{e}}_{\mathrm{G},p} : \mathrm{H}^{\mathrm{G}}_{2\epsilon(k+1)\rho}((\mathrm{B}_{\mathrm{G}}\mathrm{C}_p)_+; \underline{\mathbb{F}}_p) \longrightarrow \mathrm{H}^{\mathrm{G}}_{2\epsilon k\rho}((\mathrm{B}_{\mathrm{G}}\mathrm{C}_p)_+; \underline{\mathbb{F}}_p)$$

is an isomorphism.

Lemma 7.65. *Suppose* X *is a* G*-space,* $t \in \mathrm{H}^{\star}_{\mathrm{G}}(\mathrm{X}_+; \underline{\mathbb{F}}_p)$ *a* $\mathrm{RO}(\mathrm{G})$*-graded cohomology class, and* $x \in \mathrm{H}_*(\iota_{\mathsf{e}}\mathrm{X}_+; \mathbb{F}_p)$ *a homology class of the underlying nonequivariant space. Then:*

$$\mathsf{tr}_{\mathsf{e}}^{\mathrm{G}}(x) \frown t = \mathsf{tr}_{\mathsf{e}}^{\mathrm{G}}\,(x \frown \mathsf{res}_{\mathsf{e}}(t))).$$

Proof. We first observe that the quotient map $\pi : \mathrm{G} \longrightarrow \mathrm{G}/\mathrm{G} \cong \mathsf{e}$ induces the transfer

$$\mathsf{tr}_{\mathsf{e}}^{\mathrm{G}} = \pi_* : \mathrm{H}_{|\mathrm{V}|}(\iota_{\mathsf{e}}\mathrm{X}_+; \mathbb{F}_p) \cong \mathrm{H}^{\mathrm{G}}_{\mathrm{V}}(\mathrm{G}_+ \wedge \mathrm{X}_+; \underline{\mathbb{F}}_p) \longrightarrow \mathrm{H}^{\mathrm{G}}_{\mathrm{V}}(\mathrm{X}_+; \underline{\mathbb{F}}_p)$$

in homology, and the restriction map

$$\mathsf{res}_{\mathsf{e}} = \pi^* : \mathrm{H}^{\mathrm{W}}(\mathrm{X}_+; \underline{\mathbb{F}}_p) \longrightarrow \mathrm{H}^{\mathrm{W}}_{\mathrm{G}}(\mathrm{G}_+ \wedge \mathrm{X}_+; \underline{\mathbb{F}}_p) \cong \mathrm{H}^{|\mathrm{W}|}_{\mathrm{G}}(\mathrm{X}_+; \mathbb{F}_p)$$

in cohomology, for any $\mathrm{V}, \mathrm{W} \in \mathrm{RO}(\mathrm{G})$,. Then the result follows from the natural relation

$$\pi_*(x) \frown t = \pi_*(x \frown \pi^*(t))$$

satisfied by the cap product. ☐

Using the isomorphisms of (7.63) and (7.64) we define the $\mathrm{H}\underline{\mathbb{F}}_p$-Eulerian sequence

$$\beta_{\lambda,(p)} := (\hat{\mathsf{b}}_{0,\lambda}, \hat{\mathsf{b}}_{1,\lambda}, \dots)$$

where $\hat{\mathsf{b}}_{0,\lambda} = \mathsf{b}_{0,\lambda}$ and use induction to define $\hat{\mathsf{b}}_k$ such that

$$\hat{\mathsf{b}}_{k+1,\lambda} \frown \kappa^* \widetilde{\mathbf{e}}_{\mathrm{G},p} = \hat{\mathsf{b}}_{k,\lambda}$$

where κ is the map defined in (7.9) for p odd and identity when $p = 2$.

Remark 7.66. The elements $\hat{\mathsf{b}}_{k,\lambda}$ may differ from the element of $\mathsf{b}_{k\rho,\lambda}$ up to an element in the image of the transfer map $\mathsf{tr}_{\mathsf{e}}^{\mathrm{G}}$.

Notation 7.67. Using Main Theorem 1, we obtain a genuine stable $\mathrm{H}\underline{\mathbb{F}}_p$-cohomology operation corresponding to $\beta_{\lambda,(p)}$ and its k-shifts. When $p = 2$, we set

$$\mathrm{Sq}_{\lambda}^{k\rho_{\mathrm{G}}} := \mathfrak{S}^{\beta_{\lambda,(2)}[k]}.$$

Further, when $\lambda = 1 \in \mathrm{Irr}_1(\mathrm{G})$, we simply use the notation $\mathrm{Sq}^{k\rho_{\mathrm{G}}}$. When p is odd, we set

$$\mathrm{P}_{\lambda}^{2\epsilon k\rho_{\mathrm{G}}} := \mathfrak{S}^{\beta_{\lambda,(p)}[k]}$$

and drop the subscript when $\lambda = 1 \in \widetilde{\mathrm{Irr}}_1(\mathrm{G})$.

Proof of Main Theorem 3. Since restriction of the class $\mathsf{b}_{0,\lambda}$ to a subgroup K is

$$\mathsf{b}_{0,\iota_{\mathrm{K}}\lambda} \in \mathrm{H}_0^{\mathrm{K}}((\mathrm{B}_{\mathrm{K}}\mathrm{C}_p)_+; \underline{\mathbb{F}}_p),$$

and restriction of the Euler class $\widetilde{\mathbf{e}}_{\mathrm{G},p}$ to K is $\widetilde{\mathbf{e}}_{\mathrm{K},p}^{|\mathrm{G}/\mathrm{K}|}$, we conclude that

$$\iota_{\mathrm{K}}(\beta_{\lambda,(p)}) = \mathsf{t}_{|\mathrm{G}/\mathrm{K}|}\beta_{\iota_{\mathrm{K}}\lambda,(p)}.$$

This, combined with (4.29), (7.42) and (7.53) completes the proof. □

Proof of Main Theorem 4. First, assume that K is a normal subgroup of G. In this case, the Weyl group is $\mathrm{W}(\mathrm{K}) = \mathrm{G}/\mathrm{K}$. We observe that the modified K-geometric fixed-point of the class $\mathsf{b}_{0,\lambda}$ is

$$\mathsf{b}_{0,\lambda^{\mathrm{K}}} \in \mathrm{H}_0^{\mathrm{K}}((\mathrm{B}_{\mathrm{K}}\mathrm{C}_p)_+; \underline{\mathbb{F}}_p),$$

where we set $\mathsf{b}_{0,\lambda^{\mathrm{K}}} = 0$ when $\lambda^{\mathrm{K}} = \mathbf{0}$. Furthermore, note $\mathsf{c}_*\widetilde{\varphi}^{\mathrm{K}}(\widetilde{\mathbf{e}}_{\mathrm{G},p}) = \widetilde{\mathbf{e}}_{\mathrm{G}/\mathrm{K},p}$. Therefore, using (7.64) and the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{H}^{\mathrm{G}}_{2\epsilon(k+1)\rho_{\mathrm{G}}}((\mathrm{B}_{\mathrm{G}}\mathrm{C}_p)_+;\underline{\mathbb{F}}_p) \arrow[r, "-\frown\kappa^*\widetilde{\mathbf{e}}_{\mathrm{G},p}", "\cong"'] \arrow[d, "\mathsf{c}_*\widetilde{\varphi}^{\mathrm{K}}(-)"] & \mathrm{H}^{\mathrm{G}}_{2\epsilon k\rho_{\mathrm{G}}}((\mathrm{B}_{\mathrm{G}}\mathrm{C}_p)_+;\underline{\mathbb{F}}_p) \arrow[d, "\mathsf{c}_*\widetilde{\varphi}^{\mathrm{K}}(-)"] \\
\mathrm{H}^{\mathrm{G}}_{2\epsilon(k+1)\rho_{\mathrm{W}(\mathrm{K})}}((\mathrm{B}_{\mathrm{W}(\mathrm{K})}\mathrm{C}_p)_+;\underline{\mathbb{F}}_p) \arrow[r, "-\frown\kappa^*\widetilde{\mathbf{e}}_{\mathrm{W}(\mathrm{K}),p}", "\cong"'] & \mathrm{H}^{\mathrm{G}}_{2\epsilon k\rho_{\mathrm{G}}}((\mathrm{B}_{\mathrm{W}(\mathrm{K})}\mathrm{C}_p)_+;\underline{\mathbb{F}}_p)
\end{tikzcd}$$

we conclude that

$$\varphi^{\mathrm{K}}(\beta_{\lambda,(p)}) = \beta_{\lambda^{\mathrm{K}},(p)}.$$

Thus, the result follows from (4.33).

When K is not a normal subgroup, then we first restrict to the normalizer subgroup N(K) before calculating modified K-geometric fixed-points. In this case, the arguments above, combined with Main Theorem 3, yields the stated results. □

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New Mexico State University

Email address: `prasit@nmsu.edu`

University of Washington

Email address: `ajw48@uw.edu`

Max Plank Institute for Mathematics

Email address: `mingcongzeng@gmail.com`

Chinese Academy of Sciences

Email address: `zoufoling@amss.ac.cn`