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The stable Adams conjecture and higher associative structures on Moore spectra: Retraction

By Prasit Bhattacharya and Nitu Kitchloo

Description of errors.

The article [2] is being retracted due to two errors in the proof of the p-local stable Adams conjecture: a key tool in studying higher associative structures on Moore spectra, which was the topic of the first author's thesis [1]. The errors are described below. We have posted an erratum [3] that corrects both these errors (using an alternate argument that was already outlined in [2]). However, until our erratum has made its way through the peer review process, we believe that a retraction of [2] is warranted. The following statement is what we mean by the p-local stable Adams conjecture:

The p-local stable Adams Conjecture. (*see* [3])
Fix p, q to be any primes such that $p \neq q$. Let $\mathbf{S}_{(p)}$ denote the p-local sphere spectrum, and let $\mathrm{Pic}^{\mathrm{ev}}\,\mathbf{S}_{(p)}$ be its even Picard group generated by $\Sigma^2\mathbf{S}_{(p)}$. Recall the classical p-local J-homomorphism:

$$J : \mathbb{Z} \times \mathrm{BU}_{(p)} \longrightarrow \mathrm{Pic}^{\mathrm{ev}}\,\mathbf{S}_{(p)}.$$

Then J can be delooped to a map $\underline{J}$ of connective spectra

$$\underline{J} : \underline{\mathrm{kU}}_{(p)} \longrightarrow \mathrm{pic}^{\mathrm{ev}}\,\mathbf{S}_{(p)}$$

such that $\underline{J}$ is invariant under precomposition with the Adams operation ψ^q. More precisely, $\underline{J}$ admits a canonical factorization through a map $\underline{J}_q$

$$\begin{array}{ccc} & & \underline{\mathrm{kU}}_{(p)} \\ & \swarrow & \downarrow{\scriptstyle \underline{J}} \\ \underline{\mathrm{kU}}_{h\psi^q} & \xrightarrow{\underline{J}_q} & \mathrm{pic}^{\mathrm{ev}}\,\mathbf{S}_{(p)} \end{array},$$

where $\underline{\mathrm{kU}}_{h\psi^q}$ *denotes the homotopy orbits of the* ψ^q*-action on* $\underline{\mathrm{kU}}_{(p)}$. *Furthermore, the map* $\underline{J}_q$ *has the property that it sends a generator of* $\pi_1(\underline{\mathrm{kU}}_{h\psi^q}) = \mathbb{Z}$ *to the element* $q \in \mathbb{Z}_{(p)}^{\times} = \pi_1(\mathrm{pic}^{\mathrm{ev}}\,\mathbf{S}_{(p)})$.

A p-complete version of this conjecture was claimed by E. Friedlander in [4], which was our main reference in the original submission of [2]. Shortly after submission, a referee pointed out that the p-completed stable Adams conjecture, as stated in [4], was false (see [2] for details). Following the referee report for [2], the authors made an attempt to fill the gap left by the error in [4] by supplying a correct proof of the p-local stable Adams conjecture. Unfortunately, our proof still had two gaps: one due to an error that persisted from [4] (see Error 1 below) and another of our own making (see Error 2 below). We are grateful to E. Friedlander for flagging both these gaps in his recent preprint [6] that addresses the errors in [4].

Error 1. *In Section* 4 (*Construction* 4.1) *of* [2] *we define a simplicial scheme* $S^{2i}_{\mathbb{Z}}$ *whose complex points* (*with the analytic topology*) *are equivalent to the* $2i$*-sphere. The basepoints were incorrectly stated as the initial scheme Spec*(0). *They should have been stated as the terminal scheme Spec*$(\mathbb{Z})$ *to make this object well-defined. Since all subsequent constructions use the correct definition, this error is not consequential. However, in Remark* 4.4, *we posit the existence of a map*

$$\tau_{i,j} : S^{2i}_{\mathbb{Z}} \times S^{2j}_{\mathbb{Z}} \longrightarrow S^{2i+2j}_{\mathbb{Z}},$$

which descends to the smash product map on complex points. Unfortunately, on closer study, we see that such a map is not well-defined.

The simplicial scheme $\mathrm{S}^{2i}_{\mathbb{Z}}$ and map $\tau_{i,j}$ first appear in past work by Friedlander (see [4], page 139). However, the fact that $\tau_{i,j}$ is not well-defined had been overlooked in the literature. We too failed to notice this in [2], until the error was finally flagged by Friedlander in [6]. This is a consequential issue, since the map $\tau_{i,j}$ forms the basis of the monoidal structure one requires of all constructions that follow. In [6], Friedlander alters the definition of $\mathbb{S}^{2i}_{\mathbb{Z}}$ so that $\tau_{i,j}$ becomes well-defined.

Error 2. *Assuming the existence of a corrected map* $\tau_{i,j}$ *above, one may extend it to a* $GL_i \times GL_j$*-equivariant map of schemes, where* GL_i *denotes the general linear group scheme. This allows us to construct a map of bar constructions of the* GL_i*-action on the above simplicial schemes* (*see* [2], *diagram* 4.26):

$$\omega_{i,j} : \mathrm{SBGL}_i \times \mathrm{SBGL}_j \longrightarrow \mathrm{SBGL}_{i+j}.$$

Changing base to $\mathbb{C}$ *and applying the* (*p-completed*) *étale homotopy type functor to* $\omega_{i,j}$, *one obtains*

$$\acute{\mathrm{E}}\mathrm{t}_p(\mathrm{SBGL}_i \times \mathrm{SBGL}_j) \longrightarrow \acute{\mathrm{E}}\mathrm{t}_p(\mathrm{SBGL}_{i+j}).$$

The classical comparison theorem ([5], *Theorem* 8.4) *gives rise to a weak equivalence*

$$F : \acute{\mathrm{E}}\mathrm{t}_p(\mathrm{SBGL}_i \times \mathrm{SBGL}_j) \xrightarrow{\sim} (\mathrm{SBGL}^{\mathrm{top}}(\mathbb{C})_i)\hat{}_p \times (\mathrm{SBGL}^{\mathrm{top}}(\mathbb{C})_j)\hat{}_p,$$

where $\mathrm{SBGL}^{\mathrm{top}}(\mathbb{C})_i$ *denotes the complex points of* SBGL_i *with the analytic topology. In* [2] (*see diagram* 4.25), *we implicitly use the inverse equivalence of* F *and the map* $\omega_{i,j}$ *to construct a multicategory. However, the above equivalence* F *does not have a functorial inverse*! *This is a consequential error since one requires functoriality in subsequent categorical constructions.*

References

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Prasit Bhattacharya
Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM, USA
E-mail: prasit@nmsu.edu
https://orcid.org/0000-0002-5286-4631

Nitu Kitchloo
Department of Mathematics, Johns Hopkins University, Baltimore, MD, USA
E-mail: nitu@math.jhu.edu