ON THE BEILINSON FIBER SQUARE

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Abstract
Using topological cyclic homology, we give a refinement of Beilinson's p-adic Goodwillie isomorphism between relative continuous K*-theory and cyclic homology. As a result, we generalize results of Bloch–Esnault–Kerz and Beilinson on the p-adic deformations of* K*-theory classes. Furthermore, we prove structural results for the Bhatt–Morrow–Scholze filtration on* TC *and identify the graded pieces with the syntomic cohomology of Fontaine–Messing.*

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1. Introduction

1.1. Fiber squares

For any ring R, one has its connective algebraic K-theory $\mathrm{K}(R)$ and its negative cyclic homology $\mathrm{HC}^{-}(R)$; they are related via the Goodwillie–Jones trace map $\mathrm{tr}_{\mathrm{GJ}}\colon \mathrm{K}(R) \to \mathrm{HC}^{-}(R)$, often interpreted and referred to as a *Chern character* (see [60, Chapter 8]). Moreover, when R is a $\mathbb{Q}$-algebra, the map $\mathrm{tr}_{\mathrm{GJ}}$ induces an

DUKE MATHEMATICAL JOURNAL
Vol. 171, No. 18, © 2022 DOI 10.1215/00127094-2022-0037
Received 25 January 2021. Revision received 29 September 2021.
First published online 10 November 2022.
2020 *Mathematics Subject Classification*. Primary 14F30; Secondary 14F40, 19D55, 19E15.

isomorphism on relative theories for nilimmersions, via the following theorem of Goodwillie.

THEOREM 1.1 (Goodwillie [38])
If $I \subseteq R$ is a nilpotent ideal in an associative $\mathbb{Q}$-algebra R, then the commutative square

$$\begin{tikzcd} \mathrm{K}(R) \arrow[r] \arrow[d, "\mathrm{tr}_{\mathrm{GJ}}"] & \mathrm{K}(R/I) \arrow[d, "\mathrm{tr}_{\mathrm{GJ}}"] \\ \mathrm{HC}^{-}(R) \arrow[r] & \mathrm{HC}^{-}(R/I) \end{tikzcd}$$

is Cartesian; that is, the Goodwillie–Jones trace map induces an equivalence $\mathrm{tr}_{\mathrm{GJ}} \colon \mathrm{K}(R,I) \simeq \mathrm{HC}^{-}(R,I)$ *on relative theories.*

Here, for a pair (R,I) with $I \subseteq R$ an ideal, we write $\mathrm{K}(R,I)$ for the fiber of $\mathrm{K}(R) \to \mathrm{K}(R/I)$, and similarly for other functors such as HC^{-} and so on.

In order to extend Goodwillie's theorem to more general rings, one uses topological cyclic homology $\mathrm{TC}(R)$, introduced in [19] in the p-complete case and in [25] integrally, and the cyclotomic trace $\mathrm{tr} \colon \mathrm{K}(R) \to \mathrm{TC}(R)$, which refines the Goodwillie–Jones trace map.

THEOREM 1.2 (Dundas–Goodwillie–McCarthy [25])
If $I \subseteq R$ is a nilpotent ideal in an associative ring R, then the commutative square

$$\begin{tikzcd} \mathrm{K}(R) \arrow[r] \arrow[d, "\mathrm{tr}"] & \mathrm{K}(R/I) \arrow[d, "\mathrm{tr}"] \\ \mathrm{TC}(R) \arrow[r] & \mathrm{TC}(R/I) \end{tikzcd}$$

is Cartesian; that is, the cyclotomic trace induces an equivalence $\mathrm{tr} \colon \mathrm{K}(R,I) \simeq \mathrm{TC}(R,I)$ *on relative theories.*

Topological cyclic homology is thus the primary tool in calculations of relative K-theory (see, e.g., [47], [48], [66]), but it is a significantly more complicated invariant than cyclic homology. However, recently Beilinson [8] gave a version of Goodwillie's original result in a p-adic setting, when the ideal in question is (p) and R is assumed to be complete along (p). The first goal of this paper is to construct a variant of the Chern character and prove a strengthening of Beilinson's theorem.

Throughout this paper, we fix a prime number p. We use the convention that the modifier "$\mathbb{Z}_p$" refers to p-adic completion of an object, and "$\mathbb{Q}_p$" to the rationalization of the p-completion; for example, $\mathrm{K}(R;\mathbb{Z}_p)$ denotes the p-complete K-theory of R, and $\mathrm{K}(R;\mathbb{Q}_p)$ denotes the rationalization of $\mathrm{K}(R;\mathbb{Z}_p)$. Similarly, the modifier "$\mathbb{Q}$" refers to rationalization. We denote by HP (resp., HC) periodic cyclic (resp., cyclic) homology.

THEOREM A

For an associative ring R, there is a natural p-adic Chern character map

$$\mathrm{tr}_{\mathrm{crys}}\colon \mathrm{K}(R/p;\mathbb{Q}_p) \to \mathrm{HP}(R;\mathbb{Q}_p) \tag{1}$$

which fits into a natural commutative square

$$\begin{tikzcd} \mathrm{K}(R;\mathbb{Q}_p) \arrow[r] \arrow[d, "\mathrm{tr}_{\mathrm{GJ}}"] & \mathrm{K}(R/p;\mathbb{Q}_p) \arrow[d, "\mathrm{tr}_{\mathrm{crys}}"] \\ \mathrm{HC}^{-}(R;\mathbb{Q}_p) \arrow[r] & \mathrm{HP}(R;\mathbb{Q}_p). \end{tikzcd} \tag{2}$$

If R is commutative and Henselian along (p) then this square is Cartesian, thereby giving an equivalence $\mathrm{K}(R,(p);\mathbb{Q}_p) \simeq \Sigma\mathrm{HC}(R;\mathbb{Q}_p)$.

In [8], Beilinson constructs a natural equivalence[1] $\mathrm{K}^{\mathrm{cts}}(R,(p);\mathbb{Q}_p) \simeq \Sigma\mathrm{HC}(R;\mathbb{Q}_p)$ under the assumption that R is p-complete with bounded p-power torsion, R/p has finite stable range,[2] and the relative K-theory term $\mathrm{K}(R,(p))$ is replaced by the "continuous" relative K-theory $\mathrm{K}^{\mathrm{cts}}(R,(p)) = \varprojlim \mathrm{K}(R/p^n,(p))$; this replacement does not affect the conclusion if R is commutative thanks to [23, Theorem 5.23]. Beilinson's arguments rely on some p-adic Lie theory.

In this paper, we will construct the map (1) using the description of topological cyclic homology from Nikolaus–Scholze [73], as a consequence of Bökstedt's calculation of $\mathrm{THH}(\mathbb{F}_p)$. Together with the rigidity results of Clausen–Mathew–Morrow [23], we explain a short, homotopy-theoretic proof of Theorem A. In fact, Theorem A and all the corollaries listed below hold for any (possibly noncommutative) ring R if we replace K-theory by TC (see Theorem 2.12 and Corollary 3.9); the Henselian condition is only needed to translate between K-theory and TC.

Next, we observe some consequences of and complements to Theorem A. In [8], slightly more than an equivalence of rational spectra $\mathrm{K}(R,(p);\mathbb{Q}_p) \simeq \Sigma\mathrm{HC}(R;\mathbb{Q}_p)$

[1] Since the methods are different, we do not know if our identification on fiber terms is the same as Beilinson's.

[2] The stable range of a ring R was defined in [4] (see also [5, V.3]) and is sometimes, as in [8], called the *stable rank*.

is proved: there is a natural zigzag of "quasi-isogenies" of spectra before inverting p. By definition, a quasi-isogeny is a map which is an equivalence up to uniformly bounded denominators in any finite range of degrees. We also obtain the same conclusion in our setting and can keep track of the denominators at least in some range.

COROLLARY B

Let R be a commutative ring which is Henselian along (p). Then there is a natural zigzag of quasi-isogenies between $\mathrm{K}(R,(p);\mathbb{Z}_p)$ and $\Sigma\mathrm{HC}(R,(p);\mathbb{Z}_p)$. If R is, moreover, p-torsion-free, then there are isomorphisms $\pi_i\mathrm{K}(R,(p);\mathbb{Z}_p) \simeq \pi_i\Sigma\mathrm{HC}(R,(p);\mathbb{Z}_p)$ for $i \leq 2p-5$.

A similar result for an arbitrary nilpotent ideal, albeit in a smaller range of degrees (depending on the exponent of nilpotence of the ideal), is proved in [21]. The argument we use here seems to be special to the ideal (p).

As explained above, one could formulate Theorem A entirely in the language of topological cyclic homology, completely avoiding the mention of K-theory. At some point in the proof, however, we translate back into K-theory and use a homology argument. Therefore, we also offer an alternative, purely cyclotomic proof of this step. This relies on a study of quasi-isogenies in the homotopy theory of cyclotomic spectra, based on the t-structure introduced by Antieau–Nikolaus [3]. The key step is an extension of a theorem of Geisser–Hesselholt [36] and Land–Tamme [61].

THEOREM C

Let $f: A \to A'$ be a map of connective associative ring spectra. Suppose that:

(1) *f is a quasi-isogeny of spectra;*

(2) *the map $\pi_0(f): \pi_0(A) \to \pi_0(A')$ is surjective with nilpotent kernel.*

Then $\mathrm{THH}(A;\mathbb{Z}_p) \to \mathrm{THH}(A';\mathbb{Z}_p)$ is a quasi-isogeny in cyclotomic spectra, and, in particular, the induced map $\mathrm{TC}(A;\mathbb{Z}_p) \to \mathrm{TC}(A';\mathbb{Z}_p)$ is a quasi-isogeny.

1.2. p-adic deformations of K*-theory classes*

In our first main application of Theorem A, we generalize work of Bloch–Esnault–Kerz [17] and Beilinson [8] on the formal p-adic deformation of rational K-theory classes. Let us first recall the motivation for their work.

Fix a complete discretely valued field K of mixed characteristic $(0,p)$ with ring of integers $\mathcal{O}_K$ and perfect residue field k, as well as a proper smooth scheme $X \to \mathrm{Spec}(\mathcal{O}_K)$ with special fiber X_k and generic fiber X_K. Given X, we can consider the algebraic de Rham cohomology $H^*_{\mathrm{dR}}(X_K/K)$ of the generic fiber, together with its Hodge filtration $\mathrm{Fil}^{\geq \star} H^*_{\mathrm{dR}}(X_K/K)$; these are finite-dimensional K-vector spaces, and arise as the cohomology groups of objects $\mathrm{Fil}^{\geq\star} R\Gamma_{\mathrm{dR}}(X_K/K)$ in the derived category of K.

As usual, we have a Chern character

$$\mathrm{ch}\colon \mathrm{K}_0(X;\mathbb{Q}) \twoheadrightarrow \mathrm{K}_0(X_K;\mathbb{Q}) \to H^{\mathrm{even}}_{\mathrm{dR}}(X_K/K). \tag{3}$$

A foundational motivating question is to determine the image of this map: in other words, to determine which cohomology classes come from algebraic cycles on X_K (or, equivalently, on X). A conjecture of Fontaine–Messing, a p-adic analogue of the variational Hodge conjecture of Grothendieck [41, p. 103, note 13], predicts that this question can essentially be reduced from mixed to equal characteristic.

To formulate the conjecture, we consider also the (absolute) crystalline cohomology $H^*_{\mathrm{crys}}(X_k)$ of the special fiber, a family of finitely generated $W(k)$-modules. By the de Rham–to–crystalline comparison (see [9]), we have an isomorphism $H^*_{\mathrm{crys}}(X_k)\otimes_{W(k)} K \cong H^*_{\mathrm{dR}}(X_K/K)$. Finally, we have the crystalline Chern character map (see [43]),

$$\mathrm{ch}_{\mathrm{crys}}\colon \mathrm{K}_0(X_k) \to \bigoplus_{i\geq 0} H^{2i}_{\mathrm{crys}}(X_k;\mathbb{Q}_p), \tag{4}$$

leading to a commutative diagram

$$\begin{tikzcd} \mathrm{K}_0(X;\mathbb{Q}) \arrow[r] \arrow[d, "\mathrm{ch}"] & \mathrm{K}_0(X_k;\mathbb{Q}) \arrow[d, "\mathrm{ch}_{\mathrm{crys}}"] \\ H^{\mathrm{even}}_{\mathrm{dR}}(X_K/K) \arrow[r, "\simeq"] & H^{\mathrm{even}}_{\mathrm{crys}}(X_k)\otimes_{W(k)} K. \end{tikzcd} \tag{5}$$

Conjecture 1.3 (p-adic variational Hodge conjecture)
Let $\alpha \in H^{\mathrm{even}}_{\mathrm{dR}}(X_K/K)$. Then α belongs to the image of the Chern character from $\mathrm{K}_0(X;\mathbb{Q})$ if and only if:

(1) the image of α under the de Rham–to–crystalline isomorphism in $H^{\mathrm{even}}_{\mathrm{crys}}(X_k)\otimes_{W(k)} K$ belongs to the image of the crystalline Chern character from $\mathrm{K}_0(X_k;\mathbb{Q})$;
(2) the class α belongs to $\bigoplus_i \mathrm{Fil}^{\geq i} H^{2i}_{\mathrm{dR}}(X_K/K) \subseteq H^{\mathrm{even}}_{\mathrm{dR}}(X_K/K)$.

For further details and arithmetic applications of the p-adic variational Hodge conjecture, we refer to [27].

Motivated by Conjecture 1.3, Bloch–Esnault–Kerz [17] considered the following p-adic deformation question, which starts with a K_0-class on the special fiber (rather than a cohomology class) and asks when it lifts infinitesimally.

Question 1.4 (The p-adic deformation problem)
Given the data as above, define the "continuous" K-theory

$$\mathrm{K}^{\mathrm{cts}}(X) \stackrel{\mathrm{def}}{=} \varprojlim \mathrm{K}(X/\pi^n),$$

where π is a uniformizer of $\mathcal{O}_K$. Given a class $x \in \mathrm{K}_0(X_k;\mathbb{Q})$, when does it belong to the image of the reduction map from the continuous K-theory $\mathrm{K}_0^{\mathrm{cts}}(X;\mathbb{Q}) = \pi_0(\mathrm{K}^{\mathrm{cts}}(X))_{\mathbb{Q}}$?

Since the map $\mathrm{K}(X) \to \mathrm{K}^{\mathrm{cts}}(X)$ is generally not an equivalence, the p-adic deformation problem does not imply Conjecture 1.3. However, the p-adic deformation problem is a (pro-)infinitesimal one, so it can be studied using methods of topological cyclic homology. Using the Beilinson fiber square, we answer the p-adic deformation problem as follows; in [17], this result is proved for a smooth projective scheme of dimension $d < p + 6$ when K is unramified.

THEOREM D
Let X be a proper smooth scheme over $\mathcal{O}_K$. A class $x \in \mathrm{K}_0(X_k;\mathbb{Q})$ lifts to $\mathrm{K}_0^{\mathrm{cts}}(X;\mathbb{Q})$ if and only if $\mathrm{ch}_{\mathrm{crys}}(x) \in \bigoplus_{i\geq 0} H^{2i}_{\mathrm{crys}}(X_k;\mathbb{Q}_p)$ is carried by the de Rham–to–crystalline comparison isomorphism to a class in $\bigoplus_{i\geq 0} \mathrm{Fil}^{\geq i} H^{2i}_{\mathrm{dR}}(X_K/K) \subseteq \bigoplus_{i\geq 0} H^{2i}_{\mathrm{dR}}(X_K/K)$.

Our main observation is that Theorem A together with Hochschild–Kostant–Rosenberg comparisons between cyclic and de Rham cohomology yield a fiber square

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{cts}}(X;\mathbb{Q}) \arrow[r] \arrow[d] & \mathrm{K}(X_k;\mathbb{Q}) \arrow[d] \\
\prod_{i\in\mathbb{Z}} \mathrm{Fil}^{\geq i} R\Gamma_{\mathrm{dR}}(X_K/K)[2i] \arrow[r] & \prod_{i\in\mathbb{Z}} R\Gamma_{\mathrm{dR}}(X_K/K)[2i].
\end{tikzcd} \tag{6}$$

Moreover, on K_0, one checks that the vertical map on the right-hand side induces the crystalline Chern character (4), at least up to scalars, implying the result. For this argument, it is crucial that one has the fiber square (6), rather than a fiber sequence alone.

One can also generalize the above questions to higher K-theory. In [8], the Beilinson fiber sequence is used to prove that if $x \in \mathrm{K}_j(X_k;\mathbb{Q})$, then there exists a natural obstruction class in $\bigoplus_{i\geq 0} H^{2i-j}_{\mathrm{dR}}(X_K)/\mathrm{Fil}^{\geq i} H^{2i-j}_{\mathrm{dR}}(X_K)$ which vanishes if and only if x lifts to the continuous K-theory $\mathrm{K}_i^{\mathrm{cts}}(X;\mathbb{Q})$; however, [8] does not identify the class with the crystalline Chern character for $i = 0$. Here we also extend this result to an arbitrary quasicompact and quasiseparated (qcqs) scheme with bounded p-power torsion, using p-adic derived de Rham cohomology [10] and results of [1].

THEOREM E
Let X be a qcqs scheme with bounded p-power torsion. For each n we write X_n for $X \times_{\operatorname{Spec}\mathbb{Z}} \mathbb{Z}/p^n$, and put $\mathrm{K}^{\mathrm{cts}}(X) \stackrel{\mathrm{def}}{=} \varprojlim \mathrm{K}(X_n)$. Given a class $x \in \mathrm{K}_j(X_1;\mathbb{Q})$, there is a natural class

$$c(x) \in \bigoplus_{i \geq 0} H^{2i-j}(L\Omega_X / L\Omega_X^{\geq i})_{\mathbb{Q}_p},$$

where $L\Omega_X$ is the p-adic derived de Rham cohomology of X with the derived Hodge filtration $L\Omega_X^{\geq \star}$. The class x lifts to $\mathrm{K}_i^{\mathrm{cts}}(X;\mathbb{Q})$ if and only if $c(x) = 0$.

1.3. The motivic filtration on TC

In [12], Bhatt–Morrow–Scholze discovered a fundamental additional structure on the p-adic topological cyclic homology $\mathrm{TC}(-;\mathbb{Z}_p)$ of p-adic *commutative* rings: a "motivic filtration" $\mathrm{Fil}^{\geq\star}\mathrm{TC}(-;\mathbb{Z}_p)$ on $\mathrm{TC}(-;\mathbb{Z}_p)$ with associated graded terms denoted $\mathbb{Z}_p(i)[2i]$. The objects $\mathbb{Z}_p(i)$ thus obtained are related to integral p-adic Hodge theory and can be defined (independently of topological cyclic homology) as a type of filtered Frobenius invariants on the prismatic cohomology (see [15]). They are known explicitly in some cases: in characteristic $p > 0$ they can be identified with logarithmic de Rham–Witt sheaves (up to a shift), and, for formally smooth algebras over $\mathcal{O}_C$ (for C a complete algebraically closed nonarchimedean field), they can be identified with truncated p-adic nearby cycles.

Recall also that for p-adic (commutative) rings, $\mathrm{TC}(-;\mathbb{Z}_p)$ is p-adic étale K-theory in nonnegative degrees (see [22], [23], [34]). Therefore, it is expected (but not known in mixed characteristic) that the filtration $\mathrm{Fil}^{\geq\star}\mathrm{TC}(-;\mathbb{Z}_p)$ is the étale sheafified motivic filtration on algebraic K-theory, and that the $\mathbb{Z}_p(i)$ are p-adic étale motivic cohomology, at least where all of these objects are defined. One also has constructions of Schneider and Sato [80] of "p-adic étale Tate twists," which satisfy a type of arithmetic duality. In general, one expects that the $\mathbb{Z}_p(i)$ should be related to important foundational questions in arithmetic geometry and K-theory. An advantage of the construction of $\mathrm{Fil}^{\geq\star}\mathrm{TC}(-;\mathbb{Z}_p)$ and the $\mathbb{Z}_p(i)$ as in [12] is that it works in a much more general setting (for the quasisyntomic rings; see Section 5.1 below for a review) than existing approaches to motivic cohomology. Moreover, its definition is extremely direct: it is simply a sheafified Postnikov tower (albeit for a "large" topology).

Using Theorem A, we will give a description of the $\mathbb{Z}_p(i)$ for $i \leq p-2$ and of the $\mathbb{Q}_p(i)$ for all i in terms of syntomic cohomology as considered by Fontaine–Messing [31] and Kato [54]. In particular, this construction gives a description of the $\mathbb{Z}_p(i)$

(with the above restrictions) that relies only on derived de Rham theory, rather than prismatic theory. Our result is an analogue of a result of Geisser [33] for étale motivic cohomology for smooth schemes over Dedekind rings.

To formulate the result, we write $L\Omega_R$ for the p-adic derived de Rham cohomology for a commutative ring R equipped with its derived Hodge filtration $L\Omega_R^{\geq\star}$ (see [10]). The object $L\Omega_R$ carries a crystalline Frobenius $\varphi\colon L\Omega_R \to L\Omega_R$. For $i < p$, one has a "divided" Frobenius $\varphi/p^i : L\Omega_R^{\geq i} \to L\Omega_R$. Using the techniques of [12] (in particular, quasisyntomic sheafification) applied to the Beilinson fiber square, we deduce our next theorem.

THEOREM F

Let R be a quasisyntomic ring.

(1) *For each $i \geq 0$, there is an identification*

$$\mathbb{Q}_p(i)(R) \simeq \mathrm{fib}(\varphi - p^i : L\Omega_R^{\geq i} \to L\Omega_R)_{\mathbb{Q}_p}.$$

(2) *For $i \leq p-2$, there is an identification*

$$\mathbb{Z}_p(i)(R) \simeq \mathrm{fib}(\varphi/p^i - \mathrm{id}\colon L\Omega_R^{\geq i} \to L\Omega_R).$$

We explicitly analyze Theorem F in three cases in which one has alternate descriptions of the $\mathbb{Z}_p(i)$: rings of integers in p-adic fields, perfectoid rings, and formally smooth $\mathcal{O}_C$-algebras where C is an algebraically closed, complete non-Archimedean field of mixed characteristic. The first case recovers classical calculations of the rational p-adic K-theory of p-adic fields; the second case recovers the fundamental exact sequence in p-adic Hodge theory; and the last case recovers results of Colmez–Nizioł [24] on p-adic vanishing cycles, albeit only in the formally smooth case.

Finally, Theorem F provides a complete computation of low-degree or rationalized TC in terms of syntomic cohomology. This computation relies on the following connectivity estimate about the $\mathbb{Z}_p(i)$ and about the filtration on $\mathrm{TC}(-;\mathbb{Z}_p)$. The estimate for algebras over a perfectoid ring is stated in [12, Construction 7.4]; the argument for all quasisyntomic rings relies on the use of relative topological Hochschild homology and the spectral sequence of Krause–Nikolaus [55].

THEOREM G

If R is a quasisyntomic ring, then $\mathbb{Z}_p(i)(R) \in D^{\leq i+1}(\mathbb{Z}_p)$. If R is w-strictly local (e.g., strictly Henselian local), then $\mathbb{Z}_p(i)(R) \in D^{\leq i}(\mathbb{Z}_p)$. Consequently, $\mathrm{Fil}^{\geq i}\mathrm{TC}(R;\mathbb{Z}_p)$ is concentrated in homological degrees $\geq i-1$ (and pro-étale locally $\geq i$).

COROLLARY H

If R is any commutative ring, then there is a natural equivalence

$$\mathrm{TC}(R;\mathbb{Q}_p) \simeq \bigoplus_{i\geq 0} \mathrm{fib}(\varphi - p^i : L\Omega_R^{\geq i} \to L\Omega_R)_{\mathbb{Q}_p}.$$

Notation

Throughout, we write Sp for the ∞-category of spectra. Given a ring R, we write $D(R)$ for the derived ∞-category of R.

We will use homological indexing conventions indicated with a subscript when referring to spectra and cohomological indexing conventions indicated with a superscript when referring to objects of the derived category. For instance, given n, we write $\mathrm{Sp}_{\geq n}$ (resp., $\mathrm{Sp}_{\leq n}$) for spectra with homotopy groups concentrated in degrees $\geq n$ (resp., $\leq n$); and we write $D(R)^{\leq n}$ (resp., $D(R)^{\geq n}$) for objects of $D(R)$ with cohomology groups concentrated in degrees $\leq n$ (resp., $\geq n$).

We write $\mathrm{HH}(R)$ for the Hochschild homology of R, always relative to $\mathbb{Z}$ and always computed in a derived sense (also known as Shukla homology), and we let $\mathrm{THH}(R)$ denote the topological Hochschild homology of R. We write $\mathrm{HC}^-(R) = \mathrm{HH}(R)^{hS^1}$ for negative cyclic homology and $\mathrm{HP}(R) = \mathrm{HH}(R)^{tS^1}$ for periodic cyclic homology. For a scheme X, we let $L\Omega_X$ denote its p-completed derived de Rham cohomology (relative to $\mathbb{Z}$) and $L\Omega_X^{\geq i}$ for the ith stage of the (derived) Hodge filtration. We denote the Hodge-completed variants by $\widehat{L\Omega}_X$ and $\widehat{L\Omega}_X^{\geq i}$, respectively.

2. The Beilinson fiber square

2.1. Background

We review some background on the theory of cyclotomic spectra and topological cyclic homology as in [73], of which we will use the p-typical variant. This theory uses the ∞-category $\mathrm{Fun}(BS^1, \mathrm{Sp})$ of spectra equipped with S^1-actions. Given a spectrum X equipped with an S^1-action, we can form the homotopy S^1-orbits X_{hS^1}, the homotopy S^1-fixed points X^{hS^1}, and the S^1-Tate construction X^{tS^1}. These are related by a natural fiber sequence $\Sigma X_{hS^1} \to X^{hS^1} \to X^{tS^1}$, which we will use constantly and without further comment. See, for example, [73, Corollary I.4.3].

Definition 2.1 (Nikolaus–Scholze [73])

We let CycSp denote the symmetric monoidal, stable ∞-category of *cyclotomic spectra*.[3] An object of CycSp consists of a spectrum X equipped with an S^1-action and an S^1-equivariant map $\varphi_p : X \to X^{tC_p}$ called the *cyclotomic Frobenius*.

[3]Our conventions are slightly different from those of [73], which requires a Frobenius map for each prime number q and not only the fixed prime p. What we call a *cyclotomic spectrum* is called a *p-typical cyclotomic*

Given $X \in \mathrm{CycSp}$, we write

$$\mathrm{TC}^-(X) = X^{hS^1} \qquad \text{and} \qquad \mathrm{TP}(X) = X^{tS^1}.$$

We will define only p-complete TC for $X \in \mathrm{CycSp}$ and will assume that X is bounded below. We have two maps $\mathrm{can}\colon \mathrm{TC}^-(X;\mathbb{Z}_p) \to \mathrm{TP}(X;\mathbb{Z}_p)$ and $\varphi\colon \mathrm{TC}^-(X;\mathbb{Z}_p) \to \mathrm{TP}(X;\mathbb{Z}_p)$. By definition, can is the canonical map from S^1-invariants to the Tate construction, and φ is induced from the Frobenius φ_p. The p-complete topological cyclic homology $\mathrm{TC}(X;\mathbb{Z}_p)$, for $X \in \mathrm{CycSp}$ bounded below, can be computed as the fiber of the difference of the two maps; that is,

$$\mathrm{TC}(X;\mathbb{Z}_p) = \mathrm{fib}\big(\mathrm{can} - \varphi\colon \mathrm{TC}^-(X;\mathbb{Z}_p) \to \mathrm{TP}(X;\mathbb{Z}_p)\big). \tag{7}$$

Remark 2.2
We will use throughout the basic fact that if $X \in \mathrm{CycSp}$ has underlying n-connective spectrum, then $\mathrm{TC}(X;\mathbb{Z}_p)$ is $(n-1)$-connective (see, e.g., [23, Lemma 2.5 and Remark 2.14]).

Example 2.3

(1) Given a ring R, we can form the topological Hochschild homology $\mathrm{THH}(R)$ as a cyclotomic spectrum.
(2) Given a spectrum Y, we let Y^{triv} be the cyclotomic spectrum, where we view Y as a spectrum with trivial S^1-action and with cyclotomic Frobenius given by the natural map $Y \to Y^{hC_p} \to Y^{tC_p}$.
(3) For a spectrum X with S^1-action, we get a cyclotomic spectrum by letting $\varphi_p\colon X \to X^{tC_p}$ be zero (as an S^1-equivariant map).

Remark 2.4
Let Y be a bounded below spectrum of finite type, meaning that each $\pi_i Y$ is a finitely generated abelian group. If $X \in \mathrm{CycSp}$ is p-complete and bounded below, then there is a natural equivalence

$$\mathrm{TC}(X \otimes_{\mathbb{S}} Y^{\mathrm{triv}};\mathbb{Z}_p) \simeq \mathrm{TC}(X;\mathbb{Z}_p) \otimes_{\mathbb{S}} Y. \tag{8}$$

This is a consequence of the fact that the functor $\mathrm{TC}(-;\mathbb{Z}_p)$ commutes with geometric realizations of connective cyclotomic spectra, since it is exact and carries n-connective objects into $(n-1)$-connective objects. This even holds if Y is only

spectrum in [3], and what we write as CycSp is written as CycSp_p in [3]. For p-complete objects (the primary case of interest), there is no difference between a cyclotomic spectrum in the sense of [73] and the definition used here.

assumed to be bounded below, as long as the right-hand side of (8) is p-adically completed, by [23, Theorem 2.7].

As a simple exercise with the above definitions, we prove the following result for use below. This has been used in other references as well (see, e.g., [49, Section 1.4]).

PROPOSITION 2.5
If $X \in \mathrm{CycSp}$ *is bounded below and* $\mathrm{TP}(X;\mathbb{Z}_p) = 0$*, then we have natural equivalences* $\mathrm{TC}(X;\mathbb{Z}_p) \simeq (\Sigma X_{hS^1})^\wedge_p \simeq \mathrm{TC}^-(X;\mathbb{Z}_p)$.

Proof
In fact, the formula (7) shows that $\mathrm{TC}(X;\mathbb{Z}_p) = \mathrm{TC}^-(X;\mathbb{Z}_p)$. Now $\mathrm{TP}(X;\mathbb{Z}_p)$ is the cofiber of the norm map $(\Sigma X_{hS^1})^\wedge_p \to \mathrm{TC}^-(X;\mathbb{Z}_p)$. Since we have assumed $\mathrm{TP}(X;\mathbb{Z}_p) = 0$, we have $(\Sigma X_{hS^1})^\wedge_p \simeq \mathrm{TC}^-(X;\mathbb{Z}_p)$. Combining the two identifications, we conclude. $\square$

Next, we apply this to a specific crucial example.

Construction 2.6 (The cyclotomic spectrum $\mathbb{Z}_{hC_p}$*)*
Recall first that the cyclotomic trace (or a direct construction) gives a map

$$\mathbb{Z}^{\mathrm{triv}} \to \mathrm{THH}(\mathbb{F}_p) \tag{9}$$

in CycSp, and that as objects of $\mathrm{Fun}(BS^1, \mathrm{Sp})$ we have $\mathrm{THH}(\mathbb{F}_p) \simeq \tau_{\geq 0}(\mathbb{Z}^{tC_p})$ by [73, Sec. IV-4]; here we obtain the S^1-action on $\mathbb{Z}^{tC_p}$ via the sequence $C_p \to S^1 \xrightarrow{z \mapsto z^p} S^1$. This is a refinement of (and deduced from) Bökstedt's calculation of $\mathrm{THH}(\mathbb{F}_p)$. Consequently, there is a cofiber sequence in CycSp,

$$\mathbb{Z}_{hC_p} \to \mathbb{Z}^{\mathrm{triv}} \to \mathrm{THH}(\mathbb{F}_p), \tag{10}$$

where $\mathbb{Z}_{hC_p}$ is a cyclotomic spectrum with underlying spectrum with S^1-action $\mathbb{Z}_{hC_p} \in \mathrm{Fun}(BS^1, \mathrm{Sp})$.

Note that $(\mathbb{Z}_{hC_p})^{tC_p} \simeq 0$ by the Tate orbit lemma (see [73, Lemma I.2.1]). In particular, the map $\mathbb{Z}^{\mathrm{triv}} \to \mathrm{THH}(\mathbb{F}_p)$ induces an equivalence on $\mathrm{TP}(-;\mathbb{Z}_p)$, i.e., $\mathbb{Z}_p^{tS^1} \simeq \mathrm{THH}(\mathbb{F}_p)^{tS^1} = \mathrm{TP}(\mathbb{F}_p)$. We obtain by Proposition 2.5 that $\mathrm{TC}(\mathbb{Z}_{hC_p};\mathbb{Z}_p) \simeq \Sigma(\mathbb{Z}_p)_{hS^1}$. Our next observation is that this remains true after tensoring with any bounded below cyclotomic spectrum.

LEMMA 2.7
If $X \in \mathrm{Fun}(BS^1, \mathrm{Sp})$*, then* $(X \otimes_{\mathbb{S}} \mathbb{Z}_{hC_p})^{tS^1}$ *is* p*-adically zero (here we use the diagonal* S^1*-action).*

Proof

The spectrum $(X \otimes_{\mathbb{S}} \mathbb{Z}_{hC_p})^{tS^1}$ is a module over $(\mathbb{Z}^{hC_p})^{tS^1}$. Since $(\mathbb{Z}^{hC_p})^{tS^1}$ vanishes p-adically by the Tate fixed point lemma (see [73, Lemmas I.2.2 and II.4.2], the lemma follows. Alternatively, one can easily verify that $(\mathbb{F}_p)_{hC_p} \in \mathrm{Fun}(BS^1, \mathrm{Sp})$ is induced from the trivial subgroup, which forces the p-adic Tate vanishing. □

Combining Proposition 2.5 and Lemma 2.7, we conclude that if X is any bounded below cyclotomic spectrum, then there are equivalences

$$\mathrm{TC}(X \otimes_{\mathbb{S}} \mathbb{Z}_{hC_p}; \mathbb{Z}_p) \simeq \Sigma\big((X \otimes_{\mathbb{S}} \mathbb{Z}_{hC_p})_{hS^1}\big)^{\wedge}_p \simeq \mathrm{TC}^-(X \otimes_{\mathbb{S}} \mathbb{Z}_{hC_p}; \mathbb{Z}_p). \tag{11}$$

2.2. *Pullback squares*

Next, we establish some pullback squares involving cyclotomic spectra and give a proof of Theorem A.

PROPOSITION 2.8

Let $X \in \mathrm{CycSp}$ *be a bounded below cyclotomic spectrum. Then the commutative square*

$$\begin{tikzcd} \mathrm{TC}(X \otimes_{\mathbb{S}} \mathbb{Z}^{\mathrm{triv}}; \mathbb{Z}_p) \arrow[r] \arrow[d] & \mathrm{TC}(X \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{F}_p); \mathbb{Z}_p) \arrow[d] \\ \mathrm{TC}^-(X \otimes_{\mathbb{S}} \mathbb{Z}^{\mathrm{triv}}; \mathbb{Z}_p) \arrow[r] & \mathrm{TC}^-(X \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{F}_p); \mathbb{Z}_p) \end{tikzcd} \tag{12}$$

is Cartesian, where the horizontal maps arise from the map $\mathbb{Z}^{\mathrm{triv}} \to \mathrm{THH}(\mathbb{F}_p)$ *in* CycSp *of Construction 2.6 and the vertical maps are the canonical maps* $\mathrm{TC}(-; \mathbb{Z}_p) \to \mathrm{TC}^-(-; \mathbb{Z}_p)$ *arising from the definition of* $\mathrm{TC}(-; \mathbb{Z}_p)$.

Moreover, there is a natural fiber sequence

$$\big(\Sigma(X \otimes_{\mathbb{S}} \mathbb{Z}_{hC_p})_{hS^1}\big)^{\wedge}_p \to \mathrm{TC}(X \otimes_{\mathbb{S}} \mathbb{Z}^{\mathrm{triv}}; \mathbb{Z}_p) \to \mathrm{TC}\big(X \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{F}_p); \mathbb{Z}_p\big). \tag{13}$$

Proof

Since $\mathrm{TC}(Z; \mathbb{Z}_p)$ for a bounded-below cyclotomic spectrum Z is an equalizer of two maps $\mathrm{TC}^-(Z; \mathbb{Z}_p) \rightrightarrows \mathrm{TP}(Z; \mathbb{Z}_p)$, the statement that (12) is Cartesian follows from the fact that $X \otimes_{\mathbb{S}} \mathbb{Z}^{\mathrm{triv}} \to X \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{F}_p)$ induces an equivalence on $\mathrm{TP}(-; \mathbb{Z}_p)$, via Lemma 2.7. Moreover, the fiber sequence (13) then follows from (12) via taking fibers, and using Lemma 2.7 again to replace homotopy fixed points by homotopy orbits. Alternatively, to prove that (12) is Cartesian, one observes that the fibers of the horizontal arrows are $\mathrm{TC}(X \otimes_{\mathbb{S}} \mathbb{Z}_{hC_p}; \mathbb{Z}_p)$ and $\mathrm{TC}^-(X \otimes_{\mathbb{S}} \mathbb{Z}_{hC_p}; \mathbb{Z}_p)$ and these are naturally equivalent as in (11). □

COROLLARY 2.9

For every connective ring spectrum R we have a natural fiber sequence of p-complete spectra

$$\Sigma\big(\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}_{hC_p}\big)_{hS^1}\to \mathrm{TC}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}\to \mathrm{TC}(R\otimes_{\mathbb{S}}\mathbb{F}_p;\mathbb{Z}_p).$$

Proof

We apply Proposition 2.8 to $X=\mathrm{THH}(R;\mathbb{Z}_p)$. We have that $\mathrm{THH}(R)\otimes_{\mathbb{S}}\mathrm{THH}(\mathbb{F}_p)\simeq \mathrm{THH}(R\otimes_{\mathbb{S}}\mathbb{F}_p)$, which gives the identification of the third term. For the identification of the term in the middle, we observe that $\mathrm{TC}(X\otimes_{\mathbb{S}}\mathbb{Z}^{\mathrm{triv}};\mathbb{Z}_p)\simeq \mathrm{TC}(X;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}$ by (8). Note finally that for bounded-below spectra, tensoring with $\mathbb{Z}$ preserves p-completeness as $\mathbb{Z}$ is of finite type. ☐

Next, we study what happens in (12) after rationalization.

COROLLARY 2.10

Let $X\in\mathrm{CycSp}$ be a bounded-below, p-complete cyclotomic spectrum. Then there exists a natural map $\mathrm{TC}(X\otimes_{\mathbb{S}}\mathrm{THH}(\mathbb{F}_p);\mathbb{Z}_p)\to (X\otimes_{\mathbb{S}}\mathbb{Z}^{\mathrm{triv}})^{tS^1}$ which fits into a natural commutative square

$$\begin{tikzcd}
\mathrm{TC}(X\otimes_{\mathbb{S}}\mathbb{Z}^{\mathrm{triv}};\mathbb{Z}_p) \arrow[r] \arrow[d] & \mathrm{TC}(X\otimes_{\mathbb{S}}\mathrm{THH}(\mathbb{F}_p);\mathbb{Z}_p) \arrow[d] \\
(X\otimes_{\mathbb{S}}\mathbb{Z}^{\mathrm{triv}})^{hS^1} \arrow[r] & (X\otimes_{\mathbb{S}}\mathbb{Z}^{\mathrm{triv}})^{tS^1}.
\end{tikzcd} \tag{14}$$

Moreover, this square becomes Cartesian after inverting p.

Proof

We can vertically extend the Cartesian square (12) via the canonical maps $(-)^{hS^1}\to(-)^{tS^1}$. In this case, as we saw earlier, the map $(X\otimes_{\mathbb{S}}\mathbb{Z}^{\mathrm{triv}})^{tS^1}\to(X\otimes_{\mathbb{S}}\mathrm{THH}(\mathbb{F}_p))^{tS^1}$ is an equivalence. Using this identification, we obtain the commutative square (14). The fact that (14) is Cartesian after inverting p follows from the facts that (12) is Cartesian and that $(X\otimes_{\mathbb{S}}\mathrm{THH}(\mathbb{F}_p))^{hS^1}\to(X\otimes_{\mathbb{S}}\mathrm{THH}(\mathbb{F}_p))^{tS^1}\simeq(X\otimes_{\mathbb{S}}\mathbb{Z}^{\mathrm{triv}})^{tS^1}$ becomes an equivalence after inverting p. ☐

Remark 2.11 (Effective bounds for the denominators in Corollary 2.10)

For future reference, it will be helpful to give a more effective version of Corollary 2.10. Consider the total cofiber (cofiber of horizontal cofibers) of the square (14). This is given by $\Sigma^2(X\otimes_{\mathbb{S}}\mathrm{THH}(\mathbb{F}_p))_{hS^1}$ because (12) is homotopy Cartesian. If X is connective, then it follows that the $\tau_{\le 2i}$ of the total cofiber is annihilated by

p^i, since $\tau_{\leq 2i-2}\mathrm{THH}(\mathbb{F}_p)$ is S^1-equivariantly annihilated by p^i: indeed, this follows by the (S^1-equivariant) Postnikov filtration, since each nonzero homotopy group of $\mathrm{THH}(\mathbb{F}_p)$ is in even degree and is an $\mathbb{F}_p$-vector space.

Consequently, we can deduce the following fiber square, which is the basic TC-theoretic result from which the Beilinson fiber square is a consequence.

THEOREM 2.12
Let R be a ring (or, more generally, a connective associative $H\mathbb{Z}$-algebra spectrum). Then there is a natural commutative square of spectra

$$\begin{tikzcd} \mathrm{TC}(R;\mathbb{Z}_p) \arrow[r] \arrow[d] & \mathrm{TC}(R\otimes_{\mathbb{S}}\mathbb{F}_p) \arrow[d] \\ \mathrm{HC}^-(R;\mathbb{Z}_p) \arrow[r] & \mathrm{HP}(R;\mathbb{Z}_p) \end{tikzcd} \tag{15}$$

which becomes Cartesian after inverting p. Aside from the right vertical arrow, all the maps are the canonical ones.

Proof
Via (14) for $X = \mathrm{THH}(R;\mathbb{Z}_p)$, we obtain a natural commutative diagram

$$\begin{tikzcd} \mathrm{TC}(R;\mathbb{Z}_p) \arrow[d] & \\ \mathrm{TC}(\mathrm{THH}(R)\otimes_{\mathbb{S}}\mathbb{Z}^{\mathrm{triv}};\mathbb{Z}_p) \arrow[r] \arrow[d] & \mathrm{TC}(R\otimes_{\mathbb{S}}\mathbb{F}_p) \arrow[d] \\ (\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}^{\mathrm{triv}})^{hS^1} \arrow[r] \arrow[d] & (\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z})^{tS^1} \arrow[d] \\ \mathrm{HC}^-(R;\mathbb{Z}_p) \arrow[r] & \mathrm{HP}(R;\mathbb{Z}_p) \end{tikzcd}$$

where we use the natural cyclotomic map $\mathrm{THH}(R;\mathbb{Z}_p) \to \mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}^{\mathrm{triv}}$ and the natural S^1-equivariant map $\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z} \to \mathrm{HH}(R;\mathbb{Z}_p)$. The upper square is Cartesian after inverting p by Corollary 2.10. The map $\mathrm{TC}(R;\mathbb{Z}_p) \to \mathrm{TC}(\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}^{\mathrm{triv}}) \simeq \mathrm{TC}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}$ is an equivalence after inverting p. The induced map on the bottom horizontal fibers is $\Sigma(\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z})_{hS^1} \to$

$\Sigma\mathrm{HH}(R;\mathbb{Z}_p)_{hS^1}$, which is an equivalence after inverting p since $\mathrm{THH}(R;\mathbb{Z}_p) \otimes_{\mathbb{S}} \mathbb{Z} \to \mathrm{HH}(R;\mathbb{Z}_p)$ is an equivalence after inverting p and this property is preserved by taking S^1-homotopy orbits. Thus, the bottom square is Cartesian after inverting p. Using these identifications, the theorem follows. $\square$

Remark 2.13 (Effective bounds II)
Again, one can make effective the statement that (15) is Cartesian after inverting p, at least in the range $\leq 2p-4$. In this case, we find (via Remark 2.11) that for $i \leq p-1$, $\tau_{\leq 2i}$ of the total cofiber of (15) is annihilated by p^i. Indeed, the map on cofibers of the bottom rows of (14) and (15) is given by $\Sigma^2(\mathrm{THH}(R) \otimes_{\mathbb{S}} \mathbb{Z}^{\mathrm{triv}})_{hS^1} \to \Sigma^2\mathrm{HH}(R;\mathbb{Z}_p)_{hS^1}$. This map is an equivalence in degrees $\leq 2p-2$.

Definition 2.14 (The p-adic Chern character)
Let R be a ring. Consider the map $\mathrm{TC}(R \otimes_{\mathbb{S}} \mathbb{F}_p) \to \mathrm{HP}(R;\mathbb{Z}_p)$ from above. After inverting p, in view of Theorem 3.4 below, we have an equivalence $\mathrm{TC}(R \otimes_{\mathbb{S}} \mathbb{F}_p;\mathbb{Q}_p) \simeq \mathrm{TC}(R/p;\mathbb{Q}_p)$. We therefore obtain a map $\beta\colon \mathrm{TC}(R/p;\mathbb{Q}_p) \to \mathrm{HP}(R;\mathbb{Q}_p)$, and precomposing with the trace we obtain

$$\mathrm{tr}_{\mathrm{crys}} = \mathrm{tr} \circ \beta\colon \mathrm{K}(R/p;\mathbb{Q}_p) \to \mathrm{HP}(R;\mathbb{Q}_p).$$

We call $\mathrm{tr}_{\mathrm{crys}}$ the *p-adic Chern character* and record that it fits into a natural commutative diagram

$$\begin{tikzcd}
\mathrm{K}(R;\mathbb{Q}_p) \arrow[r] \arrow[d, "\mathrm{tr}"'] & \mathrm{K}(R/p;\mathbb{Q}_p) \arrow[d, "\mathrm{tr}"] \arrow[dd, bend left=60, "\mathrm{tr}_{\mathrm{crys}}"] \\
\mathrm{TC}(R;\mathbb{Q}_p) \arrow[r] \arrow[d] & \mathrm{TC}(R/p;\mathbb{Q}_p) \arrow[d, "\beta"] \\
\mathrm{HC}^-(R;\mathbb{Q}_p) \arrow[r] & \mathrm{HP}(R;\mathbb{Q}_p)
\end{tikzcd} \tag{16}$$

in which the bottom square is a pullback.

Remark 2.15
In recent work, Petrov–Vologodsky [76] have shown that if $p > 2$ and R is p-torsion-free, then there is a natural equivalence $\mathrm{HP}(R;\mathbb{Z}_p) \simeq \mathrm{TP}(R/p;\mathbb{Z}_p)$. Thus, one could attempt to compare the p-adic Chern character $\mathrm{tr}_{\mathrm{crys}}$ with the usual trace $\mathrm{K}(R/p;\mathbb{Z}_p) \to \mathrm{TP}(R/p;\mathbb{Z}_p)$. We have not considered this question.

We can now give a quick proof of Theorem A by combining the above results with the following theorem.

THEOREM 2.16 (Clausen–Mathew–Morrow [23])
If R is commutative and Henselian along (p), then the trace induces an equivalence $\mathrm{K}(R,(p);\mathbb{Z}_p) \simeq \mathrm{TC}(R,(p);\mathbb{Z}_p)$.

Remark 2.17
If R is only associative, but p-complete and has bounded p-power torsion,[4] then there is an equivalence $\varprojlim \mathrm{K}(R/p^n,(p)) \simeq \mathrm{TC}(R,(p);\mathbb{Z}_p)$. This follows by the Dundas–Goodwillie–McCarthy theorem [25] and the p-adic continuity of TC (see [23, Theorem 5.19]).

Proof of Theorem A
As we have already noted, the square (15) from Theorem 2.12 is a pullback after inverting p; that is, the bottom square in (16) is a pullback. But the top square in (16) is a pullback by Theorem 2.16; assembling these Cartesian squares completes the proof of the theorem. □

2.3. Fiber sequences up to quasi-isogeny
Next, we review some definitions and terminology as in [8], identify more carefully the fiber terms in the above squares, and prove Corollary B from the introduction.

Definition 2.18 (Isogenies and quasi-isogenies)
Given an additive category (or ∞-category) $\mathcal{C}$, we say that a map $f\colon X \to Y$ is an *isogeny* if there exists $g\colon Y \to X$ and an integer $N > 0$ such that $g \circ f = N\mathrm{id}_X$ and $f \circ g = N\mathrm{id}_Y$. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure which is left-complete.[5] We say that a map $f\colon X \to Y$ of bounded below objects is a *quasi-isogeny* if the following equivalent conditions are satisfied:
(1) for each n, the map $\tau_{\leq n} f\colon \tau_{\leq n}X \to \tau_{\leq n}Y$ is an isogeny in $\mathcal{C}$;
(2) for each n, the map $\pi_n X \to \pi_n Y$ in the heart $\mathcal{C}^{\heartsuit}$ is an isogeny.

We will need some elementary observations about quasi-isogenies. A map $f\colon X \to Y$ of bounded below objects in $\mathcal{C}$ is a quasi-isogeny if and only if the fiber $\mathrm{fib}(f)$ is quasi-isogenous to zero. If one restricts to $\mathcal{C}_{\geq 0}$ (i.e., connective objects), then quasi-isogenies are preserved under finite colimits and geometric realizations

[4] If R is noncommutative and p-complete, then it is natural to ask whether there is still an equivalence $\mathrm{K}(R,(p);\mathbb{Z}_p) \simeq \mathrm{TC}(R,(p);\mathbb{Z}_p)$. We do not know the answer to this question.
[5] Recall that $\mathcal{C}$ said to be left-complete (with respect to the given t-structure) if the natural map $\mathcal{C} \to \varprojlim_n \mathcal{C}_{\leq n}$ is an equivalence. This is a technical condition satisfied by many stable ∞-categories such as Sp and $\mathcal{D}(\mathbb{Z})$.

(but generally not under filtered colimits). Next, let $\mathcal{C}$, $\mathcal{D}$ be stable ∞-categories with left-complete t-structures. Given a right t-exact functor $F\colon \mathcal{C} \to \mathcal{D}$ (or just a right bounded exact functor), it is easy to see that F preserves quasi-isogenies.[6]

Given an ∞-category $\mathcal{I}$, we will say that a natural transformation $f \to g$ of functors $f, g\colon \mathcal{I} \to \mathcal{C}$ is a *quasi-isogeny* if it is a quasi-isogeny in $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ with the pointwise t-structure. We will say that two functors are *naturally quasi-isogenous* if they are are related by a zigzag of quasi-isogenies of functors.

Example 2.19
For $\mathcal{C} = \mathrm{Sp}$, the map $\mathbb{S} \to \mathbb{Z}$ is a quasi-isogeny but of course not an isogeny (as there is no nontrivial map back). In fact, in Sp one has the following formality result of Beilinson [8]: every bounded-below spectrum X is quasi-isogenous to the spectrum $\bigoplus_n H\pi_n(X)[n]$. In particular, two bounded-below spectra X and Y are quasi-isogenous precisely if for each n separately the abelian groups $\pi_n X$ and $\pi_n Y$ are isogenous. To see that every spectrum is formal in the above sense, it suffices to observe that every k-invariant of a connective spectrum X is bounded torsion (where the torsion degree only depends on the degree of the k-invariant and not on the specific homotopy groups). For explicit bounds, see [67].

This formality result of course does depend on choices and thus does not give similar results in functor categories $\mathcal{C} = \mathrm{Fun}(\mathcal{I}, \mathrm{Sp})$.

The fiber sequence of Corollary 2.9 is the key to obtain our version of Beilinson's theorem [8], as follows.

THEOREM 2.20
For any associative ring R, the following spectra are naturally quasi-isogenous to each other (i.e., related via a natural zigzag of quasi-isogenies):

$$\mathrm{TC}\big(R, (p); \mathbb{Z}_p\big) \quad \Sigma\mathrm{HC}\big(R, (p); \mathbb{Z}_p\big) \quad \Sigma\mathrm{HC}(R; \mathbb{Z}_p).$$

Moreover:

(a) *if R is p-torsion-free, then the first two are equivalent after $(2p-5)$-truncation;*
(b) *if R is p-torsion-free and*[7] $\pi_{-1}(\mathrm{TC}(R; \mathbb{Z}_p)) = 0$, *then the first two are equivalent after $(2p-4)$-truncation.*

[6] A functor $\mathcal{C} \to \mathcal{D}$ is right t-exact with respect to fixed t-structures on $\mathcal{C}$ and $\mathcal{D}$ if it restricts to a functor $\mathcal{C}_{\geq 0} \to \mathcal{D}_{\geq 0}$. It is right bounded if it restricts to a functor $\mathcal{C}_{\geq 0} \to \mathcal{D}_{\geq n}$ for some $n \in \mathbb{Z}$.
[7] This is true pro-étale locally if R is commutative, thanks to [47, Theorem F].

Proof

For every associative ring R, we have the following commutative diagram of fiber sequences

$$\begin{tikzcd}
\Sigma(\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}_{hC_p})_{hS^1} \arrow[r] & \mathrm{TC}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z} \arrow[r] & \mathrm{TC}(R\otimes_{\mathbb{S}}\mathbb{F}_p;\mathbb{Z}_p) \\
F \arrow[r] \arrow[u] \arrow[d] & \mathrm{TC}(R;\mathbb{Z}_p) \arrow[r] \arrow[u] \arrow[d, "\mathrm{id}"] & \mathrm{TC}(R\otimes_{\mathbb{S}}\mathbb{F}_p;\mathbb{Z}_p) \arrow[u, "\mathrm{id}"'] \arrow[d] \\
\mathrm{TC}(R,(p);\mathbb{Z}_p) \arrow[r] & \mathrm{TC}(R;\mathbb{Z}_p) \arrow[r] & \mathrm{TC}(R/p;\mathbb{Z}_p).
\end{tikzcd} \tag{17}$$

To form the above diagram, we use the map $\mathrm{TC}(R;\mathbb{Z}_p)\to\mathrm{TC}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}$ induced from the map $\mathbb{S}\to\mathbb{Z}$, as well as the map on $\mathrm{TC}(-;\mathbb{Z}_p)$ induced by the Postnikov section $R\otimes_{\mathbb{S}}\mathbb{F}_p\to R/p$. All horizontal sequences in (17) are fiber sequences, either by Corollary 2.9 or by definition; that is, F is defined as the fiber of $\mathrm{TC}(R;\mathbb{Z}_p)\to\mathrm{TC}(R\otimes_{\mathbb{S}}\mathbb{F}_p;\mathbb{Z}_p)$.

We claim now that all the vertical maps in diagram (17) are quasi-isogenies.

LEMMA 2.21

The map $\mathrm{TC}(R;\mathbb{Z}_p)\to\mathrm{TC}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}$ *in the diagram (17) is a natural quasi-isogeny of spectra. Moreover, its fiber is* $(2p-4)$*-connective. If* $\pi_{-1}(\mathrm{TC}(R;\mathbb{Z}_p))=0$, *then the fiber is* $(2p-3)$*-connective.*

Proof

The first part follows from the observation that tensoring a quasi-isogeny (in this case $\mathbb{S}\to\mathbb{Z}$) with a bounded below spectrum (here $\mathrm{TC}(R;\mathbb{Z}_p)$) is again a quasi-isogeny. Moreover, the fiber of $\mathrm{TC}(R;\mathbb{Z}_p)\to\mathrm{TC}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}$ is $(2p-4)$-connective since $\mathrm{TC}(R;\mathbb{Z}_p)$ is (-1)-connective and the fiber of $\mathbb{S}_{(p)}\to\mathbb{Z}_{(p)}$ is $(2p-3)$-connective. The last assertion follows similarly. $\square$

The right horizontal map $\mathrm{TC}(R\otimes_{\mathbb{S}}\mathbb{F}_p;\mathbb{Z}_p)\to\mathrm{TC}(R/p;\mathbb{Z}_p)$ in diagram (17) is also a quasi-isogeny. This follows from Theorem 3.4 that we will discuss and prove in Section 3 and which is purely internal to cyclotomic spectra. But we also want to give a direct proof here using K-theory and the Dundas–Goodwillie–McCarthy theorem.

PROPOSITION 2.22
The natural map $\mathrm{TC}(R \otimes_{\mathbb{S}} \mathbb{F}_p; \mathbb{Z}_p) \to \mathrm{TC}(R/p; \mathbb{Z}_p)$ *is a quasi-isogeny. If* R *is* p*-torsion-free, then the fiber is* $(2p-1)$*-connective.*

Proof
The map of connective ring spectra $R \otimes_{\mathbb{S}} \mathbb{F}_p \to R/p$ is an isomorphism on π_0. Thus the Dundas–Goodwillie–McCarthy theorem (for ring spectra) implies that its fiber is equivalent to the fiber of the map

$$\mathrm{K}(R \otimes_{\mathbb{S}} \mathbb{F}_p; \mathbb{Z}_p) \to \mathrm{K}(R/p; \mathbb{Z}_p).$$

But the map $R \otimes_{\mathbb{S}} \mathbb{F}_p \to R/p$ of ring spectra is a quasi-isogeny and, if R is p-torsion-free, has fiber which is $(2p-2)$-connective. Thus, the map on K-theory is a quasi-isogeny and has fiber which is $(2p-1)$-connective (cf. [61, Proposition 2.19], the proof of which shows that the map is truly a quasi-isogeny of functors). □

Now we know that the vertical maps in diagram (17) are quasi-isogenies, so we conclude that $\Sigma(\mathrm{THH}(R, \mathbb{Z}_p) \otimes_{\mathbb{S}} \mathbb{Z}_{hC_p})_{hS^1}$ and $\mathrm{TC}(R, (p); \mathbb{Z}_p)$ are quasi-isogenous to one another. Moreover, if R is p-torsion-free, then the vertical maps from F have $(2p-4)$-connective fibers by the above discussion (which upgrades to $(2p-3)$-connective fibers if $\pi_{-1}\mathrm{TC}(R; \mathbb{Z}_p) = 0$). Thus, we conclude that $\Sigma(\mathrm{THH}(R, \mathbb{Z}_p) \otimes_{\mathbb{S}} \mathbb{Z}_{hC_p})_{hS^1}$ and $\mathrm{TC}(R, (p); \mathbb{Z}_p)$ are equivalent in degrees $\leq (2p-5)$, and in degrees $\leq 2p-4$ if $\pi_{-1}\mathrm{TC}(R; \mathbb{Z}_p) = 0$. Theorem 2.20 now follows from the arguments above and the following lemma. □

LEMMA 2.23
The following spectra are naturally quasi-isogenous to each other,

$$\big(\mathrm{THH}(R; \mathbb{Z}_p) \otimes_{\mathbb{S}} \mathbb{Z}_{hC_p}\big)_{hS^1} \qquad \mathrm{HC}\big(R, (p); \mathbb{Z}_p\big) \qquad \mathrm{HC}(R; \mathbb{Z}_p),$$

and the first two are equivalent after $(2p-4)$*-truncation if* R *is* p*-torsion-free.*

Proof
We have that $\mathrm{THH}(R; \mathbb{Z}_p) \otimes_{\mathbb{S}} \mathbb{Z}_{hC_p}$ is equivalent to the fiber of

$$\mathrm{THH}(R; \mathbb{Z}_p) \otimes_{\mathbb{S}} \mathbb{Z} \to \mathrm{THH}(R \otimes_{\mathbb{S}} \mathbb{F}_p; \mathbb{Z}_p),$$

and this map sits in a commutative square

$$\begin{tikzcd}
\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z} \arrow[r] \arrow[d] & \mathrm{THH}(R\otimes_{\mathbb{S}}\mathbb{F}_p;\mathbb{Z}_p) \arrow[d] \\
\mathrm{HH}(R;\mathbb{Z}_p) \arrow[r] & \mathrm{HH}(R/p;\mathbb{Z}_p)
\end{tikzcd}$$

of spectra with S^1-action. Both vertical maps are quasi-isogenies, so that we get the desired quasi-isogeny between the first two terms of the statement by taking S^1-orbits. The term $\mathrm{HH}(R/p)$ is quasi-isogenous to 0, so that we get the last quasi-isogeny, too. If R is p-torsion-free, then the fibers of the left and right vertical maps are in degrees $\geq 2p-3$ and $\geq 2p-2$, respectively, so the last assertion follows, too. □

Corollary B follows by combining Theorem 2.16 with Theorem 2.20. In particular, we have an isomorphism $\mathrm{K}_*(R,(p);\mathbb{Z}_p)\cong \mathrm{HC}_{*-1}(R,(p);\mathbb{Z}_p)$ for $*\leq 2p-5$, for R commutative, p-torsion-free, and Henselian along (p). Note also that, with the same proof, we can deduce the following variant of Theorem 2.20 for arbitrary connective $\mathbb{Z}$-algebra ring spectra (also known as $\mathbb{Z}$-linear differential graded algebras).

PROPOSITION 2.24
If R is a connective $\mathbb{Z}$-algebra spectrum, then the fiber of $\mathrm{TC}(R;\mathbb{Z}_p)\to \mathrm{TC}(R\otimes_{\mathbb{Z}}\mathbb{F}_p;\mathbb{Z}_p)$ is quasi-isogenous to $\Sigma\mathrm{HC}(R;\mathbb{Z}_p)$ and, after $(2p-5)$-truncation, is equivalent to the fiber of

$$\Sigma\mathrm{HC}(R;\mathbb{Z}_p)\to \Sigma\mathrm{HC}(R\otimes_{\mathbb{Z}}\mathbb{F}_p;\mathbb{Z}_p).$$

Remark 2.25
In all of the above, the denominators involved in the above quasi-isogenies are uniform: they do not depend on the choice of R. More formally, one could state all of the above quasi-isogenies via the ∞-category of functors from rings R to spectra. The denominators in the next result are not independent in the same fashion.

THEOREM 2.26
Let (R,I) be a pair consisting of an associative ring R and a nilpotent ideal I. Then there is a natural zigzag of quasi-isogenies between $\mathrm{K}(R,I;\mathbb{Z}_p)$ and $\Sigma\mathrm{HC}(R,I;\mathbb{Z}_p)$.

Proof
By the Dundas–Goodwillie–McCarthy theorem, we can replace K-theory with TC. We have a natural map

$$\mathrm{TC}(R,I;\mathbb{Z}_p)\to \mathrm{fib}\big(\mathrm{TC}\big(R,(p);\mathbb{Z}_p\big)\to \mathrm{TC}\big(R/I,(p);\mathbb{Z}_p\big)\big). \tag{18}$$

Now $\mathrm{TC}(R/p;\mathbb{Z}_p) \to \mathrm{TC}(R/(I,p);\mathbb{Z}_p)$ is a quasi-isogeny in view of Theorem 3.4 below, so that (18) is a quasi-isogeny. Combining with the quasi-isogenies of Theorem 2.20 now completes the proof. □

3. Quasi-isogenies of cyclotomic spectra

In this section, we systematically study quasi-isogenies in cyclotomic spectra, give another proof of Theorem A and Corollary B, and prove Theorem C, sharpening some results of Geisser–Hesselholt [36].

3.1. Preliminaries

We will apply the notion of quasi-isogeny (Definition 2.18) to the ∞-category CycSp of cyclotomic spectra using the t-structure of [3];[8] this t-structure is defined so that the connective objects of CycSp are those whose underlying spectrum is connective, and it is checked in [3, Theorem 2.1] that the t-structure is left-complete. Note that a quasi-isogeny of bounded-below cyclotomic spectra $f: X \to Y$ is a quasi-isogeny of underlying spectra, and $\mathrm{TC}(f;\mathbb{Z}_p)\colon \mathrm{TC}(X;\mathbb{Z}_p) \to \mathrm{TC}(Y;\mathbb{Z}_p)$ is also a quasi-isogeny. However, $\mathrm{THH}(\mathbb{F}_p) \in \mathrm{CycSp}$ has underlying spectrum quasi-isogenous to zero but is not itself quasi-isogenous to zero because $\mathrm{TC}(\mathbb{F}_p) \simeq \mathrm{TC}(\mathbb{F}_p;\mathbb{Z}_p)$ is torsion-free and nonzero: $\pi_0\mathrm{TC}(\mathbb{F}_p;\mathbb{Z}_p) \cong \pi_{-1}\mathrm{TC}(\mathbb{F}_p;\mathbb{Z}_p) \cong \mathbb{Z}_p$.

In the next result, we use the notion of TR of a cyclotomic spectrum, which plays an important role in the work [3]. See [18] for an account of TR in the approach to cyclotomic spectra via genuine equivariant homotopy theory. Implicitly, TR is computed with respect to our fixed prime p, but it will not generally be p-complete unless we p-complete it forming $\mathrm{TR}(X;\mathbb{Z}_p)$ for a cyclotomic spectrum X.

PROPOSITION 3.1

A map $f: X \to Y$ of bounded-below cyclotomic spectra is a quasi-isogeny in CycSp *if and only if the map of spectra* $\mathrm{TR}(f)\colon \mathrm{TR}(X) \to \mathrm{TR}(Y)$ *is a quasi-isogeny of spectra.*

Proof

This follows from the description of the cyclotomic t-structure of [3]. In particular, the cyclotomic homotopy groups of $X \in \mathrm{CycSp}$ are precisely the homotopy groups of the spectrum $\mathrm{TR}(X)$, together with the Frobenius and Verschiebung maps. These are p-typical Cartier modules, which is to say abelian groups M with endomorphisms F and V satisfying $FV = p$, and the heart $\mathrm{CycSp}^{\heartsuit}$ is equivalent to a full subcategory of the category of p-typical Cartier modules. Now, it suffices to check that a map $h: W \to Z$ between p-typical Cartier modules is an isogeny precisely if the under-

[8]Recall that what we denote by CycSp is denoted CycSp_p in [3].

lying map of abelian groups is an isogeny. One implication is immediate, and for the other suppose that $g\colon Z \to W$ is a map of abelian groups such that $g \circ h = N\mathrm{id}_W$ and $h \circ g = N\mathrm{id}_Z$. The map g might not respect the F and V maps. However, for $z \in Z$, $F(g(z)) - g(F(z))$ and $V(g(z)) - g(F(z))$ are both in the kernel of h, which consists of N-torsion elements of W. Therefore, Ng is a map of p-typical Cartier modules, $Ng \circ h = N^2\mathrm{id}_W$, and $h \circ Ng = N^2\mathrm{id}_Z$. □

PROPOSITION 3.2
Let $X \in \mathrm{CycSp}$ be a cyclotomic spectrum such that X is bounded below, such that the Frobenius $\varphi\colon X \to X^{tC_p}$ is null-homotopic in $\mathrm{Fun}(BS^1, \mathrm{Sp})$, and such that X is quasi-isogenous to zero as a spectrum. Then X is quasi-isogenous to zero as a cyclotomic spectrum.

Proof
The assumption that the Frobenius is null-homotopic implies that $\mathrm{TR}(X)$ is a product $\prod_{n \geq 0} X_{hC_{p^n}}$, using the description of TR as an iterated pullback (see [3, Remark 2.5] and [73, Corollary II.4.7]). The assumption that X is quasi-isogenous to zero now implies that the above product is also quasi-isogenous to zero, so we conclude by Proposition 3.1. □

We observe that the theory of cyclotomic spectra admits a natural graded variant. A *graded spectrum* is an object of the functor category $\mathrm{Fun}(\mathbb{Z}^{\mathrm{ds}}_{\geq 0}, \mathrm{Sp})$, where $\mathbb{Z}^{\mathrm{ds}}_{\geq 0}$ denotes the discrete category of nonnegative integers with no nonidentity morphisms; given a graded spectrum X, we let $X_i \in \mathrm{Sp}$, $i \geq 0$ denote the ith graded piece. We let GrSp denote the ∞-category of graded spectra, which we consider as a symmetric monoidal ∞-category under Day convolution using the multiplication symmetric monoidal structure on $\mathbb{Z}^{\mathrm{ds}}_{\geq 0}$. A *graded cyclotomic spectrum* X consists of a graded spectrum $X = \{X_i\}$ equipped with an S^1-action together with a family of S^1-equivariant maps $\varphi_i\colon X_i \to X_{pi}^{tC_p}$ for $i \geq 0$. We let GrCycSp denote the ∞-category of graded cyclotomic spectra. Any graded cyclotomic spectrum $X = \{X_i\}$ has an underlying cyclotomic spectrum $\bigoplus_{i \geq 0} X_i$, and this defines a forgetful functor $\mathrm{GrCycSp} \to \mathrm{CycSp}$.

More formally, the ∞-category GrCycSp is defined as follows. We consider the ∞-category $\mathrm{Fun}(BS^1, \mathrm{GrSp})$ of graded spectra equipped with an S^1-action. This admits a natural endofunctor F which sends $\{X_i, i \geq 0\}$ to $\{X_{pi}^{tC_p}\}$, where we regard $X_{pi}^{tC_p}$ as a spectrum with an $S^1/C_p \simeq S^1$-action. Then GrCycSp is defined as the ∞-category of F-coalgebras, as in [73, Section II.5].

Given a graded ring spectrum R, there is a graded cyclotomic spectrum $\mathrm{THH}(R)$ obtained by applying the cyclic bar construction in the category of graded spectra. This refines the usual THH and admits an S^1-action in graded spectra. See Appendix A for the details of this construction. Compare also [21] for a treatment of filtered cyclotomic spectra and filtered TC using more classical methods.

PROPOSITION 3.3

Let X be a graded cyclotomic spectrum. If

(1) *the underlying spectrum of X is quasi-isogenous to zero,*

(2) *the graded piece X_0 is contractible, and*

(3) *the connectivity of the pieces X_i tends to infinity in i,*

then X is quasi-isogenous to zero as an object of CycSp.

Proof

Given a graded cyclotomic spectrum X, for each i, we can construct a graded cyclotomic spectrum $X_{\leq i} \in \mathrm{GrCycSp}$ such that $(X_{\leq i})_j = 0$ for $j > i$ and $(X_{\leq i})_j = X_i$ for $j \leq i$ and a tower of maps $X \to \cdots \to X_{\leq n} \to X_{\leq n-1} \to \cdots \to X_{\leq 1}$. This is a tower in GrCycSp, and we can consider it as a tower of underlying objects in CycSp, too.

We need to show that for each j, $\pi_j(\mathrm{TR}(X))$ is isogenous to zero. Our assumptions imply that $\pi_j(\mathrm{TR}(X)) \to \pi_j \mathrm{TR}(X_{\leq n})$ is an isomorphism for $n \gg 0$. However, the object $\mathrm{fib}(X_{\leq i} \to X_{\leq i-1})$ defines a cyclotomic spectrum with Frobenius homotopic to zero, in view of the grading. It follows from Proposition 3.2 that $\mathrm{TR}(\mathrm{fib}(X_{\leq i} \to X_{\leq i-1}))$ is quasi-isogenous to zero, and by induction $\mathrm{TR}(X_{\leq n})$ is quasi-isogenous to zero. Putting these observations together with Proposition 3.1 completes the proof. ☐

3.2. Quasi-isogenies on THH

Our main result here is the following, which restates Theorem C. On TC and for discrete rings in which p is nilpotent, it is due to Geisser–Hesselholt [36], and the main arguments are based on theirs.

THEOREM 3.4

Let $f : A \to A'$ be a map of connective associative ring spectra. If

(i) *f is a quasi-isogeny of spectra, and*

(ii) *the map $\pi_0(f) : \pi_0(A) \to \pi_0(A')$ is surjective with nilpotent kernel,*

then $\mathrm{THH}(A) \to \mathrm{THH}(A')$ *is a quasi-isogeny in* CycSp.

We will first verify some special cases.

PROPOSITION 3.5

Let R be a connective associative graded *ring spectrum. If*

(a) *each R_i, $i > 0$, is isogenous to zero as a spectrum, and*

(b) *the connectivity of the R_i tends to ∞ as $i \to \infty$,*

then the map $\mathrm{THH}(R) \to \mathrm{THH}(R_0)$ *is a quasi-isogeny in* CycSp.

Proof

Since R is a graded ring spectrum, $\mathrm{THH}(R)$ admits the structure of a graded cyclotomic spectrum (refining the usual cyclotomic structure on $\mathrm{THH}(R)$), and in degree 0 one has $\mathrm{THH}(R_0)$. For this, compare Appendix A, or the work of Brun [21], who uses the more classical approach to cyclotomic spectra.

Now we wish to apply Proposition 3.3. Consider the subcategory $\mathcal{C} \subseteq \mathrm{GrSp}_{\geq 0}$ of *connective* graded spectra spanned by graded spectra Z such that Z_i is quasi-isogenous to zero for $i > 0$ and such that the connectivity of Z_i grows without bound as $i \to \infty$. Then $\mathcal{C}$ is closed under tensor products and geometric realizations. The assumptions on R imply that $R \in \mathcal{C}$, and consequently $\mathrm{THH}(R) \in \mathcal{C}$ as well. That is, $\mathrm{THH}(R)_i$ is quasi-isogenous to zero for $i > 0$ and the connectivity of $\mathrm{THH}(R)_i$ grows without bound as $i \to \infty$. Thus, we can apply Proposition 3.3. ☐

PROPOSITION 3.6

Let A be a connective associative ring spectrum. Let M be a connective (A, A)-bimodule which is quasi-isogenous to zero. Suppose that $\widetilde{A}$ is a square-zero extension of A by M, in the sense that one has a map $f : A \to A \oplus M[1]$ in $\mathrm{Alg}_{/A}$ and a pullback diagram

$$\begin{tikzcd}
\widetilde{A} \arrow[r] \arrow[d] & A \arrow[d, "0"] \\
A \arrow[r, "f"] & A \oplus M[1].
\end{tikzcd}$$

Then the map $\mathrm{THH}(\widetilde{A}) \to \mathrm{THH}(A)$ *is a quasi-isogeny in* CycSp.

Proof

We can form the Čech nerve of the map $\widetilde{A} \to A$, that is, the simplicial object $\dots \widetilde{A} \times_A \widetilde{A} \rightrightarrows \widetilde{A}$. This yields a simplicial object $X_\bullet$ of Alg which resolves A. It follows that $|\mathrm{THH}(X_\bullet)| \simeq \mathrm{THH}(A)$ in CycSp.

Now $\widetilde{A} \times_A \widetilde{A}$ is a *trivial* square-zero extension of $\widetilde{A}$ by M. It follows that $\mathrm{THH}(\widetilde{A} \times_A \widetilde{A})$ is quasi-isogenous to $\mathrm{THH}(\widetilde{A})$ by Proposition 3.5, since a trivial square-zero extension can be given a grading. Continuing in this way, it follows that all the maps in the simplicial object $\mathrm{THH}(X_\bullet)$ are quasi-isogenies. Taking geometric realizations now, it follows that $\mathrm{THH}(\widetilde{A}) \to |\mathrm{THH}(X_\bullet)| \simeq \mathrm{THH}(A)$ is a quasi-isogeny in CycSp. $\square$

PROPOSITION 3.7
Let B be a connective associative ring spectrum, and let B' be an object of $\mathrm{Alg}_{B//B}$. Suppose that the augmentation map $B' \to B$ is a quasi-isogeny and the map $\pi_0(B') \to \pi_0(B)$ has nilpotent kernel. Then $\mathrm{THH}(B) \to \mathrm{THH}(B')$ is a quasi-isogeny.

Proof
Recall first that $\mathrm{Alg}_{B//B}$ is equivalent to the ∞-category of *nonunital* associative algebra objects in (B, B)-bimodules. In particular, $I = \mathrm{fib}(B' \to B)$ has such a structure. We can work up the Postnikov tower $\tau_{\leq \star} I$; since TR behaves well with respect to Postnikov towers, it suffices to prove the result for each $\tau_{\leq n} I$. General results as in [64, Section 7.4.1] (which go back at least to [6]) now show that $\tau_{\leq n} I$ can be obtained in finitely many steps via square-zero extensions from B, by bimodules which are quasi-isogenous to zero. Now we conclude via Proposition 3.6. $\square$

Proof of Theorem 3.4
We consider the Čech nerve of $A \to A'$. We obtain a simplicial object $X_\bullet$ in $\mathrm{Alg}_{\geq 0}$ such that $|X_\bullet| \simeq A'$ and such that each X_i is an iterated fiber product of copies of A over A'. Each X_i can be given the structure of an object of $\mathrm{Alg}_{A//A}$ (via appropriate face and degeneracy maps), whence we conclude by Proposition 3.7 that all the maps in the simplicial object $\mathrm{THH}(X_\bullet)$ are quasi-isogenies in CycSp. Finally, the result now follows by taking geometric realizations. $\square$

One important corollary of Theorem 3.4 is the following result of Geisser and Hesselholt (see [61] for generalizations).

COROLLARY 3.8 (Geisser–Hesselholt [36])
If p is nilpotent in A and $I \subseteq A$ is a two-sided nilpotent ideal, then $\mathrm{K}(A, I) \simeq \mathrm{TC}(A, I)$ is quasi-isogenous to zero.

Proof
In this case, $A \to A/I$ is a quasi-isogeny (they are both quasi-isogenous to zero) and hence Theorem 3.4 applies to prove that $\mathrm{THH}(A) \to \mathrm{THH}(A/I)$ is a quasi-isogeny of cyclotomic spectra. This implies in particular that $\mathrm{TC}(A, I)$ is quasi-isogenous to zero. $\square$

3.3. Quasi-isogenies and the Beilinson fiber sequence

Recall that Proposition 2.22, which was key in proving Theorem A, asserts that for every ring R the induced map $\mathrm{TC}(R \otimes_{\mathbb{S}} \mathbb{F}_p; \mathbb{Z}_p) \to \mathrm{TC}(R/p; \mathbb{Z}_p)$ is a quasi-isogeny. Our proof in Section 2 relied on K-theory. Alternatively, we can now also deduce this fact directly from Theorem 3.4, which implies that $\mathrm{THH}(R \otimes_{\mathbb{S}} \mathbb{F}_p) \to \mathrm{THH}(R/p)$ is a quasi-isogeny of cyclotomic spectra, and therefore $\mathrm{TC}(R \otimes_{\mathbb{S}} \mathbb{F}_p; \mathbb{Z}_p) \to \mathrm{TC}(R/p; \mathbb{Z}_p)$ is a quasi-isogeny of spectra. This immediately implies also the following variant of Theorem A.

COROLLARY 3.9

Let R be a ring. Then there is a natural pullback square

$$\begin{tikzcd} \mathrm{TC}(R;\mathbb{Q}_p) \arrow[r] \arrow[d] & \mathrm{TC}(R/p;\mathbb{Q}_p) \arrow[d] \\ \mathrm{HC}^{-}(R;\mathbb{Q}_p) \arrow[r] & \mathrm{HP}(R;\mathbb{Q}_p). \end{tikzcd} \tag{19}$$

In this section, we want to take this a step further and prove a cyclotomic version of Theorem A.

THEOREM 3.10

For every ring R, the following cyclotomic spectra are quasi-isogenous to each other,

$$\mathrm{THH}\big(R,(p);\mathbb{Z}_p\big) \qquad \mathrm{HH}(R;\mathbb{Z}_p) \quad \mathrm{HH}\big(R,(p);\mathbb{Z}_p\big),$$

where $\mathrm{HH}(R;\mathbb{Z}_p)$ and $\mathrm{HH}(R,(p);\mathbb{Z}_p)$ are equipped with the canonical S^1-actions and the zero Frobenius (see Example 2.3). Moreover, if R is p-torsion-free, then we have an equivalence of cyclotomic spectra

$$\tau^{\mathrm{cyc}}_{\le(2p-4)}\mathrm{THH}\big(R,(p);\mathbb{Z}_p\big) \simeq \tau^{\mathrm{cyc}}_{\le(2p-4)}\mathrm{HH}\big(R,(p);\mathbb{Z}_p\big).$$

We note that the last theorem immediately implies Theorem A by passing to $\mathrm{TC}(-;\mathbb{Z}_p)$ since $\mathrm{TC}(-;\mathbb{Z}_p)$ of a cyclotomic spectrum with zero Frobenius is just given by the p-completion of the shifted S^1-orbits. But Theorem 3.10 is strictly stronger than Theorem A since quasi-isogenies cannot be detected on TC. The remainder of this section is devoted to proving Theorem 3.10.

LEMMA 3.11

Suppose that $X \to Y$ is a quasi-isogeny of cyclotomic spectra and that M is any bounded-below cyclotomic spectrum. Then, $X \otimes_{\mathbb{S}} M \to Y \otimes_{\mathbb{S}} M$ is a quasi-isogeny of cyclotomic spectra.

Proof
We can assume that M is connective, in which case the functor $- \otimes_{\mathbb{S}} M$ is a right t-exact endofunctor of CycSp and hence preserves quasi-isogenies. $\square$

We now want to apply a similar proof strategy as in Section 2 and consider the diagram

$$\begin{tikzcd}
\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}_{hC_p} \arrow[r] & \mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}^{\mathrm{triv}} \arrow[r] & \mathrm{THH}(R\otimes_{\mathbb{S}}\mathbb{F}_p;\mathbb{Z}_p) \arrow[d] \\
\mathrm{THH}(R,(p);\mathbb{Z}_p) \arrow[r] & \mathrm{THH}(R;\mathbb{Z}_p) \arrow[r] \arrow[u] & \mathrm{THH}(R/p;\mathbb{Z}_p)
\end{tikzcd}$$

of cyclotomic spectra in which the horizontal rows are fiber sequences. Both vertical maps are quasi-isogenies in cyclotomic spectra: the first by Lemma 3.11 and because $\mathbb{S}^{\mathrm{triv}} \to \mathbb{Z}^{\mathrm{triv}}$ is a quasi-isogeny (since $(-)^{\mathrm{triv}}$ is right t-exact), and the second by Theorem 3.4. The right vertical map has homotopy fiber in degrees $\geq 2p-2$, while the middle vertical map has homotopy fiber in degrees $\geq 2p-3$. It follows that $\mathrm{THH}(R,(p);\mathbb{Z}_p)$ is quasi-isogenous in cyclotomic spectra to $\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}_{hC_p}$, and their cyclotomic $(2p-4)$-truncations $\tau^{\mathrm{cyc}}_{\leq 2p-4}$ are equivalent. Note that the cyclotomic Frobenius on $\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}_{hC_p}$ is null-homotopic by the Tate orbit lemma.

Now the next lemma finishes the proof of Theorem 3.10.

LEMMA 3.12
The following cyclotomic spectra are quasi-isogenous to each other:

$$\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}_{hC_p} \qquad \mathrm{HH}(R;\mathbb{Z}_p) \quad \mathrm{HH}\big(R,(p);\mathbb{Z}_p\big).$$

Moreover, the cyclotomic truncations $\tau^{\mathrm{cyc}}_{\leq 2p-4}$ *of the first and the third are naturally equivalent.*

Proof
We consider the square

$$\begin{tikzcd}
\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z} \arrow[r] \arrow[d] & \mathrm{THH}(R\otimes_{\mathbb{S}}\mathbb{F}_p;\mathbb{Z}_p) \arrow[d] \\
\mathrm{HH}(R;\mathbb{Z}_p) \arrow[r] & \mathrm{HH}(R/p;\mathbb{Z}_p)
\end{tikzcd}$$

of spectra with S^1-action, in which the vertical maps are quasi-isogenies and the terms on the right-hand side are quasi-isogenous to zero. We consider it as a square of cyclotomic spectra by equipping all spectra with the zero Frobenius map. It follows from Proposition 3.2 that the vertical maps are quasi-isogenies of cyclotomic spectra. The horizontal fibers are equivalent to $\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}_{hC_p}$ and $\mathrm{HH}(R,(p);\mathbb{Z}_p)$, which shows that they are quasi-isogenous to each other.

Finally, the induced map of cyclotomic spectra (with zero Frobenii) $\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathbb{S}}\mathbb{Z}_{hC_p}\to\mathrm{HH}(R,(p);\mathbb{Z}_p)$ has the property that it is an equivalence of underlying spectra in degrees $\leq 2p-4$ (as in Lemma 2.23) and consequently induces an equivalence on cyclotomic homotopy groups in degrees $\leq 2p-4$, for example, using the description of TR as in the proof of Proposition 3.2. □

4. Application to p-adic deformations

In this section, we prove Theorems D and E. Throughout this section, let $\mathfrak{X}$ be a quasi-compact and quasi-separated (qcqs) p-adic formal scheme with bounded p-power torsion, and write $\mathfrak{X}_n=\mathfrak{X}\times_{\mathrm{Spec}\,\mathbb{Z}_p}\mathrm{Spec}\,\mathbb{Z}/p^n$. We are interested in the following invariants of $\mathfrak{X}$, and, in particular, the p-adic deformation problem (Question 4.5 below).

Definition 4.1 (Continuous invariants of formal schemes)
Let F be an invariant of schemes (such as K, THH, HH, HC^-, HP, TC). Given the formal scheme $\mathfrak{X}$, we define $F^{\mathrm{cts}}(\mathfrak{X})$ via

$$F^{\mathrm{cts}}(\mathfrak{X})=\varprojlim_n F(\mathfrak{X}_n). \tag{20}$$

If the p-adic formal scheme $\mathfrak{X}$ arises as the p-adic completion of a scheme X, we have a natural comparison map

$$F(X)\to F^{\mathrm{cts}}(\mathfrak{X}). \tag{21}$$

PROPOSITION 4.2
Suppose that $\mathfrak{X}$ is the p-adic completion of a qcqs scheme X with bounded p-power torsion. Then the maps (21) for $F=\mathrm{HH},\mathrm{THH},\mathrm{HC}^-,\mathrm{HP},\mathrm{TC}$ are p-adic equivalences.

Proof
Using Zariski descent on X, we may assume that $X=\mathrm{Spec}(R)$, where R is a ring of bounded p-power torsion, and then $\mathfrak{X}=\mathrm{Spf}(\hat{R}_p)$. Using the cyclic bar construction, it is not difficult to show that $\mathrm{THH}^{\mathrm{cts}}(\mathfrak{X};\mathbb{Z}_p)=\mathrm{THH}(R;\mathbb{Z}_p)$, that is, that (21) is a p-adic equivalence for $F=\mathrm{THH}$ (cf. the proof of [23, Theorem 5.19]). Tensoring over $\mathrm{THH}(\mathbb{Z})$ with $\mathbb{Z}$, one deduces the result for HH, and then taking S^1-invariants and coinvariants, we find that (21) is a p-adic equivalence for $F=\mathrm{HC}^-,\mathrm{HP}$. Running

the above argument with THH instead of HH, one concludes that (21) is a p-adic equivalence for $F = \mathrm{TC}$ (see [23, Theorem 5.19]). See also [26, Corollary 4.8] for these results, when R is assumed to be Noetherian and F-finite. $\square$

By contrast, it is much more difficult to control (21) when $F = \mathrm{K}$. We mention the two following cases.

Example 4.3 (Formal affine schemes)
Suppose that $\mathfrak{X}$ is affine, that is, $\mathfrak{X} = \mathrm{Spf}(R)$, for R a *p-adically complete ring* with bounded p-power torsion. We can then write $\mathfrak{X}$ as the p-adic completion (as a formal scheme) of $X = \mathrm{Spec}(R)$. In this case, the comparison map (21) is a p-adic equivalence for $F = \mathrm{K}$ as well (cf. [23, Theorem 5.23] and [35, Theorem C]).

Example 4.4 (Proper schemes)
Suppose that R is p-complete. Suppose that $\mathfrak{X}$ is the p-completion of a proper scheme $X \to \mathrm{Spec}(R)$. The map $\mathrm{K}(X;\mathbb{Z}_p) \to \mathrm{K}^{\mathrm{cts}}(\mathfrak{X};\mathbb{Z}_p)$ is probably not an equivalence; compare [16, Appendix B] for a related counterexample in equal characteristic zero. In this case, $\mathrm{K}^{\mathrm{cts}}(\mathfrak{X};\mathbb{Z}_p)$ is generally much more tractable than $\mathrm{K}(X;\mathbb{Z}_p)$ via comparisons with topological cyclic homology.

Question 4.5 (The p-adic deformation problem)
Let $\mathfrak{X}$ be a p-adic formal scheme with special fiber $\mathfrak{X}_1$ as above. For $i \geq 0$, what is the image[9] of the map

$$\mathrm{K}_i^{\mathrm{cts}}(\mathfrak{X};\mathbb{Q}) \to \mathrm{K}_i(\mathfrak{X}_1;\mathbb{Q})? \tag{22}$$

We first observe that Question 4.5 is essentially a p-adic question in TC. For each $n \geq 1$, let $\mathrm{K}(\mathfrak{X}_n, \mathfrak{X}_1)$ be the fiber of $\mathrm{K}(\mathfrak{X}_n) \to \mathrm{K}(\mathfrak{X}_1)$. Since $\mathfrak{X}_n$ is a p-adic nilpotent thickening of $\mathfrak{X}_1$, the relative K-theory $\mathrm{K}(\mathfrak{X}_n, \mathfrak{X}_1)$ has homotopy groups which are bounded p-power torsion (cf. Corollary 3.8, due to [36]), and the spectrum is therefore p-complete. Using the Dundas–Goodwillie–McCarthy theorem from [25], and taking limits, we obtain a Cartesian square

$$\begin{tikzcd} \mathrm{K}^{\mathrm{cts}}(\mathfrak{X}) \arrow[r] \arrow[d] & \mathrm{K}(\mathfrak{X}_1) \arrow[d] \\ \mathrm{TC}^{\mathrm{cts}}(\mathfrak{X};\mathbb{Z}_p) \arrow[r] & \mathrm{TC}(\mathfrak{X}_1;\mathbb{Z}_p). \end{tikzcd} \tag{23}$$

[9]By the Milnor exact sequence, this is equivalent to describing the image of the map $(\varprojlim_n \mathrm{K}_i(\mathfrak{X}_n))_{\mathbb{Q}} \to \mathrm{K}_i(\mathfrak{X}_1;\mathbb{Q})$.

Since the $\mathfrak{X}_n, n \geq 1$ are p-power torsion schemes, their TC are already p-adically complete. Using this diagram, we see that it suffices to determine the image of the map $\mathrm{TC}_i^{\mathrm{cts}}(\mathfrak{X}; \mathbb{Q}_p) \to \mathrm{TC}_i(\mathfrak{X}_1; \mathbb{Q}_p)$.

In this section, we will describe an explicit obstruction class for Question 4.5 in case $i = 0$ (sharpening results of [17]) in certain geometric situations and construct general obstruction classes in all cases (after [8]).

4.1. The Bloch–Esnault–Kerz theorem

In [17], Bloch–Esnault–Kerz consider Question 4.5 in the case $i = 0$ and where $\mathfrak{X}$ has the following form. Let K be a complete discretely valued field of characteristic zero with ring of integers $\mathcal{O}_K$, whose residue field k is perfect of characteristic $p > 0$. We let $\pi \in \mathcal{O}_K$ be a uniformizer and denote by K_0 the ring of fractions $W(k)[\frac{1}{p}]$. We take $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_K)$ to be a smooth p-adic formal scheme, with special fiber $\mathfrak{X}_k \to \mathrm{Spec}(k)$ and rigid analytic generic fiber $\mathfrak{X}_K$ over K. The goal is to understand the image of the map $\mathrm{K}_0^{\mathrm{cts}}(\mathfrak{X}; \mathbb{Q}_p) \to \mathrm{K}_0(\mathfrak{X}_k; \mathbb{Q}_p)$.

We refer to [17] and [27] for more detailed motivation for the above question, as well as [16], [68], and [71] for discussions of the analogous question in equal characteristic. Note that when $\mathfrak{X}$ arises from a smooth proper scheme $X \to \mathrm{Spf}(\mathcal{O}_K)$, the above question says nothing about the image of the map $\mathrm{K}_0(X; \mathbb{Q}) \to \mathrm{K}_0(X_k; \mathbb{Q})$; this (at least up to homological equivalence) is the subject of the far more difficult p-adic variational Hodge conjecture of Fontaine–Messing (Conjecture 1.3).

Here we will unwind the Beilinson fiber square to answer Question 4.5 in this case in terms of the crystalline Chern character. To begin with, we need to review the crystalline Chern character and the crystalline to de Rham comparison.

Construction 4.6 (de Rham cohomology)

Given a smooth p-adic formal scheme $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_K)$, we will consider the ($p$-adic) de Rham cohomology $R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_K) \in D(\mathcal{O}_K)$, equipped with the descending, multiplicative Hodge filtration $\mathrm{Fil}^{\geq \star} R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_K)$. When $\mathfrak{X}$ is also assumed proper, all of these are perfect complexes in $D(\mathcal{O}_K)$. Furthermore, after inverting p, we write $R\Gamma_{\mathrm{dR}}(\mathfrak{X}_K/K) \in D(K)$ and $\mathrm{Fil}^{\geq \star} R\Gamma_{\mathrm{dR}}(\mathfrak{X}_K/K)$ for the induced objects.

A basic fact we will use is that when $\mathfrak{X}$ is proper, the induced spectral sequence from the Hodge filtration on $R\Gamma_{\mathrm{dR}}(\mathfrak{X}_K/K)$ degenerates after rationalization; this is the degeneration of the Hodge–to–de Rham spectral sequence for proper smooth rigid analytic varieties, proved by Scholze [82].

Construction 4.7 (Comparison between crystalline and de Rham cohomology)

Given $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_K)$ a smooth p-adic formal scheme, we can consider the crystalline cohomology $R\Gamma_{\mathrm{crys}}(\mathfrak{X}_k)$ of the special fiber as well. In the absolutely unramified case (when $\mathcal{O}_K = W(k)$), the usual de Rham to crystalline comparison theorem yields an equivalence $R\Gamma_{\mathrm{crys}}(\mathfrak{X}_k) \simeq R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_K)$. In general, by [9, Theorem 2.4], we have a natural equivalence after rationalization

$$R\Gamma_{\mathrm{dR}}(\mathfrak{X}_K/K) \simeq R\Gamma_{\mathrm{crys}}(\mathfrak{X}_k;\mathbb{Q}_p) \otimes_{K_0} K. \tag{24}$$

Construction 4.8 (The crystalline Chern character)
Let Y be a regular scheme of characteristic p. Given a vector bundle $\mathcal{V}$ on Y, we can define *Chern classes* $c_i(\mathcal{V}) \in H^{2i}_{\mathrm{crys}}(Y)$ for $i \geq 0$ satisfying the usual axioms (e.g., using the classical method of [40]; cf. [43]). The usual formula then yields a *crystalline Chern character*, that is, a natural ring homomorphism into the rationalized crystalline cohomology

$$\mathrm{ch}_{\mathrm{crys}} \colon \mathrm{K}_0(Y) \to \bigoplus_{i \geq 0} H^{2i}_{\mathrm{crys}}(Y;\mathbb{Q}_p),$$

which carries the class of a line bundle $\mathcal{L}$ to $1 + c_1(\mathcal{L})$.

Our main result is the following theorem, which extends results of Bloch–Esnault–Kerz [17]. In [17], this result is proved in the case where $K = K_0$ is absolutely unramified, X arises from a smooth projective scheme, and $p > \dim(X) + 6$. In [8], it is shown that there is an obstruction in $\bigoplus_{i\geq 0} H^{2i}_{\mathrm{dR}}(X_K)/\mathrm{Fil}^{\geq i} H^{2i}_{\mathrm{dR}}(X_K)$, but the obstruction is not identified with the Chern character (see Section 4.2 below for more discussion).

THEOREM 4.9
Let K be a complete discretely valued field of characteristic zero with ring of integers $\mathcal{O}_K$, whose residue field k is perfect of characteristic $p > 0$. Let $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_K)$ be a proper smooth p-adic formal scheme with special fiber $\mathfrak{X}_k$. A class $x \in \mathrm{K}_0(\mathfrak{X}_k;\mathbb{Q}_p)$ lifts to $\mathrm{K}_0^{\mathrm{cts}}(\mathfrak{X};\mathbb{Q}_p)$ if and only if the crystalline Chern character $\mathrm{ch}_{\mathrm{crys}}(x) \in \bigoplus_{i\geq 0} H^{2i}_{\mathrm{crys}}(\mathfrak{X}_k;\mathbb{Q}_p)$ maps (via the comparison map of (24)) to $\bigoplus_{i\geq 0} \mathrm{Fil}^{\geq i} H^{2i}_{\mathrm{dR}}(\mathfrak{X}_K/K) \subseteq \bigoplus_{i\geq 0} H^{2i}_{\mathrm{dR}}(\mathfrak{X}_K/K)$.

The proof of Theorem 4.9 will be carried out as follows. First, we give an analogous form of the Beilinson fiber square when we work relative to $\mathcal{O}_K$ (Proposition 4.10). Next, we will show that the p-adic Chern character can be defined entirely in terms of the special fiber (which will use some Kan extension techniques from Appendix B), and then identify it with the crystalline Chern character (Proposition 4.12). Theorem 4.9 will then follow directly.

In the next result, we will use the continuous Hochschild (resp., negative cyclic, periodic cyclic) homology of a formal scheme over $\mathcal{O}_K$, defined as in Definition 4.1;

note that Proposition 4.2 applies to these relative theories, too, since they can be recovered from THH.

PROPOSITION 4.10 (The fiber square relative to $\mathcal{O}_K$)
Let $\mathfrak{X}$ be a smooth formal $\mathcal{O}_K$-scheme. Then there are natural fiber squares

$$\begin{tikzcd} \mathrm{K}^{\mathrm{cts}}(\mathfrak{X};\mathbb{Q}_p) \arrow[r] \arrow[d] & \mathrm{K}(\mathfrak{X}_k;\mathbb{Q}_p) \arrow[d] \\ \mathrm{TC}^{\mathrm{cts}}(\mathfrak{X};\mathbb{Q}_p) \arrow[r] \arrow[d] & \mathrm{TC}(\mathfrak{X}_k;\mathbb{Q}_p) \arrow[d] \\ \mathrm{HC}^{-,\mathrm{cts}}(\mathfrak{X}/\mathcal{O}_K;\mathbb{Q}_p) \arrow[r] & \mathrm{HP}^{\mathrm{cts}}(\mathfrak{X}/\mathcal{O}_K;\mathbb{Q}_p). \end{tikzcd} \tag{25}$$

Proof
This will follow from the Beilinson fiber square. By Zariski descent of all terms in the formal scheme $\mathfrak{X}$, we can assume that $\mathfrak{X} = \mathrm{Spf}(R)$ for R a formally smooth, p-complete $\mathcal{O}_K$-algebra. First, $\mathrm{HH}(\mathcal{O}_K;\mathbb{Z}_p) \simeq \mathrm{HH}(\mathcal{O}_K/W(k);\mathbb{Z}_p)$. Since $L_{\mathcal{O}_K/W(k)}$ is quasi-isogenous to zero, we find that the map $\mathrm{HH}(\mathcal{O}_K;\mathbb{Q}_p) \to K$ given by truncation is an equivalence. We thus conclude (via Hochschild–Kostant–Rosenberg) that $\mathrm{HH}^{\mathrm{cts}}(R;\mathbb{Q}_p) \to \mathrm{HH}^{\mathrm{cts}}(R/\mathcal{O}_K;\mathbb{Q}_p)$ is an equivalence, whence $\mathrm{HC}(R;\mathbb{Q}_p) \to \mathrm{HC}(R/\mathcal{O}_K;\mathbb{Q}_p)$ is an equivalence, too, by taking S^1-coinvariants. Here we use that p-adic completion commutes with taking S^1-coinvariants on connective spectra, since taking S^1-coinvariants behaves as a finite colimit in any ranges of degrees. Therefore, the diagram

$$\begin{tikzcd} \mathrm{HC}^{-}(R;\mathbb{Q}_p) \arrow[r] \arrow[d] & \mathrm{HP}(R;\mathbb{Q}_p) \arrow[d] \\ \mathrm{HC}^{-}(R/\mathcal{O}_K;\mathbb{Q}_p) \arrow[r] & \mathrm{HP}(R/\mathcal{O}_K;\mathbb{Q}_p) \end{tikzcd}$$

is homotopy Cartesian. Combining with the Beilinson fiber square, the result now follows. □

Construction 4.11 (The p-adic Chern character map)
Since $\mathfrak{X}/\mathcal{O}_K$ is smooth, we obtain from Hochschild–Kostant–Rosenberg-type filtrations (as in [1], using Adams operations as in [12, Section 9.4] to split the filtration) natural decompositions

$$\mathrm{HP}^{\mathrm{cts}}(\mathfrak{X}/\mathcal{O}_K;\mathbb{Q}_p) \simeq \prod_{i\in\mathbb{Z}} R\Gamma_{\mathrm{dR}}(\mathfrak{X}_K/K)[2i]$$

and

$$\mathrm{HC}^{-,\mathrm{cts}}(\mathfrak{X}/\mathcal{O}_K;\mathbb{Q}_p) \simeq \prod_{i\in\mathbb{Z}} \mathrm{Fil}^{\geq i} R\Gamma_{\mathrm{dR}}(\mathfrak{X}_K/K)[2i].$$

It follows that we obtain from (25) a natural map

$$\mathrm{K}(\mathfrak{X}_k;\mathbb{Q}_p) \to \mathrm{TC}(\mathfrak{X}_k;\mathbb{Q}_p) \to \prod_{i\in\mathbb{Z}} R\Gamma_{\mathrm{dR}}(\mathfrak{X}_K/K)[2i] \tag{26}$$

for every smooth p-adic formal scheme $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_K)$. We observe that both the source and target actually depend only on the special fiber $\mathfrak{X}_k$ of $\mathfrak{X}$, thanks to Construction 4.7. Furthermore, to construct (26), it suffices to work with affine formal schemes over $\mathcal{O}_K$, by Zariski descent of the target, so we can assume that $\mathfrak{X} = \mathrm{Spf}(R)$ for R a formally smooth $\mathcal{O}_K$-algebra. That is, we have a natural map $\mathrm{TC}(R \otimes_{\mathcal{O}_K} k;\mathbb{Q}_p) \to \prod_{i\in\mathbb{Z}}(R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R \otimes_{\mathcal{O}_K} k);\mathbb{Q}_p) \otimes_{K_0} K)[2i]$.

Consider the functors on smooth k-algebras

$$A \mapsto \mathrm{TC}(A;\mathbb{Q}_p) \qquad \text{and} \qquad A \mapsto \prod_{i\in\mathbb{Z}}\big(R\Gamma_{\mathrm{crys}}\big(\mathrm{Spec}(A);\mathbb{Q}_p\big) \otimes_{K_0} K\big)[2i]. \tag{27}$$

The left Kan extension of $\mathrm{TC}(-;\mathbb{Q}_p)$ to almost finitely presented objects of SCR_k (as in Definition B.5) is $\mathrm{TC}(-;\mathbb{Q}_p)$ again, since this functor commutes with geometric realizations in SCR_k. By Theorem 3.4, $\mathrm{TC}(-;\mathbb{Q}_p) = \mathrm{TC}(\pi_0(-);\mathbb{Q}_p)$ on SCR_k, so hypotheses (1) and (2) of Corollary B.6 are satisfied when applied to the left Kan extensions of the functors (27) on smooth k-algebras (and Zariski sheafifying again). It follows that (26) actually upgrades to a natural transformation of functors in the special fiber alone. That is, for every smooth k-scheme Z, we obtain a natural map

$$\mathrm{K}(Z;\mathbb{Q}_p) \xrightarrow{\mathrm{tr}} \mathrm{TC}(Z;\mathbb{Q}_p) \to \prod_{i\in\mathbb{Z}}\big(R\Gamma_{\mathrm{crys}}(Z;\mathbb{Q}_p) \otimes_{K_0} K\big)[2i], \tag{28}$$

such that (26) is obtained by taking $Z = X_k$.

Next, we identify (up to scaling factors) the map (28) on π_0 with the crystalline Chern character.

PROPOSITION 4.12

There exists a scalar $\lambda \in K^\times$ such that for every smooth separated k-scheme Z, the map $\mathrm{K}_0(Z;\mathbb{Q}_p) \to \prod_{i\geq 0} H^{2i}_{\mathrm{crys}}(Z;\mathbb{Q}_p) \otimes_{K_0} K$ of (28) is given by the crystalline Chern character composed with the automorphism that multiplies the ith factor by λ^i.

Proof
It suffices (by the resolution property) to evaluate (28) on the class of a vector bundle on Z, and for this we will reduce to the universal case. For this it will be convenient to extend to stacks over k as well. Given a smooth scheme or stack Z over k, let $\mathrm{Vect}_n(Z)$ denote the groupoid of n-dimensional vector bundles on Z. It follows that we obtain from (28) a natural transformation (of spaces) for all smooth k-schemes Z,

$$f_n : \mathrm{Vect}_n(Z) \to \Omega^\infty\Big(\prod_{i\in\mathbb{Z}}\big(R\Gamma_{\mathrm{crys}}(Z;\mathbb{Q}_p)\otimes_{K_0} K\big)[2i]\Big),$$

such that the $\{f_n\}$ are additive and multiplicative.

Both the source and target of the f_n are sheaves of spaces for the smooth or étale topology on smooth k-schemes. Sheafifying for the smooth topology, we obtain such a natural transformation for any smooth Artin stack, which still satisfies the additivity and multiplicativity properties. By naturality, it suffices to show that for $Z = BGL_n$ and for $\mathcal{E}$ the tautological n-dimensional vector bundle, $f_n(\mathcal{E})$ is given by (up to scalars) the crystalline Chern character of $\mathcal{E}$. It follows that for each n, $f_n(\mathcal{E})$ is given by a power series (with K coefficients) in the crystalline Chern classes of $\mathcal{E}$, since $R\Gamma_{\mathrm{crys}}(BGL_n;\mathbb{Q}_p)$ (defined via sheafification) is the polynomial ring $K_0[c_1,\dots,c_n]$, for example, as in the calculations of de Rham and Hodge cohomology of BGL_n in [86]. By additivity, multiplicativity, and the splitting principle to reduce to the case of line bundles, we find easily that f_n must be the Chern character up to normalization by powers of some constant λ. Moreover, $\lambda \neq 0$ by comparison with the left-hand side of (25). ☐

Proof of Theorem 4.9
We use the fiber square of (25). As before, we have identifications $\mathrm{HP}^{\mathrm{cts}}(\mathfrak{X}/\mathcal{O}_K;\mathbb{Q}_p) \simeq \prod_{i\in\mathbb{Z}} R\Gamma_{\mathrm{dR}}(\mathfrak{X}_K/K)[2i]$ and $\mathrm{HC}^{-,\mathrm{cts}}(\mathfrak{X}/\mathcal{O}_K;\mathbb{Q}_p) \simeq \prod_{i\in\mathbb{Z}} \mathrm{Fil}^{\geq i} R\Gamma_{\mathrm{dR}}(\mathfrak{X}_K/K)[2i]$. Using the crystalline–to–de Rham comparison (Construction 4.7) and Proposition 4.12, we see that the map

$$\mathrm{K}_0(\mathfrak{X}_k;\mathbb{Q}_p) \to \prod_{i\in\mathbb{Z}} H^{2i}_{\mathrm{dR}}(\mathfrak{X}_K/K) \simeq \prod_{i\in\mathbb{Z}} H^{2i}_{\mathrm{crys}}(\mathfrak{X}_k;\mathbb{Q}_p)\otimes_{K_0} K$$

is given up to scalar factors by the crystalline Chern character. The result now follows from Proposition 4.10. ☐

4.2. *Generalization of Beilinson's obstruction and proof of Theorem E*

Let K, $\mathcal{O}_K$, k be as in the preceding subsection. Consider a proper scheme $X \to \mathrm{Spec}(\mathcal{O}_K)$ with smooth generic fiber X_K and possibly singular special fiber X_k. In [8], Beilinson considers more generally the deformation problem for classes in higher K-theory, and proves the following.

THEOREM 4.13 (Beilinson [8])
Given $x \in \mathrm{K}_i(X_k)_{\mathbb{Q}}$, there is a natural obstruction class in $\bigoplus_{r\geq 0} H^{2r-i}_{\mathrm{dR}}(X_K/K)/\mathrm{Fil}^r H^{2r-i}_{\mathrm{dR}}(X_K/K)$ which vanishes if and only if x lifts to $(\varprojlim \mathrm{K}_i(X/\pi^n))_{\mathbb{Q}}$. More precisely, there is a natural equivalence of spectra

$$\mathrm{cofib}\big(\mathrm{K}^{\mathrm{cts}}(X;\mathbb{Q}_p) \to \mathrm{K}(X_k;\mathbb{Q}_p)\big) \simeq \bigoplus_{r\geq 0} R\Gamma_{\mathrm{dR}}(X_K/K)/\mathrm{Fil}^{\geq r} R\Gamma_{\mathrm{dR}}(X_K/K)[2r].$$

In particular, Theorem 4.13 applies for $i = 0$ and overlaps with the results of [17], although it does not identify the obstruction class with the crystalline Chern character. In this subsection, we observe that Theorem 4.13 can be extended to essentially arbitrary formal schemes, using comparisons between cyclic and de Rham cohomology as in [1]. This argument will not essentially rely on having a fiber square as in Theorem A (versus a fiber sequence), and could be deduced from the results of [8].

THEOREM 4.14
Let $\mathfrak{X}$ be a qcqs p-adic formal scheme with bounded p-power torsion. Given $i \in \mathbb{Z}$ and a class $x \in \mathrm{K}_i(\mathfrak{X}_1;\mathbb{Q})$ there is a natural class

$$c(x) \in \bigoplus_{r\geq 0} H^{2r-i}(L\Omega_{\mathfrak{X}}/L\Omega^{\geq r}_{\mathfrak{X}})_{\mathbb{Q}_p}$$

with the property that x lifts to $\mathrm{K}^{\mathrm{cts}}_i(\mathfrak{X};\mathbb{Q})$ if and only if $c(x) = 0$. More precisely, there is a natural equivalence of spectra

$$\mathrm{cofib}\big(\mathrm{K}^{\mathrm{cts}}(\mathfrak{X};\mathbb{Q}) \to \mathrm{K}(\mathfrak{X}_1;\mathbb{Q})\big) \simeq \bigoplus_{r\geq 0}\big(L\Omega_{\mathfrak{X}}/L\Omega^{\geq r}_{\mathfrak{X}}[2r]\big)_{\mathbb{Q}_p}. \tag{29}$$

Proof of Theorem 4.14
It clearly suffices to exhibit the natural equivalence (29), and for this we may assume that $\mathfrak{X} = \mathrm{Spf}(R)$ is affine, since all terms satisfy Zariski descent. Now we have seen that the cofiber in (29) can be identified with the cofiber of $\mathrm{TC}(R;\mathbb{Q}_p) \to \mathrm{TC}(R/p;\mathbb{Q}_p)$ or, equivalently, with $\mathrm{HC}(R;\mathbb{Q}_p)[2]$ by Theorem 2.20. We invoke the result of [1] which constructs on $\mathrm{HC}(R;\mathbb{Z}_p)[2]$ a natural exhaustive decreasing filtration $\mathrm{Fil}^{\geq\star}\mathrm{HC}(R;\mathbb{Z}_p)[2]$ with graded pieces

$$\mathrm{gr}^n\mathrm{HC}(R;\mathbb{Z}_p)[2] \simeq L\Omega_R/L\Omega^{\geq n}_R[2n],$$

where, as before, $L\Omega_R$ is the p-adic derived de Rham cohomology of R and $L\Omega^{\geq\star}_R$ is the Hodge filtration on the derived de Rham cohomology (see specifically the proof of [1, Corollary 4.11]).[10] It follows that on $\mathrm{HC}(R;\mathbb{Q}_p)[2]$ there is a natural exhaustive

[10]The work of [1] was essentially motivated by that of [12], which, among many other things, established such filtrations for quasisyntomic rings by descent.

decreasing filtration $\mathrm{Fil}^{\geq\star}\mathrm{HC}(R;\mathbb{Q}_p)[2]$ with graded pieces

$$\mathrm{gr}^n\mathrm{HC}(R;\mathbb{Q}_p)[2] \simeq (L\Omega_R/L\Omega_R^{\geq n})_{\mathbb{Q}_p}[2n].$$

An argument as in [12, Section 9.4] can be used to show that there is an action of Adams operations on $\mathrm{HC}(R;\mathbb{Z}_p)[2]$ where $\lambda \in \mathbb{Z}_p^\times$ acts via λ^n on $\mathrm{gr}^n\mathrm{HC}(R;\mathbb{Z}_p)$. In particular, these split $\mathrm{HC}(R;\mathbb{Q}_p)[2]$ into eigenspaces so that there is a natural decomposition

$$\mathrm{HC}(R;\mathbb{Q}_p)[2] \simeq \bigoplus_n (L\Omega_R/L\Omega_R^{\geq n})_{\mathbb{Q}_p}[2n].$$

The result now follows from the Beilinson fiber sequence. □

Remark 4.15 (Changing the base ring)
In the above work, $\mathbb{Z}_p$ was used as the base for cyclic and de Rham cohomology, but often this is not essential. Suppose now that A is a commutative $\mathbb{Z}$-algebra with $\widehat{L_{A/\mathbb{Z}}}$ quasi-isogenous to zero. Then it is not difficult to see that, for formal schemes over A, we can replace all occurrences of derived de Rham cohomology relative to $\mathbb{Z}_p$ with such occurrences relative to A.

5. The motivic filtration on TC

In this section (which will not use the Beilinson fiber square), we prove some general structural results on topological cyclic homology TC (in particular, Theorem G from the introduction) and on the "motivic" filtration constructed by Bhatt–Morrow–Scholze [12].

Recall that, according to [12], for R a quasisyntomic ring (see Definition 5.4 below for a review), $\mathrm{TC}(R;\mathbb{Z}_p)$ admits a complete descending $\mathbb{Z}_{\geq 0}$-indexed filtration $\mathrm{Fil}^{\geq\star}\mathrm{TC}(R;\mathbb{Z}_p)$ with associated graded terms given as $\mathrm{gr}^i\mathrm{TC}(R;\mathbb{Z}_p) \simeq \mathbb{Z}_p(i)(R)[2i]$. In this section, we will prove some structural properties of this filtration. Our main results are as follows.

THEOREM 5.1 (Connectivity properties)

(1) *Let $R \in \mathrm{QSyn}$ be a quasisyntomic ring. Then $\mathbb{Z}_p(i)(R) \in D^{\leq i+1}(\mathbb{Z}_p)$. Consequently, we have $\mathrm{Fil}^{\geq i}\mathrm{TC}(R;\mathbb{Z}_p) \in \mathrm{Sp}_{\geq i-1}$.*
(2) *The functors $R \mapsto \mathbb{Z}_p(i)(R)$ and $R \mapsto \mathrm{Fil}^{\geq i}\mathrm{TC}(R;\mathbb{Z}_p)$ are left Kan extended from finitely generated p-complete polynomial $\mathbb{Z}_p$-algebras.*

Part (2) was indicated to us by Scholze. In view of it, we can extend the construction of the $\mathbb{Z}_p(i)$ to all (p-complete) rings.

THEOREM 5.2 (Rigidity)
Let (R, I) be a Henselian pair, where R and R/I are p-complete. Then

$$\mathrm{fib}\big(\mathbb{Z}_p(i)(R) \to \mathbb{Z}_p(i)(R/I)\big) \in D^{\leq i}(\mathbb{Z}_p).$$

In particular, using the known description in characteristic p, we obtain that for any R there is a complete description of the top cohomology $H^{i+1}(\mathbb{F}_p(i)(R))$ and that this vanishes étale locally.

5.1. Review of [12]
Here we recall some of the major results and techniques of [12]. We recall first the quasisyntomic site QSyn (a non-Noetherian version of the syntomic site used by Fontaine–Messing [31]) and the subcategory QRSPerfd $\subset$ QSyn of quasiregular semiperfectoid rings.

Definition 5.3 (p-complete (faithful) flatness and Tor*-amplitude; [12, Definition 4.1])*
Let R be a commutative ring. An R-module M is called *p-completely flat* (resp., *p-completely faithfully flat*) if $M \otimes_R^{\mathbb{L}} (R/p) \in D(R/p)$ is a flat (resp., faithfully flat) R/p-module concentrated in degree 0. Similarly, an object $N \in D(R)$ has *p-complete* Tor*-amplitude in* $[a, b]$ if $N \otimes_R^{\mathbb{L}} R/p \in D(R/p)$ has Tor-amplitude in $[a, b]$.

Definition 5.4 (The quasisyntomic site; cf. [12, Section 4])

(1) A commutative ring R is called *quasisyntomic* if it is p-complete, has bounded p-power torsion, and $L_{R/\mathbb{Z}_p}$ has p-complete Tor-amplitude in $[-1, 0]$ (indexing conventions for the derived category are cohomological). We let QSyn be the category of quasisyntomic rings, with all ring homomorphisms.
(2) The category QSyn (or, more precisely, its opposite) acquires the structure of a site as follows: a map $A \to B$ in QSyn is a cover if $A \to B$ is p-completely faithfully flat and if $L_{B/A} \in D(B)$ has p-complete Tor-amplitude in $[-1, 0]$. We call a map with all of the above properties, except that $A \to B$ only assumed p-completely flat (rather than p-completely faithfully flat), a *quasisyntomic map*.
(3) An object $R \in$ QSyn is *quasiregular semiperfectoid* if R admits a map from a perfectoid ring and the Frobenius on R/p is surjective. We let QRSPerfd $\subset$ QSyn be the full subcategory spanned by quasiregular semiperfectoid rings. If R is additionally an $\mathbb{F}_p$-algebra, then R is called *quasiregular semiperfect.*

For future reference, we will also need the relative versions qSyn_A and $\mathscr{Q}\mathrm{Syn}_A$ of the quasisyntomic sites (of which the first is considered in [12]).

Definition 5.5 (Relative quasisyntomic sites; cf. [12, Section 4.5])
Fix a quasisyntomic ring $A \in \mathrm{QSyn}$. We define the sites qSyn_A and $\mathscr{Q}\mathrm{Syn}_A$ as follows.

(1) We let $\mathscr{Q}\mathrm{Syn}_A$ denote the category of A-algebras B which are quasisyntomic as underlying rings and such that $L_{B/A} \in D(B)$ has p-complete Tor-amplitude in $[-1, 0]$. We let $\mathrm{qSyn}_A \subset \mathscr{Q}\mathrm{Syn}_A$ be the full subcategory spanned by the quasisyntomic A-algebras (i.e., those B such that B is additionally p-completely flat over A).
(2) We make $\mathscr{Q}\mathrm{Syn}_A$ and qSyn_A into sites by declaring a cover to be a map which is a cover in QSyn.
(3) We let $\mathscr{Q}\mathrm{RSPerfd}_A$ (resp., $\mathrm{qrsPerfd}_A$) denote the subcategory of $\mathscr{Q}\mathrm{Syn}_A$ (resp., qSyn_A) spanned by A-algebras whose underlying ring is quasiregular semiperfectoid. Note that if $B \in \mathscr{Q}\mathrm{RSPerfd}_A$, then the p-completion of $L_{B/A}[-1]$ is a p-completely flat, discrete B-module by [12, Lemma 4.7(1)].

Note that in the case $A = \mathbb{Z}_p$, $\mathrm{qSyn}_{\mathbb{Z}_p}$ is the category of p-torsion-free quasisyntomic rings and $\mathscr{Q}\mathrm{Syn}_{\mathbb{Z}_p} = \mathrm{QSyn}$. For $A = \mathbb{F}_p$, $\mathscr{Q}\mathrm{Syn}_{\mathbb{F}_p}$ and $\mathrm{qSyn}_{\mathbb{F}_p}$ are both simply the subcategory of QSyn spanned by those quasisyntomic rings which are $\mathbb{F}_p$-algebras (see [12, Lemma 4.34]); more generally, $\mathscr{Q}\mathrm{Syn}_A$ is the category of A-algebras which are quasisyntomic for any perfectoid ring A.

The site QSyn has a basis given by QRSPerfd, and similarly in the relative cases. All the functors below will be sheaves on QSyn; to describe them, it therefore suffices to describe them as sheaves on QRSPerfd (see [12, Proposition 4.31]).

We now review the prismatic sheaves on QSyn, constructed via topological Hochschild and cyclic homology. A purely algebraic construction via the prismatic cohomology of Bhatt–Scholze is given in [15] (at least for algebras over a base perfectoid ring), which also produces objects before Nygaard completion.

Definition 5.6 (Prismatic sheaves on QSyn*; [12, Section 7])*
The objects $\widehat{\mathbf{\Delta}}_R\{i\}$ and $\mathcal{N}^{\geq n}\widehat{\mathbf{\Delta}}_R$ define sheaves on QSyn with values in $D(\mathbb{Z}_p)^{\geq 0}$. Each of these sheaves is constructed via descent (see [12, Proposition 4.31]) from $\mathrm{QRSPerfd} \subset \mathrm{QSyn}$, on which they take discrete values defined via topological Hochschild homology.

(1) For $R \in \mathrm{QRSPerfd}$, $\mathrm{THH}(R; \mathbb{Z}_p)$ is concentrated in even degrees, so the homotopy fixed point and Tate spectral sequences for $\mathrm{TC}^-(R; \mathbb{Z}_p)$ and $\mathrm{TP}(R; \mathbb{Z}_p)$ degenerate and $\mathrm{TP}(R; \mathbb{Z}_p)$ is 2-periodic. For $R \in \mathrm{QRSPerfd}$, we

have

$$\widehat{\mathbf{\Delta}}_R = \pi_0\big(\mathrm{TC}^-(R;\mathbb{Z}_p)\big) = \pi_0\big(\mathrm{TP}(R;\mathbb{Z}_p)\big). \tag{30}$$

(2) For $R \in$ QRSPerfd and $n \in \mathbb{Z}$, the ideal $\mathcal{N}^{\geq n}\widehat{\mathbf{\Delta}}_R \subset \widehat{\mathbf{\Delta}}_R$ is the one defined by the homotopy fixed point spectral sequence; that is,

$$\mathcal{N}^{\geq n}\widehat{\mathbf{\Delta}}_R = \mathrm{im}\Big(\pi_0\Big(\big(\tau_{\geq 2n}\mathrm{THH}(R;\mathbb{Z}_p)\big)^{hS^1}\Big) \to \pi_0\big(\mathrm{THH}(R;\mathbb{Z}_p)^{hS^1}\big)\Big). \tag{31}$$

(3) For $i \in \mathbb{Z}$ we further have the invertible $\widehat{\mathbf{\Delta}}$-modules (as sheaves on QSyn) $\widehat{\mathbf{\Delta}}\{i\}$, called *Breuil–Kisin twists*. For $R \in$ QRSPerfd,

$$\widehat{\mathbf{\Delta}}_R\{i\} = \pi_{2i}\big(\mathrm{TP}(R;\mathbb{Z}_p)\big), \tag{32}$$

and by 2-periodicity $\widehat{\mathbf{\Delta}}_R\{i\} = \widehat{\mathbf{\Delta}}_R\{1\}^{\otimes i}$. We have a natural isomorphism

$$\mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_R\{i\} \simeq \pi_{2i}\big(\mathrm{TC}^-(R;\mathbb{Z}_p)\big). \tag{33}$$

More generally, $\mathcal{N}^{\geq n+i}\widehat{\mathbf{\Delta}}_R\{i\}$ is the image of the injection

$$\pi_{2i}\Big(\big(\tau_{\geq 2n+2i}\mathrm{THH}(R;\mathbb{Z}_p)\big)^{hS^1}\Big) \to \pi_{2i}\Big(\big(\mathrm{THH}(R;\mathbb{Z}_p)\big)^{hS^1}\Big) \quad \text{for } n \in \mathbb{Z}.$$

(4) There are two maps of sheaves on QSyn,

$$\mathrm{can}, \varphi_i \colon \mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_R\{i\} \rightrightarrows \widehat{\mathbf{\Delta}}_R\{i\} \tag{34}$$

arising from the canonical and Frobenius maps $\mathrm{TC}^-(R;\mathbb{Z}_p) \rightrightarrows \mathrm{TP}(R;\mathbb{Z}_p)$; in particular, we obtain an endomorphism $\varphi = \varphi_0 \colon \widehat{\mathbf{\Delta}}_R \to \widehat{\mathbf{\Delta}}_R$.

(5) Finally, the map $\mathrm{TC}^-(R;\mathbb{Z}_p) \to R$ yields a projection map $a_R \colon \widehat{\mathbf{\Delta}}_R \to R$, a surjection with kernel $\mathcal{N}^{\geq 1}\widehat{\mathbf{\Delta}}_R \subset \widehat{\mathbf{\Delta}}_R$.

Example 5.7 (Perfectoid rings; [12, Section 6])
Let R_0 be a perfectoid ring. In this case, we have Fontaine's ring $\mathrm{A}_{\mathrm{inf}}(R_0) = W(R_0^\flat)$ and the surjective map $\theta \colon \mathrm{A}_{\mathrm{inf}}(R_0) \to R_0$, whose kernel is a principal ideal generated by a non-zero-divisor $\xi \in \mathrm{A}_{\mathrm{inf}}(R_0)$; θ is the universal pro-nilpotent, p-complete thickening of R_0. We have a canonical isomorphism $\widehat{\mathbf{\Delta}}_{R_0} \simeq \mathrm{A}_{\mathrm{inf}}(R_0)$ such that the projection map $a_R \colon \widehat{\mathbf{\Delta}}_{R_0} \to R_0$ is θ. There are also noncanonical isomorphisms $\widehat{\mathbf{\Delta}}_{R_0}\{i\} \simeq \mathrm{A}_{\mathrm{inf}}(R_0)$ for each i. The map $\varphi = \varphi_0 \colon \widehat{\mathbf{\Delta}}_{R_0} \to \widehat{\mathbf{\Delta}}_{R_0}$ is given by the Witt vector Frobenius on $\mathrm{A}_{\mathrm{inf}}(R_0)$. The Nygaard filtration on $\widehat{\mathbf{\Delta}}_R = \mathrm{A}_{\mathrm{inf}}(R_0)$ is the ξ-adic filtration. The map φ_i is injective (and φ_0-semilinear), and its image is given by $(\widetilde{\xi}^{-\min(i,0)})$ for $\widetilde{\xi} = \varphi(\xi)$.

Definition 5.8 (The sheaves $\mathbb{Z}_p(i)$)
For $i \geq 0$, the sheaf $\mathbb{Z}_p(i)$ is defined as the homotopy equalizer of can, φ_i, that is, via

$$\mathbb{Z}_p(i) \simeq \mathrm{fib}\big(\mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_R\{i\} \xrightarrow{\mathrm{can}-\varphi_i} \widehat{\mathbf{\Delta}}_R\{i\}\big).$$

Consequently, since TC is itself a homotopy equalizer, we also have for $R \in$ QRSPerfd,

$$\mathbb{Z}_p(i)(R) = \big(\tau_{[2i-1,2i]}\mathrm{TC}(R;\mathbb{Z}_p)\big)[-2i]. \tag{35}$$

We also define quasisyntomic sheaves $\mathbb{F}_p(i)$ and $\mathbb{Q}_p(i)$ by reducing $\mathbb{Z}_p(i)$ modulo p or inverting p on $\mathbb{Z}_p(i)$, respectively.

Using the result [12, Section 3] that TC defines a sheaf for the fpqc topology on rings, one sheafifies the Postnikov filtration and obtains the following fundamental result.

THEOREM 5.9 ("Motivic filtrations"; [12, Theorem 1.12])
Let $R \in$ QSyn. Then $\mathrm{THH}(R;\mathbb{Z}_p)$, $\mathrm{TC}^-(R;\mathbb{Z}_p)$, $\mathrm{TP}(R;\mathbb{Z}_p)$, and $\mathrm{TC}(R;\mathbb{Z}_p)$ naturally upgrade to filtered spectra with complete, multiplicative descending filtrations $\mathrm{Fil}^{\geq\star}\mathrm{THH}(R;\mathbb{Z}_p)$, $\mathrm{Fil}^{\geq\star}\mathrm{TC}^-(R;\mathbb{Z}_p)$, $\mathrm{Fil}^{\geq\star}\mathrm{TP}(R;\mathbb{Z}_p)$, and $\mathrm{Fil}^{\geq\star}\mathrm{TC}(R;\mathbb{Z}_p)$, indexed by $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}$, $\mathbb{Z}$, and $\mathbb{Z}_{\geq 0}$, respectively, such that the associated graded pieces are given by:

(1) $\mathrm{gr}^i\mathrm{THH}(R;\mathbb{Z}_p) \simeq \mathcal{N}^i\widehat{\mathbf{\Delta}}_R\{i\}[2i] \overset{\mathrm{def}}{=} (\mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_R\{i\}/\mathcal{N}^{\geq i+1}\widehat{\mathbf{\Delta}}_R\{i\})[2i]$ *for all $i \geq 0$; moreover, in this case the Breuil–Kisin twists can be trivialized, so also* $\mathrm{gr}^i\mathrm{THH}(R;\mathbb{Z}_p) \simeq \mathcal{N}^i\widehat{\mathbf{\Delta}}_R[2i]$.

(2) $\mathrm{gr}^i\mathrm{TC}^-(R;\mathbb{Z}_p) = \mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_R\{i\}[2i]$ *for all $i \in \mathbb{Z}$.*

(3) $\mathrm{gr}^i\mathrm{TP}(R;\mathbb{Z}_p) = \widehat{\mathbf{\Delta}}_R\{i\}[2i]$ *for all $i \in \mathbb{Z}$.*

(4) $\mathrm{gr}^i\mathrm{TC}(R;\mathbb{Z}_p) = \mathbb{Z}_p(i)(R)[2i]$ *for all $i \geq 0$.*

Remark 5.10 (Comparison with K-theory)
Recall that for p-adic rings, TC and p-adic étale K-theory agree in nonnegative degrees. Compare [34] for smooth algebras in characteristic p, and [23] and [22] for smooth algebras in general. One may thus expect the filtration of Theorem 5.9 to be the étale sheafification of the filtration on algebraic K-theory with associated graded motivic cohomology (cf. [32], [58] for smooth schemes over fields). In particular, one expects the $\mathbb{Z}_p(i)$ to be some form of p-adic étale motivic cohomology. This is essentially understood in equal characteristic (already by [12]), as we review below, but has not yet appeared in mixed characteristic. In mixed characteristic and under finiteness assumptions (e.g., smooth schemes over a discrete valuation ring), many authors have studied étale motivic cohomology [33] and similar "p-adic étale

Tate twists” (e.g., those of [31], [80], [81]), though the construction is very different from that of [12]; one ultimately hopes to compare all of them, and we will at least offer some information in this and the next section.

We review the discreteness property of the $\mathbb{Z}_p(i)$. By construction, the objects $\mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_R\{i\}$ are sheaves on QSyn with values in $D(\mathbb{Z}_p)^{\geq 0}$ (recall from [65, Corollary 2.1.2.3] that such sheaves form the coconnective part of the derived ∞-category of the category of abelian sheaves on QSyn); as objects of this category, they are in fact *discrete*, since they take discrete values on the basis QRSPerfd. A deep result of Bhatt–Scholze (conjectured in [12] and proved in the characteristic p case there) is that this discreteness also holds for the $\mathbb{Z}_p(i)$, although they in general take values in cohomological degrees [0, 1] for rings in QRSPerfd.

THEOREM 5.11 (Bhatt–Scholze [15, Theorem 14.1])
The $D(\mathbb{Z}_p)^{\geq 0}$-valued sheaf $\mathbb{Z}_p(i)$ on QSyn *is discrete and torsion-free. More precisely, given $R \in$* QSyn*, there is a cover $R \to R'$ in* QSyn *such that $\mathbb{Z}_p(i)(R')$ is discrete and torsion-free.*

Finally, we review the prism structure on $\widehat{\mathbf{\Delta}}_R$, for R quasiregular semiperfectoid. For simplicity, we will assume R to be p-torsion-free.

PROPOSITION 5.12
Let $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$. Suppose that R is an algebra over the perfectoid ring R_0, with notation as in Example 5.7. Then $\mathrm{TP}(R;\mathbb{Z}_p)/\widetilde{\xi} \simeq \mathrm{THH}(R;\mathbb{Z}_p)^{tC_p}$, and this is concentrated in even degrees and p-torsion-free.

Proof
We have that

$$\mathrm{TP}(R;\mathbb{Z}_p)/\xi \simeq \mathrm{HP}(R/R_0;\mathbb{Z}_p)$$

by [12, Theorem 6.7]. Since R is p-torsion-free and quasiregular semiperfectoid, we find that $\mathrm{HP}(R/R_0;\mathbb{Z}_p)$ is concentrated in even degrees and is p-torsion-free, where it is given by Hodge-complete p-adic derived de Rham cohomology by [12, Proposition 5.15]. In particular, it follows that (ξ, p) defines a regular sequence on $\widehat{\mathbf{\Delta}}_R$. Since $\widehat{\mathbf{\Delta}}_R$ is complete with respect to this ideal, it follows that (p,ξ) is a regular sequence; since $\xi^p \equiv \widetilde{\xi} (\mathrm{mod}\ p)$, we get that $(p,\widetilde{\xi})$ is a regular sequence, and hence so is $(\widetilde{\xi}, p)$. Now, the equivalence $\mathrm{TP}(R;\mathbb{Z}_p)/\widetilde{\xi} \simeq \mathrm{THH}(R;\mathbb{Z}_p)^{tC_p}$ is [12, Proposition 6.4], from which the remainder now follows. □

Construction 5.13 (The prismatic structure on $\widehat{\mathbf{\Delta}}_R$; cf. [15, Section 13])
Let $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$. Suppose that R is an algebra over the perfectoid ring R_0. Then the ring $\widehat{\mathbf{\Delta}}_R = \pi_0(\mathrm{TP}(R;\mathbb{Z}_p))$ has the structure of a prism (in the sense of [15]).

(1) We have the endomorphism $\varphi = \varphi_0$, which is congruent to the Frobenius modulo p, by [15, Section 13], and thus defines a δ-structure on $\widehat{\mathbf{\Delta}}_R$ by p-torsion-freeness.

(2) We have the ideal $I \subset \widehat{\mathbf{\Delta}}_R$ given by $I = (\widetilde{\xi})$; I is the kernel of $\pi_0(\mathrm{TP}(R;\mathbb{Z}_p)) \to \pi_0(\mathrm{THH}(R;\mathbb{Z}_p)^{tC_p})$ and therefore does not depend on the choice of R_0.

Finally, there is a natural map

$$\eta_R \colon R \to \widehat{\mathbf{\Delta}}_R/I,$$

given via the cyclotomic Frobenius $\mathrm{THH}(R;\mathbb{Z}_p) \to \mathrm{THH}(R;\mathbb{Z}_p)^{tC_p} = \mathrm{TP}(R;\mathbb{Z}_p)/\widetilde{\xi}$ upon applying π_0.

Remark 5.14
In fact, by [15, Theorem 13.1], $\widehat{\mathbf{\Delta}}_R$ is the Nygaard completion of the absolute prismatic cohomology of R, although we will not need this fact.

5.2. *Relative* THH *and its filtration*

In this subsection and the next, we will prove connectivity bounds for the motivic filtration on THH. We will prove that for any $R \in \mathrm{QSyn}$, we have $\mathrm{Fil}^{\geq n}\mathrm{THH}(R;\mathbb{Z}_p) \in \mathrm{Sp}_{\geq n}$ and $\mathcal{N}^n\widehat{\mathbf{\Delta}}_R \in D^{\leq n}(\mathbb{Z})$. It is not difficult to deduce the above connectivity bound in the case where R is an algebra over a fixed perfectoid ring, using methods as in [12, Sections 6–7] (see, in particular, [12, Construction 7.4]. To verify the connectivity bound in the general case, we will use additionally a fiber sequence which arises from the work of Krause–Nikolaus [55], which gives a comparison between relative and absolute THH.

Let $\mathcal{O}_K$ denote a complete discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field k, and let $\pi \in \mathcal{O}_K$ be a uniformizer. The primary case of interest is $\mathcal{O}_K = \mathbb{Z}_p$ and $\pi = p$.

Construction 5.15 (Relative topological Hochschild homology)
Let $R \in \mathcal{Q}\mathrm{Syn}_{\mathcal{O}_K}$. We consider the $\mathbb{E}_\infty$-ring $\mathbb{S}[z]$ and we consider R as an $\mathbb{S}[z]$-algebra via $z \mapsto \pi$. Using this, we can form the relative topological Hochschild homology (with p-adic coefficients) $\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$. The construction $R \mapsto \mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$ defines a sheaf of spectra on $\mathcal{Q}\mathrm{Syn}_{\mathcal{O}_K}$, thanks to [12, Section 3] (which gives that $R \mapsto \mathrm{HH}(R/\mathcal{O}_K;\mathbb{Z}_p)$ is a sheaf) and (37) below. We observe the following two comparisons for relative THH.

(1) When we base-change along the map $\mathbb{S}[z] \to \mathbb{S}$ where $z \mapsto 0$, we find that

$$\mathrm{THH}\big(R/\mathbb{S}[z];\mathbb{Z}_p\big) \otimes_{\mathcal{O}_K} k \simeq \mathrm{THH}(R \otimes^L_{\mathcal{O}_K} k;\mathbb{Z}_p). \tag{36}$$

(2) We have an equivalence

$$\mathrm{THH}\big(R/\mathbb{S}[z];\mathbb{Z}_p\big) \otimes_{\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)} \mathcal{O}_K \simeq \mathrm{HH}(R/\mathcal{O}_K;\mathbb{Z}_p). \tag{37}$$

Thus, $\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$ is a deformation of Hochschild homology relative to $\mathcal{O}_K$.

Next, we need an analogue of the Hochschild–Kostant–Rosenberg theorem for relative THH (in the absolute case for algebras over a perfectoid ring; this is [44, Theorem B] and [12, Corollary 6.9]).

PROPOSITION 5.16
Let R be a formally smooth $\mathcal{O}_K$-algebra. Then we have a natural isomorphism of graded rings

$$\mathrm{THH}_*\big(R/\mathbb{S}[z];\mathbb{Z}_p\big) \simeq \widehat{\Omega^*_{R/\mathcal{O}_K}}[\sigma], \quad |\sigma| = 2,$$

where $\widehat{\Omega^*_{R/\mathcal{O}_K}}$ denotes the p-completion of the de Rham complex of R over $\mathcal{O}_K$.

Proof
In the case $R = \mathcal{O}_K$, this follows from Bökstedt's calculation of $\mathrm{THH}(\mathbb{F}_p)$ (cf. [12, Proposition 11.10], [2, Theorem 3.5], or [55, Theorem 3.1]). Now (37) shows that $\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)/\sigma \simeq \mathrm{HH}(R/\mathcal{O}_K;\mathbb{Z}_p)$, and the Hochschild–Kostant–Rosenberg theorem yields $\mathrm{HH}_*(R/\mathcal{O}_K;\mathbb{Z}_p) \simeq \widehat{\Omega^*_{R/\mathcal{O}_K}}$. It remains to show that the induced Bockstein spectral sequence for $\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$ (with respect to taking the cofiber of σ) degenerates, or, equivalently, that the map of the HKR isomorphism lifts to a map $\widehat{\Omega^*_{R/\mathcal{O}_K}} \to \mathrm{THH}_*(R/\mathbb{S}[z];\mathbb{Z}_p)$. Indeed, since $\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$ is an $\mathbb{E}_\infty$-algebra with an S^1-action receiving a map from $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)$, we obtain the structure of a commutative differential graded algebra (CDGA) on $\mathrm{THH}_*(R/\mathbb{S}[z];\mathbb{Z}_p)$, and it receives a map (of CDGAs) from $\mathcal{O}_K[\sigma]$ with trivial differential. The universal property of the de Rham complex now produces the desired map $\widehat{\Omega^*_{R/\mathcal{O}_K}} \to \mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$. □

Left-Kan-extending from finitely generated p-complete polynomial $\mathcal{O}_K$-algebras, we obtain the following result, which is proved exactly as in [12, Proposition 7.5]; the key point is that any $R \in \mathcal{Q}\mathrm{RSPerfd}_{\mathcal{O}_K}$ has the property that the p-completion of $L_{R/\mathcal{O}_K}$ is the shift of a p-completely flat R-module.

COROLLARY 5.17
If $R \in \mathcal{Q}\mathrm{RSPerfd}_{\mathcal{O}_K}$, then $\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$ is concentrated in even degrees, and each $\pi_{2n}\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$ is a p-completely flat R-module. Furthermore, the R-module $\pi_{2n}\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$ admits a natural finite increasing filtration with graded pieces the (discrete and p-completely flat) R-modules $(\widehat{\bigwedge^j L_{R/\mathcal{O}_K}})[-j]$ for $j \le n$, where $\widehat{\bigwedge^j L_{R/\mathcal{O}_K}}$ denotes the p-completion of $\bigwedge^j L_{R/\mathcal{O}_K}$.

Construction 5.18 (The filtration on relative THH*)*
Let $R \in \mathcal{Q}\mathrm{Syn}_{\mathcal{O}_K}$. In [12, Section 11], a multiplicative, convergent $\mathbb{Z}_{\ge 0}$-indexed filtration $\mathrm{Fil}^{\ge\star}\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$ on $\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$ in sheaves of spectra on $\mathcal{Q}\mathrm{Syn}_{\mathcal{O}_K}$ is defined.[11] This filtration is defined such that it restricts to the double-speed Postnikov filtration for $R \in \mathcal{Q}\mathrm{RSPerfd}_{\mathcal{O}_K}$; that is, $\mathrm{Fil}^{\ge n}\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p) = \tau_{\ge 2n}\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$ for such R. By Corollary 5.17 and [12, Theorem 3.1], the associated graded pieces of the Postnikov filtration on $\mathrm{THH}(-/\mathbb{S}[z];\mathbb{Z}_p)$ on $\mathcal{Q}\mathrm{RSPerfd}_{\mathcal{O}_K}$ are sheaves; thus, one unfolds and obtains the filtration for all $R \in \mathcal{Q}\mathrm{Syn}_{\mathcal{O}_K}$.

COROLLARY 5.19
Let $R \in \mathcal{Q}\mathrm{Syn}_{\mathcal{O}_K}$. Then $\mathrm{gr}^n\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$ admits a natural finite increasing filtration with associated graded $(\widehat{\bigwedge^j L_{R/\mathcal{O}_K}})[2n-j]$ for $j \le n$. In particular, we find that $\mathrm{gr}^n\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p) \in \mathrm{Sp}_{\ge n}$ and $\mathrm{Fil}^{\ge n}\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p) \in \mathrm{Sp}_{\ge n}$. Furthermore, the constructions

$$R \mapsto \mathrm{gr}^n\mathrm{THH}\big(R/\mathbb{S}[z];\mathbb{Z}_p\big) \qquad \textit{and} \qquad R \mapsto \mathrm{Fil}^{\ge n}\mathrm{THH}\big(R/\mathbb{S}[z];\mathbb{Z}_p\big)$$

(as functors on $\mathcal{Q}\mathrm{Syn}_{\mathcal{O}_K}$ to p-complete spectra) are left Kan extended from finitely generated p-complete polynomial $\mathcal{O}_K$-algebras.

Proof
The first assertion follows from Corollary 5.17 by unfolding; the connectivity assertions then follow in turn. Since the cotangent complex and its wedge powers are left Kan extended from finitely generated polynomial algebras, the last assertion follows, too. □

5.3. *Preliminary connectivity bounds*
We use the spectral sequence of Krause–Nikolaus [55] to obtain a relationship between the relative and absolute THH.

[11] Actually, in [12, Section 11], the filtration is defined only on those objects which are flat over $\mathcal{O}_K$, but the arguments do not require this.

PROPOSITION 5.20 (Relative versus absolute THH)
If $R \in \mathcal{Q}\mathrm{RSPerfd}_{\mathcal{O}_K}$*, then there exist natural surjective maps* $f_n\colon \pi_{2n}\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p) \to \pi_{2n-2}\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$ *and natural isomorphisms* $\pi_{2n}\mathrm{THH}(R;\mathbb{Z}_p) \simeq \ker(f_n)$.

Proof
Recall that both $\mathrm{THH}(R;\mathbb{Z}_p)$ and $\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$ are concentrated in even degrees since $R \in \mathcal{Q}\mathrm{RSPerfd}_{\mathcal{O}_K}$ (see [12, Theorem 7.1] and Corollary 5.17). Therefore, the result follows directly from [55, Proposition 4.1]; the spectral sequence of [55, Proposition 4.1] must degenerate after the first differential, and the maps of the first differential must be surjective or one would have odd degree contributions to $\mathrm{THH}(R;\mathbb{Z}_p)$. ∎

The following fiber sequence (38) will be the basic tool in obtaining connectivity bounds on the filtration on THH and its variants.

COROLLARY 5.21 (Connectivity of the filtration on THH)
If $R \in \mathcal{Q}\mathrm{Syn}_{\mathcal{O}_K}$*, then for each* n *there is a natural fiber sequence*

$$\mathrm{gr}^n\mathrm{THH}(R;\mathbb{Z}_p) \to \mathrm{gr}^n\mathrm{THH}\big(R/\mathbb{S}[z];\mathbb{Z}_p\big) \to \mathrm{gr}^{n-1}\mathrm{THH}\big(R/\mathbb{S}[z];\mathbb{Z}_p\big)[2]. \tag{38}$$

In particular, we have $\mathrm{gr}^n\mathrm{THH}(R;\mathbb{Z}_p)$, $\mathrm{Fil}^{\geq n}\mathrm{THH}(R;\mathbb{Z}_p) \in \mathrm{Sp}_{\geq n}$ *for any* $R \in \mathcal{Q}\mathrm{Syn}_{\mathcal{O}_K}$*. Finally, the functors* $R \mapsto \mathrm{gr}^n\mathrm{THH}(R;\mathbb{Z}_p)$ *and* $R \mapsto \mathrm{Fil}^{\geq n}\mathrm{THH}(R;\mathbb{Z}_p)$ *on* QSyn *are left Kan extended from finitely generated* p*-complete polynomial* $\mathbb{Z}_p$*-algebras, as functors to* p*-complete spectra.*

Proof
The fiber sequence follows from Proposition 5.20 by unfolding in R. The connectivity assertion for $\mathrm{gr}^n\mathrm{THH}(R;\mathbb{Z}_p)$ then follows from Corollary 5.19; the assertion for $\mathrm{Fil}^{\geq n}\mathrm{THH}(R;\mathbb{Z}_p)$ then follows since the filtration is complete. The Kan extension assertion for $\mathrm{Fil}^{\geq n}\mathrm{THH}(R;\mathbb{Z}_p)$ also follows from the one for $\mathrm{Fil}^{\geq n}\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$ as in Corollary 5.19 (taking $\mathcal{O}_K = \mathbb{Z}_p$). ∎

COROLLARY 5.22 (Connectivity bounds for $\mathcal{N}^i\widehat{\mathbf{\Delta}}_R$)

(1) *If* $R \in \mathrm{QSyn}$*, then* $\mathcal{N}^n\widehat{\mathbf{\Delta}}_R \in D^{\leq n}(\mathbb{Z}_p)$.
(2) *If* $R \to R'$ *is a surjective map in* QSyn*, then* $\mathrm{fib}(\mathcal{N}^n\widehat{\mathbf{\Delta}}_R \to \mathcal{N}^n\widehat{\mathbf{\Delta}}_{R'}) \in D^{\leq n}(\mathbb{Z}_p)$.

Proof
Part (1) is a special case of Corollary 5.21 (take $\mathcal{O}_K = \mathbb{Z}_p$ and $\pi = p$).

For part (2), note that the hypothesis implies that $R \to R'$ induces a surjection on H^0 of p-completed cotangent complexes over $\mathbb{Z}_p$, and similarly on any wedge power. It then follows from Corollary 5.19 that $\mathrm{fib}(\mathrm{gr}^n\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)[-2n] \to \mathrm{gr}^n\mathrm{THH}(R'/\mathbb{S}[z];\mathbb{Z}_p)[-2n]) \in D^{\leq n}(\mathbb{Z}_p)$, whence we conclude by (38). $\square$

Our main general connectivity bound is Proposition 5.25 below. To formulate it, we need to be able to twist the ideal $I \subset \widehat{\mathbf{\Delta}}_R$ from Construction 5.13. First, we observe that this ideal is also trivialized after a base change along a_R.

LEMMA 5.23
Let $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$, and let $I \subset \widehat{\mathbf{\Delta}}_R$ denote the ideal defining the prism structure. Then there is a natural isomorphism $I \otimes_{\widehat{\mathbf{\Delta}}_R} R \simeq R$; that is, the ideal is naturally trivialized after a base change along $a_R\colon \widehat{\mathbf{\Delta}}_R \to R$.

Proof
Observe that the base change $I \otimes_{\widehat{\mathbf{\Delta}}_R} R$ defines a functorial choice of invertible R-module, for any $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$. By faithfully flat descent, we obtain for any $R \in \mathrm{qSyn}_{\mathbb{Z}_p}$ a choice of invertible R-module, which is functorial in R. Choosing a trivialization over $R = \mathbb{Z}_p$, we obtain a functorial trivialization everywhere. $\square$

Definition 5.24 (Twisting by I)
For $s,i,n \geq 0$, we let $R \mapsto I^s\mathcal{N}^{\geq n}\widehat{\mathbf{\Delta}}_R\{i\}$ denote the $D(\mathbb{Z}_p)^{\geq 0}$-valued sheaf on $\mathrm{qSyn}_{\mathbb{Z}_p}$ defined by unfolding the discrete sheaf on $\mathrm{qrsPerfd}_{\mathbb{Z}_p}$ defined by the aforementioned formula, for $I \subset \widehat{\mathbf{\Delta}}_R$ the ideal defining the prismatic structure. For $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$, since I defines a Cartier divisor in $\widehat{\mathbf{\Delta}}_R$, we have $I^s\mathcal{N}^{\geq n}\widehat{\mathbf{\Delta}}_R\{i\} \simeq I^s \otimes_{\widehat{\mathbf{\Delta}}_R} \mathcal{N}^{\geq n}\widehat{\mathbf{\Delta}}_R\{i\}$.

PROPOSITION 5.25 (Connectivity of Nygaard quotients)
Let $R \in \mathrm{qSyn}_{\mathbb{Z}_p}$ and $i,n,s \geq 0$. Then the cofiber $I^s\widehat{\mathbf{\Delta}}_R\{i\}/I^s\mathcal{N}^{\geq n}\widehat{\mathbf{\Delta}}_R\{i\}$ belongs to $D^{\leq n-1}(\mathbb{Z}_p)$. Moreover, this cofiber is left Kan extended from finitely generated p-complete polynomial $\mathbb{Z}_p$-algebras.

Proof
By dévissage it suffices to show that $I^s\mathcal{N}^n\widehat{\mathbf{\Delta}}_R\{i\} \in D^{\leq n}(R)$ for each $n \geq 0$ and that this is left Kan extended from finitely generated p-complete polynomial $\mathbb{Z}_p$-algebras. Here we write $I^s\mathcal{N}^n\widehat{\mathbf{\Delta}}_R\{i\}$ for the unfolding from $\mathrm{qrsPerfd}_{\mathbb{Z}_p}$ of $I^s \otimes_{\widehat{\mathbf{\Delta}}_R} \mathcal{N}^n\widehat{\mathbf{\Delta}}_R\{i\}$. However, the twists here are trivialized by Lemma 5.23 since $\mathcal{N}^n\widehat{\mathbf{\Delta}}_R$ is an R-module, so that $I^s\mathcal{N}^n\widehat{\mathbf{\Delta}}_R\{i\} \simeq \mathcal{N}^n\widehat{\mathbf{\Delta}}_R$. Thus, the result follows from Corollary 5.22 as well as Corollary 5.21 (for the left Kan extension assertion). $\square$

We finish this subsection by recording a connectivity bound that depends on the number of generators of the cotangent complex (we will not use this result in the paper, but we note that it implies in particular that $\widehat{\mathbf{\Delta}}_{\mathcal{O}_K} \in D^{\leq 1}(\mathbb{Z}_p)$).

LEMMA 5.26
Let R be a commutative ring, and let $M \in D^{\leq 0}(R)$. Suppose that $H^0(M)$ is generated by d elements. Then for all j, $(\bigwedge^j M)[-j] \in D^{\leq d}(R)$.

Proof
The result is clear if $M = R^d$ itself. In general, we have a map $R^d \to M$ inducing a surjection on H^0, so the cofiber F of the map satisfies $F \in D(R)^{\leq -1}$. It follows that $\bigwedge^{j'} F[-j'] \in D(R)^{\leq 0}$ for all j' by standard connectivity estimates (see [65, Section 25.2.4] for an account). Using the natural filtration on $\bigwedge^j M[-j]$ with associated graded terms $\bigwedge^{j'} F[-j'] \otimes_R \bigwedge^{j-j'} R^d[-(j-j')]$, the result easily follows. $\square$

PROPOSITION 5.27
Let $R \in \mathcal{Q}\mathrm{Syn}_{\mathcal{O}_K}$, let $n, i \geq 0$, and suppose that $H^0(\widehat{L_{R/\mathcal{O}_K}})$ is generated by d elements. Then $\mathcal{N}^n \widehat{\mathbf{\Delta}}_R$ and $\mathcal{N}^{\geq n} \widehat{\mathbf{\Delta}}_R\{i\}$ lie in $D^{\leq d+1}(R)$.

Proof
By Corollary 5.19, $\mathrm{gr}^n \mathrm{THH}(R/\mathbb{S}[z]; \mathbb{Z}_p)$ has a finite filtration with graded pieces $\widehat{\bigwedge^j L_{R/\mathcal{O}_K}}[2n-j]$ for $0 \leq j \leq n$. By Lemma 5.26, we find that $\mathrm{gr}^n \mathrm{THH}(R/\mathbb{S}[z]; \mathbb{Z}_p) \in D^{\leq d-2n}(R)$. Using the fiber sequence (38), we find now that $\mathrm{gr}^n \mathrm{THH}(R; \mathbb{Z}_p) \in D^{\leq d-2n+1}(R)$. Shifting by $2n$ the result now follows for $\mathcal{N}^n \widehat{\mathbf{\Delta}}_R$. The same connectivity bound then follows for each $\mathcal{N}^{\geq n} \widehat{\mathbf{\Delta}}_R\{i\}/\mathcal{N}^{\geq n+r} \widehat{\mathbf{\Delta}}_R\{i\}$ by dévissage, and then for $\mathcal{N}^{\geq n} \widehat{\mathbf{\Delta}}_R\{i\}$ by passing to the limit. $\square$

5.4. Frobenius nilpotence on $\widehat{\mathbf{\Delta}}_R/p$ and proof of Theorem 5.1(2)
In this subsection, we record some results about the contracting property of Frobenius on $\widehat{\mathbf{\Delta}}_R/p$ and use it to prove part of Theorem 5.1. If $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$ is a p-torsion-free quasiregular semiperfectoid ring, then both $\widehat{\mathbf{\Delta}}_R$ and all graded steps $\mathcal{N}^n \widehat{\mathbf{\Delta}}_R$ of the Nygaard filtration are p-torsion-free (e.g., because $\mathrm{THH}_*(R; \mathbb{Z}_p)$ is p-torsion-free and concentrated in even degrees). For $i, r \geq 0$, we will consider the maps

$$\mathrm{can}, \varphi_i : \mathcal{N}^{\geq i+r} \widehat{\mathbf{\Delta}}_R\{i\}/p \to \widehat{\mathbf{\Delta}}_R\{i\}/p. \tag{39}$$

and show that both maps respect the I-adic filtration from Definition 5.24, with φ_i inducing the zero map on associated graded pieces in positive degrees (Proposition 5.30). We will show in addition that $\mathrm{can} - \varphi_i$ induces an automorphism of $\mathcal{N}^{\geq i+r} \widehat{\mathbf{\Delta}}_R\{i\}/p$ for $r \gg 0$ (Corollary 5.31).

PROPOSITION 5.28
Let $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$ and $i, r \geq 0$. Then the map $\varphi_i : \mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_R\{i\} \to \widehat{\mathbf{\Delta}}_R\{i\}$ carries $\mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\}$ into $I^r\widehat{\mathbf{\Delta}}_R\{i\}$.

Proof
Let R_0 be a perfectoid ring mapping to R, and fix $\xi, \widetilde{\xi} \in \mathrm{A}_{\mathrm{inf}}(R_0)$ as usual. Then by [12, Section 6] we have an isomorphism $\mathrm{TC}^-_*(R_0;\mathbb{Z}_p) \simeq \mathrm{A}_{\mathrm{inf}}(R_0)[u,v]/(uv-\xi)$ for $|u| = 2$, $|v| = -2$. In this case, the filtration on $\mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_R\{n\} \simeq \pi_{2i}(\mathrm{TC}^-(R;\mathbb{Z}_p))$ is the filtration by powers of v:

$$\mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\} = v^r\pi_{2i+2r}\mathrm{TC}^-(R;\mathbb{Z}_p) \subset \pi_{2i}\mathrm{TC}^-(R;\mathbb{Z}_p).$$

But (as in [12, Section 6]) the cyclotomic Frobenius carries v to a multiple of $\varphi(\xi) = \widetilde{\xi}$ in $\pi_{-2}\mathrm{TP}(R_0;\mathbb{Z}_p)$; recalling that $I = (\tilde{\xi})$, the result follows. □

Construction 5.29 (The I-adic filtrations modulo p)
Let $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$. For each $i, s, r \geq 0$, the map $\varphi_i : \mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\}/p \to \widehat{\mathbf{\Delta}}_R\{i\}/p$ is Frobenius semilinear by Construction 5.13(1), and so carries $I^s(\mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\}/p)$ to $I^{ps+r}(\widehat{\mathbf{\Delta}}_R\{i\}/p)$ by Proposition 5.28. But, we have seen in Proposition 5.12 that $(p, \tilde{\xi})$ and $(\tilde{\xi}, p)$ are regular sequences on $\widehat{\mathbf{\Delta}}_R$, whence the canonical maps are isomorphisms $I \otimes_{\widehat{\mathbf{\Delta}}_R} \widehat{\mathbf{\Delta}}_R/p \simeq I(\widehat{\mathbf{\Delta}}_R/p) \simeq I/p$, and similarly for any power of I and Breuil–Kisin twist of $\widehat{\mathbf{\Delta}}_R$. We thus get maps

$$\mathrm{can}, \varphi_i : I^s \otimes_{\widehat{\mathbf{\Delta}}_R} \mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\}/p \to I^s \otimes_{\widehat{\mathbf{\Delta}}_R} \widehat{\mathbf{\Delta}}_R\{i\}/p. \tag{40}$$

For convenience, we record what we have proved about the interaction of the Frobenius and the I-adic filtration, as it will be used to prove Proposition 5.35.

PROPOSITION 5.30 (The canonical and Frobenius map are I-adically filtered modulo p)
Let $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$, and let $i, r \geq 0$. The maps (39) upgrade to the structure of filtered maps with respect to the I-adic filtrations on both sides; that is, there are compatible maps for each $s \geq 0$,

$$\mathrm{can}, \varphi_i : I^s \otimes_{\widehat{\mathbf{\Delta}}_R} \mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\}/p \to I^s \otimes_{\widehat{\mathbf{\Delta}}_R} \widehat{\mathbf{\Delta}}_R\{i\}/p.$$

Furthermore, the map φ_i induces the zero map on associated graded pieces unless $s = r = 0$.

Proof
In Construction 5.29 we constructed the maps and showed that in fact φ_i has image in $I^{ps+r} \otimes_{\widehat{\mathbf{\Delta}}_R} \widehat{\mathbf{\Delta}}_R\{i\}/p$. □

For the moment, we need the following consequence of our arguments.

COROLLARY 5.31 (The Nygaard filtrations modulo p)
Let $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$ and $i \geq 0$. For $r \gg 0$ (independent of R), the map $\mathrm{can} - \varphi_i : \mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_R\{i\}/p \to \widehat{\mathbf{\Delta}}_R\{i\}/p$ carries $\mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\}/p$ isomorphically onto itself. Consequently, for such r, one has a natural isomorphism

$$\begin{aligned}
&\mathbb{F}_p(i)(R) \\
&\quad \simeq \mathrm{fib}\big(\mathrm{can} - \varphi_i : \\
&\qquad \big(\mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_R\{i\}/\mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\}\big)/p \to \big(\widehat{\mathbf{\Delta}}_R\{i\}/\mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\}\big)/p\big). \quad (41)
\end{aligned}$$

Proof
This is [12, Lemma 7.22]. By Proposition 5.28 above (and choosing as usual a perfectoid ring R_0 mapping to R), we find that φ_i carries $\mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\}/p$ (equivalently, $v^r\mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i+r\}/p$) into multiples of $\tilde{\xi}^r\widehat{\mathbf{\Delta}}_R\{i\}/p$. Since we are working modulo p, we have $\tilde{\xi}^r\widehat{\mathbf{\Delta}}_R\{i\}/p = \xi^{rp}\widehat{\mathbf{\Delta}}_R\{i\}/p \subset \mathcal{N}^{\geq rp}\widehat{\mathbf{\Delta}}_R\{i\}/p$. For $r \gg 0$, this is contained in $\mathcal{N}^{\geq i+r+1}\widehat{\mathbf{\Delta}}_R\{i\}/p$, whence we have shown that φ_i carries $\mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\}/p$ to $\mathcal{N}^{\geq i+r+1}\widehat{\mathbf{\Delta}}_R\{i\}/p$.

It follows that $\mathrm{can} - \varphi_i$ carries $\mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\}/p$ into itself, and it differs from the identity by a topologically nilpotent endomorphism of $\mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\}/p$ with respect to the Nygaard filtration. Therefore, it is an isomorphism and the result follows. □

PROPOSITION 5.32 (A criterion for being left Kan extended)
Let $F, G : \mathrm{QSyn} \to D(\mathbb{Z}_p)$ be p-complete quasisyntomic sheaves equipped with complete descending $\mathbb{Z}_{\geq 0}$-indexed filtrations $\mathrm{Fil}^{\geq \star} F$ and $\mathrm{Fil}^{\geq \star} G$. Let $F \to G$ be a map of functors (not necessarily filtration-preserving). If

(1) *for $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$, the objects $\mathrm{gr}^r F(R)$ and $\mathrm{gr}^r G(R)$ are discrete, p-complete, and p-torsion-free (and therefore so are $F(R)$, $G(R)$),*

(2) *each of the associated graded terms $\mathrm{gr}^r F$ and $\mathrm{gr}^r G$ is left Kan extended from finitely generated p-complete polynomial $\mathbb{Z}_p$-algebras to the p-complete derived category, and*

(3) *there exists N such that for $r \geq N$ and for $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$, the map $F(R)/p \to G(R)/p$ carries the submodule $\mathrm{Fil}^{\geq r} F(R)/p$ isomorphically to $\mathrm{Fil}^{\geq r} G(R)/p$ (so that $\mathrm{Fil}^{\geq N} F(R)/p \to \mathrm{Fil}^{\geq N} G(R)/p$ is an equivalence of filtered abelian groups),*

then $\mathrm{fib}(F \to G)$ is p-completely left Kan extended from finitely generated p-complete polynomial $\mathbb{Z}_p$-algebras.

Proof

It suffices to check that $\mathrm{fib}(F \to G)/p$ is left Kan extended from finitely generated p-complete polynomial algebras. Let F', G' denote the functors on QSyn obtained by restricting F, G to finitely generated p-complete polynomial algebras and then left Kan extending to QSyn, with their left Kan extended filtrations. Our assumptions imply that F, G are the respective completions of F', G' with respect to their filtrations.

It suffices to check that the natural map induces an equivalence $\mathrm{fib}(F' \to G')/p \simeq \mathrm{fib}(F \to G)/p$. We claim that for any $R \in \mathrm{QSyn}$, there are natural commutative diagrams, compatible in $r \geq N$,

$$\begin{tikzcd} \mathrm{Fil}^{\geq r} F'(R)/p \arrow[r, "\simeq"] \arrow[d] & \mathrm{Fil}^{\geq r} G'(R)/p \arrow[d] \\ F'(R)/p \arrow[r] & G'(R)/p. \end{tikzcd} \tag{42}$$

In fact, it suffices to prove this by left Kan extension for R a finitely generated p-complete polynomial over $\mathbb{Z}_p$, and then we can replace F', G' by F, G. By descent for F, G, we can then reduce to $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$, whence we have the desired diagrams by hypothesis.

Using the diagrams (42), we find that there is a natural commutative diagram

$$\begin{tikzcd} F'(R)/p \arrow[r] \arrow[d] & G'(R)/p \arrow[d] \\ F(R)/p \arrow[r] & G(R)/p \end{tikzcd}$$

which is homotopy Cartesian (taking the inverse limit over r). We finally find that $\mathrm{fib}(F \to G) = \mathrm{fib}(F' \to G')$, which is left Kan extended from finitely generated p-complete polynomial algebras, as desired. ◻

Proof of Theorem 5.1(2)

We show that $R \mapsto \mathbb{Z}_p(i)(R)$, as a functor on QSyn, is left Kan extended from finitely generated p-complete polynomial $\mathbb{Z}_p$-algebras. Since $\mathbb{Z}_p(i)(R) = \mathrm{fib}(\mathrm{can} - \varphi_i : \mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_R\{i\} \to \widehat{\mathbf{\Delta}}_R\{i\})$, we will apply Proposition 5.32 with $F = \mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}\{i\}$, $G = \widehat{\mathbf{\Delta}}\{i\}$ using the Nygaard filtrations $R \mapsto \mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\}$ on QSyn. Indeed, the associated graded terms for the Nygaard filtration are left Kan extended from finitely generated p-complete polynomial algebras (Proposition 5.25), and they are torsion-free on $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$. The last hypothesis follows from Corollary 5.31. Then Proposition 5.32 gives that the $\mathbb{Z}_p(i)$ are left Kan extended from finitely generated p-complete polynomial algebras, as desired. Since $R \mapsto \mathrm{TC}(R;\mathbb{Z}_p)$ is left Kan extended from finitely generated p-complete polynomial rings (by [23, Theorem G] and since $\mathrm{TC}(-;\mathbb{Z}_p)$ commutes with geometric realizations on simplicial commutative rings), it follows inductively that the constructions $R \mapsto \mathrm{Fil}^{\geq i}\mathrm{TC}(R;\mathbb{Z}_p)$ are also left Kan extended from finitely generated p-complete $\mathbb{Z}_p$-algebras. □

For future reference, we observe that we can obtain a motivic filtration on TC for any simplicial commutative ring, by left Kan extension. We let SCR denote the ∞-category of simplicial commutative rings.

Construction 5.33 (Left-Kan-extending to SCR*)*
We have seen that the functor which sends $R \in \mathrm{QSyn}$ to the filtration $\mathrm{Fil}^{\geq\star}\mathrm{TC}(R;\mathbb{Z}_p)$ is left Kan extended from finitely generated p-complete polynomial algebras. Thus, we can left Kan extend to all p-complete simplicial commutative rings to obtain a functor $R \mapsto \mathrm{Fil}^{\geq\star}\mathrm{TC}(R;\mathbb{Z}_p)$, from SCR to p-complete filtered spectra, which commutes with sifted colimits. We define functors $\mathbb{Z}_p(i)$ on SCR as $\mathbb{Z}_p(i)(R) = \mathrm{gr}^i\mathrm{TC}(R;\mathbb{Z}_p)[-2i]$, or in other words by left Kan extending $\mathbb{Z}_p(i)$ from finitely generated p-complete polynomial algebras.

Once we complete the proof of Theorem 5.1 in the next subsection, it will follow that $\mathrm{Fil}^{\geq i}\mathrm{TC}(R;\mathbb{Z}_p)$ (resp., $\mathbb{Z}_p(i)$) belongs to $\mathrm{Sp}_{\geq i-1}$ (resp., $D^{\leq i+1}(\mathbb{Z}_p)$) for each i, by left Kan extending the connectivity estimate from the quasisyntomic case.

We also emphasize that the above proof of Theorem 5.1(2) shows the following. Given $i \geq 0$, there is $r \gg 0$ such that for all p-complete rings R there is a natural expression

$$\begin{aligned}\mathbb{F}_p(i)(R) & \\ &\simeq \mathrm{fib}\big(\mathrm{can} - \varphi_i : \big(\mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_R\{i\}/\mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\}\big) \otimes^L_{\mathbb{Z}} \mathbb{F}_p \\ &\quad \to \big(\widehat{\mathbf{\Delta}}_R\{i\}/\mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\}\big) \otimes^L_{\mathbb{Z}} \mathbb{F}_p\big), \end{aligned} \tag{43}$$

where the two Nygaard quotients on the right-hand side are defined by left Kan extension from finitely generated p-complete polynomial algebras.

5.5. Proofs of the connectivity bounds (Theorem 5.1(1)) for the $\mathbb{Z}_p(i)$
In this subsection, we complete the proof of Theorem 5.1.

LEMMA 5.34 (Connectivity lemma)
Let $\mathrm{can}, \varphi : \mathrm{Fil}^{\geq\star}M \to \mathrm{Fil}^{\geq\star}N$ *be maps of filtered objects in* $D(\mathbb{Z})$ *(with underlying objects* M, N*). Suppose that:*

(1) *both filtrations are complete;*
(2) *φ induces the zero map on associated graded pieces;*
(3) *there is a fixed r such that, for each s, the induced map* $\mathrm{can}\colon \mathrm{Fil}^{\geq s} M \to \mathrm{Fil}^{\geq s} N$ *has fiber in $D(\mathbb{Z})^{\leq r}$.*

Then $\mathrm{can} - \varphi\colon M \to N$ *has fiber in $D(\mathbb{Z})^{\leq r}$.*

Proof
The fiber $\mathrm{fib}(\mathrm{can} - \varphi\colon M \to N)$ acquires the natural structure of a filtered spectrum, since can, φ are filtered maps. On graded pieces, we find $\mathrm{gr}^s \mathrm{fib}(\mathrm{can} - \varphi\colon M \to N) \simeq \mathrm{gr}^s \mathrm{fib}(\mathrm{can}\colon M \to N)$ since φ vanishes on associated graded terms. In particular, the associated graded terms belong to $D(\mathbb{Z})^{\leq r}$. Since the filtration on $\mathrm{fib}(\mathrm{can} - \varphi)$ is complete, the connectivity assertion on the fiber now follows from the analogous assertion on associated graded terms. □

PROPOSITION 5.35 (The $\mathbb{Z}_p(i)$ connectivity bound for $\mathrm{qSyn}_{\mathbb{Z}_p}$)
Let $R \in \mathrm{qSyn}_{\mathbb{Z}_p}$. Then $\mathbb{Z}_p(i)(R) \in D^{\leq i+1}(\mathbb{Z}_p)$ for each $i \geq 0$.

Proof
First, we recall from the proof of Proposition 5.25 that the inclusion $\mathcal{N}^{\geq i+1}\widehat{\mathbf{\Delta}}_R\{i\} \to \mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_R\{i\}$ has cofiber in $D^{\leq i}(\mathbb{Z}_p)$. Using the resulting cofiber sequence, it thus suffices to show that the fiber of $\mathrm{can} - \varphi_i\colon \mathcal{N}^{\geq i+1}\widehat{\mathbf{\Delta}}_R\{i\} \to \widehat{\mathbf{\Delta}}_R\{i\}$ belongs to $D^{\leq i+1}(\mathbb{Z}_p)$. Since everything is p-complete, it suffices to check this with mod p coefficients.

We consider the two maps

$$\mathrm{can}, \varphi_i : \mathcal{N}^{\geq i+1}\widehat{\mathbf{\Delta}}_R\{i\}/p \to \widehat{\mathbf{\Delta}}_R\{i\}/p.$$

Unfolding Proposition 5.30 shows that these upgrade to maps $\mathrm{can}, \varphi_i : I^s \times \mathcal{N}^{\geq i+1}\widehat{\mathbf{\Delta}}_R\{i\}/p \to I^s\widehat{\mathbf{\Delta}}_R\{i\}/p$ for all $s \geq 0$ (i.e., of I-adically filtered objects), and that the map φ_i acts trivially on associated graded pieces. Furthermore, for each $s \geq 0$, the fiber of $\mathrm{can}\colon I^s\mathcal{N}^{\geq i+1}\widehat{\mathbf{\Delta}}_R\{i\}/p \to I^s\widehat{\mathbf{\Delta}}_R\{i\}/p$ belongs to $D^{\leq i+1}(\mathbb{Z}_p)$ by Proposition 5.25. Lemma 5.34 (whose hypothesis (1) is satisfied by $\tilde{\xi}$-adic completeness in the case of $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$) now shows that $\mathrm{can} - \varphi_i : \mathcal{N}^{\geq i+1}\widehat{\mathbf{\Delta}}_R\{i\} \to \widehat{\mathbf{\Delta}}_R\{i\}$ belongs to $D^{\leq i+1}(\mathbb{Z}_p)$, as desired. □

Proof of Theorem 5.1(1)
We wish to show that $\mathbb{Z}_p(i) \in D^{\leq i+1}(\mathbb{Z}_p)$. But we have already proved part (2) of Theorem 5.1, so the problem reduces to the case of finitely generated p-completely polynomial rings over $\mathbb{Z}_p$, which is covered by Proposition 5.35. □

5.6. Rigidity and proof of Theorem 5.2

In this subsection, we give a proof of Theorem 5.2. Our strategy is to first prove a continuity statement, after which Néron–Popescu and left Kan extension arguments reduce the general case to that of a square-zero extension. In that case, we use the automatic gradings that exist and argue with the pro-nilpotence of Frobenius. Recall that a ring R is said to be *F-finite* if R/p is finitely generated over its subring of pth-powers. The next result is an analogue on graded pieces of [26, Theorem 4.5].

PROPOSITION 5.36 (Continuity)

Let R be a Noetherian, F-finite, p-complete ring, and let $I \subseteq R$ be an ideal such that R is I-adically complete. Then the natural map $\mathbb{Z}_p(i)(R) \to \varprojlim_s \mathbb{Z}_p(i)(R/I^s)$ is an equivalence for any $i \geq 0$.

Proof

From Definition 5.8 and completeness of the Nygaard filtration, it is enough to prove the analogous continuity for each $\mathcal{N}^n \widehat{\mathbf{\Delta}}\{i\} \simeq \mathrm{gr}^n \mathrm{THH}(-;\mathbb{Z}_p)[-2n]$. Then Corollary 5.21 reduces us further to continuity for each $\mathrm{gr}^n\mathrm{THH}(-/\mathbb{S}[z];\mathbb{Z}_p)$, and finally the filtration of Corollary 5.19 reduces the problem to continuity for each $\widehat{\bigwedge^n L_{-/\mathbb{Z}_p}}$.

A cell attachment lemma of André and Quillen (see [69, Theorem 4.4(i)] for a presentation in this context) shows that all cohomology groups of all wedge powers of $L_{(R/I^s)/R}$ are pro zero in s (except H^0 of $\bigwedge^0 L$). So the transitivity fiber sequence shows that $\bigwedge^n L_{R/\mathbb{Z}_p} \otimes^L_R R/I^s \to \bigwedge^n L_{(R/I^s)/\mathbb{Z}_p}$ is a pro isomorphism on all cohomology groups. This reduces the problem to showing that $\bigwedge^n L_{R/\mathbb{Z}_p} \to \varprojlim_s \bigwedge^n L_{R/\mathbb{Z}_p} \otimes^L_R R/I^s$ is an equivalence after p-adic completion. Since R has bounded p-power torsion, this derived p-adic completion may be equivalently computed as $\varprojlim_r(- \otimes^L_R R/p^r)$. Exchanging the limits, it is enough to show that $M \simeq \varprojlim_s M \otimes^L_R R/I^s$, where $M = \bigwedge^n L_{R/\mathbb{Z}_p} \otimes^L_R R/p^r$ for any $n \geq 0$, $r \geq 1$. But this follows from the facts that R is Noetherian, that M is bounded above (cohomologically), and that its cohomology groups are finitely generated R-modules (cf. [26, Theorem 3.6], which uses F-finiteness). □

Remark 5.37

As usual, one can prove stronger continuity statements when $I = (p)$. For example, given a p-complete ring R with bounded p-power torsion, we claim that $\mathbb{Z}_p(i)(R)/p^r \to \{\mathbb{Z}_p(i)(R/p^s)/p^r\}_s$ induces an isomorphism of pro groups on all cohomology groups for any fixed $r \geq 1$. To prove this we reduce to the case $r = 1$ and appeal to (43) (instead of the completeness of the Nygaard filtration used at beginning of the proof of Proposition 5.36) to again reduce to the analogous assertion for graded pieces of the Nygaard filtration. Then argue as in the proof of Proposition 5.36 (the assumption that R has bounded p-power torsion implies that the ideal (p) is pro Tor-unital in the sense of [69], which is needed to verify pro vanishing of $\bigwedge^n L_{(R/p^s)/R})$ to reduce to the analogous statement about $(\bigwedge^n L_{R/\mathbb{Z}_p})/p \to \{(\bigwedge^n L_{R/\mathbb{Z}_p})/p \otimes_R^L R/p^s\}_s$, which follows from boundedness of the p-power torsion in R.

PROPOSITION 5.38
Theorem 5.2 holds in the special case of Henselian pairs of the form $R = A \oplus N$, $I = N$, where A is the p-completion of a finitely generated $\mathbb{Z}_p$-polynomial algebra, N is a finitely generated A-module, and R is the trivial square-zero extension of A by N.

The proof of Proposition 5.38 will be given below. Using Proposition 5.38, we explain how to deduce Theorem 5.2.

Proof of Theorem 5.2
First, note that it is equivalent to prove that the homotopy fiber of Theorem 5.2 mod p belongs to $D^{\leq i}(\mathbb{Z}_p)$; that is, we may replace $\mathbb{Z}_p(i)$ by $\mathbb{F}_p(i) = \mathbb{Z}_p(i)/p$.

We consider the functor on $\mathbb{Z}$-algebras, $R \mapsto F(R) \overset{\text{def}}{=} \mathbb{F}_p(i)(\widehat{R}_p)$ (where $\hat{R}_p$ denotes the derived p-completion of R). By Theorem 5.1, the functor F commutes with filtered colimits in R. It suffices to show that if (R, I) is a Henselian pair, then

$$\operatorname{fib}\bigl(F(R) \to F(R/I)\bigr) \in D^{\leq i}(\mathbb{F}_p). \tag{44}$$

First, we prove (44) in case $I \subset R$ is a nilpotent ideal. By transitivity and an easy induction, it suffices to assume that $I^2 = 0$. Next we apply a standard trick to reduce to the case that $R \to R/I$ is split: choose a simplicial resolution $P_\bullet \to R/I$ by polynomial $\mathbb{Z}$-algebras (possibly in infinitely many variables), and let $Q_\bullet$ be the fiber product along $R \to R/I$. Then each $Q_j \to P_j$ has kernel I and admits a section, since P_j is a polynomial algebra. Taking the geometric realization using Theorem 5.1(2), we thus reduce to proving (44) for each pair $(Q_j = P_j \oplus I, I)$. But we can write P_j as a filtered colimit of polynomial $\mathbb{Z}$-algebras on finitely many variables and I as a filtered colimit of finitely generated modules. Using Theorem 5.1(2) again (which shows that F commutes with filtered colimits), and Proposition 5.38, we conclude (44) for $I \subset R$ nilpotent.

Second, we prove (44) in the case where R is Noetherian and F-finite, and R is I-adically complete. In this case, Proposition 5.36 shows that $F(R) \simeq \varprojlim F(R/I^n)$. We consider the tower (in n), $T_n = \operatorname{fib}(F(R/I^n) \to F(R/I))$; the fiber of each successive map $T_n \to T_{n-1}$ belongs to $D^{\leq i}(\mathbb{F}_p)$, and thus we get that $\varprojlim_n T_n = \operatorname{fib}(F(R) \to F(R/I)) \in D^{\leq i}(\mathbb{F}_p)$, as desired.

Finally, suppose that (R, I) is a general Henselian pair; we prove (44). Since F commutes with filtered colimits, it suffices to assume that the pair (R, I) is the Henselization of a finitely generated $\mathbb{Z}$-algebra R_0 along an ideal $I_0 \subset R_0$. By the previous paragraph, we have $\mathrm{fib}(F(\hat{R}_I) \to F(R/I)) \in D^{\le i}(\mathbb{F}_p)$, since $\hat{R}_I$ is an F-finite, Noetherian ring. Now R_0 is an excellent ring as a finitely generated $\mathbb{Z}$-algebra; since $R_0 \to R$ is ind-étale, R is also excellent (see [39]). It follows that $R \to \widehat{R}_I$ is geometrically regular (see [42, 7.8.4(v)]) and is therefore a filtered colimit of smooth maps by Néron–Popescu desingularization (see [77], [78] [84, Tag 07BW]); each of these maps necessarily admits a section. In particular, the map $\mathrm{fib}(F(R) \to F(R/I)) \to \mathrm{fib}(F(\widehat{R}_I) \to F(R/I))$ is a filtered colimit of maps, each of which admits a section. Since we have just seen that the target of this map belongs to $D^{\le i}(\mathbb{F}_p)$, the source does as well. □

To complete the proof of Theorem 5.2 by treating the pairs in the statement of Proposition 5.38, we will exploit the presence of the grading induced by the variables.

*Remark 5.39 (*THH *for graded rings)*
In the remainder of this subsection, we will systematically use graded objects indexed over a commutative monoid M (which will be $\mathbb{Z}[1/p]_{\ge 0}$ or $\mathbb{Z}_{\ge 0}$): an *M-graded object* in a (∞-)category $\mathcal{C}$ is by definition a functor $R_\star \colon M \to \mathcal{C}$. When $\mathcal{C}$ is symmetric monoidal, then so is the resulting category $\mathrm{Fun}(M, \mathcal{C})$ of M-graded objects, under the Day convolution product, and an *M-graded ring* is then a ($\mathbb{E}_\infty$-, etc.) monoid object in M-graded objects.

The *underlying object* R of a graded object $R_\star$ is by definition $\bigoplus_{m \in M} R_m$, when it exists. When all direct sums exist, the functor $R_\star \mapsto R$ is conservative (and faithful when $\mathcal{C}$ is an ordinary category). We will be particularly interested in *p-complete graded rings*, by which we mean a graded ring $R_\star$ in the category of p-complete abelian groups; the underlying object R is then the p-completed direct sum $\widehat{\bigoplus}_{m \in M} R_m$, which is itself a p-complete ring. We sometimes abusively identify $\widehat{\bigoplus}_{m \in M} R_m$ with $R_\star$ itself; this is a mild abuse of notation given that the functor $R_\star \mapsto \widehat{\bigoplus}_{m \in M} R_m$ is conservative and faithful.

An M-graded commutative ring $R_\star$ may of course be viewed as an M-graded $\mathbb{E}_\infty$-ring in spectra, that is, an $\mathbb{E}_\infty$-monoid in the symmetric monoidal stable ∞-category $\mathrm{Fun}(M, \mathrm{Sp})$. By Appendix A, we may then form the S^1-equivariant object $\mathrm{THH}(R_\star) \in \mathrm{Fun}(M, \mathrm{Sp})^{BS^1}$ and the associated homotopy fixed points $\mathrm{TC}^-(R_\star)$, homotopy orbits $\mathrm{THH}(R_\star)_{hS^1}$, and Tate construction $\mathrm{TP}(R_\star)$, all of which are M-graded spectra. Note that the underlying spectrum of $\mathrm{THH}(R_\star)$ is the THH of the underlying ring spectrum of $R_\star$ because the underlying spectrum functor preserves tensor products and colimits; this is not true for TC^-, TP because the underlying spectrum functor need not preserve limits (e.g., S^1-homotopy fixed points).

Construction 5.40 ([12] for graded rings)
Assume that the monoid M is uniquely p-divisible, such as $\mathbb{Z}[1/p]_{\geq 0}$. Then the main constructions and results of [12] extend to M-graded rings.

We will say that a p-complete M-graded ring $R_\star$ is *quasisyntomic* (resp., *quasiregular semiperfectoid*) if the underlying ring $R = \widehat{\bigoplus}_{m \in M} R_m$ is quasisyntomic (resp., quasiregular semiperfectoid). One has a natural graded analogue of the quasisyntomic site, and similarly quasiregular semiperfectoids form a basis (e.g., by extracting p-power roots of homogeneous elements as in [12, Lemma 4.27]; this is why p-divisibility of M is required); one obtains an analogue of unfolding in this context.

For such $R_\star$, we have seen in Remark 5.39 that we have natural M-graded spectra, $\mathrm{THH}(R_\star; \mathbb{Z}_p)$, $\mathrm{TC}^-(R_\star; \mathbb{Z}_p)$, $\mathrm{THH}(R_\star; \mathbb{Z}_p)_{hS^1}$, and $\mathrm{TP}(R_\star; \mathbb{Z}_p)$. Moreover, the latter are naturally filtered objects in M-graded p-complete spectra, by carrying over the construction of the motivic filtration of [12] to the graded context. It follows that we get M-graded p-complete objects $\widehat{\mathbf{\Delta}}_{R_\star}\{i\}$, $\mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_{R_\star}\{i\}$, and so on. In general, the underlying (p-complete) objects of $\widehat{\mathbf{\Delta}}_{R_\star}\{i\}$, $\mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_{R_\star}\{i\}$ do *not* agree with those of $\widehat{\mathbf{\Delta}}_R\{i\}$, $\mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_R\{i\}$, because the underlying object of $\mathrm{TC}^-(R_\star; \mathbb{Z}_p)$ is not $\mathrm{TC}^-(R; \mathbb{Z}_p)$. However, for each $j \geq i$, the underlying p-complete object of $\mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_{R_\star}\{i\}/\mathcal{N}^{\geq j}\widehat{\mathbf{\Delta}}_{R_\star}\{i\}$ is $\mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_R\{i\}/\mathcal{N}^{\geq j}\widehat{\mathbf{\Delta}}_R\{i\}$. This follows because the underlying object of $\mathrm{THH}(R_\star; \mathbb{Z}_p)$ is $\mathrm{THH}(R; \mathbb{Z}_p)$ and the forgetful functor from M-graded spectra to spectra commutes with *finite* homotopy limits.

Throughout [12], a basic tool is the cotangent complex and its wedge powers; here we implicitly use that if $R_\star$ is a p-complete M-graded ring, then we have natural p-complete graded objects $\bigwedge^i L_{R_\star/\mathbb{Z}_p}$ (defined in the usual manner as a left derived functor of differential forms). The Hochschild–Kostant–Rosenberg theorem remains valid in the graded context, and from there the results of Section 5.2 carry over to this context as well.

We now turn to the interaction of the Frobenius with the grading, in the case which interests us.

PROPOSITION 5.41 (Frobenius multiplies grading by p)
Let $R_\star$ be a quasisyntomic $\mathbb{Z}[1/p]_{\geq 0}$-graded (resp., $\mathbb{Z}_{\geq 0}$-graded) ring (with underlying quasisyntomic ring R). Then we claim that:
(1) $\widehat{\mathbf{\Delta}}_R\{i\}/\mathcal{N}^{\geq n}\widehat{\mathbf{\Delta}}_R\{i\}$ *naturally upgrades to have the structure of a $\mathbb{Z}[1/p]_{\geq 0}$- (resp., $\mathbb{Z}_{\geq 0}$-) graded object in the p-complete derived ∞-category $\widehat{D(\mathbb{Z}_p)}$;*

(2) *the Frobenius map modulo p, as in (43)*

$$\varphi_i : \mathcal{N}^{\geq i} \widehat{\mathbf{\Delta}}_R\{i\}/\mathcal{N}^{i+r} \widehat{\mathbf{\Delta}}_R\{i\} \otimes^L_{\mathbb{Z}} \mathbb{F}_p \to \widehat{\mathbf{\Delta}}_R\{i\}/\mathcal{N}^{\geq i+r} \widehat{\mathbf{\Delta}}_R\{i\} \otimes^L_{\mathbb{Z}} \mathbb{F}_p$$

multiplies degrees by p.

Proof

In the case in which $R_\star$ is $\mathbb{Z}[1/p]_{\geq 0}$-graded, part (1) is covered by the general Construction 5.40: arguing locally on the graded version of the quasisyntomic site, we require a natural graded version of Definition 5.6 for any graded quasiregular semiperfectoid ring. But this follows from THH of a graded ring being a graded spectrum with S^1-action, as explained in Remark 5.39.

When R is actually $\mathbb{Z}_{\geq 0}$-graded, we claim that the same is true of $\widehat{\mathbf{\Delta}}_R\{i\}/\mathcal{N}^{\geq n} \widehat{\mathbf{\Delta}}_R\{i\}$. By dévissage, it suffices to prove this for each $\mathcal{N}^t \widehat{\mathbf{\Delta}}_R$, which in turn follows from the filtrations of Corollaries 5.19 and 5.21. Here we use that the p-completed cotangent complex $\widehat{L_{R/\mathbb{Z}_p}}$ and its wedge powers are naturally $\mathbb{Z}_{\geq 0}$-graded, and the filtrations of the aforementioned corollaries respect these gradings.

For part (2) we apply a left Kan extension argument to assume that R is p-torsion-free, and then argue locally as above to reduce to the case that R is a p-torsion-free, graded, quasiregular semiperfectoid. Then both sides of φ_i are discrete, and so the problem reduces to verifying the *property* that it multiplies degrees by p. But this follows from the treatment of graded cyclotomic spectra in Appendix A. ☐

Proof of Proposition 5.38

Let $R = A \oplus N$ and $I = N$ be a Henselian pair of the form of Proposition 5.38. We view R as being $\mathbb{Z}_{\geq 0}$-graded, with A in degree 0 and N in degree 1. We must prove that $\mathrm{fib}(\mathbb{F}_p(i)(R) \to \mathbb{F}_p(i)(A)) \in D^{\leq i}(\mathbb{F}_p)$. Choose $r \gg 0$ so that expression (43) is valid.

The degree 0 part of $\widehat{\bigwedge^i L_{R/\mathbb{Z}_p}}$ identifies with $\widehat{\bigwedge^i L_{A/\mathbb{Z}_p}}$, so by dévissage using Corollary 5.19 and Corollary 5.21, we see that the same is true of the $\mathbb{Z}_{\geq 0}$-graded object $\widehat{\mathbf{\Delta}}_R\{i\}/\mathcal{N}^{\geq i+r} \widehat{\mathbf{\Delta}}_R\{i\}$. Namely, its degree 0 part is $\widehat{\mathbf{\Delta}}_A\{i\}/\mathcal{N}^{\geq i+r} \widehat{\mathbf{\Delta}}_A\{i\}$. The same is true modulo p, and so $\mathrm{fib}(\mathbb{F}_p(i)(R) \to \mathbb{F}_p(i)(A))$ identifies with the fiber of

$$\begin{aligned} \mathrm{can} - \varphi_i : \big(\mathcal{N}^{\geq i} \widehat{\mathbf{\Delta}}_R\{i\}/\mathcal{N}^{\geq i+r} \widehat{\mathbf{\Delta}}_R\{i\}\big)_{>0} \otimes^L_{\mathbb{Z}} \mathbb{F}_p \\ \to \big(\widehat{\mathbf{\Delta}}_R\{i\}/\mathcal{N}^{\geq i+r} \widehat{\mathbf{\Delta}}_R\{i\}\big)_{>0} \otimes^L_{\mathbb{Z}} \mathbb{F}_p, \end{aligned}$$

where the subscript > 0 denotes the $\mathbb{Z}_{>0}$-subobject of a $\mathbb{Z}_{\geq 0}$-graded object.

To complete the proof, we must verify the conditions of Lemma 5.42 below. First, the Frobenius multiplies degrees by p by Proposition 5.41(2). Next, the fiber of can is $(\widehat{\mathbf{\Delta}}_R\{i\}/\mathcal{N}^{\geq i} \widehat{\mathbf{\Delta}}_R\{i\})_{>0}[-1]$, which lies in $D^{\leq i}(\mathbb{Z}_p)$ by Proposition 5.25. Finally, to verify condition (2), observe that each cohomology group of $\widehat{\bigwedge^i L_{R/\mathbb{Z}_p}}$ is a $\mathbb{Z}_{\geq 0}$-graded, finitely generated R-module, and so necessarily zero, except in finitely many degrees of the grading. The same then holds for $\widehat{\mathbf{\Delta}}_R\{i\}/\mathcal{N}^{\geq i+r}\widehat{\mathbf{\Delta}}_R\{i\}$ by another dévissage through Corollaries 5.19 and 5.21. ☐

LEMMA 5.42
Let M, N be $\mathbb{Z}_{>0}$-graded objects of $D(\mathbb{F}_p)$, and let $i \geq 0$. Let $\mathrm{can}\colon M \to N$ *be a map of graded objects, and let $\varphi\colon M \to N$ be a map which multiplies degrees by p. Suppose that:*
(1) *the fiber of* can *belongs to $D(\mathbb{F}_p)^{\leq i}$;*
(2) *for any fixed n, the cohomologies $H^n(M)$, $H^n(N)$ vanish except in finitely many degrees of the grading.*

Then $\mathrm{fib}(\mathrm{can} - \varphi\colon M \to N)$ *belongs to $D(\mathbb{F}_p)^{\leq i}$.*

Proof
By (2), we can replace the direct sums $\bigoplus M_i$ and $\bigoplus N_i$ with the corresponding infinite products. Therefore, the result follows from Lemma 5.34. ☐

Finally, we use the rigidity theorem to give a description of the top cohomology of the $\mathbb{F}_p(i)$. For B an $\mathbb{F}_p$-algebra, let Ω^i_B be the (underived) module of i-forms on B (relative to $\mathbb{Z}_p$, or $\mathbb{F}_p$), and let

$$C^{-1}\colon \Omega^i_B \to \Omega^i_B/d\Omega^{i-1}_B$$

be the inverse Cartier operator.

COROLLARY 5.43 (Top cohomology of $\mathbb{F}_p(i)$)
Let R be a p-complete ring. Then there is a natural isomorphism

$$H^{i+1}\big(\mathbb{F}_p(i)(R)\big) \simeq \mathrm{coker}(1 - C^{-1}\colon \Omega^i_{R/p} \to \Omega^i_{R/p}/d\Omega^{i-1}_{R/p}). \tag{45}$$

In particular, if R is w-strictly local, then $H^{i+1}(\mathbb{F}_p(i)(R)) = H^{i+1}(\mathbb{Z}_p(i)(R)) = 0$.

Proof
Without loss of generality, we can assume that R is an $\mathbb{F}_p$-algebra via Theorem 5.2. We now use a standard argument (cf., e.g., [50]) to reduce to the case where R is ind-smooth over $\mathbb{F}_p$. Choose a polynomial $\mathbb{F}_p$-algebra P surjecting onto R and Henselize P along the kernel of $P \twoheadrightarrow R$, forming a P-algebra P' augmented over R; another use of Theorem 5.2 gives $H^{i+1}(\mathbb{F}_p(i)(R)) = H^{i+1}(\mathbb{F}_p(i)(P'))$. Moreover, [23, Proposition 4.31] gives that the right-hand-side of equation (45) is also invariant

under passage from P' to R. Replacing R by P', we thus reduce the case that R is ind-smooth over $\mathbb{F}_p$. But then

$$\mathbb{F}_p(i)(R) \simeq \mathrm{fib}(\Omega^i_R \xrightarrow{1-C^{-1}} \Omega^i_R/d\Omega^{i-1}_R)[-i]. \tag{46}$$

This follows because of the expression of $\mathbb{F}_p(i)$ as the shifted étale cohomology of logarithmic forms $\Omega^i_{\log}$ (cf. [12, Corollary 8.21] and reduction modulo p; we also review this in the next section) and the short exact sequence of étale sheaves (cf. [53, Section 2.4] and [70, Corollary 4.2])

$$0 \to \Omega^i_{\log} \to \Omega^i \xrightarrow{1-C^{-1}} \Omega^i/d\Omega^{i-1} \to 0.$$

Expression (46) implies the claim. Note that this short exact sequence (in the étale topology) implies also that, for any ind-smooth $\mathbb{F}_p$-algebra which has no nonsplit étale covers (e.g., R could be w-strictly local), $H^{i+1}(\mathbb{F}_p(i)(R)) = 0$. □

Theorem 5.1, Theorem 5.2, and Corollary 5.43 imply Theorem G from the introduction.

6. The comparison with syntomic cohomology

In this section, we show that the $\mathbb{Z}_p(i)$ for $i \le p-2$ and the $\mathbb{Q}_p(i)$ for all i can be described purely in terms of derived de Rham (instead of prismatic) cohomology, using a form of syntomic cohomology (see [31], [54]). The strategy is to use the description of the $\mathbb{Z}_p(i)$ in equal characteristic p from [12] together with the Beilinson fiber square to relate the $\mathbb{Z}_p(i)$ in mixed and equal characteristic. In particular, we prove Theorem F.

6.1. Syntomic cohomology

To begin, we define another form of syntomic cohomology via the quasisyntomic site, by descent from quasiregular semiperfectoids.

Definition 6.1 (p-adic derived de Rham cohomology)

For a map of rings $A \to R$, we let $L\Omega_{R/A} \in D(A)$ denote the *p-adic derived de Rham cohomology* of R relative to A (see [10]). By definition, when R is a finitely generated polynomial A-algebra, $L\Omega_{R/A}$ is given by the p-completed relative de Rham complex $\Omega^\bullet_{R/A}$, and, in general, $L\Omega_{R/A}$ is defined via p-complete left Kan extension. One can show [10, Corollary 3.10] that $L\Omega_{R/A}$ more generally agrees with the p-completed (underived) relative de Rham complex when R is smooth over A. When $A = \mathbb{Z}$, we omit A from the notation.

The p-adic derived de Rham cohomology $L\Omega_{R/A}$ is equipped with the derived Hodge filtration $L\Omega^{\geq\star}_{R/A}$, obtained by left Kan extending the naive filtration in the polynomial (or more generally smooth) case.

Example 6.2 (Derived de Rham cohomology and divided powers)
We recall the following basic calculation: for the map $\mathbb{Z}[x] \to \mathbb{Z}$, we have that $L\Omega_{\mathbb{Z}/\mathbb{Z}[x]} \simeq \widehat{\Gamma(x)}$ is the p-complete divided power algebra on the class x, and the derived Hodge filtration is the divided power filtration. This is a special case of [10, Corollary 3.40]. See also [85, Proposition 3.16] for an account.

Definition 6.3 (Derived de Rham–Witt cohomology)
For an $\mathbb{F}_p$-algebra S, we let $LW\Omega_S \in D(\mathbb{Z}_p)$ denote the p-adic derived de Rham–Witt cohomology or derived crystalline cohomology of S (defined via p-complete left Kan extension from finitely generated polynomial $\mathbb{F}_p$-algebras; see [12, Section 8]); for ind-smooth $\mathbb{F}_p$-algebras S, this agrees with Illusie's usual $W\Omega_S$. It comes equipped with the derived Nygaard filtration $\mathcal{N}^{\geq\star}LW\Omega_S$ obtained via left Kan extension of the usual Nygaard filtration in the finitely generated polynomial case (see [11, Section 8] for an account); we write $\widehat{LW\Omega}_S$ for the completion of $LW\Omega_S$ with respect to the Nygaard filtration. We have by [12, Lemma 8.2] an identification of the associated graded terms of the Nygaard filtration $\mathcal{N}^{\geq i}LW\Omega_S/\mathcal{N}^{\geq i+1}LW\Omega_S \simeq L(\tau^{\leq i}\Omega_{S/\mathbb{F}_p})$ with the derived functor of $S \mapsto \tau^{\leq i}\Omega_{S/\mathbb{F}_p}$. The Frobenius $\varphi\colon S \to S$ induces an endomorphism of $LW\Omega_S$; on $\mathcal{N}^{\geq i}LW\Omega_S$, it becomes divisible by p^i, and indeed we have divided Frobenius maps

$$\varphi_i : \mathcal{N}^{\geq i}LW\Omega_S \to LW\Omega_S. \tag{47}$$

Finally, using the de Rham–to–crystalline comparison, we find that if R is a p-torsion-free ring, then there is a natural equivalence

$$L\Omega_R \simeq LW\Omega_{R/p}; \tag{48}$$

in particular, $L\Omega_R$ naturally carries a Frobenius operator φ.

Remark 6.4 (Sheaf properties)
The functors $S \mapsto LW\Omega_S$ and $S \mapsto \widehat{LW\Omega}_S$ are sheaves on $\mathrm{qSyn}_{\mathbb{F}_p}$. For $LW\Omega_S$, it suffices to work modulo p, and then use the conjugate filtration on derived de Rham cohomology (see [10]) and the flat descent for the wedge powers of the cotangent complex (see [12, Section 3]). For the Nygaard-completed $\widehat{LW\Omega}_S$, this follows since the associated graded terms have this property, by a similar argument. Similarly, the filtration pieces $S \mapsto \mathcal{N}^{\geq i}LW\Omega_S$ and $S \mapsto \mathcal{N}^{\geq i}\widehat{LW\Omega}_S$ are sheaves on $\mathrm{qSyn}_{\mathbb{F}_p}$.

Construction 6.5 (Derived de Rham–Witt cohomology of quasiregular semiperfect rings; cf. [12, Section 8.2])
Let $S \in \mathrm{qrsPerfd}_{\mathbb{F}_p}$ be a quasiregular semiperfect $\mathbb{F}_p$-algebra. In this case, one forms the ring $\mathrm{A}_{\mathrm{crys}}(S)$ (defined by Fontaine [30]), which is the universal p-adically complete divided power thickening of S, with divided powers compatible with those on $(p) \subset \mathbb{Z}_p$; quasiregularity ensures that it is p-torsion-free. Then, one has a natural identification

$$LW\Omega_S = \mathrm{A}_{\mathrm{crys}}(S),$$

and the Nygaard filtration becomes the filtration

$$\mathcal{N}^{\geq i}\mathrm{A}_{\mathrm{crys}}(S) = \left\{x \in \mathrm{A}_{\mathrm{crys}}(S) \colon \varphi(x) \in p^i \mathrm{A}_{\mathrm{crys}}(S)\right\}.$$

Here $\varphi\colon \mathrm{A}_{\mathrm{crys}}(S) \to \mathrm{A}_{\mathrm{crys}}(S)$ denotes the endomorphism induced by the Frobenius on S; it has the further property that $\varphi(x) \equiv x^p \pmod p$ for $x \in \mathrm{A}_{\mathrm{crys}}(S)$; that is, φ defines the structure of a δ-ring on the p-torsion-free ring $\mathrm{A}_{\mathrm{crys}}(S)$.

Construction 6.6 (Derived de Rham and de Rham–Witt cohomology of quasiregular semiperfect rings)
Let $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$. We consider the following rings.

(1) The derived de Rham–Witt cohomology $LW\Omega_{R/p}$ of R/p. Since the ring R/p is a quasiregular semiperfect $\mathbb{F}_p$-algebra, it follows from Construction 6.5 that there is an isomorphism

$$LW\Omega_{R/p} \simeq \mathrm{A}_{\mathrm{crys}}(R/p).$$

(2) The (p-adic) derived de Rham cohomology $L\Omega_R$. Here $L\Omega_R$ is a discrete, p-complete and p-torsion-free ring, as follows from the previous point and (48). The ring $L\Omega_R$ is also equipped with the multiplicative, descending Hodge filtration $L\Omega_R^{\geq \star}$.

Via the de Rham–to–crystalline comparison, we have equivalences $L\Omega_R \simeq LW\Omega_{R/p} \simeq \mathrm{A}_{\mathrm{crys}}(R/p)$. In particular, we find via (1) and (2) above that the p-complete, p-torsion-free ring $L\Omega_R$ is equipped with both a Frobenius operator and a Hodge filtration.

LEMMA 6.7
Let $r \geq 0$ be an integer.

(1) *The p-adic valuation of $\frac{(pr)!}{r!}$ is equal to r.*

(2) *The p-adic valuation of $\frac{p^r}{r!}$ is at least* $\min(r, p-1)$.

Proof
Both assertions follow from Legendre's formula, $v_p(n!) = \sum_{j>0} \lfloor n/p^j \rfloor$ for v_p the p-adic valuation. $\square$

PROPOSITION 6.8 (Divisibility of Frobenius; cf. also [87, Lemma A1.4])
Let $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$. Then for $i \leq p-1$, the Frobenius $\varphi\colon L\Omega_R \to L\Omega_R$ carries $L\Omega_R^{\geq i}$ into $p^i L\Omega_R$, or, in other words, the de Rham–to–crystalline comparison carries $L\Omega_R^{\geq i}$ into $\mathcal{N}^{\geq i}\mathrm{A}_{\mathrm{crys}}(R/p)$.

Proof
For any $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$, we can write $R = W(A)/I$, where A is a perfect $\mathbb{F}_p$-algebra and $I \subset W(A)$ is an ideal. We have an identification of the p-complete cotangent complex, $\widehat{L_{R/\mathbb{Z}_p}} \simeq \widehat{I/I^2}[1]$.

We first verify the assertion when the ideal I as above can be written as $I = (f)$, for f a non-zero-divisor, so $R = W(A)/(f)$. In this case, in view of Example 6.2 and base change, we find that $L\Omega_R = L\Omega_{R/W(A)}$ is the p-completion of the divided power envelope of the regular ideal (f), that is, the ring $W(A)[f^n/n!]_{n\geq 1}$; furthermore, for each i, the Hodge filtered piece $L\Omega_R^{\geq i}$ identifies with the corresponding divided power filtration, that is, the ideal $(f^j/j!)_{j\geq i}$. Now the Frobenius φ on $L\Omega_R \simeq LW\Omega_{R/p} \simeq \mathrm{A}_{\mathrm{crys}}(R/p)$ is a Frobenius lift coming from a δ-structure, so

$$\varphi\Big(\frac{f^j}{j!}\Big) = \frac{(f^p + p\delta(f))^j}{j!} = \frac{\sum_{0\leq l\leq j}\binom{j}{l} f^{pl} p^{j-l}\delta(f)^{j-l}}{j!}. \tag{49}$$

The lth term in the sum above is divisible (in the ring $L\Omega_R$) by $\frac{f^{pl}p^{j-l}}{l!(j-l)!} = \frac{f^{pl}}{(pl)!}\frac{(pl)!p^{j-l}}{l!(j-l)!}$, where we use the divided powers on (f) to see $\frac{f^{pl}}{(pl)!} \in L\Omega_R$. Now the p-adic valuation of $\frac{(pl)!p^{j-l}}{l!(j-l)!}$ is at least $l + \min(j-l, p-1)$ thanks to Lemma 6.7. So if $i \leq p-1$, then it follows that φ carries $L\Omega_R^{\geq i}$ into $p^i L\Omega_R$.

Now suppose that R is a p-complete tensor product over $\alpha \in \mathcal{A}$ of rings of the form $W(A_\alpha)/(f_\alpha)$, for A_α perfect $\mathbb{F}_p$-algebras and $f_\alpha \in W(A_\alpha)$ regular elements. In this case, we have an isomorphism (after p-completion) of filtered rings $L\Omega_R^{\geq\star} \simeq \bigotimes_{\alpha\in\mathcal{A}} L\Omega^{\geq\star}_{W(A_\alpha)/f_\alpha}$ by the Künneth formula, which is compatible with the Frobenius operators. The assertion $\varphi(L\Omega_R^{\geq i}) \subset p^i L\Omega_R$ for $i \leq p-1$ for such R thus follows by taking tensor products.

Finally, let $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$ be arbitrary, and write $R = W(A)/I$ for A a perfect $\mathbb{F}_p$-algebra. To prove the claim $\varphi(L\Omega_R^{\geq i}) \subset p^i L\Omega_R$ for $i \leq p-1$, we will reduce to the previous cases, following the strategy of [12, Theorem 8.14]. Let $\{x_t\}_{t\in T}$ be a system of generators for the ideal I and for each t, we write $x_t = \sum_{i\geq 0} p^i [y_{t,i}]$ for some $y_{t,i} \in A$. For each $t \in T$, we have a map

$$W\left(\mathbb{F}_p[u_1, u_2, \ldots,]_{\text{perf}}\right)/\left([u_1] + p[u_2] + \cdots\right) \to W(A)/I = R \tag{50}$$

sending $[u_i] \mapsto [y_{t,i}]$; note that the source belongs to $\text{qrsPerfd}_{\mathbb{Z}_p}$, and its cotangent complex is the shift of a free module of rank 1. The map (50) has image on p-completed cotangent complexes given by the class of x_t.

We consider the p-completed tensor product

$$R' \stackrel{\text{def}}{=} W(A) \hat{\otimes} \bigotimes_{t \in T} \left(W\left(\mathbb{F}_p[u_1, u_2, \ldots,]_{\text{perf}}\right)/[u_1] + p[u_2] + \cdots\right),$$

which maps surjectively to R (via the above maps) and induces a surjection on $H^0(\widehat{L_{-/\mathbb{Z}_p}}[-1])$. Comparing with the reduction mod p and using the Hodge and conjugate filtrations on derived de Rham cohomology, we find that $L\Omega^{\geq i}_{R'} \to L\Omega^{\geq i}_R$ is a surjection for each i. Since the previous discussion shows that $\varphi(L\Omega^{\geq i}_{R'}) \subset p^i L\Omega_{R'}$ for $i \leq p-1$, we can now conclude the claim for R by naturality, as desired. $\square$

Using the divisibility property of Frobenius, we can define, for $R \in \text{qrsPerfd}_{\mathbb{Z}_p}$ and $i \leq p-1$, a *divided Frobenius* $\varphi/p^i : L\Omega^{\geq i}_R \to L\Omega_R$ (of discrete, p-torsion-free abelian groups). Using the divided Frobenius, we now define syntomic cohomology; this definition is based on the ideas of [31] and [54] (and can be compared with it using the comparison between derived de Rham and crystalline cohomology in the local complete intersection case; cf. [10, Section 3]).

Definition 6.9 (Syntomic cohomology)
We define sheaves $\mathbb{Z}_p(i)^{\text{FM}}$ for $0 \leq i \leq p-2$, and $\mathbb{Q}_p(i)^{\text{FM}}$ for $i \geq 0$, on $\text{qrsPerfd}_{\mathbb{Z}_p}$ via

$$\mathbb{Z}_p(i)^{\text{FM}}(R) = \text{fib}(\varphi/p^i - 1 \colon L\Omega^{\geq i}_R \to L\Omega_R), \tag{51}$$

$$\mathbb{Q}_p(i)^{\text{FM}}(R) = \text{fib}(\varphi - p^i \colon L\Omega^{\geq i}_R \to L\Omega_R)_{\mathbb{Q}_p}. \tag{52}$$

These are sheaves on $\text{qrsPerfd}_{\mathbb{Z}_p}$ because $R \mapsto L\Omega^{\geq i}_R$ is a sheaf. Unfolding, we obtain sheaves $\mathbb{Z}_p(i)^{\text{FM}}$ for $0 \leq i \leq p-2$ and $\mathbb{Q}_p(i)^{\text{FM}}$ for all $i \geq 0$ on $\text{qSyn}_{\mathbb{Z}_p}$.

Remark 6.10
While one could define $\mathbb{Z}_p(p-1)^{\text{FM}}(R)$ via the same formula, this does not give the correct integral theory in weight $(p-1)$.

6.2. The $\mathbb{Z}_p(i)$ in equal characteristic p

In equal characteristic p, the $\mathbb{Z}_p(i)$ can be determined via the theory of the de Rham–Witt complex and its derived versions (cf. [52, Section VIII.2], [10], and, in particular, [12, Section 8]).[12] We review this next.

THEOREM 6.11 ($\mathbb{Z}_p(i)$ in equal characteristic p; cf. [12, Section 8])
Suppose that S is a quasisyntomic $\mathbb{F}_p$-algebra. Then, for each i:

(1) *$\widehat{\mathbf{\Delta}}_S\{i\}$ is the Nygaard-completed derived de Rham–Witt cohomology $\widehat{LW\Omega}_S$ of S;*

(2) *the Nygaard filtration $\mathcal{N}^{\geq i}\widehat{\mathbf{\Delta}}_S\{i\}$ identifies with the de Rham–Witt Nygaard filtration $\mathcal{N}^{\geq i}\widehat{LW\Omega}_S$, and the prismatic Frobenius φ_i identifies with the divided Frobenius (47).*

Consequently,

$$\mathbb{Z}_p(i)(S) = \operatorname{fib}(\varphi_i - \mathrm{can}\colon \mathcal{N}^{\geq i}\widehat{LW\Omega}_S \to \widehat{LW\Omega}_S), \tag{53}$$

where $\varphi_i : \mathcal{N}^{\geq i}\widehat{LW\Omega}_S \to \widehat{LW\Omega}_S$ is the divided Frobenius operator, so that $p^i\varphi_i$ is the Frobenius.

Remark 6.12
The Nygaard completion is redundant in the formula (53) for $\mathbb{Z}_p(i)(S)$. This follows easily from the fact that φ_i acts by zero on $\mathcal{N}^{\geq i+1}\widehat{LW\Omega}_S/p$. In particular, we can write

$$\begin{aligned}\mathbb{F}_p(i)(S) = \operatorname{fib}\big(\varphi_i - \mathrm{can}\colon & \\ &(\mathcal{N}^{\geq i}LW\Omega_S/\mathcal{N}^{\geq i+1}LW\Omega_S)\otimes^L_{\mathbb{Z}_p}\mathbb{F}_p \\ &\to (LW\Omega_S/\mathcal{N}^{\geq i+1}LW\Omega_S)\otimes^L_{\mathbb{Z}_p}\mathbb{F}_p\big).\end{aligned}$$

Example 6.13
If S is a quasiregular semiperfect $\mathbb{F}_p$-algebra, then, for $i > 0$, $\mathbb{Z}_p(i)(S)$ is discrete and p-torsion-free, and there is a natural isomorphism of abelian groups

$$\mathbb{Z}_p(i)(S) \simeq \ker\big(\varphi - p^i : \mathrm{A}_{\mathrm{crys}}(S) \to \mathrm{A}_{\mathrm{crys}}(S)\big), \tag{54}$$

where φ is induced by the Frobenius. For $i = 0$, we should instead take the homotopy fiber of $\varphi - 1$ on $\mathrm{A}_{\mathrm{crys}}(S)$, so it may have terms in cohomological degree 1.

In the ind-smooth case, one has an identification with logarithmic de Rham–Witt forms.

[12] See also [34], [37] for the identification with with p-adic étale motivic cohomology.

Definition 6.14 (Logarithmic de Rham–Witt forms)
For S an ind-smooth $\mathbb{F}_p$-algebra, we let $W\Omega^\bullet_{S,\log}$ denote the graded subring of the de Rham–Witt complex $W\Omega^\bullet_S$ consisting of fixed points for F. When S is local, one knows that $W\Omega^\bullet_{S,\log}$ is generated, modulo any power of p, in degree 1 by elements of the form $d[x]/[x]$, for $x \in S^\times$ and $[x] \in W(S)$ the Teichmüller representative (cf. [53, Theorem 5.7.2], which proves this étale locally, and [70, Theorem 0.10] for a very general Zariski local result). Note that for each i, $W\Omega^i_{-,\log}$ defines a pro-étale sheaf on $\mathrm{Spec}(S)$.

THEOREM 6.15 (Cf. [12, Cor. 8.21])
Let S be an ind-smooth $\mathbb{F}_p$-algebra. Then there are natural identifications

$$\mathbb{Z}_p(i)(S) \simeq R\Gamma_{\mathrm{proet}}\big(\mathrm{Spec}(S), W\Omega^i_{-,\log}\big)[-i].$$

6.3. *The Beilinson fiber square on graded pieces*

Our goal is to relate the $\mathbb{Z}_p(i)$ in mixed and in equal characteristic. We use the Beilinson fiber sequence to prove a basic fiber square which gives a version of Theorem A on associated graded terms for the motivic filtrations.

Construction 6.16 (The trace on graded pieces)
Let R be any commutative ring. Then we have the trace maps

$$\mathrm{K}(R;\mathbb{Z}_p) \to \mathrm{TC}(R;\mathbb{Z}_p) \to \mathrm{HC}^-(R;\mathbb{Z}_p).$$

When $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$, we have that $\mathrm{HC}^-(R;\mathbb{Z}_p)$ is concentrated in even degrees and π_{2i} is given by $\widehat{L\Omega}^{\geq i}_R$ (cf. [12, Section 5] and [1]). Unfolding, we conclude that, on graded pieces, we obtain a natural map $\mathbb{Z}_p(i)(R) \to \widehat{L\Omega}^{\geq i}_R$ for $R \in \mathrm{qSyn}_{\mathbb{Z}_p}$. This naturally factors through $L\Omega^{\geq i}_R$ since $R \mapsto \mathbb{Z}_p(i)(R)$ is left Kan extended from p-complete polynomial algebras (Theorem 5.1).

THEOREM 6.17 (The Beilinson fiber square on graded terms)
Let $R \in \mathrm{qSyn}_{\mathbb{Z}_p}$. Then, for each $i \geq 0$, there exists a natural map $\chi_i : \mathbb{Q}_p(i)(R/p) \to (L\Omega_R)_{\mathbb{Q}_p}$ and a functorial pullback square

$$\begin{tikzcd}
\mathbb{Q}_p(i)(R) \arrow[r] \arrow[d] & \mathbb{Q}_p(i)(R/p) \arrow[d, "\chi_i"] \\
(L\Omega^{\geq i}_R)_{\mathbb{Q}_p} \arrow[r] & (L\Omega_R)_{\mathbb{Q}_p}
\end{tikzcd} \tag{55}$$

in the derived ∞-category $D(\mathbb{Q}_p)$. The map χ_i arises from a natural map $\mathbb{Z}_p(i)(R/p) \to p^{-N} L\Omega_R$ for some $N \gg 0$ (depending only on i), fitting into an analogous commutative diagram.

Furthermore, the associated fiber sequence holds up to isogeny: $\mathrm{cofib}(\mathbb{Z}_p(i)(R) \to \mathbb{Z}_p(i)(R/p))$ *and $L\Omega_R/L\Omega_R^{\geq i}$ are naturally isogenous to each other. Finally, for $i \leq p-2$, we have natural equivalences for* $\mathrm{qSyn}_{\mathbb{Z}_p}$,

$$\mathrm{fib}\big(\mathbb{Z}_p(i)(R) \to \mathbb{Z}_p(i)(R/p)\big) \simeq \mathrm{fib}(L\Omega_R/L\Omega_R^{\geq i} \to L\Omega_{R/p}/L\Omega_{R/p}^{\geq i})[-1]. \quad (56)$$

Proof

Note that the hypothesis that $R \in \mathrm{qSyn}_{\mathbb{Z}_p}$ ensures that $R/p \in \mathrm{QSyn}$. The square (55) will be constructed in the ∞-category of $D(\mathbb{Q}_p)^{\geq 0}$-valued sheaves on $\mathrm{qSyn}_{\mathbb{Z}_p}$. It suffices to construct the above pullback square for $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$, by unfolding. For $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$, we have a pullback square by Corollary 3.9:

$$\begin{tikzcd}
\mathrm{TC}(R;\mathbb{Q}_p) \arrow[r] \arrow[d] & \mathrm{TC}(R/p;\mathbb{Q}_p) \arrow[d] \\
\mathrm{HC}^-(R;\mathbb{Q}_p) \arrow[r] & \mathrm{HP}(R;\mathbb{Q}_p).
\end{tikzcd} \quad (57)$$

Note since $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$, the terms in the bottom row of the above fiber square are concentrated in even degrees (see [12, Lemma 5.14]). Consequently, for each i, we can apply $\tau_{[2i-1,2i]}$ and still obtain a fiber square. By definition of the $\mathbb{Q}_p(i)$ and by the corresponding description of derived de Rham cohomology, as in [12, Theorem 1.17] (or using the filtration of [1]), we obtain (55) for R (after a shift), albeit with a Hodge completion. In particular, instead of χ_i, we obtain a completed version

$$\hat{\chi}_i : \mathbb{Q}_p(i)(R/p) \to (\widehat{L\Omega}_R)_{\mathbb{Q}_p},$$

as well as a Hodge-completed version of the fiber square (55).

We can refine $\hat{\chi}_i$ to χ_i (and obtain (55)) as follows. First, by construction of these maps via the Beilinson fiber square, a multiple of $\hat{\chi}_i$ lifts to a map $\mathbb{Z}_p(i)(R/p) \to \widehat{L\Omega}_R$. Since the source is left Kan extended (as a functor to the p-complete derived ∞-category) from finitely generated p-complete polynomial algebras, we can restrict and left Kan extend to obtain that a multiple of $\hat{\chi}_i$ lifts to $\mathbb{Z}_p(i)(R/p) \to L\Omega_R$. Inverting p, we obtain that $\hat{\chi}_i$ factors through a map $\chi_i : \mathbb{Q}_p(i)(R/p) \to (L\Omega_R)_{\mathbb{Q}_p}$.

From the quasi-isogeny between $\mathrm{TC}(R,(p);\mathbb{Z}_p)$ and $\Sigma\mathrm{HC}(R,(p);\mathbb{Z}_p)$ as in Theorem 2.20, we obtain the isogeny claim.

Finally, we verify (56); again, we can assume that $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$ by unfolding. Since everything is a pro-étale sheaf, we can even assume that R is w-strictly local in the sense of [13], so that $\pi_{-1}\mathrm{TC}(R;\mathbb{Z}_p) = \mathrm{coker}(F-1\colon W(R)\to W(R))$ (by [47, Theorem F]) vanishes. Recall that we have an equivalence $\tau_{\leq 2p-4}\mathrm{TC}(R,(p);\mathbb{Z}_p)\simeq \tau_{\leq 2p-4}\Sigma\mathrm{HC}(R,(p);\mathbb{Z}_p)$ by Theorem 2.20. It follows that, for $i\leq p-2$, we have an equivalence

$$\tau_{[2i-1,2i]}\mathrm{TC}\big(R,(p);\mathbb{Z}_p\big)\simeq \tau_{[2i-1,2i]}\Sigma\mathrm{HC}\big(R,(p);\mathbb{Z}_p\big).$$

Now $\mathrm{TC}(R/p;\mathbb{Z}_p)$, $\mathrm{HC}(R;\mathbb{Z}_p)$, and $\mathrm{HC}(R/p;\mathbb{Z}_p)$ are concentrated in even degrees since $R\in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$. For the first claim, see [12, Proposition 8.20]. The second and third follow from the filtrations constructed in [1] and [12, Section 5].

It follows from the above definitions that

$$\tau_{[2i-1,2i]}\mathrm{TC}\big(R,(p);\mathbb{Z}_p\big)\simeq \mathrm{fib}\big(\mathbb{Z}_p(i)(R)\to \mathbb{Z}_p(i)(R/p)\big)[2i],$$

and from [12, Section 5] and [1] that

$$\tau_{[2i-1,2i]}\Sigma\mathrm{HC}\big(R,(p);\mathbb{Z}_p\big)\simeq \mathrm{fib}(L\Omega_R/L\Omega_R^{\geq i}\to L\Omega_{R/p}/L\Omega_{R/p}^{\geq i})[2i-1].$$

Using these identifications, we deduce (56). $\square$

We next identify the p-adic Chern character $\chi_i\colon \mathbb{Q}_p(i)(R/p)\to (L\Omega_R)_{\mathbb{Q}_p}$ on graded pieces (from Theorem 6.17) more explicitly. To this end, we prove the following basic result.

PROPOSITION 6.18 (The image of χ_i)
Let $R\in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$. Then for each $i>0$, the map (of $\mathbb{Q}_p$-vector spaces) $\chi_i\colon \mathbb{Q}_p(i)(R/p)\to (L\Omega_R)_{\mathbb{Q}_p} = \mathrm{A}_{\mathrm{crys}}(R/p)_{\mathbb{Q}_p}$ is injective and has image given by the $\varphi = p^i$ eigenspace.

The main issue is the following: both the source and target of χ_i are functors of R/p, thanks to de Rham–Witt theory. We have seen that the p-adic Chern character χ_i induces a natural map $\mathbb{Z}_p(i)(R/p)\to p^{-N}L\Omega_R$ for $R\in \mathrm{qSyn}_{\mathbb{Z}_p}$ for some N. However, it is not a priori obvious that the map χ_i arises from a natural transformation of functors on $\mathbb{F}_p$-algebras (which would force it to commute with Frobenius operators, for example). Our first goal is to verify this.

LEMMA 6.19 (The Frobenius action on $\mathbb{Z}_p(i)(R)$)
For any $R\in \mathrm{qSyn}_{\mathbb{F}_p}$, the Frobenius on R acts as multiplication by p^i on $\mathbb{Z}_p(i)(R)$.

Proof
This reduces to the case of a quasiregular semiperfect $\mathbb{F}_p$-algebra by descent. But, in this case, the identification of Example 6.13 clearly proves the claim. $\square$

COROLLARY 6.20
The natural map $\chi_i\colon \mathbb{Z}_p(i)(R/p) \to p^{-N}(L\Omega_R) \to (L\Omega_R)_{\mathbb{Q}_p}$, *for* $R \in \mathrm{qSyn}_{\mathbb{Z}_p}$ *arises (by precomposition with reduction mod* p*) from a unique natural transformation* $\chi_i\colon \mathbb{Z}_p(i) \to p^{-N'} LW\Omega_{(-)}$ *on* $\mathrm{qSyn}_{\mathbb{F}_p}$ *for some* $N' \geq N$.

Proof
This follows from Corollary B.4 and Lemma 6.19 (the latter shows that the hypotheses of the former are satisfied), and then left Kan extension from finitely generated p-complete polynomial rings. Uniqueness follows since these sheaves are torsion-free. $\square$

Next, we consider the sheaf of graded $\mathbb{E}_\infty$-rings $\bigoplus_{i=0}^{\infty} \mathbb{Z}_p(i)$ on $\mathrm{qSyn}_{\mathbb{F}_p}$. For each $N \geq 0$, we can also truncate to obtain a sheaf of graded $\mathbb{E}_\infty$-rings $\bigoplus_{i=0}^{N} \mathbb{Z}_p(i)$.

PROPOSITION 6.21
Let $f\colon \bigoplus_{i=0}^{N} \mathbb{Z}_p(i) \to \bigoplus_{i=0}^{N} \mathbb{Z}_p(i)$ *be a natural map of sheaves of graded* $\mathbb{E}_\infty$ *rings on* $\mathrm{qSyn}_{\mathbb{F}_p}$. *Then there exists* $\lambda \in \mathbb{Z}_p$ *such that in degree* i, f *is given by multiplication by* λ^i.

Proof
We first observe that the only endomorphisms of $\mathbb{Z}_p(1)$ (as a functor on $\mathrm{qSyn}_{\mathbb{F}_p}$) are given by scalars. It suffices to verify this on quasiregular semiperfect algebras, and there $\mathbb{Z}_p(1)$ is corepresentable (cf. [12, Propositions 7.17, 8.20]) by $\mathbb{F}_p[x^{1/p^\infty}]/(x-1)$, on which $\mathbb{Z}_p(1)(\mathbb{F}_p[x^{1/p^\infty}]/(x-1)) \simeq \mathbb{Z}_p$. So the endomorphism f is given by a scalar action at least on $\mathbb{Z}_p(1)$.

Note that all these functors are left Kan extended from smooth algebras (to the p-complete category), so f is determined by the values on smooth $\mathbb{F}_p$-algebras. Furthermore, the map f is determined by its values modulo p^n for each n. However, classes in $H^i(\mathbb{Z}/p^n(i))$ are étale locally written as sums of products of classes in $H^1(\mathbb{Z}/p^n(1))$ (thanks to Theorem 6.15), so the value of f on $\mathbb{Z}_p(1)$ determines the value of f in general. The result now follows because, on smooth algebras, $\mathbb{Z}/p^n(i)$ is concentrated in cohomological degree i étale locally. $\square$

Proof of Proposition 6.18
Recall that the map χ_i is actually a special case of a map $\mathbb{Z}_p(i)(R_0) \to p^{-N'}(LW\Omega_{R_0})$ defined on $R_0 \in \mathrm{qSyn}_{\mathbb{F}_p}$, by Corollary 6.20. For R_0 quasiregular semiperfect, we know that the Frobenius acts as p^i on $\mathbb{Z}_p(i)(R_0)$, so we obtain a natural, multiplicative map $\mathbb{Q}_p(i)(R_0) \to (\mathrm{A}_{\mathrm{crys}}(R_0)^{\varphi=p^i})_{\mathbb{Q}_p}$. We wish to see that these maps are isomorphisms.

Now we know independently that $\mathbb{Q}_p(i)(R_0)$ is identified (for $i > 0$) with $\mathrm{A}_{\mathrm{crys}}(R_0)_{\mathbb{Q}_p}^{\varphi=p^i}$ via the theory of topological cyclic homology (Theorem 6.11, following [12, Section 8]). Thus, we actually obtain natural, multiplicative (in i) maps $\mathbb{Q}_p(i)(R_0) \to \mathbb{Q}_p(i)(R_0)$ for $R_0 \in \mathrm{qSyn}_{\mathbb{F}_p}$, and we wish to see that these are isomorphisms. Up to rescaling by a power of p, furthermore, they carry $\mathbb{Z}_p(i)(R_0)$ into $\mathbb{Z}_p(i)(R_0)$. As we saw in Proposition 6.21, these maps are necessarily all given by scalar multiplication by some λ^i in degree i, for some $\lambda \in \mathbb{Z}_p$; we know that $\lambda \neq 0$ (by comparing with $i = 1$, say), so the result now follows. □

6.4. *Comparison of the $\mathbb{Z}_p(i)^{\mathrm{FM}}$ and $\mathbb{Z}_p(i)$*

Our main result is the following comparison, which establishes Theorem F.

THEOREM 6.22

For $R \in \mathrm{qSyn}_{\mathbb{Z}_p}$, there are natural, multiplicative identifications $\mathbb{Z}_p(i)^{\mathrm{FM}}(R) \simeq \mathbb{Z}_p(i)(R)$ for $i \leq p-2$ and $\mathbb{Q}_p(i)^{\mathrm{FM}}(R) \simeq \mathbb{Q}_p(i)(R)$ for all $i \geq 0$.

By [33, Theorem 1.3], for $i \leq p-2$ and for formally smooth schemes over discrete valuation rings, syntomic cohomology in the above form (see also [54], [57]) is p-adic étale motivic cohomology.

Proof of the rational case of Theorem 6.22

Fix $i \geq 0$. It is enough to prove the equivalences for all $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$. Thanks to the odd vanishing conjecture proved in [15, Section 14], we may, moreover, assume that $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$ is such that $\mathbb{Z}_p(i)(R)$ is concentrated in degree 0. In the homotopy Cartesian square of Theorem 6.17, the terms $\mathbb{Q}_p(i)(R)$, $(L\Omega_R^{\geq i})_{\mathbb{Q}_p}$, and $(L\Omega_R)_{\mathbb{Q}_p}$ are all concentrated in degree 0, whence the same is true of the remaining term $\mathbb{Q}_p(i)(R/p)$ (i.e., $\varphi - p^i : (L\Omega_R)_{\mathbb{Q}_p} \to (L\Omega_R)_{\mathbb{Q}_p}$ is surjective) and the fiber square is simply a Cartesian and co-Cartesian square of abelian groups:

$$\begin{tikzcd} \mathbb{Q}_p(i)(R) \arrow[r] \arrow[d] & \mathbb{Q}_p(i)(R/p) \arrow[d] \\ (L\Omega_R^{\geq i})_{\mathbb{Q}_p} \arrow[r] & (L\Omega_R)_{\mathbb{Q}_p}. \end{tikzcd}$$

Note that all the arrows are injections: the bottom since it is the inclusion of the Hodge filtration, the right by Proposition 6.18, and the others since the diagram is Cartesian.

We claim that the map $\varphi - p^i : (L\Omega_R^{\geq i})_{\mathbb{Q}_p} \to (L\Omega_R)_{\mathbb{Q}_p}$ is surjective. Indeed, given $x \in (L\Omega_R)_{\mathbb{Q}_p}$, we can write $x = (\varphi - p^i)(x')$ for some $x' \in (L\Omega_R)_{\mathbb{Q}_p}$; as we noted above, $\varphi - p^i : (L\Omega_R)_{\mathbb{Q}_p} \to (L\Omega_R)_{\mathbb{Q}_p}$ is surjective. Using Proposition 6.18 to identify the image of the vertical map, the diagram being co-Cartesian means that $(L\Omega_R)_{\mathbb{Q}_p}^{\varphi=p^i} \oplus (L\Omega_R^{\geq i})_{\mathbb{Q}_p} \to (L\Omega_R)_{\mathbb{Q}_p} = \mathrm{A}_{\mathrm{crys}}(R/p)_{\mathbb{Q}_p}$ is surjective. So we can write $x' = y' + z'$ for $y' \in \ker(\varphi - p^i)$ and $z' \in (L\Omega_R)_{\mathbb{Q}_p}^{\geq i}$. Applying $\varphi - p^i$, we get that $x = (\varphi - p^i)(z')$, proving the claim as desired.

Combining these observations, we have established a natural identification

$$\mathbb{Q}_p(i)(R) = (L\Omega_R)_{\mathbb{Q}_p}^{\varphi=p^i} \cap (L\Omega_R^{\geq i})_{\mathbb{Q}_p} \simeq \mathrm{fib}\big(\varphi - p^i : (L\Omega_R^{\geq i})_{\mathbb{Q}_p} \to (L\Omega_R)_{\mathbb{Q}_p}\big),$$

as desired. □

COROLLARY 6.23 (A description of $\mathrm{TC}(R;\mathbb{Q}_p)$)
Let R be any simplicial commutative ring. Then there is a natural equivalence

$$\mathrm{TC}(R;\mathbb{Q}_p) \simeq \bigoplus_{i\geq 0} \mathrm{fib}(\varphi - p^i : L\Omega_R^{\geq i} \to L\Omega_R)_{\mathbb{Q}_p}.$$

Proof
The map from R to its derived p-adic completion induces an equivalence on all the terms appearing in the statement: for derived de Rham cohomology and its Hodge filtration, this follows from base change, while it holds for $\mathrm{THH}(-;\mathbb{Z}_p)$ (and hence $\mathrm{TC}(-;\mathbb{Q}_p)$) by [23, Lemma 5.2]. We may therefore assume R is p-complete, at which point we know from Construction 5.33 that $\mathrm{TC}(R;\mathbb{Q}_p)$ admits a complete descending filtration with associated graded given by $\mathbb{Q}_p(i)(R)[2i]$, for $i \geq 0$. Using Adams operations on TC as in [12, Section 9.4], we can split the filtration functorially. Combining with the rational part of Theorem 6.22 (or, more precisely, its left Kan extension to p-complete simplicial commutative rings), the claim follows. □

Next, we will prove the integral case of Theorem 6.22. The main step is to show that the $\mathbb{Z}_p(i)^{\mathrm{FM}}(-)$ for $i \leq p-2$ are discrete, as sheaves on $\mathrm{qSyn}_{\mathbb{Z}_p}$; this is the analogue of Theorem 5.11, that is, of the odd vanishing conjecture. To see this, we will use the odd vanishing conjecture itself and some cases of the results of Li–Liu [59].

PROPOSITION 6.24
As $D(\mathbb{Z}_p)^{\geq 0}$-valued sheaves on $\mathrm{qSyn}_{\mathbb{Z}_p}$*:*
(1) $\mathbb{Z}_p(i)^{\mathrm{FM}}(\cdot)$ *is discrete and torsion-free for* $0 \leq i \leq p-2$*;*
(2) $\mathbb{Q}_p(i)^{\mathrm{FM}}(\cdot)$ *is discrete.*

Proof
Item (2) has already been proved above (in light of the odd vanishing conjecture [15,

Theorem 14.1]), so here we prove (1). It suffices to show that $\mathbb{Z}_p(i)^{\mathrm{FM}}(\cdot)$ is discrete and torsion-free as a $D(\mathbb{Z}_p)^{\geq 0}$-valued sheaf on $\mathrm{qSyn}_{\mathcal{O}_C}$, for $\mathcal{O}_C$ the ring of integers in $C = \mathbb{C}_p$. In particular, we will show that, for any $R \in \mathrm{qSyn}_{\mathcal{O}_C}$, there exists a quasisyntomic cover $R \to R'$ such that R' is quasiregular semiperfectoid and such that $\varphi/p^i - 1\colon L\Omega^{\geq i}_{R'} \to L\Omega_{R'}$ is surjective for all $i \leq p-2$.

To this end, on $\mathrm{qrsPerfd}_{\mathcal{O}_C}$, we consider the (non-Nygaard-complete) sheaf $R \mapsto \mathbf{\Delta}_R$ (cf. [15, Section 7]) together with its Nygaard filtration $\mathcal{N}^{\geq \star}\mathbf{\Delta}_R$ and divided Frobenius maps $\frac{\varphi}{\widetilde{\xi}^i}\colon \mathcal{N}^{\geq i}\mathbf{\Delta}_R \to \mathbf{\Delta}_R$, for $\widetilde{\xi}$ a generator of the ideal defining the prism structure on $\mathbf{\Delta}_{\mathcal{O}_C}$. For any $R \in \mathrm{qrsPerfd}_{\mathcal{O}_C}$, we use [59, Theorem 3.5, Remark 3.6] (applied to the perfect prism $(\mathrm{A}_{\mathrm{inf}}(\mathcal{O}_C), \widetilde{\xi})$) or the crystalline comparison (see [15, Theorem 5.2]) to obtain a natural map of δ-rings

$$\mathbf{\Delta}_R \to \mathrm{A}_{\mathrm{crys}}(R/p) \simeq L\Omega_R. \tag{58}$$

This map of δ-rings carries $\widetilde{\xi}$ to an element of the form $p\varphi(u)$, for $u \in (L\Omega_R)^\times$ and $\varphi\colon L\Omega_R \to L\Omega_R$ the crystalline Frobenius. By [59, Theorem 4.13], the map (58) canonically refines to a filtered map $\mathcal{N}^{\geq\star}\mathbf{\Delta}_R \to L\Omega^{\geq\star}_R$. Moreover, the filtered map $\mathcal{N}^{\geq\star}\mathbf{\Delta}_R \to L\Omega^{\geq\star}_R$ induces an isomorphism on associated graded terms in degrees $\leq p-1$ (see [59, Theorem 4.13]).

By the odd vanishing conjecture (see [15, Section 14]), there exists a basis of objects $R \in \mathrm{qrsPerfd}_{\mathcal{O}_C}$ for which the map $\varphi/\widetilde{\xi}^i - 1\colon \mathcal{N}^{\geq i}\mathbf{\Delta}_R \to \mathbf{\Delta}_R$ is surjective. Choose any such R. Consider now the evident commutative diagram

$$\begin{tikzcd} \mathcal{N}^{\geq i}\mathbf{\Delta}_R \arrow[r, "\varphi/\widetilde{\xi}^i - 1"] \arrow[d] & \mathbf{\Delta}_R \arrow[d] \\ L\Omega^{\geq i}_R \arrow[r, "\varphi/\widetilde{\xi}^i - 1"] & L\Omega_R. \end{tikzcd} \tag{59}$$

For $i \leq p-2$, it follows from the surjectivity of the top horizontal arrow in (59) (and that $\widetilde{\xi}$ maps to $p\varphi(u)$ in $L\Omega_R$, for $u \in (L\Omega_R)^\times$) that the composite

$$L\Omega^{\geq i}_R \xrightarrow{\varphi/p^i - 1} L\Omega_R \to L\Omega_R/L\Omega^{\geq i+1}_R \simeq \mathbf{\Delta}_R/\mathcal{N}^{\geq i+1}\mathbf{\Delta}_R \tag{60}$$

is surjective. Now the map

$$L\Omega^{\geq i+1}_R \subset L\Omega^{\geq i}_R \xrightarrow{\varphi/p^i - 1} L\Omega_R \to L\Omega_R/p$$

is simply the inclusion composed with reduction mod p, since φ/p^i is divisible by p on $L\Omega^{\geq i+1}_R$. This observation combined with the surjectivity of (60) and p-completeness now gives the surjectivity of $L\Omega^{\geq i}_R \xrightarrow{\varphi/p^i - 1} L\Omega_R$, as desired. □

Proof of Theorem 6.22 for $i \leq p-2$
Fix $i \leq p-2$ and $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$ such that $\mathbb{Z}_p(i)(R)$ and $\mathbb{Z}_p(i)^{\mathrm{FM}}(R)$ are concentrated in degree 0 for all $i \leq p-2$; we can do this by the odd vanishing conjecture and Proposition 6.24. In this case, the map of discrete abelian groups $\mathbb{Z}_p(i)(R) \to \mathbb{Z}_p(i)(R/p)$ is injective and has torsion-free cokernel, thanks to the equivalence (56). So from the Beilinson fiber square on graded pieces (Theorem 6.17) and the description of the image of χ_i (Proposition 6.18), we find that $\mathbb{Z}_p(i)(R) \subset \mathbb{Z}_p(i)(R/p) = (L\Omega_R)^{\varphi=p^i}$ is the submodule consisting of those elements such that the image in $L\Omega_R$ belongs to $L\Omega_R^{\geq i}$. In particular, it is precisely the kernel of $\varphi/p^i - 1\colon L\Omega_R^{\geq i} \to L\Omega_R$. Since this map is surjective, we get the natural equivalence $\mathbb{Z}_p(i)(R) \simeq \mathbb{Z}_p(i)(R)^{\mathrm{FM}}$ as desired. $\square$

7. Examples

7.1. K-theory of p-adic fields

Let F be a complete discretely valued field of characteristic 0 with ring of integers $\mathcal{O}_F \subset F$ and perfect residue field k of characteristic p. In this subsection, we will use the Beilinson fiber sequence to recover various calculations of the p-adic K-theory of F. All these results are previously known, at least in the case of F local (see [90, Theorem 61] for a detailed survey).

THEOREM 7.1
The homotopy groups of $\mathrm{K}(F;\mathbb{Q}_p)$ are given (as $\mathbb{Q}_p$-vector spaces) as follows:
(1) $\mathrm{K}_{2s}(F;\mathbb{Q}_p) = 0$ *for* $s > 0$*;*
(2) *there is a natural isomorphism* $\mathrm{K}_{2s-1}(F;\mathbb{Q}_p) \simeq F$ *for each* $s > 1$*;*
(3) *there is a natural short exact sequence* $0 \to F \to \mathrm{K}_1(F;\mathbb{Q}_p) \to \mathbb{Q}_p \to 0$.

Proof
Since k is perfect, we have that $\mathrm{K}_i(k;\mathbb{Z}_p) = \mathbb{Z}_p$ for $i = 0$ and 0 otherwise (cf. [45, Theorem 5.4] and [56, Corollary 5.5]). Taking the dévissage cofiber sequence $\mathrm{K}(k) \to \mathrm{K}(\mathcal{O}_F) \to \mathrm{K}(F)$ with $\mathbb{Z}_p$-coefficients shows that $\mathrm{K}_i(\mathcal{O}_F;\mathbb{Z}_p) \cong \mathrm{K}_i(F;\mathbb{Z}_p)$ for $i \neq 1$ and that there is an exact sequence

$$0 \to \mathrm{K}_1(\mathcal{O}_F;\mathbb{Z}_p) \to \mathrm{K}_1(F;\mathbb{Z}_p) \to \mathbb{Z}_p \to 0,$$

where the map $\mathrm{K}_1(F;\mathbb{Z}_p) \cong F^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathbb{Z}_p$ is induced by the p-adic valuation.

Next, since $\mathrm{K}(\mathcal{O}_F/p;\mathbb{Q}_p) \simeq \mathrm{K}(k;\mathbb{Q}_p) \simeq \mathbb{Q}_p$, the Beilinson fiber square (Theorem A) for $\mathcal{O}_F$ yields a fiber sequence

$$\Sigma\mathrm{HC}(\mathcal{O}_F;\mathbb{Q}_p) \to \mathrm{K}(\mathcal{O}_F;\mathbb{Q}_p) \to \mathbb{Q}_p.$$

But note that the cyclic homology term may be equivalently written as $\Sigma\mathrm{HC}(F/F_0)$, where $F_0 := W(k)[\frac{1}{p}]$; indeed, the vanishing of the p-adic completion of the cotangent complex $L_{W(k)/\mathbb{Z}}$ implies that $\mathrm{HC}(\mathcal{O}_F;\mathbb{Z}_p) \simeq \mathrm{HC}(\mathcal{O}_F/\mathcal{O}_{F_0};\mathbb{Z}_p)$, but $\mathrm{HC}(\mathcal{O}_F/\mathcal{O}_{F_0})$ is already derived p-adically complete since its homology groups are all finitely generated $\mathcal{O}_F$-modules. The Beilinson fiber sequence therefore implies that

$$\tau_{\geq 2}\Sigma\mathrm{HC}(F/F_0) \simeq \tau_{\geq 2}\mathrm{K}(F;\mathbb{Q}_p)$$

and

$$\mathrm{HC}_0(F/F_0) \simeq \mathrm{K}_1(\mathcal{O}_F;\mathbb{Q}_p).$$

The proof is completed by noting that, since F is an étale F_0-algebra, its cyclic homology is given by $\mathrm{HC}_i(F/F_0) = F$ for $i \geq 0$ even and is 0 otherwise. $\square$

Example 7.2 (Local fields)
If F is a finite extension of $\mathbb{Q}_p$, of degree d, then the theorem shows that

$$\dim_{\mathbb{Q}_p} \mathrm{K}_{2s-1}(F;\mathbb{Q}_p) = \begin{cases} d+1 & \text{if } s = 1, \\ d & \text{otherwise.} \end{cases} \tag{61}$$

This dimension calculation is a classical result, arising from Wagoner's [89] calculation of the ranks of the continuous K-groups and Panin's [75] proof of an early case of the p-adic continuity of K-theory.

In addition, the dimension calculation (61) is in accordance with the Beilinson–Lichtenbaum conjecture for F. Recall that the Beilinson–Lichtenbaum conjecture, prior to its general proof by Rost–Voevodsky for all fields, was proved by Hesselholt–Madsen [48] in this case when $p > 2$ using TC-theoretic methods. Since F has cohomological dimension 2, the Beilinson–Lichtenbaum conjecture predicts $\mathrm{K}_{2s-1}(F;\mathbb{Q}_p) \simeq H^1_{\text{ét}}(F,\mathbb{Q}_p(s))$ and $\mathrm{K}_{2s-2}(F;\mathbb{Q}_p) \simeq H^2_{\text{ét}}(F,\mathbb{Q}_p(s))$ for $s > 0$. Then the dimensions in (61) agree with the dimensions of the $\mathbb{Q}_p$-cohomology of F, as computed via Tate's local duality and Euler characteristic formula (see [74, VII.3] for an account).

Example 7.3 (Integral calculation, unramified case)
Assume in this example that $p > 3$ so that the results hold in a nonempty range. We will show that, in the range $1 \leq i \leq 2p-5$, the p-adic K-groups of $W(k)$ are given by

$$\mathrm{K}_i\big(W(k);\mathbb{Z}_p\big) \simeq \begin{cases} W(k) & \text{if } i = 2s-1, \\ 0 & \text{if } i \text{ is even.} \end{cases} \tag{62}$$

Note that for k finite, the calculation of the entire homotopy type of $\mathrm{K}(W(k);\mathbb{Z}_p)$ is carried out by Bökstedt–Madsen [20] at odd primes and Rognes [79] at $p=2$ (for $k=\mathbb{F}_2$); again, see [90, Theorem 61] for a survey for all of these results.

The integral form of the Beilinson fiber sequence (Corollary B) takes the form of a natural fiber sequence

$$\tau_{\leq 2p-5}\Sigma\mathrm{HC}\big(W(k),(p);\mathbb{Z}_p\big)\to\tau_{\leq 2p-5}\mathrm{K}\big(W(k);\mathbb{Z}_p\big)\to\mathbb{Z}_p.$$

As in the proof of Theorem 7.1, the cyclic homology term may be replaced by the cyclic homology $\tau_{\leq 2p-5}\Sigma\mathrm{HC}(W(k),(p)/W(k))$ over $W(k)$.

A standard calculation of derived de Rham cohomology with divided powers (as in [85, Proposition 3.16]) gives $L\Omega_k/L\Omega_k^{\geq s}\simeq W(k)/p^s$ for $s\leq p-1$; in general, $L\Omega_k/L\Omega_k^{\geq s}$ is discrete for all s. Using the filtrations of [1], we conclude for $i\leq 2p-1$,

$$\pi_i\mathrm{HC}\big(k/W(k)\big)\simeq\begin{cases}W(k)/p^{s+1} & \text{if } i=2s\geq 0,\\ 0 & \text{otherwise.}\end{cases}\tag{63}$$

See also [21, Proposition 7.2] for this calculation. Also, $\mathrm{HC}_i(W(k);\mathbb{Z}_p)\cong\mathrm{HC}_i(W(k)/W(k))\cong W(k)$ for $i\geq 0$ even and is zero otherwise. Thus, in the range $0\leq i\leq 2p-2$, we deduce that

$$\pi_i\mathrm{HC}(\big(W(k),(p)/W(k)\big)\simeq\begin{cases}p^{s+1}W(k) & \text{if } i=2s,\\ 0 & \text{if } i \text{ is odd.}\end{cases}$$

This completes the proof.

Remark 7.4 (Integral calculation, ramified case)
Assume now that F is ramified so that $\mathcal{O}_F/p\cong k[x]/(x^e)$, where e is the absolute ramification degree of F. Corollary B implies, by rewriting the p-adic cyclic homology as noncompleted cyclic homology with respect to $W(k)$, that $\tau_{\leq 2p-5}\Sigma\mathrm{HC}(\mathcal{O}_F,(p)/W(k))\simeq\tau_{\leq 2p-5}\mathrm{K}(\mathcal{O}_F,(p);\mathbb{Z}_p)$.

We now appeal to the fact that the algebraic K-theory of truncated polynomial rings over fields is known (see [46], [83]). The positive even p-adic K-groups of $k[x]/(x^e)$ vanish so that we get five-term exact sequences

$$\begin{aligned}0&\to\pi_{2s-1}\Sigma\mathrm{HC}\big(\mathcal{O}_F,(p)/W(k)\big)\to\mathrm{K}_{2s-1}(\mathcal{O}_F;\mathbb{Z}_p)\\&\to\mathbb{W}_{se}(k)/V_e\mathbb{W}_s(k)\to\pi_{2s-2}\Sigma\mathrm{HC}\big(\mathcal{O}_F,(p)/W(k)\big)\to\mathrm{K}_{2s-2}(\mathcal{O}_F;\mathbb{Z}_p)\to 0\end{aligned}$$

for $2\leq s\leq p-2$. In low degrees, this gives a computation of the integral p-adic K-groups of $\mathcal{O}_F$ which is independent of [48]; on the other hand, using this calculation and [48], we can view the computation as giving information about the low-degree étale cohomology of F.

We can also carry out previous types of calculations for the syntomic complexes $\mathbb{Z}_p(i)$ rather than K-theory.

THEOREM 7.5 (Syntomic cohomology of discrete valuation rings)
Let $\mathcal{O}_F$ be a complete discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field k.

(1) *We have natural identifications*

$$\mathbb{Q}_p(i)(\mathcal{O}_F) \simeq \begin{cases} R\Gamma_{\mathrm{proet}}(\mathrm{Spec}(k), \mathbb{Q}_p) & \text{if } i = 0, \\ F[-1] & \text{if } i > 0. \end{cases} \tag{64}$$

(2) *In the unramified case $\mathcal{O}_F = W(k)$,*

$$\mathbb{Z}_p(i)\big(W(k)\big) \simeq \begin{cases} R\Gamma_{\mathrm{proet}}(\mathrm{Spec}(k), \mathbb{Z}_p) & \text{if } i = 0, \\ W(k)[-1] & \text{if } 0 < i \leq p-2. \end{cases} \tag{65}$$

Proof
For (1), we have (where the first equivalence follows from Lemma 6.19 and taking powers of Frobenius)

$$\mathbb{Q}_p(i)(\mathcal{O}_F/p) \simeq \mathbb{Q}_p(i)(k) \simeq \begin{cases} R\Gamma_{\mathrm{proet}}(\mathrm{Spec}(k), \mathbb{Q}_p) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the fiber sequence from Theorem 6.17 yields equivalences

$$\mathbb{Q}_p(i)(\mathcal{O}_F) \simeq \begin{cases} R\Gamma_{\mathrm{proet}}(\mathrm{Spec}(k), \mathbb{Q}_p) & \text{if } i = 0, \\ (L\Omega_{\mathcal{O}_F}/L\Omega^{\geq i}_{\mathcal{O}_F})_{\mathbb{Q}_p}[-1] & \text{if } i > 0. \end{cases}$$

As in the proof of Theorem 7.1, the latter truncated p-adic derived de Rham cohomologies may be computed as the analogous uncompleted derived de Rham cohomologies for $F_0 \to F$; since $L_{F/F_0} \simeq 0$, we conclude that $\mathbb{Q}_p(i)(\mathcal{O}_F) \simeq F[-1]$ for $i > 0$.

The integral claim follows from Theorem 6.22. Indeed, we find that $\mathbb{Z}_p(0)(W(k)) = \mathrm{fib}(\varphi - 1\colon W(k) \to W(k)) \simeq R\Gamma_{\mathrm{proet}}(\mathrm{Spec}(k), \mathbb{Z}_p)$. For $i > 0$, we get $\mathbb{Z}_p(i)(W(k)) = \mathrm{fib}(\varphi/p^i - 1\colon 0 \to W(k))$, so the claim follows. □

7.2. Perfectoid rings

In this section, we apply the Beilinson fiber square to a perfectoid ring. The main result (which was indicated to us by Scholze) is that it recovers the fundamental exact sequence in p-adic Hodge theory.

Let R be a perfectoid ring. We review the period rings associated to R and their interpretation via derived de Rham theory (cf. [7], [10]).

Construction 7.6 (Period rings)

Let R be a perfectoid ring.

(1) As before, we have Fontaine's ring $\mathrm{A}_{\mathrm{inf}}(R)$ equipped with the canonical map $\theta\colon \mathrm{A}_{\mathrm{inf}}(R) \to R$ with kernel (ξ). Here $\mathrm{A}_{\mathrm{inf}}(R)$ is also the prismatic cohomology $\widehat{\mathbf{\Delta}}_R$; the Nygaard filtration is the ξ-adic filtration.

(2) We have $\mathrm{A}_{\mathrm{crys}}(R) = \mathrm{A}_{\mathrm{crys}}(R/p)$, the p-adic completion of the divided power envelope of $(\xi) \subset \mathrm{A}_{\mathrm{inf}}(R)$; we have that $\mathrm{A}_{\mathrm{crys}}(R) \simeq L\Omega_R$ is the derived de Rham cohomology of R. The Hodge filtration is given by the divided power filtration. We let $\mathrm{B}^+_{\mathrm{crys}}(R) = \mathrm{A}_{\mathrm{crys}}(R)[1/p] = (L\Omega_R)_{\mathbb{Q}_p}$; the ring $\mathrm{B}^+_{\mathrm{crys}}(R)$ also inherits a Frobenius operator φ.

(3) We have $\mathrm{B}^+_{\mathrm{dR}}(R) = \varprojlim(\mathrm{A}_{\mathrm{inf}}(R)/\xi^n[1/p])$. The ring $\mathrm{B}^+_{\mathrm{dR}}(R)$ can also be obtained as the Hodge completion of $(L\Omega_R)_{\mathbb{Q}_p}$. The Hodge filtration yields a filtration (the ξ-adic filtration) on $\mathrm{B}^+_{\mathrm{dR}}(R)$.

Our goal is now to recover the following result in p-adic Hodge theory (cf. [30, Theorem 5.3.7] and [28, Theorem 6.4.1]).

THEOREM 7.7 (The fundamental exact sequence)

For any $R \in \mathrm{Perfd}$ and $i > 0$, there is a natural pullback square in $D(\mathbb{Q}_p)$,

$$\begin{tikzcd} \mathrm{R}\Gamma_{\mathrm{proet}}(\mathrm{Spec}(R[1/p]), \mathbb{Q}_p(i)) \arrow[r] \arrow[d] & \mathrm{B}^+_{\mathrm{crys}}(R)^{\varphi = p^i} \arrow[d] \\ \mathrm{Fil}^{\geq i}\mathrm{B}^+_{\mathrm{dR}}(R) \arrow[r] & \mathrm{B}^+_{\mathrm{dR}}(R). \end{tikzcd} \tag{66}$$

Example 7.8

When $R = \mathcal{O}_C$, where C is a complete algebraically closed non-Archimedean field, the fundamental exact sequence is often written as the exact sequence

$$0 \to \mathbb{Q}_p(i) \to \mathrm{B}^+_{\mathrm{crys}}(\mathcal{O}_C)^{\varphi = p^i} \to \mathrm{B}^+_{\mathrm{dR}}(\mathcal{O}_C)/\mathrm{Fil}^{\geq i}\mathrm{B}^+_{\mathrm{dR}}(\mathcal{O}_C) \to 0$$

of abelian groups.

Proof of Theorem 7.7
We apply Theorem 6.17 to the perfectoid ring R and obtain a fiber square

$$\begin{tikzcd} \mathbb{Q}_p(i)(R) \arrow[r] \arrow[d] & \mathbb{Q}_p(i)(R/p) \arrow[d] \\ (L\Omega_R^{\geq i})_{\mathbb{Q}_p} \arrow[r] & (L\Omega_R)_{\mathbb{Q}_p}. \end{tikzcd} \tag{67}$$

By [15, Theorem 9.4], the first term is identified with $\mathrm{R}\Gamma_{\mathrm{proet}}(\mathrm{Spec}(R[1/p]), \mathbb{Q}_p(i))$. The ring R/p is quasiregular semiperfect, so we have $\mathbb{Q}_p(i)(R/p) \simeq \mathrm{B}^+_{\mathrm{crys}}(R)^{\varphi=p^i}$ (see [12, Section 8]).

Note that we can replace the square (67) by Hodge completing the bottom row and it will still remain Cartesian, since the homotopy fibers do not change. This yields a new homotopy Cartesian square where one identifies the rings as in Construction 7.6, and then the result follows. ☐

Remark 7.9 (Identifying the maps)
Unfortunately, in general we do not know a good way of identifying the map $\mathrm{K}(\mathcal{O}_C/p;\mathbb{Q}_p) \to \mathrm{HP}(\mathcal{O}_C;\mathbb{Q}_p)$ with the usual map in the fundamental exact sequence. However, we can argue that it has to match with the usual map, at least for $C = \mathbb{C}_p$, by appealing to some general results. For simplicity, in this example, we drop the argument of the perfectoid ring; that is, we write $\mathrm{B}^+_{\mathrm{dR}}$ for $\mathrm{B}^+_{\mathrm{dR}}(\mathcal{O}_{\mathbb{C}_p})$, and so on.

Our first goal is to identify the map obtained from π_2 in (66),

$$(\mathrm{B}^+_{\mathrm{crys}})^{\varphi=p} \to \mathrm{B}^+_{\mathrm{dR}}; \tag{68}$$

by construction, it is $\mathrm{Gal}(\mathbb{Q}_p)$-equivariant. Now we have a (Galois-equivariant) short exact sequence

$$0 \to \mathbb{Q}_p(1) \to (\mathrm{B}^+_{\mathrm{crys}})^{\varphi=p} \to \mathbb{C}_p \to 0,$$

as above. Furthermore, the map (68) when restricted to the submodule $\mathbb{Q}_p(1) \subset (\mathrm{B}^+_{\mathrm{crys}})^{\varphi=p}$ is essentially determined: it is given by the dlog map to derived de Rham cohomology as it comes from the usual Chern character $\mathrm{K}(\mathcal{O}_C;\mathbb{Q}_p) \to \mathrm{HC}^-(\mathcal{O}_C;\mathbb{Q}_p)$, for $\mathcal{O}_C$. As is proved in [29, Proposition 2.17], the image of a generator of $\mathbb{Q}_p(1)$ gives a uniformizer of $\mathrm{B}^+_{\mathrm{dR}}$ (which is a discrete valuation ring).

Recall that $\mathrm{B}^+_{\mathrm{dR}}$ has a complete, exhaustive filtration (via powers of the augmentation ideal) with associated graded given by $\mathbb{C}_p, \mathbb{C}_p(1), \mathbb{C}_p(2), \ldots$. Moreover, there are no $\mathrm{Gal}(\mathbb{Q}_p)$-equivariant maps $\mathbb{C}_p \to \mathrm{B}^+_{\mathrm{dR}}$ (see [30, Remark 1.5.8]). Thus there is at most one (and hence exactly one, by construction) Galois-equivariant map

$(\mathrm{B}^+_{\mathrm{crys}})^{\varphi=p} \to \mathrm{B}^+_{\mathrm{dR}}$ which extends the dlog map. This shows that the map (68) is actually completely determined by its behavior on $\mathbb{Q}_p(1)$.

Now by a deep result of Fargues–Fontaine, the graded ring $\bigoplus_{i\geq 0} \mathrm{B}^+_{\mathrm{crys}}(\mathcal{O}_C)^{\varphi=p^i}$ is generated in degree 1 (see [28, Theorem 6.2.1]), so the maps for higher i are determined by their behavior for $i=1$ by multiplicativity. In particular, these observations show that the maps in the fundamental exact sequence, although here they are produced by topological means, are entirely determined by their value on $\mathbb{Q}_p(1)$, as long as they are Galois-equivariant.

7.3. Application to p-adic nearby cycles

Let C be an algebraically closed, complete non-Archimedean field of mixed characteristic $(0,p)$. In [12, Section 10], an explicit description of the $\mathbb{Z}_p(i)$ sheaves is given for smooth formal schemes over $\mathcal{O}_C$. Using this, we can recover some cases of comparison results of Colmez–Nizioł [24] and Tsuji [88] (cf. also Kato [54]).

Definition 7.10 (p-adic nearby cycles)
Let $\mathfrak{X}$ be a formal scheme over $\mathcal{O}_C$. We consider the pro-étale site $\mathfrak{X}_{\mathrm{proet}}$ of $\mathfrak{X}$ (equivalently, of its special fiber; cf. [13]).

For each i, we consider[13] the sheaf of *p-adic nearby cycles* $R\psi_*(\mathbb{Z}_p(i))$, which is a $D(\mathbb{Z}_p)^{\geq 0}$-valued sheaf on $\mathfrak{X}_{\mathrm{proet}}$. Explicitly, given an affine pro-étale open $\mathrm{Spf}\, A \to \mathfrak{X}$, we have that $R\Gamma(\mathrm{Spf}\, A, R\psi_*(\mathbb{Z}_p(i))) \simeq R\Gamma_{\mathrm{proet}}(\mathrm{Spec}\, A[1/p], \mathbb{Z}_p(i))$ is the pro-étale cohomology of $A[1/p]$ with values in the (usual) sheaf $\mathbb{Z}_p(i)$.[14]

THEOREM 7.11 (Bhatt–Morrow–Scholze [12])
Let R be a formally smooth $\mathcal{O}_C$-algebra, and let $\mathfrak{X} = \mathrm{Spf}(R)$. Then, as sheaves on $\mathfrak{X}_{\mathrm{proet}}$, we have a natural equivalence $\mathbb{Z}_p(i) \simeq \tau^{\leq i} R\psi_*(\mathbb{Z}_p(i))$. In particular, it follows that

$$\mathbb{Z}_p(i)(R) \simeq R\Gamma\big(\mathfrak{X}_{\mathrm{proet}}, \tau^{\leq i} R\psi_*\big(\mathbb{Z}_p(i)\big)\big).$$

To apply this, let K be a *discretely* valued field with perfect residue field k, ring of integers $\mathcal{O}_K \subset K$, and uniformizer $\pi \in \mathcal{O}_K$; suppose that $K \subset C$ (e.g., we could take $C = \widehat{\overline{K}}$). Let $\mathfrak{X}_0$ be a smooth proper formal scheme over $\mathcal{O}_K$ with generic fiber X_0, a smooth proper rigid space over K.

Construction 7.12 (de Rham cohomology of formal schemes and rigid spaces)
We let $L\Omega_{\mathfrak{X}_0/\mathcal{O}_K}$ denote the (p-adic) derived de Rham cohomology of $\mathfrak{X}_0$ over

[13] The functor ψ here should refer to the generic fiber functor, but we do not define it here to avoid technicalities.
[14] Here we can consider either the scheme $\mathrm{Spec}(A[1/p])$ or the rigid analytic generic fiber by the affinoid comparison theorem (see [51, Corollary 3.2.2]).

$\mathcal{O}_K$ equipped with its Hodge filtration. In fact, $L\Omega_{\mathfrak{X}_0/\mathcal{O}_K}$ is also the p-complete *usual* de Rham complex since $\mathfrak{X}_0$ is formally smooth over $\mathcal{O}_K$ (cf. [10]), and the Hodge filtration is a finite filtration. We let (by a slight abuse of notation) $\Omega_{X_0/K} = (L\Omega_{\mathfrak{X}_0/\mathcal{O}_K})_{\mathbb{Q}_p}$ denote the rationalization, which we can interpret as the de Rham cohomology of the rigid generic fiber X_0. Note that $L\Omega_{\mathfrak{X}_0/\mathcal{O}_K}$ is a perfect $\mathcal{O}_K$-module and $\Omega_{X_0/K}$ is a perfect K-module.

We also consider the ring $\mathrm{B}_{\mathrm{dR}}^+ = \mathrm{B}_{\mathrm{dR}}^+(\mathcal{O}_C)$ with its ξ-adic filtration. Together, it follows that $\Omega_{X_0/K} \otimes_K \mathrm{B}_{\mathrm{dR}}^+$ admits a filtration in the derived ∞-category $D(K)$. Then one has the following result, a special case of results of [24] and [88] in the case of good reduction; note that [24] and [88] treat the more general semistable case, which we do not consider here. In the following, all references to $\mathrm{A}_{\mathrm{crys}}$, $\mathrm{B}_{\mathrm{dR}}^+$, and so on will implicitly be with respect to the perfectoid ring $\mathcal{O}_C$.

THEOREM 7.13 (cf. Colmez–Nizioł [24], Tsuji [88])
Let $\mathfrak{X}_0/\mathcal{O}_K$ be a smooth proper formal scheme. Let $\mathfrak{X}$ denote the base change of $\mathfrak{X}_0$ to $\mathcal{O}_C$, and let $\overline{\mathfrak{X}_0}$ be its reduction modulo π. For each $i \geq 0$, we have a natural pullback square in $D(\mathbb{Q}_p)$:

$$\begin{tikzcd}
R\Gamma(\mathfrak{X}_{\mathrm{proet}}, \tau^{\leq i} R\psi_*(\mathbb{Q}_p(i))) \arrow[r] \arrow[d] & (\mathrm{A}_{\mathrm{crys}} \otimes_{W(k)} R\Gamma_{\mathrm{crys}}(\overline{\mathfrak{X}_0}/W(k)))^{\varphi = p^i}[1/p] \arrow[d] \\
\mathrm{Fil}^{\geq i}(\Omega_{X_0/K} \otimes_K \mathrm{B}_{\mathrm{dR}}^+) \arrow[r] & (\Omega_{X_0/K} \otimes_K \mathrm{B}_{\mathrm{dR}}^+).
\end{tikzcd}$$

Proof
We claim that this follows from Theorem 6.17, applied to $\mathfrak{X}$. Note that (55) gives a fiber square

$$\begin{tikzcd}
\mathbb{Q}_p(i)(\mathfrak{X}) \arrow[r] \arrow[d] & \mathbb{Q}_p(i)(\mathfrak{X}/\pi) \arrow[d] \\
(L\Omega^{\geq i}_{\mathfrak{X}/\mathcal{O}_K})_{\mathbb{Q}_p} \arrow[r] & (L\Omega_{\mathfrak{X}/\mathcal{O}_K})_{\mathbb{Q}_p}.
\end{tikzcd} \tag{69}$$

In fact, $\mathbb{Q}_p(i)(\mathfrak{X}/p) \to \mathbb{Q}_p(i)(\mathfrak{X}/\pi)$ is an isomorphism thanks to Lemma 6.19.

The top left term in (69) is identified via Theorem 7.11. For the top right, we observe that there is a natural equivalence

$$\mathfrak{X} \otimes_{\mathcal{O}_C} \mathcal{O}_C/\pi \simeq \overline{\mathfrak{X}_0} \otimes_k \mathcal{O}_C/\pi.$$

For $\mathbb{F}_p$-algebras, the construction $LW\Omega_{(-)}$ satisfies a Künneth formula, so we get

$$LW\Omega_{\mathfrak{X}/\pi} \simeq LW\Omega_{\overline{\mathfrak{X}_0}} \otimes_{W(k)} \mathrm{A}_{\mathrm{crys}}.$$

Note that we do not need to p-complete again, since $\mathfrak{X}_0$ is smooth and proper. Taking Frobenius fixed points, we identify the top right term now rationally, thanks to (54).

We can replace the bottom row of (69) with its completion with respect to the (rationalized) Hodge filtration. Recall that p-adic derived de Rham cohomology together with its Hodge filtration (so as a filtered $\mathbb{E}_\infty$-algebra) satisfies a Künneth formula. Therefore, we have

$$L\Omega_{\mathfrak{X}/\mathcal{O}_K} \simeq L\Omega_{\mathfrak{X}_0/\mathcal{O}_K} \otimes_{\mathcal{O}_K} L\Omega_{\mathcal{O}_C/\mathcal{O}_K},$$

since the filtration on $L\Omega_{\mathfrak{X}_0/\mathcal{O}_K}$ is finite, and it is by perfect $\mathcal{O}_K$-modules. It follows that the Hodge completion of the rationalization of $L\Omega_{\mathfrak{X}/\mathcal{O}_K}$ is equivalent, in the filtered derived ∞-category of K, to

$$\Omega_{X_0/K} \otimes_K \mathrm{B}_{\mathrm{dR}}^+,$$

where we use Construction 7.6 for the identification with $\mathrm{B}_{\mathrm{dR}}^+$. $\square$

Appendix A. Twisted Tate diagonals

In this section, we investigate under which conditions Hochschild homology in a general symmetric monoidal ∞-category admits a (twisted) cyclotomic structure. The main result is Corollary A.9, and the fact that it applies to graded and filtered THH is recorded in Examples A.10 and A.11.

As usual, we fix a prime p. Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category, and let $L: \mathcal{C} \to \mathcal{C}$ be a symmetric monoidal, left adjoint functor.

Definition A.1
An *L-twisted diagonal* on $\mathcal{C}$ is a symmetric monoidal natural transformation

$$\Delta\colon L(C) \to (C \otimes \cdots \otimes C)^{hC_p}$$

of lax symmetric monoidal functors $\mathcal{C} \to \mathcal{C}$. Assume that $\mathcal{C}$ is additionally stable;[15] then an L-twisted *Tate diagonal* is a symmetric monoidal natural transformation

$$\Delta\colon L(C) \to T_p(C) := (C \otimes \cdots \otimes C)^{tC_p}.$$

[15]In fact, semiadditive (so that the Tate construction is defined) suffices.

Example A.2

(1) The ∞-category of spaces admits a (unique) id-twisted diagonal, and the ∞-category of spectra admits a (unique) id-twisted Tate diagonal (see [73, Section III.1]).

(2) More generally, if $\mathcal{C}$ admits the Cartesian symmetric monoidal structure, then it admits a canonical id-twisted diagonal induced by the actual diagonal.

Example A.3

Suppose that R is an $\mathbb{E}_\infty$-ring and $\mathcal{C} = \mathrm{Mod}_R$, considered as a symmetric monoidal ∞-category with the R-linear tensor product. Then every left adjoint, symmetric monoidal functor L is given by an induction along an $\mathbb{E}_\infty$-map $l\colon R \to R$, and we will prove below that the datum of an L-twisted Tate diagonal on $\mathcal{C}$ is equivalent to an $\mathbb{E}_\infty$-homotopy between the composition

$$R \xrightarrow{l} R \xrightarrow{\mathrm{triv}} R^{tC_p}$$

and the Tate-valued Frobenius $\varphi\colon R \to R^{tC_p}$ of R (see [73, IV.1]). We shall refer to an $\mathbb{E}_\infty$-ring R with such a datum as a *cyclotomic base*. An example is $R = \mathbb{S}[z]$ (see Example A.12 below).

Proof

To see that the datum of a twisted Tate diagonal is equivalent to such an equivalence, we first note that a symmetric monoidal natural transformation $L \to T_p$ of functors $\mathrm{Mod}_R \to \mathrm{Mod}_R$ is determined by its restriction to the perfect modules $\mathrm{Mod}_R^\omega \subset \mathrm{Mod}_R$ since L preserves filtered colimits. Since T_p is lax symmetric monoidal, we get a factorization

$$\mathrm{Mod}_R \xrightarrow{T_p} \mathrm{Mod}_{T_p(R)} \xrightarrow{\mathrm{res}_{\mathrm{triv}}} \mathrm{Mod}_R$$

as lax symmetric monoidal functors. Upon restriction to Mod_R^ω, the first functor is given by base change along the Tate-valued Frobenius $\varphi\colon R \to R^{tC_p}$ such that we get a factorization $T_p\,|_{\mathrm{Mod}_R^\omega} = \mathrm{res}_{\mathrm{triv}} \circ \mathrm{ind}_\varphi$. Now a symmetric monoidal transformation

$$L\,|_{\mathrm{Mod}_R^\omega} = \mathrm{ind}_l \to \mathrm{res}_{\mathrm{triv}} \circ \mathrm{ind}_\varphi$$

is by adjunction equivalent to a natural transformation

$$\mathrm{ind}_{\mathrm{triv}\circ l} = \mathrm{ind}_{\mathrm{triv}}\mathrm{ind}_l \to \mathrm{ind}_\varphi.$$

But every object in Mod_R^ω is dualizable, and both functors $\mathrm{ind}_{\mathrm{triv}\circ l}$ and ind_φ are symmetric monoidal. Thus every such symmetric monoidal transformation is necessarily an equivalence and thus induced by an equivalence of maps of $\mathbb{E}_\infty$-rings $R \to R^{tC_p}$. This shows the claim. $\square$

Remark A.4
Note that a general symmetric monoidal, stable ∞-category $\mathcal{C}$ does not admit L-twisted Tate diagonals for arbitrary L. For example, if we consider $\mathcal{C} = \mathcal{D}(\mathbb{Z}) \simeq \mathrm{Mod}_{H\mathbb{Z}}$, then by Example A.3 above a twisted Tate diagonal would be the same as a factorization of the Tate-valued Frobenius

$$H\mathbb{Z} \to (H\mathbb{Z})^{tC_p}$$

through the triv-map $H\mathbb{Z} \to H\mathbb{Z}^{tC_p}$. These two maps, however, differ by Steenrod operations, as shown in [73, IV.1]. But any twist would be the identity.

In the following, we use the notation $\mathrm{HH}A$ to denote the Hochschild homology object of an algebra object $A \in \mathrm{Alg}(\mathcal{C})$ internal to $\mathcal{C}$; for example, if $\mathcal{C} = \mathrm{Sp}$, then this recovers THH.

PROPOSITION A.5
Assume that $\mathcal{C}$ is equipped with an L-twisted diagonal (resp., Tate diagonal). Then we get for each algebra object $A \in \mathrm{Alg}(\mathcal{C})$ an induced S^1-equivariant map

$$L(\mathrm{HH}A) \to (\mathrm{HH}A)^{hC_p} \quad \textit{resp. } L(\mathrm{HH}A) \to (\mathrm{HH}A)^{tC_p}$$

which is functorial and symmetric monoidal in A.

Proof
We closely follow the construction of the cyclotomic structure on THH given in [73, Section III.2]. We will mostly indicate the necessary changes and thus recommend that the reader take a look at the construction there first. We treat the case of the Tate diagonal, which is the only case that we will need in this paper, although the case of the diagonal works exactly the same.

We first recall that $\mathrm{HH}A$ is the geometric realization of the cyclic object in $\mathcal{C}$ informally written as

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A \otimes A \otimes A \arrow[loop above, "C_3"] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A \otimes A \arrow[loop above, "C_2"] \arrow[r, shift left=1] \arrow[r, shift right=1] & A . \end{tikzcd}$$

Thus $L(\mathrm{HH}A)$ is the geometric realization of the cyclic object

$$
\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & L(A \otimes A \otimes A) \arrow[loop above, "C_3"] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & L(A \otimes A) \arrow[loop above, "C_2"] \arrow[r, shift left=1] \arrow[r, shift right=1] & L(A)\,.
\end{tikzcd}
$$

For a given L-twisted Tate diagonal

$$\Delta\colon L(C) \to (C \otimes \cdots \otimes C)^{tC_p} = T_p(C)$$

we want to construct a natural map of cyclic objects

$$
\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & L(A^{\otimes 3}) \arrow[loop above, "C_3"] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\Delta"] & L(A^{\otimes 2}) \arrow[loop above, "C_2"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d, "\Delta"] & LA \arrow[d, "\Delta"] \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & (A^{\otimes 3p})^{tC_p} \arrow[loop below, "C_3"] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & (A^{\otimes 2p})^{tC_p} \arrow[loop below, "C_2"] \arrow[r, shift left=1] \arrow[r, shift right=1] & (A^{\otimes p})^{tC_p}
\end{tikzcd}
$$

and obtain the desired map $L(\mathrm{HH}A) \to (\mathrm{HH}A)^{tC_p}$ as the geometric realization of this map of cyclic objects followed by the canonical interchange map from the realization of the Tate constructions to the Tate construction of the realization.

In order to construct such a natural transformation of cyclic objects, we proceed as in [73]: we eventually need to show that we can extend the symmetric monoidal natural transformation $\Delta\colon L \to T_p$ of functors $\mathcal{C} \to \mathcal{C}$ to a BC_p-equivariant symmetric monoidal natural transformation of functors from the functor

$$\tilde{L}\colon N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} \mathcal{C}^{\otimes}_{\mathrm{act}} \xrightarrow{\mathrm{pr}} \mathcal{C}^{\otimes}_{\mathrm{act}} \xrightarrow{\otimes} \mathcal{C} \xrightarrow{L} \mathcal{C}$$

given by

$$\big(S, (X_{\overline{s} \in \overline{S} = S/C_p})\big) \mapsto L\Big(\bigotimes_{\overline{s} \in \overline{S}} X_{\overline{s}}\Big)$$

to the functor

$$\tilde{T}_p\colon N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} \mathrm{Sp}^{\otimes}_{\mathrm{act}} \to (\mathcal{C}^{\otimes}_{\mathrm{act}})^{BC_p} \xrightarrow{\otimes} \mathcal{C}^{BC_p} \xrightarrow{-^{tC_p}} \mathcal{C}$$

given by

$$\big(S, (X_{\overline{s} \in \overline{S} = S/C_p})\big) \mapsto \Big(\bigotimes_{s \in S} X_{\overline{s}}\Big)^{tC_p}.$$

Here Free_{C_p} is the category of finite free C_p-sets equipped with the co-Cartesian symmetric monoidal structure. The group object BC_p-acts on this category in the obvious way and acts trivially on $\mathcal{C}$. For a precise construction of these functors, we refer to [73, Section III.3], specifically Proposition III.3.6 and the construction around that.

The inclusion

$$\mathrm{Fun}_{\otimes}\big(N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} \mathcal{C}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big) \subseteq \mathrm{Fun}_{\mathrm{lax}}\big(N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} \mathcal{C}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big)$$

admits a right adjoint by Lemma III.3.3 (resp., Remark III.3.5) in [73]. Using the same construction and argument as in the proof of [73, Lemma III.3], we see that the ∞-category $\mathrm{Fun}_{\otimes}(N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} \mathcal{C}^{\otimes}_{\mathrm{act}}, \mathcal{C})$ is equivalent to the ∞-category $\mathrm{Fun}(N\mathrm{Tor}_{C_p}, \mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}, \mathcal{C}))$, where Tor_{C_p} denotes the category of C_p-torsors. Under this equivalence, the right adjoint to the inclusion is given by restricting a functor in $\mathrm{Fun}_{\mathrm{lax}}(N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} \mathcal{C}^{\otimes}_{\mathrm{act}}, \mathcal{C})$ to $N\mathrm{Tor}_{C_p} \times \mathcal{C} \subseteq N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} \mathcal{C}^{\otimes}_{\mathrm{act}}$ and forming the adjunct.

Now the functor $\tilde{L}$ is symmetric monoidal rather than lax symmetric monoidal. Thus to construct a map from $\tilde{L}$ to $\tilde{T}_p$ is by adjunction equivalent to constructing a transformation in $\mathrm{Fun}(N\mathrm{Tor}_{C_p}, \mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}, \mathcal{C}))$ between the respective restrictions. Moreover, BC_p-acts on all those categories; that is, constructing a BC_p-equivariant transformation between $\tilde{L}$ and $\tilde{T}_p$ is equivalent to constructing a transformation in

$$\mathrm{Fun}^{BC_p}\big(N\mathrm{Tor}_{C_p}, \mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}, \mathcal{C})\big).$$

Now the category Tor_{C_p} is in fact equivalent to BC_p. Since the BC_p-action on $\mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}, \mathcal{C})$ is trivial, it follows that the above ∞-category of BC_p-equivariant functors is equivalent to $\mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}, \mathcal{C})$.

Taking everything together, we see that there is a unique symmetric monoidal transformation $\tilde{L} \to \tilde{T}_p$ extending the transformation $\Delta\colon L \to T_p$. Together with the constructions above, this finishes the proof. □

We shall refer to the map $L(\mathrm{HH}A) \to (\mathrm{HH}A)^{tC_p}$ as a *twisted cyclotomic structure* on $\mathrm{HH}A$. Thus the last result shows that, for ∞-categories with a twisted Tate diagonal, we find that Hochschild homology admits a twisted cyclotomic structure.

LEMMA A.6
For a given L-twisted diagonal on $\mathcal{C}$, the stabilization $\mathrm{Sp}(\mathcal{C})$ *admits a canonical induced* $\mathrm{Sp}(L)$*-twisted Tate diagonal.*

Proof
We would like to construct a symmetric monoidal natural transformation

$$\mathrm{Sp}(L)(C) \to (C \otimes \cdots \otimes C)^{tC_p} = T_p(C).$$

Such a transformation is by adjunction the same as a symmetric monoidal transformation

$$\mathrm{id} \to R' T_p,$$

where $R'\colon \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$ is the right adjoint to $\mathrm{Sp}(L)$. We now use that the functor Ω^∞ induces an equivalence

$$\mathrm{Fun}^{\mathrm{Ex}}_{\mathrm{lax}}\big(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{C})\big) \to \mathrm{Fun}^{\mathrm{Ex}}_{\mathrm{lax}}\big(\mathrm{Sp}(\mathcal{C}), \mathcal{C}\big)$$

by [72]. It follows that it suffices to construct a symmetric monoidal transformation

$$\Omega^\infty \to \Omega^\infty R' T_p$$

of functors $\mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$. We denote by $R\colon \mathcal{C} \to \mathcal{C}$ the right adjoint to the functor $L\colon \mathcal{C} \to \mathcal{C}$. Then we have an equivalence $\Omega^\infty R' \simeq R\Omega^\infty$ of lax symmetric monoidal functors which follows from the fact that the left adjoint diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "L"] \arrow[d, "\Sigma^\infty"] & \mathcal{C} \arrow[d, "\Sigma^\infty"] \\
\mathrm{Sp}(\mathcal{C}) \arrow[r, "\mathrm{Sp}(L)"] & \mathrm{Sp}(\mathcal{C})
\end{tikzcd}$$

commutes (up to symmetric monoidal equivalence). As a result, we need to construct a symmetric monoidal natural transformation

$$\Omega^\infty \to R\Omega^\infty T_p. \tag{70}$$

Now we use that we have canonical symmetric monoidal transformations

$$\gamma\colon (\Omega^\infty C \otimes \cdots \otimes \Omega^\infty C)^{hC_p} \to \Omega^\infty\big((C \otimes \cdots \otimes C)^{hC_p}\big) \to \Omega^\infty\big((C \otimes \cdots \otimes C)^{tC_p}\big),$$

where the first one is induced by the lax symmetric monoidal structure of Ω^∞ together with the fact that it commutes with limits and the second by the canonical map from homotopy fixed points to the Tate construction.

Now we use the unstable diagonal on $\mathcal{C}$ to get as the adjoint a symmetric monoidal natural transformation

$$\Omega^\infty C \to R(\Omega^\infty C \otimes \cdots \otimes \Omega^\infty C)^{hC_p}$$

and compose it with the map $R(\gamma)$ above to get a symmetric monoidal natural transformation as in (70). $\square$

For every symmetric monoidal ∞-category I, we consider the symmetric monoidal functor $l_p\colon I \to I$ given by sending i to $i^{\otimes p}$. We let

$$L_p\colon \mathrm{Fun}(I, \mathcal{S}) \to \mathrm{Fun}(I, \mathcal{S})$$

be a left Kan extension along l_p. We equip the category $\mathrm{Fun}(I, \mathcal{S})$ with the Day convolution symmetric monoidal structure. Then the left Kan extension L_p becomes symmetric monoidal.

LEMMA A.7
Assume that the ∞-category I has the following property: for every pair of objects $i, j \in I$ we have that the canonical forgetful map

$$\mathrm{Map}_I(i^{\otimes p}, j)^{hC_p} \to \mathrm{Map}_I(i^{\otimes p}, j) \tag{71}$$

is an equivalence of spaces.[16] *Then the inverse of the map (71) induces a canonical L_p-twisted diagonal on* $\mathrm{Fun}(I, \mathcal{S})$.

Proof
We consider the symmetric monoidal (co)Yoneda embedding

$$I^{\mathrm{op}} \to \mathrm{Fun}(I, \mathcal{S}).$$

Then symmetric monoidal transformations

$$L_p(C) \to (C \otimes \cdots \otimes C)^{hC_p}$$

as functors $\mathrm{Fun}(I, \mathcal{S}) \to \mathrm{Fun}(I, \mathcal{S})$ are the same as symmetric monoidal transformations between the restrictions of the functors along the Yoneda embedding. The restricted functors $I^{\mathrm{op}} \to \mathrm{Fun}(I, \mathcal{S})$ are given by the lax symmetric monoidal assignments

$$i \mapsto \big(j \mapsto \mathrm{Map}_I(i^{\otimes p}, j)\big) \qquad \text{and} \qquad i \mapsto \big(j \mapsto \mathrm{Map}_I(i^{\otimes p}, j)^{hC_p}\big).$$

The canonical map $\mathrm{Map}_I(i^{\otimes p}, j)^{hC_p} \to \mathrm{Map}_I(i^{\otimes p}, j)$ is a symmetric monoidal natural transformation. By assumption, it is an equivalence so that the inverse induces the required transformation. $\square$

Remark A.8
For a general symmetric monoidal ∞-category I, the category $\mathrm{Fun}(I, \mathcal{S})$ does not

[16]Note that an equivalent way of stating this condition is to say that the homotopy orbits $(i^{\otimes p})_{hC_p}$ exist in I and the map $i^{\otimes p} \to (i^{\otimes p})_{hC_p}$ is an equivalence.

admit an L_p-twisted diagonal. As an example, consider any co-Cartesian symmetric monoidal ∞-category I. Then the Day convolution structure on $\mathrm{Fun}(I,\mathrm{Sp})$ is Cartesian.[17] Thus an L_p-twisted diagonal would amount to a natural symmetric monoidal transformation

$$F^{\times p} \to (F^{\times p})^{hC_p},$$

which does not exist.

But note that this category admits an id-twisted diagonal. This raises the question if for every symmetric monoidal ∞-category I there is a twist on $\mathrm{Fun}(I,\mathcal{S})$ and a twisted diagonal. The answer to this question is also "no" in general, but we will not go into the intricacies of concrete counterexamples here.

COROLLARY A.9

If I is a symmetric monoidal ∞-category satisfying the condition of Lemma A.7, then we have for every algebra A in $\mathrm{Fun}(I,\mathrm{Sp})$ *a twisted cyclotomic structure on* $\mathrm{HH}A$*, that is, an S^1-equivariant map*

$$L_p(\mathrm{HH}A) \to (\mathrm{HH}A)^{tC_p}.$$

This map is natural and symmetric monoidal in A.

Proof

Combine Proposition A.5 with Lemmas A.6 and A.7. □

Example A.10

We consider the category $I = \mathbb{Z}_{\geq 0}^{\mathrm{ds}}$. Then $\mathrm{Fun}(I,\mathrm{Sp})$ is the ∞-category of graded spectra. The category I obviously satisfies the condition of Lemma A.7. Thus we get that for a graded ring $R_\bullet$, graded THH admits an L_p-twisted cyclotomic structure or, equivalently, a sequence of S^1-equivariant maps

$$\mathrm{THH}(R)_i \to \mathrm{THH}(R)_{pi}^{tC_p}.$$

The same logic applies to spectra graded over any discrete monoid in place of $\mathbb{Z}_{\geq 0}^{\mathrm{ds}}$.

Example A.11

Consider the ∞-category $I = \mathbb{Z}_{\geq 0}^{\mathrm{op}}$ associated to the poset of positive integers. Then this also satisfies the condition of Lemma A.7. The category of functors $\mathrm{Fun}(I,\mathrm{Sp})$ is given by filtered spectra, and thus filtered THH of a filtered ring spectrum R admits a

[17] This follows from the fact that, generally, Day convolution for a co-Cartesian source is given by the pointwise tensor product, which, in our case, happens to agree with the Cartesian product.

filtered cyclotomic structure, that is, S^1-equivariant maps

$$\mathrm{Fil}^{\geq i}\mathrm{THH}(R) \to \big(\mathrm{Fil}^{\geq pi}\mathrm{THH}(R)\big)^{tC_p}.$$

Example A.12
Consider the category $I = B\mathbb{Z}_{\geq 0}$. This category also obviously satisfies the condition of Lemma A.7. Thus the category

$$\mathrm{Fun}(I, \mathrm{Sp}) \simeq \mathrm{Mod}_{\mathbb{S}[z]}$$

admits a twisted Tate diagonal, and thus relative THH admits a (twisted) cyclotomic structure, as is used in [12, Section 11]. The twist L_p corresponds to the map $l\colon \mathbb{S}[z] \to \mathbb{S}[z]$ sending z to z^p.

We want to end this section by noting some functorialities of the twisted cyclotomic structures.

Definition A.13
A *symmetric monoidal category with (Tate) diagonals* consists of a triple $(\mathcal{C}, L, \Delta)$ as in Definition A.1. A map of symmetric monoidal categories with (Tate) diagonals

$$(\mathcal{C}, L, \Delta) \to (\mathcal{C}', L', \Delta')$$

is given by a left adjoint symmetric monoidal functor $F\colon \mathcal{C} \to \mathcal{C}'$ together with a symmetric monoidal equivalence $L' \circ F \simeq F \circ L$ and a natural symmetric monoidal equivalence between the two maps

$$L'(FX) \to (FX \otimes \cdots \otimes FX)^{tC_p}$$

induced from Δ and Δ' (both sides considered as lax symmetric monoidal functors $\mathcal{C} \to \mathcal{C}'$).

Form the construction of the twisted cyclotomic structure in Proposition A.5, we see immediately that for such a map of symmetric monoidal ∞-categories with Tate diagonals we get an equivalence of twisted cyclotomic objects

$$F(\mathrm{HH}A) \simeq \mathrm{HH}(FA)$$

for every algebra A in $\mathcal{C}$. Here the first object $F(\mathrm{HH}A)$ is twisted cyclotomic by the composition

$$LF(\mathrm{HH}A) \xrightarrow{\simeq} FL(\mathrm{HH}A) \xrightarrow{F\varphi} F(\mathrm{HH}A^{tC_p}) \to F(\mathrm{HH}A)^{tC_p}.$$

We also have a relative analogue of Lemma A.6: every map of symmetric monoidal ∞-categories with diagonals induces upon stabilization a map of symmetric monoidal ∞-categories with Tate diagonals. This is straightforward to prove. Finally, there is also an analogue of Lemma A.7, which we will state and prove now.

LEMMA A.14
Assume that $f : I \to I'$ is a symmetric monoidal functor such that I and I' satisfy the condition of Lemma A.7. Then left Kan extension along f induces a map of symmetric monoidal ∞-categories with diagonals

$$\bigl(\mathrm{Fun}(I, \mathcal{S}), L_p, \Delta\bigr) \to \bigl(\mathrm{Fun}(I', \mathcal{S}), L'_p, \Delta'\bigr),$$

where L_p, L'_p, Δ and Δ' are as in Lemma A.7.

Proof
We have a commutative square

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d, "l_p"] & I' \arrow[d, "l'_p"] \\ I \arrow[r, "f"] & I' \end{tikzcd}$$

for the functors $l_p(i) = i^{\otimes p}$ and $l'_p(j) = j^{\otimes p}$. Thus we get an induced square of the left Kan extensions

$$\begin{tikzcd} \mathrm{Fun}(I, \mathcal{S}) \arrow[r, "F"] \arrow[d, "L_p"] & \mathrm{Fun}(I', \mathcal{S}) \arrow[d, "L'_p"] \\ \mathrm{Fun}(I, \mathcal{S}) \arrow[r, "F"] & \mathrm{Fun}(I', \mathcal{S}). \end{tikzcd}$$

This provides the first part of the datum of a map of symmetric monoidal ∞-categories with Tate diagonals. We now also have to provide an equivalence of two different natural transformations between two functors

$$\mathrm{Fun}(I, \mathcal{S}) \to \mathrm{Fun}(I', \mathcal{S}).$$

Such a transformation is determined by its restriction to $I^{\mathrm{op}} \subseteq \mathrm{Fun}(I, \mathcal{S})$, and there the functors are given by

$$i \mapsto \bigl(j \mapsto \mathrm{Map}_{I'}\bigl(f(i)^{\otimes p}, j\bigr)\bigr)$$

and

$$i \mapsto \big(j \mapsto \mathrm{Map}_{I'}\big(f(i)^{\otimes p}, j\big)^{hC_p}\big).$$

Unravelling the constructions, we see that both of the two transformations are given by the inverse of the canonical forgetful map

$$\mathrm{Map}_{I'}\big(f(i)^{\otimes p}, j\big)^{hC_p} \to \mathrm{Map}_{I'}\big(f(i)^{\otimes p}, j\big)$$

and thus are canonically equivalent. □

From these statements together we can deduce the following corollary.

COROLLARY A.15
Assume that $f: I \to I'$ is a symmetric monoidal functor such that I and I' satisfy the condition of Lemma A.7. Then, for every algebra $A \in \mathrm{Fun}(I, \mathrm{Sp})$, we have an equivalence of L'_p twisted cyclotomic objects

$$F(\mathrm{HH}A) \simeq \mathrm{HH}(FA),$$

where F is left Kan extension along f.

Example A.16
For a graded ring spectrum $R_\bullet$ we have that the direct sum

$$\bigoplus_i \mathrm{THH}(R_\bullet)_i$$

is equivalent to $\mathrm{THH}(\bigoplus_i R)$ as cyclotomic spectra. Similarly, for a filtered ring spectrum R, we have that the filtered cyclotomic structure refines the cyclotomic structure on $\mathrm{THH}(R)$.

Example A.17
We finally note that one can also look at the functor

$$\mathrm{ev}_0 : \mathrm{Fun}(\mathbb{Z}^{\mathrm{ds}}_{\geq 0}, \mathrm{Sp}) \to \mathrm{Sp}$$

given by restriction to the 0th component. We claim that this also refines to a map of symmetric monoidal ∞-categories with Tate diagonals. This can be seen by verifying the corresponding unstable statement, which is straightforward using an argument similar to the one in the proof of Corollary A.15. This then shows that the cyclotomic structure on the 0th graded component $\mathrm{THH}(R_\bullet)_0$ agrees with the one on $\mathrm{THH}(R_0)$ for every graded ring spectrum $R_\bullet$.

Appendix B. Categorical lemmas

Construction B.1 (Left Kan extensions)
Let R be a ring, and let Poly_R be the category of finitely generated polynomial R-algebras. Given a presentable ∞-category $\mathcal{C}$ and an accessible functor $f: \mathrm{Poly}_R \to \mathcal{C}$, we can left Kan extend to obtain a functor $Lf: \mathrm{SCR}_R \to \mathcal{C}$ which commutes with geometric realizations, for SCR_R the ∞-category of simplicial commutative R-algebras (cf. [62, Section 5.5.8] and [63, Section 4.2]).

Let $(\mathcal{L}, \mathcal{R})\colon \mathcal{C} \rightleftarrows \mathcal{D}$ be an adjunction of ∞-categories. Then, for any ∞-category $\mathcal{E}$, we obtain an adjunction

$$(\mathcal{R}^*, \mathcal{L}^*) = (f \mapsto f \circ \mathcal{R}, f' \mapsto f \circ \mathcal{L})\colon \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \rightleftarrows \mathrm{Fun}(\mathcal{D}, \mathcal{E}). \tag{72}$$

Remark B.2
Let $f_1, f_2\colon \mathcal{D} \to \mathcal{E}$ be functors. Suppose that, for any $x \in \mathcal{D}$, the natural map $f_1(\mathcal{L}\mathcal{R}x) \to f_1(x)$ is an equivalence. Then we find

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{D},\mathcal{E})}(f_1, f_2) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{E})}(f_1 \circ \mathcal{L}, f_2 \circ \mathcal{L}). \tag{73}$$

This follows from the adjunction (72).

Now we specialize to the case where $\mathcal{C} = \mathrm{SCR}$ is the ∞-category of simplicial commutative rings and $\mathcal{D} = \mathrm{SCR}_{\mathbb{F}_p}$ is the ∞-category of simplicial commutative $\mathbb{F}_p$-algebras. We have an adjunction $(\mathcal{L}, \mathcal{R})\colon \mathrm{SCR} \rightleftarrows \mathrm{SCR}_{\mathbb{F}_p}$, where the left adjoint is $R \mapsto R \otimes^L_{\mathbb{Z}} \mathbb{F}_p$ and the right adjoint is simply the forgetful functor.

For any $R \in \mathrm{SCR}_{\mathbb{F}_p}$, we have a canonical endomorphism $\varphi\colon R \to R$, the Frobenius.

LEMMA B.3
Let $R \in \mathrm{SCR}_{\mathbb{F}_p}$. There is a natural map $f\colon R \to R \otimes^L_{\mathbb{Z}} \mathbb{F}_p$ in $\mathrm{SCR}_{\mathbb{F}_p}$ such that the composites $R \xrightarrow{f} R \otimes^L_{\mathbb{Z}} \mathbb{F}_p \to R$ and $R \otimes^L_{\mathbb{Z}} \mathbb{F}_p \to R \to R \otimes^L_{\mathbb{Z}} \mathbb{F}_p$ are the respective Frobenius endomorphisms.

Proof
It suffices to assume that R is discrete (even a finitely generated polynomial ring) via left Kan extension. In this case, $R \otimes^L_{\mathbb{Z}} \mathbb{F}_p$ is concentrated in homological degrees 0 and 1 (with $\pi_0 = R$ itself), and one knows that the Frobenius endomorphism annihilates π_1 (cf. [14, Proposition 11.6]). Thus, the Frobenius map $R \otimes^L_{\mathbb{Z}} \mathbb{F}_p \to R \otimes^L_{\mathbb{Z}} \mathbb{F}_p$ factors canonically through the truncation map $R \otimes^L_{\mathbb{Z}} \mathbb{F}_p \to \pi_0(R \otimes^L_{\mathbb{Z}} \mathbb{F}_p) \cong R$. This gives the map f as desired. $\square$

COROLLARY B.4
Let $F_1, F_2\colon \mathrm{SCR}_{\mathbb{F}_p} \to D(\mathbb{Z})$ *be two functors. Suppose that* F_1 *has the property that the natural map* $F_1(R) \to F_1(R)$ *given by Frobenius is multiplication by* p^i. *Then for any natural transformation* $u\colon F_1(- \otimes^L_{\mathbb{Z}} \mathbb{F}_p) \to F_2(- \otimes^L_{\mathbb{Z}} \mathbb{F}_p)$ *of functors* $\mathrm{SCR}_{\mathbb{Z}} \to D(\mathbb{Z})$, *we have that* $p^i u$ *arises from a natural transformation* $F_1 \to F_2$. *In fact, we have*

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Fun}(\mathrm{SCR}_{\mathbb{Z}}, D(\mathbb{Z}))}\big(F_1(- \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p), F_2(- \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p)\big)[1/p] \\ \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathrm{SCR}_{\mathbb{F}_p}, D(\mathbb{Z}))}(F_1, F_2)[1/p].\end{aligned}$$

Proof
This follows from (72) in the case of the adjunction $(\mathcal{L}, \mathcal{R})\colon \mathrm{SCR}_{\mathbb{Z}} \rightleftarrows \mathrm{SCR}_{\mathbb{F}_p}$. By construction, we are given a map $\mathcal{L}^* F_1 \to \mathcal{L}^* F_2$ of functors $\mathrm{SCR}_{\mathbb{F}_p} \to D(\mathbb{Z})$, or, equivalently, by adjointness a map $\mathcal{R}^*\mathcal{L}^* F_1 \to F_2$ of functors $\mathrm{SCR}_{\mathbb{Z}} \to D(\mathbb{Z})$. Now we have a natural map $F_1 \to \mathcal{R}^*\mathcal{L}^* F_1$ given by the natural map $f\colon R \to R \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p$ of Lemma B.3; it has the property that the composites in either order with the adjunction map $\mathcal{R}^*\mathcal{L}^* F_1 \to F_1$ are given by multiplication by p^i. The composition $F_1 \to \mathcal{R}^*\mathcal{L}^* F_1 \to F_2$ defines the desired map $F_1 \to F_2$. This argument also proves the displayed equation. $\square$

Let K be a complete discretely valued field with ring of integers $\mathcal{O}_K \subset K$ and residue field k; let $\pi \in \mathcal{O}_K$ be a uniformizer. Let $\mathrm{FSmooth}_{\mathcal{O}_K}$ denote the category of topologically finitely generated, formally smooth $\mathcal{O}_K$-algebras, and let Smooth_k denote the category of smooth k-algebras.

We now give a similar result for functors defined on a restricted class of simplicial commutative k-algebras. For the next result, we will argue similarly, but with a smaller set of ∞-categories. For these finiteness conditions, see [64, Section 7.2] (in the slightly more complicated $\mathbb{E}_\infty$-case).

Definition B.5

(1) Let $\mathrm{SCR}_k^{\mathrm{afp}}$ denote the ∞-category of simplicial commutative k-algebras R which are almost finitely presented: equivalently, $\pi_0(R)$ is finitely generated as a k-algebra and each $\pi_i(R)$ is a finitely generated $\pi_0(R)$-module. Equivalently, R belongs to $\mathrm{SCR}_k^{\mathrm{afp}}$ if and only if R can be written as the geometric realization of a simplicial diagram of finitely generated polynomial k-algebras.

(2) Similarly, we define $\widehat{\mathrm{SCR}}_{\mathcal{O}_K}^{\mathrm{afp}}$ to be the ∞-category of π-complete simplicial commutative $\mathcal{O}_K$-algebras R such that $\pi_0(R)$ is topologically finitely generated over $\mathcal{O}_K$ (i.e., a quotient of a π-completed polynomial ring) and each

$\pi_i(R)$ is finitely generated over R. Equivalently, R belongs to $\widehat{\mathrm{SCR}}^{\mathrm{afp}}_{\mathcal{O}_K}$ if and only if R can be written as the geometric realization of a simplicial diagram of π-completed finitely generated polynomial $\mathcal{O}_K$-algebras. Yet another characterization is that R should be almost finitely presented over the π-completion of a finitely generated polynomial algebra over $\mathcal{O}_K$ with a structure map that is surjective on π_0.

COROLLARY B.6
Let $\mathcal{E}$ be an ∞-category admitting sifted colimits. Let $F_1, F_2\colon \mathrm{SCR}^{\mathrm{afp}}_k \to \mathcal{E}$ be functors. If

(1) *F_1 commutes with geometric realizations, and*

(2) *$F_1(R) \simeq F_1(\pi_0 R)$ for $R \in \mathrm{SCR}^{\mathrm{afp}}_k$,*

then

$$\mathrm{Hom}_{\mathrm{Fun}(\mathrm{Smooth}_k,\mathcal{E})}(F_1, F_2) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathrm{FSmooth}_{\mathcal{O}_K},\mathcal{E})}\big(F_1(-\otimes_{\mathcal{O}_K} k), F_2(-\otimes_{\mathcal{O}_K} k)\big).$$

Proof
Since F_1 is left Kan extended from smooth (even finite-type polynomial) k-algebras as it commutes with geometric realizations, we have

$$\mathrm{Hom}_{\mathrm{Fun}(\mathrm{Smooth}_k,\mathcal{E})}(F_1, F_2) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathrm{SCR}^{\mathrm{afp}}_k,\mathcal{E})}(F_1, F_2).$$

Similarly,

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{Fun}(\mathrm{FSmooth}_{\mathcal{O}_K},\mathcal{E})}\big(F_1(-\otimes_{\mathcal{O}_K} k), F_2(-\otimes_{\mathcal{O}_K} k)\big)\\ &\quad\simeq \mathrm{Hom}_{\mathrm{Fun}(\widehat{\mathrm{SCR}}^{\mathrm{afp}}_{\mathcal{O}_K},\mathcal{E})}\big(F_1(-\otimes_{\mathcal{O}_K} k), F_2(-\otimes_{\mathcal{O}_K} k)\big),\end{aligned}$$

because $F_1(-\otimes_{\mathcal{O}_K} k)\colon \widehat{\mathrm{SCR}}^{\mathrm{afp}}_{\mathcal{O}_K} \to \mathcal{E}$ is left Kan extended from $\mathrm{FSmooth}_{\mathcal{O}_K}$. Now we have an adjunction $\widehat{\mathrm{SCR}}^{\mathrm{afp}}_{\mathcal{O}_K} \rightleftarrows \mathrm{SCR}^{\mathrm{afp}}_k$ given by base change and restriction of scalars. Thus, the result follows as in Remark B.2. □

Acknowledgments. We are very grateful to Peter Scholze for suggesting this question to us and for sharing many insights. We would also like to thank Johannes Anschütz, Alexander Beilinson, Bhargav Bhatt, Hélène Esnault, Lars Hesselholt, Luc Illusie, Moritz Kerz, Arthur-César Le Bras, Wiesława Nizioł, and Sam Raskin for helpful conversations. The third author would like to thank Uwe Jannsen, Moritz Kerz, and Guido Kings for organizing and inviting him to the 2014 Kleinwalsertal workshop on Beilinson's paper [8].

Two referees provided detailed, invaluable comments on the paper which has been improved as a result.

The first two authors would like to thank the University of Bonn and the University of Münster for their hospitality. This material is based upon work supported by National Science Foundation (NSF) grant DMS-1440140 while the first three authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the spring 2019 semester. The first author was supported by NSF grant DMS-1552766. This work was done while the second author was a Clay Research Fellow. The fourth author was funded by the Deutsche Forschungsgemeinschaft under Germany's Excellence Strategy EXC 2044-390685587, Mathematics Münster: Dynamics–Geometry–Structure.

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