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Generalized Tate Homology, Homotopy Fixed Points and the Transfer

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Abstract. We define and study the *generalized Tate homology* of a compact Lie group G with coefficients in a spectrum upon which G acts.

§1. Introduction

Let G be a finite group and X a G–CW complex. The cellular chain complex $\mathcal{S}_*(X)$ is a graded $\mathbf{Z}[G]$-module, and the hyperhomology of G with coefficients in $\mathcal{S}_*(X)$ (denoted $H_*^G(X)$) is known as the equivariant homology of X. The groups $H_*^G(X)$ can be identified with the ordinary homology groups of the Borel construction $EG \times_G X$ or equivalently with the homotopy groups of the spectrum $\mathbf{H} \wedge (EG \times_G X)_+$, where $\mathbf{H}$ is the Eilenberg-MacLane spectrum $\{K(\mathbf{Z}, n)\}$.

Define the *equivariant Tate homology* of X, denoted $\hat{H}_*^G(X)$, to be the Tate hyperhomology of G with coefficients in $\mathcal{S}_*(X)$. (Up to regrading, the *Tate hyperhomology* of G with coefficients in a $\mathbf{Z}[G]$ chain complex $\mathcal{S}_*$ is the Tate hypercohomology [Sw] of the cochain complex S^* obtained by reversing the indices of S_* in sign. The regrading is such that the Tate hyperhomology of G with coefficients in a single module M concentrated in degree 0 agrees in strictly positive dimensions with the ordinary group homology of G with coefficients in M.) One goal of this paper is to prove that the equivariant Tate homology of X can also be expressed in a natural way as the homotopy of a certain spectrum. In fact, we will make a much more general construction. Let G be a compact Lie group. A *spectrum* $\mathbf{E}$ *with an action of* G is by definition a spectrum $\{E_n\}$ in the usual sense (cf. [A]) together with an action of G on each space with respect to which the suspension maps are equivariant. A *weak equivalence* between two such objects is an equivariant map which is a weak equivalence on the underlying spectra.

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THEOREM 1.1. *Let G be a compact Lie group and $\mathbf{E}$ a spectrum with an action of G. Then there exists a spectrum $\hat{\mathbf{H}}^G(\mathbf{E})$ with the following properties:*

(1) *$\mathbf{E} \mapsto \hat{\mathbf{H}}^G(\mathbf{E})$ is a functor from the category of spectra with an action of G to the category of spectra. This functor carries weak equivalences to weak equivalences.*

(2) *If G is a finite group, X is a G–CW complex, and $\mathbf{E}$ is the spectrum $\mathbf{H} \wedge (X_+)$ with the induced action of G, then there is a natural isomorphism*

$$\pi_*(\hat{\mathbf{H}}^G(\mathbf{E})) \cong \hat{H}_*^G(X) .$$

(3) *If G is a finite group, X is a G–CW complex, and $\mathbf{E}$ is the suspension spectrum of X_+ with the induced action of G, then $\hat{\mathbf{H}}^G(\mathbf{E})$ is equivalent to the fiber of the Kahn-Priddy transfer map [KP]*

$$Q(EG \times_G X_+) \to Q(X_+)^{hG} .$$

(4) *If G is the circle group S^1, X is a G–CW complex, and $\mathbf{E}$ is the spectrum $\mathbf{H} \wedge X_+$ with the induced action of G, then $\pi_*(\hat{\mathbf{H}}^G(\mathbf{E}))$ is the periodic cyclic homology as defined by Jones [J] and Goodwillie [Go2] of the the chain complex $S_*(X)$.*

REMARK: In (3) above, $Q(Y) = \varinjlim \Omega^n \Sigma^n(Y)$. For any G-space Z, Z^{hG} denotes the *homotopy fixed point set* $\mathrm{Map}_G(EG, Z)$.

Part (3) of this theorem reflects the fact that the spectra $\hat{\mathbf{H}}^G(\mathbf{E})$ are in fact constructed in terms of homotopy fibers of certain transfer maps. To be more precise, assume that G is a compact Lie group with Lie algebra g. Given a G-space X, one can use the adjoint representation of G on g to construct an induced vector bundle

$$ad : EG \times_G (g \times X) \to EG \times_G X .$$

Let $(EG \times_G X)^{ad}$ denote the Thom space of this vector bundle. The Becker-Schultz transfer (or *umkehr*) map [BS] can be viewed as a map of spectra

$$\tau : \Sigma^\infty (EG \times_G X)^{ad} \to (\Sigma^\infty(X_+))^{hG}$$

which when G is finite agrees with the Kahn-Priddy transfer mentioned in 1.1(3). (The definition of the above homotopy fixed point spectrum will be given in §2.) Now let $\mathbf{E}$ be any spectrum with an action of G. Using the Becker-Schultz construction we will produce *an* $\mathbf{E}$-*transfer map*

$$\tau_{\mathbf{E}} : (EG_+ \wedge_G \mathbf{E})^{ad} \to \mathbf{E}^{hG}$$

which agrees with the Becker-Schultz transfer when E is $\Sigma^\infty X$. The spectrum $\hat{\mathbf{H}}^G(\mathbf{E})$ is defined to be the stable fiber of $\tau_{\mathbf{E}}$. The generalized Tate homology groups of G with coefficients in E are defined by

$$\hat{\mathbf{H}}_*^G(\mathbf{E}) = \pi_*(\hat{\mathbf{H}}^G(\mathbf{E})) .$$

REMARK: Greenlees was the first to prove that Tate cohomology is a representable functor, and he has studied its representing spectrum in some detail. We urge the reader to consult his paper [**Gr**]. We would like to thank him for making his work available to us and for pointing out common results. Our work complements his in some ways; one of our intentions is to show that defining Tate homology in terms of a classical stable homotopy transfer map naturally leads to a more general notion that allows for compact Lie groups and arbitrary spectra.

Organization of the paper. Section 2 contains the definition of generalized Tate homology and the proof of 1.1(3). Section 3 is concerned with the proof of 1.1(2). Finally, Section 4 treats in detail one example of generalized Tate homology, describes the proof of 1.1(4), and concludes with some remarks about generalized cyclic homology.

§2. GENERALIZED TATE HOMOLOGY

In this section we will give the construction of the Tate homology spectrum $\hat{\mathbf{H}}^G(\mathbf{E})$ and prove 1.1(3). Our basic idea is to use the transfer construction of Becker and Schultz [**BS**] and keep track of equivariance.

First we will establish some notation. Let G be a compact Lie group. If W is a finite dimensional representation of G we will denote by S^W the one point compactification of the underlying vector space of W. The space S^W is a sphere, and the action of G on W induces an action of G on S^W which fixes the (base-) point at infinity. Given a based G-space Y let $\Sigma^W Y$ be the smash product $S^W \wedge Y$ and $\Omega^W Y$ the space $\mathrm{Map}_*(S^W, Y)$ of basepoint preserving maps from S^W to Y. It is clear that $\Sigma^W Y$ has a diagonal G-action and that $\Omega^W Y$ has a G-action given by conjugation of maps. (The fixed point set $(\Omega^W Y)^G$ of this latter action is the space of G-equivariant maps $S^W \to Y$.) We define $Q_G(Y)$ by the formula

$$Q_G(Y) = \varinjlim_{W} \Omega^W \Sigma^W (Y)$$

where the limit is taken over all finite dimensional representations W of G. As usual the space QY is obtained by restricting the indexing set in the above direct limit to be the set of trivial finite-dimensional representations of G.

If $\zeta : E \to B$ is a vector bundle over B, let B^ζ denote the Thom space of ζ.

Now suppose that G is a compact Lie group and that X is a finite, free, G–CW complex. Let $B = X/G$ be the orbit space, and $p : X \to B$ the natural projection map. Choose an equivariant embedding $e : X \hookrightarrow V$ of X into a finite dimensional representation space V of G. This induces an embedding $p \times e : X \to B \times V$ with normal bundle, say, ν. (Note that the notion of normal bundle makes sense here. Over any point b of B the map $p \times e$ induces an equivariant embedding $e_b : p^{-1}(b) \hookrightarrow V$. Since $p^{-1}(b)$ is a free, transitive G-space, it is clear that e_b has a well-defined normal bundle ν_b. As b varies the bundles ν_b can be assembled into a global normal bundle ν for X in $B \times V$.)

Denote the projection map $B \times V \to B$ by π. There is evidently a Thom-Pontryagin collapse map $\Sigma^V B_+ = B^\pi \to X^\nu$ but (cf. [**BS**]) unless G is finite this is not quite the transfer map we want.

Let g be the Lie algebra of G and ad the vector bundle over B associated to the adjoint action of G on g. We will denote the total space $X \times_G g$ of ad by E. Let $\pi' : E \times V \to E \to B$ be the composite of ad with the product projection. By using the zero-section of ad it is possible to lift the map $p : X \to B$ to a map $p' : X \to E$. This induces an embedding $p' \times e : X \to E \times V$ with normal bundle, say, ν'. It is immediate that the Thom space $B^{\pi'}$ is $\Sigma^V B^{ad}$, and it is well-known (see [BS]) that the corresponding Thom space $X^{\nu'}$ is equivariantly homeomorphic to $\Sigma^V(X_+)$. The *umkehr* map we are seeking is the Thom-Pontryagin collapse map $\Sigma^V B^{ad} \to \Sigma^V(X_+)$. One observes that this collapse map is equivariant, The adjoint map $B^{ad} \to \Omega^V \Sigma^V(X_+)$ consequently factors through a map $B^{ad} \to (\Omega^V \Sigma^V(X_+))^G$ which stabilizes to

$$t''_X : B^{ad} \to (Q_G(X_+))^G$$

The map t''_X may be composed with the inclusion of the fixed point set in the homotopy fixed point set to give a map

$$t'_X : B^{ad} \to (Q_G(X_+))^{hG} .$$

The natural map $Q(X_+) \to Q_G(X_+)$ is a *weak G-equivalence*, in other words, it is an equivariant map that is a (nonequivariant) homotopy equivalence. Such weak G-equivalences become equivalences when the homotopy fixed point functor is applied.

DEFINITION 2.1: If X is a finite, free G–CW complex, the *transfer map* $t_X : (X/G)^{ad} \to (Q(X_+))^{hG}$ is the composite

$$t_X : (X/G)^{ad} \to (Q_G(X_+))^{hG} \to (Q(X_+))^{hG}$$

where the first map is t'_X and the second map is the inverse of the above natural homotopy equivalence of homotopy fixed point sets.

We now need to extend this definition. Suppose first that X is a free G–CW complex which is not necessarily finite. In this case one restricts at first to the finite G-subcomplexes K of X, constructs maps t_K as above, and then checks that these transfers fit together and extend to a transfer map t_X. This procedure is standard and we leave its verification to the reader (see [Cl] or [LMS] for details). If X is a G-space which is not a G–CW complex, we will tacitly assume that X has been replaced by a G–CW approximation [LMS], eg. by the equivariant version of the realization of its singular complex. If X is a G-space which is not free, we can make the above construction with the equivalent free G-space $EG \times X$. The upshot of this is to obtain for any G-space X a transfer map

$$t_X : (EG \times_G X)^{ad} \to (Q(X_+))^{hG} .$$

Suppose now that E is an *infinite loop space with an action of* G, that is, an infinite loop space [M2] together with an action of G that respects all of the infinite loop structure. In this case the infinite loop structure gives a map $Q(E_+) \to E$ which is G-equivariant and induces a map $(Q(E_+))^{hG} \to E^{hG}$;

moreover, if e_0 is the basepoint of E (necessarily fixed by the action of G), the inclusion $S^0 \to E_+$ determined by e_0 induces maps $Q(S^0) \to Q(E_+) \to E$ and $(Q(S^0))^{hG} \to (Q(E_+))^{hG} \to E^{hG}$ which are null homotopic. It follows that the map t_E induces a map

$$\tau_E : (EG_+ \wedge_G E)^{ad} \to E^{hG} .$$

where $(EG_+ \wedge_G E)^{ad}$ denotes the quotient of $(EG \times_G E)^{ad}$ by $(EG \times \{e_0\})^{ad} = BG^{ad}$. If $\mathbf{E} = \{E_n\}$ is an omega spectrum with an action of G then the source spaces of the maps τ_{E_n} fit together to form a spectrum which we will denote $(EG_+ \wedge_G \mathbf{E})^{ad}$. Similarly, the target spaces fit together to form a spectrum whose structure maps are induced in the obvious way by the structure maps of the spectrum $\mathbf{E}$. We call this second spectrum the *homotopy fixed point spectrum* of the action of G on $\mathbf{E}$ and denote it by $\mathbf{E}^{hG}$. Notice furthermore that the maps τ_{E_n} respect the structure maps of these two spectra and hence induce a map of spectra

$$\tau_{\mathbf{E}} : (EG_+ \wedge_G \mathbf{E})^{ad} \to \mathbf{E}^{hG} .$$

If $\mathbf{E}$ is simply a spectrum with an action of G (not necessarily an omega spectrum) we define $\mathbf{E}^{hG}$ to be $(\mathbf{E}')^{hG}$, where $\mathbf{E}'$ is the associated omega spectrum, and obtain as before a transfer map $\tau_{\mathbf{E}}$.

DEFINITION 2.2: If G is a compact Lie group and $\mathbf{E}$ a spectrum with an action of G, we define

(1) the *generalized Tate homology spectrum* $\hat{\mathbf{H}}^G(\mathbf{E})$ to be the (stable) fiber of the map of spectra given by the transfer $\tau_{\mathbf{E}} : (EG_+ \wedge_G \mathbf{E})^{ad} \to \mathbf{E}^{hG}$, and
(2) the generalized Tate homology groups $\hat{\mathbf{H}}^G_*(\mathbf{E})$ to be the homotopy groups of $\hat{\mathbf{H}}^G(\mathbf{E})$.

We will now prove 1.1(3). To do this it is convenient to use configuration space approximations to loop spaces, as described in [M2]. To be more precise, if V is a finite dimensional vector space let

$$F(V,n) = \{(v_1, \cdots, v_n) \in V^n : v_i \neq v_j \quad \text{if} \quad i \neq j\}.$$

This configuration spaces has a free symmetric group (Σ_n) action given by permuting coordinates. Given a based space Z, let $C(V,Z)$ be the complex

$$C(V,Z) = \bigcup_{n>0} F(V,n) \times_{\Sigma_n} Z^n / \sim$$

where the components in this union are patched together by a basepoint relation as described in [M2]. For Z path connected, there is a well known weak homotopy equivalence [M2] [H] $\alpha : C(V,Z) \to \Omega^V \Sigma^V(Z)$. If V is a G-orthogonal representation space and Z is a G-space with a basepoint that is fixed under the action, then $C(V,Z)$ has an induced G-action and it is easily observed that the map α is an equivariant. Hence if Z is connected, α is a weak G-equivalence and so induces an equivalence on homotopy fixed points.

PROOF OF 1.1(3): Assume that X is finite and that G acts freely on X. (If the action on X is not free, replace X as above by $EG \times X$. If X is not finite, restrict to finite subcomplexes and make a direct limit argument. We leave it to the reader to fill in these details [Cl].) Choose as above an equivariant embedding $e : X \hookrightarrow V$ of X into a representation space V. Let B be the quotient space X/G.

Suppose the order of the finite group G is n. Consider the Kahn-Priddy transfer map [KP]

$$\tau_{KP} : B \to F(V,n) \times_{\Sigma_n} X^n \hookrightarrow C(V, X_+)$$

defined by $\tau_{KP}(b) = (e(x_1), \cdots, e(x_n)) \times (x_1, \cdots, x_n)$, where the x_i's run through the orbit in X represented by $b \in B$. It is clear that the image of τ_{KP} lies in the fixed points of $C(V, X_+)$. Moreover, a check of the definition of the map $\alpha : C(V, X_+) \to \Omega^V \Sigma^V (X_+)$ as given in [M2] shows that the composition $B \xrightarrow{\tau_{KP}} C(V, X_+)^G \xrightarrow{\alpha} (\Omega^V \Sigma^V (X_+))^G \to Q_G(X_+)$ is the map t''_X defined above. (Note that since G is a finite group $B^{ad} = B_+$.) The desired result is immediate.

§3. RELATIONSHIP TO CLASSICAL TATE HOMOLOGY

In this section we will prove 1.1(2). It is convenient to work simplicially [M1], and so we will assume that X is a simplicial set with an action of the finite group G and that EG is a contractible simplicial set on which G acts freely [M1, p. 83]. The first step is to give a homotopy-theoretic construction of the positive-dimensional part of the classical Tate homology of (the realization of) X. Consider the simplicial free abelian group $\mathbf{Z} \otimes X$ which in each dimension n is the free abelian group on the set X_n of n-simplices of X. As in [M1, p. 98], the homotopy groups of the realization $|\mathbf{Z} \otimes X|$ are the integral homology groups of $|X|$.

One can associate to an n-simplex σ of $EG \times_G X$ the sum of the n-simplices in $EG \times X$ which are in the free G-orbit σ represents. This association extends additively to a simplicial map $tr : \mathbf{Z} \otimes (EG \times_G X) \to \mathbf{Z} \otimes (EG \times X)$ whose image clearly lies in the G-fixed-points of $\mathbf{Z} \otimes (EG \times X)$. Let γ denote the composition of tr with the obvious map $(\mathbf{Z} \otimes (EG \times X))^G \to (\mathbf{Z} \otimes X)^G$ and let $\mathcal{T}^G_{\mathbf{Z}}(X)$ denote the homotopy fiber of the map $|\mathbf{Z} \otimes (EG \times_G X)| \to |\mathbf{Z} \otimes X|^{hG}$ given by composing $|\gamma|$ with the inclusion of the fixed point set into the homotopy fixed point set.

PROPOSITION 3.1. *For any simplicial set X with an action of the finite group G there are natural isomorphisms*

$$\pi_i \mathcal{T}^G_{\mathbf{Z}}(X) \cong \hat{H}^G_i(|X|)$$

for all $i \geq 0$.

This requires a lemma. Let $N_\bullet$ be the normalization functor which assigns to each simplicial abelian group its normalized chain complex and $N^\bullet$ the dual functor which assigns to each cosimplicial abelian group its normalized cochain complex [**BK1**, §2]. Suppose that A is a cosimplicial simplicial abelian group and that $B = N_\bullet N^\bullet A$ is the double chain complex obtained by normalizing A in both directions [**BK1**, 2.4]. Define a chain complex $\mathrm{t}(A)$ by letting $\mathrm{t}(A)_n$ be the product $\prod_{i-j=n} B_i^j$ and using the standard formula [**BK1**, §3] for the differential. The chain complex $\mathrm{t}(A)$ in general has entries in both positive and negative degrees; we will let $\mathrm{t}_+(A)$ stand for the sub chain complex which is zero in negative dimensions, agrees with $\mathrm{t}(A)$ in dimensions greater than 0, and contains in dimension 0 the cycles of $\mathrm{t}(A)_0$.

LEMMA 3.2. *For any cosimplicial simplicial abelian group A there is a natural chain map*

$$\eta(A) : N_\bullet Tot(A) \to \mathrm{t}_+(A)$$

which induces an isomorphism on homology.

REMARK: Here $\mathrm{Tot}(A)$ is defined as in [**BK2**, X, §3]; it is clear that applying the Tot functor to a cosimplicial simplicial abelian group yields a simplicial abelian group.

PROOF OF 3.2: This result is in some sense implicit in [**BK1**] and [**BK2**, X, §6] and is in any case very similar to the Eilenberg-Zilber theorem [**M1**, p. 129]. We will only sketch the proof. The first step is to construct natural maps η_n : $\mathrm{Tot}(A)_n \to \mathrm{t}(A)_n$, $(n \geq 0)$. Since $\mathrm{Tot}(-)_n$ is represented by the cosimplical simplicial abelian group $\mathbf{Z} \otimes (\underset{\sim}{\Delta} \times \Delta[n])$ [**BK2**, §3], such a map η_n is determined universally by an element $e_n \in \mathrm{t}(\mathbf{Z} \otimes (\underset{\sim}{\Delta} \times \Delta[n])_n$. By definition, such an e_n is specified by a collection $e_n(k)$ $(k \geq 0)$, where $e_n(k)$ lies in $N_\bullet\mathbf{Z}\otimes(\Delta[k]\times\Delta[n])_{k+n}$ and $e_n(k)$ maps trivially to $N_\bullet\mathbf{Z} \otimes (\Delta[k-1] \times \Delta[n])$ under the maps induced by the k standard collapses $s^j : \Delta[k] \to \Delta[k-1]$. Choose $e_n(K)$ to be the fundamental cycle of $\Delta[n] \times \Delta[k]$ provided by the Eilenberg-MacLane shuffle formula [**M1**, p. 133]. A short calculation then shows that the maps η_n vanish on degenerate elements of $\mathrm{Tot}(A)$ and combine to produce the desired chain map $\eta(A) : N_\bullet\mathrm{Tot}(A) \to \mathrm{t}_+(A)$. The fact that $\eta(A)$ induces an isomorphism on homology follows from the methods of [**BK1**]; both $\mathrm{Tot}(A)$ and $\mathrm{t}_+(A)$ can be expressed as inverse limits in a natural way, and $\eta(A)$ induces equivalences between the constituents of the corresponding towers of chain complexes.

PROOF OF 3.1: Let A denote the cosimplicial simplicial abelian group with $A_q^p = \mathrm{Hom}_{\mathrm{Sets}}(EG_p, \mathbf{Z} \otimes (EG \times X)_q)$ and B the cosimplicial simplicial abelian group defined in a corresponding way with EG replaced by $\Delta[0]$. Note that all of the cosimplicial operators of B are isomorphisms, so that $N_\bullet\mathrm{Tot}(B) \cong N_\bullet(\mathbf{Z} \otimes (EG \times X)) \cong \mathrm{t}_+(B) \cong \mathrm{t}(B)$. The unique map $EG \to \Delta[0]$ induces a G–map $h : B \to A$ of cosimplicial simplicial objects. By [**BK2**, X, 3.3(i)] the fixed point set $\mathrm{Tot}(A)^G$ is isomorphic to the simplicial function complex of G–maps $EG \to EG \times X$ and thus $|\mathrm{Tot}(A)^G|$ is weakly equivalent to $|\mathbf{Z} \otimes (EG \times X)|^{hG}$. Let F_1 be the homotopy fiber (in the category of simplicial abelian groups) of the composite $\mathbf{Z} \otimes (EG \times_G X) \xrightarrow{tr} (\mathbf{Z} \otimes (EG \times X))^G \xrightarrow{\mathrm{Tot}(h)} \mathrm{Tot}(A)^G$. It follows that $\mathcal{T}_{\mathbf{Z}}^G(X)$ is naturally homotopy equivalent to $|F_1|$. Let F_2 be the homotopy fiber (in the category of chain complexes) of the composite $N_\bullet(\mathbf{Z}\otimes(EG\times_G X)) \xrightarrow{N_\bullet(tr)}$

$N_\bullet(\mathbf{Z}\otimes(EG\times X))^G \overset{t(h)}{\to} \mathrm{t}(A)^G$, It follows essentially by inspection that the homology groups of F_2 are the classical Tate hyperhomology groups of G with coefficients in $N_\bullet(\mathbf{Z}\otimes X)\sim \mathcal{S}_*(|X|)$. The proposition is now a consequence of the existence of a commutative diagram

$$\begin{array}{ccc} N_\bullet((\mathbf{Z}\otimes(EG\times X))^G) & \xrightarrow{N_\bullet(h)} & N_\bullet\mathrm{Tot}(A^G) \\ {\scriptstyle \eta(B^G)}\downarrow & & \downarrow \\ N_\bullet((\mathbf{Z}\otimes(EG\times X))^G) & \xrightarrow{\mathrm{t}(h)} & \mathrm{t}(A^G) \end{array}$$

in which the right-hand vertical arrow (which is the composite of $\eta(A^G)$ and the inclusion $\mathrm{t}_+(A^G)\to \mathrm{t}(A^G)$) is a homology equivalence in non-negagtive dimensions.

Let SP^∞ denote the infinite symmetric product construction, either in the category of topological spaces or in the category of simplicial sets. Let X_+ be the space obtained by adjoining a disjoint G-fixed basepoint to X. By [Sp] there is a weak G-equivalence between $|SP^\infty(X_+)|$ and $SP^\infty|X_+|$

Define a simplicial transfer $tr' : SP^\infty(EG\times_G X_+)\to SP^\infty(EG\times X_+)$ as follows: for each n-simplex in $EG\times_G X_+$ take the G-orbit in $SP^\infty(EG\times X_+)$ which it represents. The map tr' is a simplicial analogue of the transfer defined by L. Smith [Sm]. There is an evident natural G–equivariant group completion map $gp : SP^\infty(X_+)\to \mathbf{Z}\otimes X$ which sends the added basepoint "+" to 0 (this map is an isomorphism on homotopy in strictly positive dimensions).

The following diagram then commutes :

$$\begin{array}{ccc} SP^\infty(EG\times_G X_+) & \xrightarrow{tr'} & SP^\infty(EG\times X_+) \\ {\scriptstyle gp}\downarrow & & \downarrow{\scriptstyle gp} \\ \mathbf{Z}\otimes(EG\times_G X) & \xrightarrow[tr]{} & \mathbf{Z}\otimes(EG\times X)\,. \end{array}$$

Both transfers fall into the fixed-point sets. Let $\mathcal{T}^G_{SP}(X)$ denote the homotopy fiber of the map $SP^\infty|EG\times_G X_+|\to SP^\infty|EG\times X|^{hG}\to SP^\infty|X|^{hG}$ corresponding to tr'. By using the homotopy invariance property of homotopy fixed point sets [BK2, XI, 5.6] we obtain by 3.1 the following proposition.

PROPOSITION 3.3. *For any simplicial set X with an action of the finite group G there are natural isomorphisms*

$$\pi_i\mathcal{T}^G_{SP}(X)\cong \hat{H}^G_i(|X|)$$

for all $i>0$.

PROOF OF 1.1(2): Assume as in the proof of 1.1(3) (§2) that X is a finite G–CW complex on which G acts freely and that $X\hookrightarrow V$ is an embedding of X into an orthogonal representation space of G. Let B be the quotient X/G. Take as a model of the Eilenberg-MacLane space $K(\mathbf{Z},m)$ the space $SP^\infty(S^m)$. The

Kahn-Priddy transfer τ_{KP} combines with the diagonal map of $SP^\infty(S^m)$ to give a transfer map

$$\tau(m,V) : SP^\infty(S^m) \wedge B_+ \to C(V, SP^\infty(S^m) \wedge X_+)$$

whose image lies in the fixed point set. It follows from the proof of 1.1(3) and a little manipulation that passing to the limit in m and V with $\tau(m,V)$ and extracting homotopy fixed points in the range will give the transfer map of §2 whose fiber is $\hat{\mathbf{H}}^G(\mathbf{H} \wedge X_+)$. (A similar result holds with $\mathbf{H}$ replaced by any other spectrum on which G acts trivially.) For any based space Z there is a natural map $h : C(V,Z) \to SP^\infty(Z)$ obtained by adding coordinates. One can check that the following diagram commutes on the point set level:

$$\begin{tikzcd}
SP^\infty(S^m) \wedge B_+ \arrow[r, "\tau(m,V)"] \arrow[d, "\delta"'] & C(V, SP^\infty(S^m) \wedge X_+) \arrow[d, "h"] \\
SP^\infty(S^m \wedge B_+) \arrow[r, "tr'(m)"] & SP^\infty(S^m \wedge X_+)
\end{tikzcd}$$

where $tr'(m)$ is constructed from the map tr' above in the obvious way and the map labeled δ is the standard pairing. Now loop the diagram down m times and pass to the limit in m and V. The vertical arrows become weak equivalences and, as noted above, the upper horizontal map (after passing to homotopy fixed points in the range) determines the zero space in the Ω-spectrum corresponding to $\hat{\mathbf{H}}^G(\mathbf{H} \wedge X_+)$. It follows that this zero space can be computed as the homotopy fiber of the map

$$\varinjlim_m \Omega^m SP^\infty(S^m \wedge B_+) \xrightarrow{\varinjlim tr'(m)} \varinjlim_m (\Omega^m SP^\infty(S^m \wedge X_+))^{hG}.$$

However, the standard pairing δ produces by adjointness a commutative diagram

$$\begin{tikzcd}
SP^\infty(B_+) \arrow[r, "tr'"] \arrow[d, "\delta"'] & SP^\infty(X_+) \arrow[d, "\delta"'] \\
\varinjlim_m \Omega^m SP^\infty(S^m \wedge B_+) \arrow[r, "\varinjlim tr'(m)"] & \varinjlim_m (\Omega^m SP^\infty(S^m \wedge X_+))
\end{tikzcd}$$

in which the vertical arrows induce isomorphisms on homotopy in positive dimensions. Proposition 3.4 then implies that $\hat{\mathrm{H}}_i^G(\mathbf{H} \wedge X_+)$ is isomorphic to $\hat{H}_i^G(X)$ for $i > 0$. The general theorem is proven by applying this result to an arbitrarily high suspension of X; it is easy to see that suspending X has essentially the effect of shifting both the classical Tate homology and the generalized Tate homology up by one in dimension.

§4. Two examples

In this section we analyze two examples of generalized Tate homology. In the first example the group G is a finite p-group and the spectrum involved is the suspension spectrum of a finite G–CW complex. In this case we will use Carlsson's theorem (the Segal conjecture [Ca]) to give a complete p-primary calculation of the generalized Tate homology. In the second example the group is the circle group S^1 and the spectrum involved is $\mathbf{H} \wedge X_+$ for X a finite G–CW complex. In this case we relate the corresponding generalized Tate homology theory to the periodic cyclic homology theory of Goodwillie [Go2] and Jones [J].

We begin with the assumption that G is a finite p-group, where p is a prime number. Let X be a finite G–CW complex. We recall tom Dieck's theorem [**tD**] describing the fixed points $Q_G(X_+)^G$.

Theorem 4.1 [**tD**]. *Suppose that G is a finite p-group and that X is a finite G–CW complex. For H a subgroup of G, let $N(H) < G$ be the normalizer subgroup of H, and let $W(H) = N(H)/H$ be the corresponding Weyl group. Then there is a homotopy equivalence of infinite loop spaces*

$$\phi : \prod_H Q((EW(H) \times_{W(H)} X^H)_+) \to Q_G(X_+)^G$$

where the product is taken over all conjugacy classes of subgroups $H < G$. Moreover the restriction of ϕ to the factor corresponding to the trivial subgroup is given by the Kahn-Priddy transfer map

$$\tau : Q((EG \times_G X)_+) \to Q_G(EG \times X_+)^G \to Q_G(X_+)^G.$$

We now recall Carlsson's theorem [Ca].

Theorem 4.2 [Ca]. *For G a finite p - group and X a finite G–CW complex, the natural map*

$$\alpha : Q_G(X_+)^G \to Q_G(X_+)^{hG}$$

is a weak equivalence when completed at the prime p.

Consider the Tate homology spectrum $\hat{\mathbf{H}}^G(\Sigma^\infty X_+)$ as defined in §2. By definition, its corresponding zero space $\Omega^\infty \hat{\mathbf{H}}^G(\Sigma^\infty X_+)$ is the fiber of the map of infinite loop spaces $\tau : Q(EG \times_G X_+) \to Q_G(X_+)^{hG} \simeq Q(X_+)^{hG}$. Thus by combining theorems 4.1 and 4.2 we obtain the following calculation of the p-adic completion of the Tate homology spectrum:

Theorem 4.3. *For G a finite p-group and X a finite G–CW complex there is an isomorphism*

$$\hat{\mathbf{H}}^G_*(\Sigma^\infty X_+) \cong_p \bigoplus_{H \neq \{e\}} \pi^s_*(EW(H) \times_{W(H)} X^H).$$

Here $\cong_p$ denotes isomorphism after p-completion, and the sum is taken over all conjugacy classes of nontrivial subgroups $H < G$.

REMARK: Notice that this theorem says that the Tate homology of $\Sigma^\infty X_+$ depends at least p-adically only on the singular subspace of X. In particular, if X is a free G-space, the p-adic Tate homology of its suspension spectrum is zero.

We now turn our attention to the case in which $G = S^1$, the circle group, and to the proof of 1.1(4). Our machinery actually leads to the definition of generalized cyclic homology theories, and we will make some observations about this.

Assume X is a finite G-CW complex. As observed in [DHK] and [J], the chain complex $\mathcal{S}_*(X)$ of X has a natural cyclic structure. We will write $HC_*(X)$, $HC_*^-(X)$, and $H\hat{C}_*(X)$ respectively to denote the cyclic, negative cyclic, and periodic cyclic homology groups of the cyclic module $\mathcal{S}_*(X)$. We refer the reader to [Go2] and [J] for a description of these theories. It has been proved by Goodwillie [Go1] that the groups $HC_*(X)$ are the homotopy groups of the spectrum $\mathbf{H}\wedge(ES^1 \times_{S^1} X_+)$. We will now describe the other two types of cyclic homology in terms of the homotopy groups of spectra.

LEMMA 4.4. *There is a natural isomorphism*

$$HC_*^-(X) \cong \pi_*(\mathbf{H} \wedge X_+)^{hS^1}.$$

PROOF: This lemma is proved in two steps. First, by [DHK] [Go1] we may assume without loss of generality that X is the realization of a cyclic set K. By applying the free abelian group functor $\mathbf{Z} \otimes -$ from the category of cyclic sets to the category of cyclic modules (cf. §3) and then taking realization, we end up with an S^1-space $|\mathbf{Z} \otimes K|$ which is homotopy equivalent to the zero space $\Omega^\infty(\mathbf{H} \wedge X_+)$. Moreover we can make this equivalence a weak equivariant equivalence, and in particular produce an equivalence of infinite loop spaces

$$\phi : \Omega^\infty(\mathbf{H} \wedge X_+)^{hS^1} \to |\mathbf{Z} \otimes K|^{hS^1}.$$

The construction of ϕ is carried out using the techniques in the proof of 1.1(2).

The second step in the proof is then to show that $\pi_*|\mathbf{Z}\otimes K|^{hS^1}$ is isomorphic in positive dimensions to $HC_*^-(X)$ or equivalently to the negative cyclic homology of the cyclic module $\mathbf{Z}\otimes K$. This is done by recalling the second quadrant double chain complex for computing HC_*^- constructed in [J]. One compares this double complex with the one obtained by first giving ES^1 a cyclic decomposition and then using this to construct a cocyclic cyclic abelian group $\mathrm{Hom}_{\mathrm{cyc}}(ES^1, \mathbf{Z} \otimes K)$ whose total space is $|\mathbf{Z} \otimes K|^{hS^1}$. Here "$\mathrm{Hom}_{\mathrm{cyc}}$" denotes morphisms preserving the cyclic structure. An argument analogous to the one which was used to prove 1.1(2) proves that in positive total dimensions the above double complex computes

$$\pi_*(\mathrm{Tot}\ \mathrm{Hom}_{\mathrm{cyc}}(ES^1, \mathbf{Z} \otimes K) \cong \pi_*(|\mathbf{Z} \otimes K|^{hS^1}) \cong \pi_*(\mathbf{H} \wedge X_+).$$

As in the proof of 1.1(2), the restriction to positive dimensions can be removed by working with arbitrarily high suspensions of X. This completes the proof of the lemma.

REMARK: A special case of this lemma, namely the case in which X is weakly equivalent to the free loop space of a finite complex, was proved by Cohen and Jones in [**CJ**].

PROOF OF 1.1(4): The complex used to compute HC_*^- is a subcomplex of the one used to compute $H\hat{C}_*$. One therefore has a long exact sequence (see [**Go2**] or [**J**])

$$\cdots \xrightarrow{\delta} HC_{q+1}^-(X) \xrightarrow{i} H\hat{C}_{q+1}(X) \xrightarrow{p} HC_{q-1}(X) \xrightarrow{\delta} HC_q^-(X) \to \cdots$$

As was discussed in [**J**], the connecting homomorphism δ is induced on the chain level by the S^1-transfer map. Translating that discussion to the language we are using here gives that the connecting map δ is given by the composition

$$\begin{aligned}\delta : HC_{q-1}(X) \cong \pi_q(\mathbf{H} \wedge \Sigma(ES^1 \times_{S^1} X_+)) &= \pi_q(ES^1_+ \wedge_{S^1} (\mathbf{H} \wedge X_+))^{ad} \\ &\xrightarrow[\tau]{} \pi_q((\mathbf{H} \wedge X_+)^{hS^1}) \cong HC_q^-(X)\,.\end{aligned}$$

Here τ is the transfer map defined in §2 for the Lie group S^1. Note that $(-)^{ad}$ is just suspension because S^1 is abelian. By the definition of the Tate homology spectrum, this immediately gives the desired result.

Given the above it makes sense to define generalized cyclic homology groups as follows. Let $\mathbf{E}$ be a spectrum representing a generalized homology theory $\mathbf{E}_*$. Let X be a space with an S^1-action. Define the $\mathbf{E}_*$ *generalized cyclic homology* groups of X by the formulas

$$\mathbf{E}C_q(X) = \pi_q(\mathbf{E} \wedge (ES^1 \times_{S^1} X)_+), \quad \mathbf{E}C_q^-(X) = \pi_q(\mathbf{E} \wedge X_+)^{hS^1}$$

and

$$\mathbf{E}\hat{C}_q(X) = \pi_{q+1}\hat{\mathbf{H}}^{S^1}(E \wedge X_+)\,.$$

By [**Go1**], lemma 3.4 and theorem 1.1(3) these definitions agree with the definitions of the usual cyclic homology theories when $\mathbf{E} = \mathbf{H}$. Moreover the homotopy exact sequence for the transfer map gives an exact sequence

$$\cdots \to \mathbf{E}C_{q+1}^- \to \mathbf{E}\hat{C}_{q+1}(X) \to \mathbf{E}C_{q-1}(X) \to \mathbf{E}C_q^-(X) \to \cdots$$

analogous to the standard one above. However, the generalized "periodic cyclic homology" $\mathbf{E}\hat{C}_*(X)$ need no longer be periodic — periodicity fails in general, for instance, if $\mathbf{E}$ is the sphere spectrum.

We end by remarking that the case in which $\mathbf{E}$ is the sphere spectrum $\mathbf{S}$ has proved very important in the study of Waldhausen's K-theory [**CCGH**], [**CJ**], [**B**]. In particular the negative cyclic stable homotopy of the free loop space $\mathbf{S}C_*^-(\Lambda X_+)$ is the target of the Chern character map defined in [**CJ**] and of the cyclic trace map defined in [**B**].

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