

A simple universal property of Thom ring spectra

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ABSTRACT

We give a simple universal property of the multiplicative structure on the Thom spectrum of an n -fold loop map, obtained as a special case of a characterization of the algebra structure on the colimit of a lax \mathcal{O} -monoidal functor. This allows us to relate Thom spectra to \mathbb{E}_n -algebras of a given characteristic in the sense of Szymik. As applications, we recover the Hopkins–Mahowald theorem realizing $H\mathbb{F}_p$ and $H\mathbb{Z}$ as Thom spectra, and compute the topological Hochschild homology and the cotangent complex of various Thom spectra.

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1. Introduction

1.1. Motivation

Many spectra of interest fall in one of two classes: either they contain important geometric or arithmetic information and are difficult to compute with, for example, the sphere spectrum or various algebraic K -theory spectra, or they belong to a family of what Hopkins calls ‘designer homotopy types’, like Eilenberg–Mac Lane spectra or Brown–Gitler spectra.

Thom spectra are remarkable in that they often belong to the intersection: Classical examples include cobordism spectra, whose homotopy groups and other invariants often turn out to be completely computable. As another beautiful example, Mahowald [20] proves that the Eilenberg–Mac Lane spectra $H\mathbb{F}_2$ and $H\mathbb{Z}$ can be realized as Thom spectra, an observation which he applies in his construction of new elements in the homotopy groups of spheres [19]. Similar ideas also play a role in the proof of the nilpotence theorem by Devinatz, Hopkins, and Smith [11].

From a more conceptual point of view, work of Lewis and May [16], Sullivan [26], and May and Sigurdsson [24] places the theory of Thom spectra in the context of parametrized spectra and thus reveals the underlying geometry of the construction. We are interested here in how this framework accounts for some of the good algebraic properties of Thom spectra.

1.2. *Outline and main results*

In [1, 2], the authors describe a convenient model of Thom spectra as homotopy colimits of diagrams whose shape is given by the base space. Their approach is closely related and essentially equivalent to the earlier models cited above, but is expressed in the language of ∞ -categories. The study of multiplicative structures on Thom spectra can thus take place in the setting of \mathcal{O} -monoidal functors between \mathcal{O} -monoidal ∞ -categories.

Section 2 first explains the situation in ordinary category theory that we then extend to \mathcal{O} -monoidal ∞ -categories, proving our general characterization of the algebra structure of colimits of lax \mathcal{O} -monoidal functors in Theorem 2.13. This is used in Section 3 to deduce the following new universal property of the multiplicative structure of Thom spectra of \mathbb{E}_n -maps, see Theorem 3.5.

THEOREM. *If X is an \mathbb{E}_n -space and $f: X \rightarrow \text{Pic}(S^0)$ an \mathbb{E}_n -map, then the space $\text{Map}_{\text{Alg}_{S^0}^{\mathbb{E}_n}}(Mf, A)$ is equivalent to the space of \mathbb{E}_n -lifts of f indicated below:*

$$\begin{array}{ccc} & & \text{Pic}(S^0)_{\downarrow A} \\ & \nearrow \text{---} & \downarrow \\ X & \xrightarrow{f} & \text{Pic}(S^0). \end{array}$$

This generalizes earlier structural results about Thom spectra proven by Lewis [2, 16]. We then show that any n -fold loop map with Thom spectrum Mf is canonically \mathbb{E}_{n-1} Mf -orientable, thereby establishing a structured version of the Thom isomorphism. Moreover, we deduce a theorem of Chadwick and Mandell [9], describing \mathbb{E}_n -orientations.

We start Section 4 by introducing a notion of characteristic for \mathbb{E}_n -algebras in spectra, a straightforward extension of the \mathbb{E}_∞ -case studied previously by Szymik [27, 28]; see also [4] for related ideas. It is easy to construct a weakly initial example of an algebra of characteristic χ , denoted $S^0 //_{\mathbb{E}_n} \chi$. We then show that these algebras satisfy the same universal property as certain Thom spectra naturally corresponding to them, which gives our second main result, Theorem 4.10.

THEOREM. *For any n and any $k \geq 1$ and $f: S^k \rightarrow BGL_1 S^0$ with corresponding n -fold loop map $\bar{f}: \Omega^n \Sigma^n S^k \rightarrow BGL_1 R$ and with associated characteristic $\chi = \chi(f)$, there is an equivalence $M\bar{f} = S^0 //_{\mathbb{E}_n} \chi$.*

This theorem establishes a connection between Thom spectra and versal algebras which allows to transfer results proven for one class to the other. We illustrate this idea in Section 5. For instance, by specializing to $n = 2$ and $(1 - p) = f_p: S^1 \rightarrow BGL_1 S^0_p$ and using the computation of the mod p cohomology of the free \mathbb{E}_2 -algebra on S^1 due to Araki and Kudo [15, Theorem 7.1] and Dyer and Lashof [12, Theorem 5.2] as well as the determination of the corresponding Dyer–Lashof operations by Steinberger [8, Chapter 3, Theorems 2.2, 2.3], we recover the Hopkins–Mahowald theorem

$$M\bar{f}_p = H\mathbb{F}_p.$$

This exhibits $H\mathbb{F}_p$ as the versal \mathbb{E}_2 -algebra of characteristic p . Furthermore, our argument allows us to directly deduce the identification of $H\mathbb{Z}$ as an \mathbb{E}_2 -Thom ring spectrum. Finally, we describe the topological Hochschild homology and the \mathbb{E}_n -cotangent complex of the \mathbb{E}_n -algebras considered above.

2. Colimits of lax functors

2.1. Ordinary lax functors

Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a lax monoidal functor between two (ordinary) monoidal categories. Recall that this means that we are given a family of morphisms $FX \otimes FY \rightarrow F(X \otimes Y)$ natural in X and Y which are compatible with the associativity constraints in \mathcal{C} and \mathcal{D} . If \mathcal{C} is small and \mathcal{D} is *monoidally cocomplete*, by which we mean that \mathcal{D} is cocomplete and $\otimes: \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ preserves colimits in each variable separately, then $M := \operatorname{colim} F$ acquires the structure of an algebra object in \mathcal{D} . The multiplication $\mu: M \otimes M \rightarrow M$ is obtained from the lax structure of F as follows: the composite morphisms

$$FX \otimes FY \rightarrow F(X \otimes Y) \rightarrow M$$

induce a morphism

$$M \otimes M \cong \operatorname{colim}_{(X,Y) \in \mathcal{C} \times \mathcal{C}} (FX \otimes FY) \rightarrow M,$$

where the first isomorphism comes from the assumption that tensor products in \mathcal{D} commute with colimits in each variable separately. This is an analogue in ordinary category theory of Lewis result that the Thom spectrum of a loop map is an \mathbb{E}_1 -ring spectrum.

If we are additionally given an algebra object A in \mathcal{D} , the slice category \mathcal{D}/A acquires a monoidal structure in which $(W \xrightarrow{f} A) \otimes (Z \xrightarrow{g} A)$ is given by the composite

$$W \otimes Z \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu_A} A.$$

The projection functor $\pi: \mathcal{D}/A \rightarrow \mathcal{D}$ is, of course, monoidal.

The universal property we will give for the multiplicative structure of Thom spectra is an analogue of the following fact: morphisms $M \rightarrow A$ of algebra objects in \mathcal{D} are in one to one correspondence with lax monoidal lifts of F through π :

$$\begin{array}{ccc} & & \mathcal{D}/A \\ & \nearrow & \downarrow \pi \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

In this setting, this fact is straightforward to verify: lax monoidal lifts are ‘lax monoidal cocones’ over F with vertex A , and such cocones induce algebra morphisms $M = \operatorname{colim} F \rightarrow A$.

In Section 2.3 we will generalize this discussion in two ways: first, we will work with ∞ -categories instead of ordinary categories; second, instead of considering only monoidal ∞ -categories we will work with \mathcal{O} -monoidal ∞ -categories for an arbitrary ∞ -operad \mathcal{O} . (For applications we will take \mathcal{O} to be the \mathbb{E}_n operad for some $0 \leq n \leq \infty$.) Before doing that, however, let us recall some definitions from the theory of ∞ -operads.

2.2. A few words about ∞ -operads

We recall some definitions about ∞ -operads from chapters 2 and 3 of Lurie’s *higher algebra* [18]. The theory of ∞ -operads developed there is the $(\infty, 1)$ -generalization of the theory of colored operads or symmetric multicategories. So an ∞ -operad \mathcal{O} has a collection of objects and for every finite (unordered) family $\{X_i\}_{i \in I}$ of objects and any object Y there is a space of morphisms $\operatorname{Mul}_{\mathcal{O}}(\{X_i\}, Y)$. The definitions in [18] encode this data indirectly: an ∞ -operad \mathcal{O} is specified by an ∞ -category \mathcal{O}^{\otimes} and a functor $\mathcal{O}^{\otimes} \rightarrow N(\operatorname{Fin}_*)$ to the category of finite pointed sets satisfying certain conditions. Morphisms of ∞ -operads $\mathcal{O} \rightarrow \mathcal{O}'$ are functors over $N(\operatorname{Fin}_*)$ (that is, functors $\mathcal{O}^{\otimes} \rightarrow \mathcal{O}'^{\otimes}$ that form a commutative triangle with the structure

maps to $N(\text{Fin}_*)$) which satisfy one additional condition. See [18, Definitions 2.1.1.10, 2.1.2.7] for the precise definition.

REMARK 2.1. To give some intuition for how the definition captures the notion of colored operad, think of the case of symmetric monoidal categories. These can be regarded as a special kind of operad in which $\text{Mul}_{\mathcal{C}}(\{X_i\}_{i \in I}, Y) = \text{Hom}_{\mathcal{C}}(\bigotimes_{i \in I} X_i, Y)$; but they can also be thought of as commutative monoids in the category of all categories, which using Segal’s idea of Γ -spaces, can be encoded as appropriately weak functors $\text{Fin}_* \rightarrow \text{Cat}$ satisfying certain conditions. Applying the (covariant) Grothendieck construction to such a functor one obtains a coCartesian fibration $\mathcal{C}^{\otimes} \rightarrow \text{Fin}_*$. The definition of symmetric monoidal ∞ -category is exactly what this would suggest: a coCartesian fibration of ∞ -categories $\mathcal{C}^{\otimes} \rightarrow N(\text{Fin}_*)$ satisfying an analogue of the Segal condition, and the definition of ∞ -operad is a generalization of this.

When representing symmetric monoidal categories as colored operads, one needs to be aware of a subtlety relating to morphisms: maps of operads correspond not to symmetric monoidal functors (that is, functors with a natural isomorphism $F(X) \otimes F(Y) \cong F(X \otimes Y)$, compatible with the unit, associativity, and symmetry), but to *lax* symmetric monoidal functors (that is, functors with just a compatible natural *transformation* $F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$). In terms of coCartesian fibrations $\mathcal{C}^{\otimes} \rightarrow \text{Fin}_*$, symmetric monoidal functors correspond to functors over Fin_* which send coCartesian morphisms to coCartesian morphisms, and lax symmetric monoidal functors correspond to a more general type of functor over Fin_* which is analogous to the definition of morphism of ∞ -operad cited above.

An \mathcal{O} -monoidal ∞ -category is defined to be a coCartesian fibration $\mathcal{C}^{\otimes} \rightarrow \mathcal{O}^{\otimes}$ such that the composite $\mathcal{C}^{\otimes} \rightarrow \mathcal{O}^{\otimes} \rightarrow N(\text{Fin}_*)$ presents \mathcal{C} as an ∞ -operad. If \mathcal{O} is the \mathbb{E}_1 or \mathbb{E}_{∞} operad, this recovers the notion of monoidal ∞ -category or symmetric monoidal ∞ -category, respectively.

REMARK 2.2. Again, the intuition for this definition comes from the Grothendieck construction: morally, one would want to say that \mathcal{O} -monoidal categories are \mathcal{O} -algebras in categories, or more precisely, morphisms of operads from \mathcal{O} to the $(\infty, 2)$ -category of ∞ -categories equipped with its symmetric monoidal structure. But instead of appealing to the theory of $(\infty, 2)$ -categories, it is much easier to describe what the result of applying a version of the Grothendieck construction would be and adopt the resulting coCartesian fibration as the definition.

WARNING 2.3. In the case that \mathcal{O} is an operad with a single color X , an \mathcal{O} -monoidal ∞ -category $p: \mathcal{C}^{\otimes} \rightarrow \mathcal{O}^{\otimes}$ can be thought of as an ∞ -category $\mathcal{C} = p^{-1}(X)$ equipped with functors $\mathcal{C}^n \rightarrow \mathcal{C}$ for each n -ary operation in \mathcal{O} . Of course, something similar is true for multicolored \mathcal{O} , except that instead of a single underlying ∞ -category, there is an ∞ -category $\mathcal{C}_X = p^{-1}(X)$ for each object X of \mathcal{O} , and a functor $\prod \mathcal{C}_{X_i} \rightarrow \mathcal{C}_Y$ for each operation in $\text{Mul}_{\mathcal{O}}(\{X_i\}, Y)$. The reader can pretend that whenever we mention an \mathcal{O} -monoidal ∞ -category, \mathcal{O} has a single object without missing out on anything essential.

Between two \mathcal{O} -monoidal ∞ -categories \mathcal{C} and \mathcal{D} we can consider \mathcal{O} -monoidal functors and *lax* \mathcal{O} -monoidal functors, these two notions follow the pattern described in the second paragraph of Remark 2.1. Lax \mathcal{O} -monoidal functors are simply morphisms of ∞ -operads $\mathcal{C} \rightarrow \mathcal{D}$ over \mathcal{O} ; they form an ∞ -category denoted $\text{Fun}_{\mathcal{O}}^{\text{lax}}(\mathcal{C}, \mathcal{D})$ or $\text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$. The lax \mathcal{O} -monoidal functors that send coCartesian morphisms to coCartesian morphisms are the \mathcal{O} -monoidal functors; they form an ∞ -category $\text{Fun}_{\mathcal{O}}^{\otimes}(\mathcal{C}, \mathcal{D})$.

DEFINITION 2.4. Let \mathcal{O}^{\otimes} be an ∞ -operad. An \mathcal{O} -monoidally cocomplete ∞ -category is an \mathcal{O} -monoidal category, that is, a coCartesian fibration of ∞ -operads $q: \mathcal{C}^{\otimes} \rightarrow \mathcal{O}^{\otimes}$, such that q

is compatible with all small colimits in the sense of [18, Definition 3.1.1.18]. This means that for every object X of the underlying ∞ -category \mathcal{O} of \mathcal{O}^\otimes , the ∞ -category $\mathcal{C}_X := q^{-1}(X)$ is cocomplete, and for every morphism $f \in \text{Mul}_{\mathcal{O}}(\{X_i\}_{1 \leq i \leq n}, Y)$, the functor $\otimes_f: \prod_{1 \leq i \leq n} \mathcal{C}_{X_i} \rightarrow \mathcal{C}_Y$ preserves colimits in each variable separately.

Recall that the core of an ∞ -category \mathcal{C} is the maximal ∞ -groupoid contained in \mathcal{C} . If \mathcal{C} is incarnated as a quasi-category, then the core \mathcal{C}^\simeq is easily described as a Kan complex: it is the subsimplicial set of \mathcal{C} consisting of simplices all of whose edges are invertible morphisms of \mathcal{C} . If \mathcal{C} has an \mathcal{O} -monoidal structure, then, as expected from the theory of ordinary monoidal categories, \mathcal{C}^\simeq inherits an \mathcal{O} -monoidal structure as well. More precisely, we have:

PROPOSITION 2.5. *Let \mathcal{O} be an ∞ -operad, and let $q: \mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$ exhibit \mathcal{C} as an \mathcal{O} -monoidal ∞ -category. Define $\mathcal{C}_{\text{coCart}}^\otimes$ to be subcategory of \mathcal{C}^\otimes spanned by the q -coCartesian morphisms. The restriction of q to this subcategory, $\tilde{q}: \mathcal{C}_{\text{coCart}}^\otimes \rightarrow \mathcal{O}^\otimes$, then exhibits $\mathcal{C}_{\text{coCart}}^\otimes$ as an \mathcal{O} -monoidal category such that for each object X of \mathcal{O} , the underlying ∞ -category $(\mathcal{C}_{\text{coCart}}^\otimes)_X$ is the core of \mathcal{C}_X .*

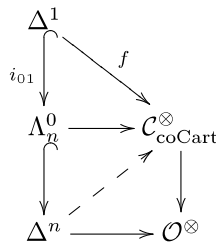
Proof. We will show that:

- (1) \tilde{q} is a coCartesian fibration;
- (2) $(\mathcal{C}_{\text{coCart}}^\otimes)_X = \mathcal{C}_X^\simeq$ for any object X of \mathcal{O} or even of \mathcal{O}^\otimes ;
- (3) the composite $\mathcal{C}_{\text{coCart}}^\otimes \xrightarrow{\tilde{q}} \mathcal{O}^\otimes \rightarrow N(\text{Fin}_*)$ exhibits $\mathcal{C}_{\text{coCart}}^\otimes$ as an ∞ -operad.

The items 1 and 3 guarantee that $\mathcal{C}_{\text{coCart}}^\otimes$ is an \mathcal{O} -monoidal category. We have placed item 2 in the middle because it will be used in the proof of 3.

Proof of 1. First we need to show that \tilde{q} is an inner fibration of simplicial sets, that is, we must show it has the right lifting property against inner horn inclusions. But q has this lifting property and the fillers only have 1-simplices not present in the horn in a single case: $\Lambda_1^2 \hookrightarrow \Delta^2$. Since coCartesian morphisms are closed under composition, the filler for q will also serve as a filler for \tilde{q} .

Now we will show that all morphisms in $\mathcal{C}_{\text{coCart}}^\otimes$ are \tilde{q} -coCartesian. This is very similar to the above proof that \tilde{q} is an inner fibration. A morphism f is coCartesian if and only if for every $n \geq 2$ and every commutative diagram



there is a dotted arrow that makes the diagram commute (this is dual to [17, Remark 2.4.1.4]). As above, we can always take the same filler as for the corresponding diagram with \mathcal{C}^\otimes in place of $\mathcal{C}_{\text{coCart}}^\otimes$: indeed, the simplex Δ^n has no 1-simplices absent from Λ_n^0 unless $n = 2$; and if $n = 2$, we can still use the same filler because if g and $h \circ g$ are coCartesian then so is h (this is the dual of [17, Proposition 2.4.1.7]).

Proof of 2. The morphisms in $(\mathcal{C}_{\text{coCart}}^\otimes)_X$ project to the identity morphism of $X \in \mathcal{O}^\otimes$ by definition. Morphisms projecting to the identity are coCartesian if and only if they are invertible [17, Proposition 2.4.1.5].

Proof of 3. If $p: \mathcal{O}^\otimes \rightarrow N(\text{Fin}_*)$ is an ∞ -operad and $\langle n \rangle \in \text{Fin}_*$ denotes the pointed set $\{*, 1, 2, \dots, n\}$, there is an equivalence $\mathcal{O}_{\langle n \rangle}^\otimes \rightarrow \mathcal{O}^n$. What we need to check according to

[18, Proposition 2.1.2.12] is that for every $X \in \mathcal{O}_{\langle n \rangle}^{\otimes}$ with corresponding sequence $(X_1, \dots, X_n) \in \mathcal{O}^n$, we get an equivalence $(\mathcal{C}_{\text{coCart}}^{\otimes})_X \cong \prod_{i=1}^n (\mathcal{C}_{\text{coCart}}^{\otimes})_{X_i}$ induced by taking coCartesian lifts of the inert morphisms $X \rightarrow X_i$. For the purposes of this argument, the definition of inert does not matter: we simply note that by the same proposition applied to \mathcal{C}^{\otimes} , we *do* have such an equivalence for \mathcal{C}^{\otimes} instead of $\mathcal{C}_{\text{coCart}}^{\otimes}$. This equivalence restricts to the required equivalence because coCartesian morphisms are closed under composition. \square

REMARK 2.6. This proposition encodes the \mathcal{O} -monoidal structure of the space \mathcal{C}^{\simeq} as an ∞ -category $\mathcal{C}_{\text{coCart}}^{\otimes}$ with a coCartesian fibration over \mathcal{O}^{\otimes} . A more direct way to encode an \mathcal{O} -space is as an \mathcal{O} -algebra in the ∞ -category of spaces. These two forms are equivalent: the coCartesian fibration \tilde{q} constructed in the proposition has the feature that all morphisms in the domain are coCartesian, and thus by [18, Proposition 2.4.2.4], \tilde{q} is a left fibration, and therefore classifies a functor $\mathcal{O}^{\otimes} \rightarrow \text{Spaces}$ which exhibits \mathcal{C}^{\simeq} as an \mathcal{O} -monoid in spaces. This is the correspondence of [18, Example 2.4.2.4] restricted to left fibrations.

REMARK 2.7. That the core of an \mathcal{O} -monoidal ∞ -category inherits an \mathcal{O} -monoidal structure has been used implicitly before (see for example, [1, Definition 8.5, Remark 8.6; 23, Proposition 2.2.3]), but as far the authors know, no explicit description has appeared in the literature, which is why we feel justified in giving it in such detail.

2.3. *The \mathcal{O} -algebra structure on the colimit of a lax \mathcal{O} -monoidal functor*

For \mathcal{C} an \mathcal{O} -monoidal category, the ∞ -category of \mathcal{O} -algebras in \mathcal{C} , denoted by $\text{Alg}_{/\mathcal{O}}(\mathcal{C})$, is defined to be $\text{Alg}_{\mathcal{O}/\mathcal{O}}(\mathcal{C})$, the ∞ -category of lax \mathcal{O} -monoidal functors $\mathcal{O} \rightarrow \mathcal{C}$. Unwinding the definitions we see that \mathcal{O} -algebras in \mathcal{C} are sections of the coCartesian fibration $\mathcal{C}^{\otimes} \rightarrow \mathcal{O}^{\otimes}$ presenting the \mathcal{O} -monoidal structure on \mathcal{C} ; more precisely \mathcal{O} -algebras are those sections which are also maps of ∞ -operads, that is, maps sending inert morphisms to inert morphisms.

THEOREM 2.8. *Let \mathcal{O} be an ∞ -operad, \mathcal{C} be a small \mathcal{O} -monoidal ∞ -category, and \mathcal{D} be an \mathcal{O} -monoidally cocomplete ∞ -category. If $F: \mathcal{C}^{\otimes} \rightarrow \mathcal{D}^{\otimes}$ is a lax \mathcal{O} -monoidal functor, then there is an \mathcal{O} -algebra in \mathcal{D} given by a functor $M: \mathcal{O} \rightarrow \mathcal{D}$ such that for every object X of \mathcal{O} , $M(X) = \text{colim}(F_X: \mathcal{C}_X \rightarrow \mathcal{D}_X)$.*

REMARK 2.9. In the case that \mathcal{O} has a single object, the conclusion should be thought of as saying that the colimit of the functor $F: \mathcal{C} \rightarrow \mathcal{D}$ has a canonical structure of an \mathcal{O} -algebra in \mathcal{D} .

REMARK 2.10. This theorem is due to Lewis [16, Section IX.7] in the case that \mathcal{C} is an ∞ -groupoid and \mathcal{D} is the category of spectra — and \mathcal{O} has a single color. In [1], the authors prove a similar result under slightly stronger assumptions: that \mathcal{O} is coherent and \mathcal{D} is an \mathcal{O} -algebra object in the ∞ -category of presentable ∞ -categories (and functors which are left adjoints).

Proof. Let $p: \mathcal{C}^{\otimes} \rightarrow \mathcal{O}^{\otimes}$ and $q: \mathcal{D}^{\otimes} \rightarrow \mathcal{O}^{\otimes}$ be the coCartesian fibrations of ∞ -operads presenting \mathcal{C} and \mathcal{D} as \mathcal{O} -monoidal categories. We will use [18, Theorem 3.1.2.3 (A)] to show that our assumption that \mathcal{D} is \mathcal{O} -monoidally cocomplete guarantees the existence of an operadic left Kan extension M of F along p relative to q . Using that theorem requires repackaging the functor $p: \mathcal{C}^{\otimes} \rightarrow \mathcal{O}^{\otimes}$ we wish to extend along as a Δ^1 -family of operads $\mathcal{M}^{\otimes} \rightarrow \text{Fin}_* \times \Delta^1$. This family is simply the mapping cylinder of F , given by

$$\mathcal{M}^{\otimes} = (\mathcal{C}^{\otimes} \times \Delta^1) \coprod_{\mathcal{C}^{\otimes} \times \{1\}} \mathcal{O}^{\otimes}.$$

(A word of motivation for readers unfamiliar with Lurie's approach to defining Kan extensions: the idea is that a functor $G : \mathcal{M}^\otimes \rightarrow \mathcal{D}^\otimes$ comprises the data of a functor $G_1 : \mathcal{O}^\otimes \rightarrow \mathcal{D}^\otimes$, namely $G_1 = G|_{\mathcal{M}^\otimes \times_{\Delta^1} \{1\}}$, and a natural transformation $\mathcal{C}^\otimes \times \Delta^1 \rightarrow \mathcal{M}^\otimes \xrightarrow{G} \mathcal{D}^\otimes$ from G_0 to $G_1 \circ p$. This way, the problem of finding the left Kan extension of F along p together with the universal natural transformation $F \rightarrow \text{Lan}_p F \circ p$, becomes the problem of left Kan extending F along the inclusion $\mathcal{C}^\otimes = \mathcal{M}^\otimes \times_{\Delta^1} \{0\} \hookrightarrow \mathcal{M}^\otimes$.)

According to [18, Theorem 3.1.2.3 (A)], there exists an operadic left Kan extension $L : \mathcal{M}^\otimes \rightarrow \mathcal{D}^\otimes$ of F relative to q making the following diagram commute

$$\begin{array}{ccc} \mathcal{C}^\otimes & \xrightarrow{F} & \mathcal{D}^\otimes \\ i_0 \downarrow & \nearrow L & \downarrow q \\ \mathcal{M}^\otimes & \xrightarrow{p'} & \mathcal{O}^\otimes \end{array}$$

(where the map $p' : \mathcal{M}^\otimes \rightarrow \mathcal{O}^\otimes$ is the one induced by $p \circ \pi_1$ on $\mathcal{C}^\otimes \times \Delta^1$ and the identity on \mathcal{O}^\otimes), if and only if for each object X of $\mathcal{O}^\otimes = \mathcal{M}^\otimes \times_{\Delta^1} \{1\}$, the diagram

$$(\mathcal{M}_{\text{act}}^\otimes)_{/X} \times_{\Delta^1} \{0\} \rightarrow \mathcal{M}^\otimes \times_{\Delta^1} \{0\} \xrightarrow{F} \mathcal{D}^\otimes$$

can be extended to an operadic q -colimit lifting the map

$$((\mathcal{M}_{\text{act}}^\otimes)_{/X} \times_{\Delta^1} \{0\})^\triangleright \rightarrow \mathcal{M}^\otimes \rightarrow \mathcal{O}^\otimes;$$

we refer the reader to [18, Definition 2.1.2.3, Remark 2.2.4.3] for the definition of $\mathcal{M}_{\text{act}}^\otimes$.

Since \mathcal{D} is assumed to be \mathcal{O} -monoidally cocomplete, [18, Proposition 3.1.1.20], provides the required operadic q -colimits.

The restriction M of L to $\mathcal{M}^\otimes \times_{\Delta^1} \{1\} = \mathcal{O}^\otimes$ is the desired algebra structure on colim F . First of all, it is indeed an \mathcal{O} -algebra, because L is a map of families of operads, and according to the commutative diagram above $q \circ M = \text{id}_{\mathcal{O}^\otimes}$. Next we need to check that for each object X in \mathcal{O} , $M(X) \cong \text{colim}(F|_{\mathcal{C}_X} : \mathcal{C}_X \rightarrow \mathcal{D}_X)$. But by definition of operadic relative left Kan extension, the diagram $((\mathcal{C}_{\text{act}}^\otimes)_{/X})^\triangleright \rightarrow \mathcal{D}^\otimes$ induced by L is an operadic colimit diagram relative to q . Finally [18, Proposition 3.1.1.16] states that an operadic colimit relative to a coCartesian fibration (here, q) becomes a colimit in each underlying ∞ -category \mathcal{D}_X . \square

COROLLARY 2.11. *There exists an algebra structure on colim F for each F . In fact, there is a left adjoint to the functor $p^* = (- \circ p) : \text{Alg}_{/\mathcal{O}}(\mathcal{D}) \rightarrow \text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$ that gives this algebra structure functorially.*

Proof. This follows from the proof of Theorem 2.8 by [18, Corollary 3.1.3.4]. \square

Recall that for an \mathcal{O} -algebra $A \in \text{Alg}_{/\mathcal{O}}(\mathcal{D})$, the slice category $\mathcal{D}_{/A}$ has the structure of an \mathcal{O} -monoidal ∞ -category; in other words, $\mathcal{D}_{/A}$ is the underlying ∞ -category of an \mathcal{O} -monoidal ∞ -category $\mathcal{D}_{/A\mathcal{O}}^\otimes \rightarrow \mathcal{O}^\otimes$ (see [18, Theorem 2.2.2.4]). The slice category has the following universal property:

LEMMA 2.12. *Let \mathcal{C} and \mathcal{D} be \mathcal{O} -monoidal ∞ -categories, A be an \mathcal{O} -algebra in \mathcal{D} , and $F : \mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ be a lax \mathcal{O} -monoidal functor. Lax \mathcal{O} -monoidal lifts of F through the projection $\mathcal{D}_{/A\mathcal{O}}^\otimes \rightarrow \mathcal{D}^\otimes$ then correspond to \mathcal{O} -monoidal natural transformations $F \rightarrow A \circ p$, where $p : \mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$ exhibits \mathcal{C} as \mathcal{O} -monoidal. More precisely, there is a homotopy equivalence*

$$\text{Map}_{\text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})}(F, A \circ p) \cong \{F\} \times_{\text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})} \text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}_{/A}).$$

Proof. The definition of the \mathcal{O} -monoidal structure of the slice category [18, Section 2.2.2], says that morphisms of simplicial sets over \mathcal{O}^\otimes from $Y \rightarrow \mathcal{O}^\otimes$ to $\mathcal{D}_{/A\mathcal{O}}^\otimes \rightarrow \mathcal{O}^\otimes$ are in bijection with commutative diagrams of the form:

$$\begin{array}{ccccc} Y & \xrightarrow{i_1} & Y \times \Delta^1 & \xrightarrow{\pi_1} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}^\otimes & \xrightarrow{A} & \mathcal{D}^\otimes & \xrightarrow{q} & \mathcal{O}^\otimes, \end{array}$$

where i_1 includes Y in $Y \times \Delta^1$ as $Y \times \{1\}$, q is the coCartesian fibration of ∞ -operads that exhibits \mathcal{D} as \mathcal{O} -monoidal, and π_1 denotes the projection onto the first factor.

This implies that maps from a simplicial set Z into the quasi-category $\text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}_{/A})$ are in bijection with morphisms $\tilde{F}: Z \times \mathcal{C}^\otimes \times \Delta^1 \rightarrow \mathcal{D}^\otimes$ such that:

- (1) the following diagram commutes:

$$\begin{array}{ccccc} Z \times \mathcal{C}^\otimes & \xrightarrow{i_1} & Z \times \mathcal{C}^\otimes \times \Delta^1 & \xrightarrow{\pi_{1,2}} & Z \times \mathcal{C}^\otimes \\ p \circ \pi_2 \downarrow & & \tilde{F} \downarrow & & \downarrow p \circ \pi_2 \\ \mathcal{O}^\otimes & \xrightarrow{A} & \mathcal{D}^\otimes & \xrightarrow{q} & \mathcal{O}^\otimes \end{array}$$

where

- $p: \mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$ and $q: \mathcal{D}^\otimes \rightarrow \mathcal{O}^\otimes$ are the coCartesian fibrations of ∞ -operads that exhibit \mathcal{C} and \mathcal{D} as \mathcal{O} -monoidal ∞ -categories,
 - i_1 includes $Z \times \mathcal{C}^\otimes$ as $Z \times \mathcal{C}^\otimes \times \{1\}$,
 - $\pi_{1,2}$ denotes the projection onto the first two factors, and
- (2) \tilde{F} sends triples (z, f, σ) with f an inert morphism of \mathcal{C}^\otimes to inert morphisms of \mathcal{D}^\otimes .

The forgetful map $\text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}_{/A}) \rightarrow \text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$ induces the map restricting \tilde{F} to $Z \times \mathcal{C}^\otimes \times \{0\}$. So, putting it all together, we have that morphisms of simplicial sets from Z to $\{F\} \times_{\text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})} \text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}_{/A})$ are given by maps $\tilde{F}: Z \times \mathcal{C}^\otimes \times \Delta^1 \rightarrow \mathcal{D}^\otimes$ such that:

- (1) $\tilde{F}|_{Z \times \mathcal{C}^\otimes \times \{0\}} = F \circ \pi_2$,
- (2) $\tilde{F}|_{Z \times \mathcal{C}^\otimes \times \{1\}} = A \circ p \circ \pi_2$,
- (3) \tilde{F} sends triples (z, f, σ) with f an inert morphism of \mathcal{C}^\otimes to inert morphisms of \mathcal{D}^\otimes and satisfies $q \circ \tilde{F} = p \circ \pi_2$, where π_2 projects onto the second factor of $Z \times \mathcal{C}^\otimes \times \Delta^1$.

With the above description, $\{F\} \times_{\text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})} \text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}_{/A})$ is readily seen to be isomorphic to the simplicial set $\text{Hom}_{\mathcal{A}}(F, A \circ p) = \{F\} \times_{\mathcal{A}} \mathcal{A}^{\Delta^1} \times_{\mathcal{A}} \{A \circ p\}$ (where $\mathcal{A} := \text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$), which is one of the basic models for the mapping space $\text{Map}_{\text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})}(F, A \circ p)$. \square

THEOREM 2.13. *The \mathcal{O} -algebra M from Theorem 2.8 is characterized by the following universal property: For any \mathcal{O} -algebra A in \mathcal{D} the space of \mathcal{O} -algebra maps $\text{Map}_{\text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})}(M, A)$ is homotopy equivalent to the space of lax \mathcal{O} -monoidal lifts*

$$\begin{array}{ccc} & & \mathcal{D}_{/A\mathcal{O}}^\otimes \\ & \nearrow & \downarrow \\ \mathcal{C}^\otimes & \longrightarrow & \mathcal{D}^\otimes. \end{array}$$

Proof. Just combine Corollary 2.11 and Lemma 2.12. \square

3. The universal multiplicative property of the Thom spectrum

Let $n \geq 0$ and R be an \mathbb{E}_{n+1} -ring spectrum, then the ∞ -category of (left) R -module spectra, Mod_R , can be equipped with the structure of an \mathbb{E}_n -monoidal ∞ -category, see [18, Corollary 5.1.2.6]. This means that we can talk about \mathbb{E}_n -algebra objects in Mod_R . We will call these \mathbb{E}_n R -algebras and denote the ∞ -category they form by $\text{Alg}_R^{\mathbb{E}_n}$, that is, we set $\text{Alg}_R^{\mathbb{E}_n} = \text{Alg}_{/\mathbb{E}_n}(\text{Mod}_R)$. Our multiplicative Thom spectra will be \mathbb{E}_n R -algebras.

As in [2], we associate two ∞ -groupoids to R .

- Let $\text{Pic}(R)$ be the subcategory of invertible R -modules and all equivalences between them, that is, the core of the subcategory of Mod_R on the invertible objects.
- Let BGL_1R denote the subcategory of modules equivalent to R and all equivalences between them. This is a full subcategory of $\text{Pic}(R)$, namely, the component of the R -module R .

By Proposition 2.5 $(\text{Mod}_R)^\simeq$ inherits an \mathbb{E}_n -monoidal structure from Mod_R . Since the proposed sets of objects for both $\text{Pic}(R)$ and BGL_1R are closed under tensor products, by [18, Proposition 2.2.1.1], both categories inherit the structure of \mathbb{E}_n -monoidal ∞ -groupoids, or equivalently, \mathbb{E}_n -spaces. By construction both are grouplike.

DEFINITION 3.1. A local system of invertible R -modules on a space X is simply a map $f: X \rightarrow \text{Pic}(R)$. The Thom spectrum of f is given by

$$M_R f = Mf := \text{colim}(X \rightarrow \text{Pic}(R) \rightarrow \text{Mod}_R).$$

This construction of Thom spectrum appears in [2] where it is shown to agree with the definitions in [16, 24]. We will often apply the above definition in the special case that the map f factors through BGL_1R .

3.1. The universal property and its consequences

Applying Theorem 2.8 to Mod_R we directly obtain Lewis’s theorem, or rather the generalization from the sphere spectrum to arbitrary R given in [1]:

COROLLARY 3.2. If X is an \mathbb{E}_n -space and $f: X \rightarrow \text{Pic}(R)$ is an \mathbb{E}_n -map, then Mf becomes an \mathbb{E}_n R -algebra.

DEFINITION 3.3. Given an \mathbb{E}_n R -algebra A , we define \mathbb{E}_n -spaces $\text{Pic}(R)_{\downarrow A}$ and $BGL_1R_{\downarrow A}$ by requiring the following squares to be pullbacks of \mathbb{E}_n -monoidal categories:

$$\begin{array}{ccccc} BGL_1R_{\downarrow A} & \longrightarrow & \text{Pic}(R)_{\downarrow A} & \longrightarrow & (\text{Mod}_R)_{/A} \\ \downarrow & & \downarrow & & \downarrow \\ BGL_1R & \longrightarrow & \text{Pic}(R) & \longrightarrow & \text{Mod}_R, \end{array}$$

where $(\text{Mod}_R)_{/A}$ is the slice of Mod_R over the underlying R -module of A .

Alternatively, we could define $BGL_1R_{\downarrow A}$ and $\text{Pic}(R)_{\downarrow A}$ merely as ∞ -groupoids by requiring the above diagram to consist of pullback squares of ∞ -categories. They would then inherit \mathbb{E}_n -monoidal structures from $(\text{Mod}_R)_{/A}$ in the same way that BGL_1R and $\text{Pic}(R)$ inherit their structure from Mod_R .

We can think of the objects of $\text{Pic}(R)_{\downarrow A}$ as invertible R -modules M equipped with a map $M \rightarrow A$ of R -modules, and of the morphisms as commuting triangles where the arrow $M \rightarrow M'$ is an equivalence.

WARNING 3.4. Our choice of notation for $\text{Pic}(R)_{\downarrow A}$ and $BGL_1 R_{\downarrow A}$ is meant to distinguish these categories from the usual slice categories $\text{Pic}(R)_{/A}$ and $BGL_1 R_{/A}$, which are only defined if $R \rightarrow A$ is an equivalence. Even when $R = A$ and all of these are defined, they differ. Indeed, $\text{Pic}(R)_{/R}$ is a slice of an ∞ -groupoid and thus contractible, but $\text{Pic}(R)_{\downarrow R}$ has many components. For example, the component of the identity map $R \rightarrow R$ is equivalent to $\text{Pic}(R)_{/R}$ and thus contractible; but the component of the zero map $R \rightarrow R$ is equivalent to $BGL_1 R$.

We can now state a characterization of the \mathbb{E}_n -structure on Mf :

THEOREM 3.5. *Let X be an \mathbb{E}_n -space and $f: X \rightarrow \text{Pic}(R)$ be an \mathbb{E}_n -map. The \mathbb{E}_n -algebra structure of Mf is characterized by the following universal property: the space of \mathbb{E}_n R -algebra maps $\text{Map}_{\text{Alg}_R^{\mathbb{E}_n}}(Mf, A)$ is equivalent to the space of \mathbb{E}_n -lifts of f indicated below:*

$$\begin{array}{ccc} & & \text{Pic}(S^0)_{\downarrow A} \\ & \nearrow & \downarrow \\ X & \xrightarrow{f} & \text{Pic}(S^0). \end{array}$$

Proof. Theorem 2.13 tells us directly that $\text{Map}_{\text{Alg}_R^{\mathbb{E}_n}}(Mf, A)$ is the space of lifts of $X \xrightarrow{f} \text{Pic}(R) \rightarrow \text{Mod}_R$ to a lax \mathbb{E}_n -functor $X \rightarrow (\text{Mod}_R)_{/A}$. Now, edges in $(\text{Mod}_R)_{/A}^{\otimes}$ are coCartesian for the projection $(\text{Mod}_R)_{/A}^{\otimes} \rightarrow \mathbb{E}_n^{\otimes}$ if and only if their image in Mod_R^{\otimes} is coCartesian. So a lax \mathbb{E}_n -functor lifting f is automatically monoidal and factors through $\text{Pic}(R)_{\downarrow A}$. \square

NOTATION 3.6. If A is an \mathbb{E}_{n+1} -ring spectrum under R , that is an \mathbb{E}_{n+1} -ring spectrum equipped with an \mathbb{E}_{n+1} -morphism $\eta: R \rightarrow A$, we use Ind_R^A to denote the cocontinuous \mathbb{E}_n -monoidal functor $\text{Mod}_R \rightarrow \text{Mod}_A$ induced by η (see [18, Proposition 7.1.2.6]). Since this functor is \mathbb{E}_n -monoidal, there is also an induced functor $\text{Alg}_R^{\mathbb{E}_n} \rightarrow \text{Alg}_A^{\mathbb{E}_n}$ for which we also use the same notation.

REMARK 3.7. Let us explain the relation between the notions of \mathbb{E}_{n+1} -ring spectrum under R and \mathbb{E}_n R -algebra. First, if A is an \mathbb{E}_{n+1} -ring spectrum under R , we can canonically equip it with the structure of an \mathbb{E}_n R -algebra: by [18, Corollary 7.3.2.7], the right adjoint of $\text{Ind}_R^A: \text{Mod}_R \rightarrow \text{Mod}_A$ is lax \mathbb{E}_n -monoidal, so it preserves \mathbb{E}_n -algebra objects. In particular, this right adjoint lets us view the unit object A as an \mathbb{E}_n R -algebra.

Now, for $n = \infty$, the concepts of \mathbb{E}_∞ R -algebra and \mathbb{E}_∞ -ring spectrum under R actually coincide by [18, Variant 7.1.3.8]. They do not for $n < \infty$, but an \mathbb{E}_n R -algebra structure on A still makes A an \mathbb{E}_n -ring spectrum under R [18, Warning 7.1.3.9], so for such an A there is still an \mathbb{E}_{n-1} -monoidal functor Ind_R^A .

As a corollary, we easily obtain another of Lewis' results [16, IX.7.1]. To state it we will use the following notation.

NOTATION 3.8. For $m \leq n$, let $\text{Ind}_{\mathbb{E}_m}^{\mathbb{E}_n}$ be the free \mathbb{E}_n R -algebra on an \mathbb{E}_m R -algebra, that is, the left adjoint to the forgetful functor $\text{Alg}_R^{\mathbb{E}_n} \rightarrow \text{Alg}_R^{\mathbb{E}_m}$.

COROLLARY 3.9. *For any pointed map of spaces $f: X \rightarrow BGL_1 R$, there is a natural equivalence*

$$\text{Ind}_{\mathbb{E}_0}^{\mathbb{E}_n}(Mf) \xrightarrow{\sim} M\bar{f},$$

where $\bar{f}: \Omega^n \Sigma^n X \rightarrow BGL_1 R$ is the n -fold loop map adjoint to the pointed map $\Sigma^n f$.

Proof. We will check that both sides represent the same functor on $\text{Alg}_{\mathbb{E}_n}^R$; this yields the claim. Let A be an arbitrary \mathbb{E}_n R -algebra. By adjunction, we have that $\text{Map}_{\text{Alg}_{\mathbb{E}_n}^R}(\text{Ind}_{\mathbb{E}_0}^{\mathbb{E}_n}(Mf), A) \cong \text{Map}_{\text{Alg}_{\mathbb{E}_0}^R}(Mf, A)$ and by Theorem 3.5, this is the space L_0 of lifts of f to an \mathbb{E}_0 -map (that is, a pointed map) $X \rightarrow BGL_1 R_{\downarrow A}$. Moreover, and again by Theorem 3.5, $\text{Map}_{\text{Alg}_{\mathbb{E}_n}^R}(M\bar{f}, A)$ is the space L_n of lifts of \bar{f} to an \mathbb{E}_n -map $\Omega^n \Sigma^n X \rightarrow BGL_1 R_{\downarrow A}$.

Since $\Omega^n \Sigma^n X$ is the free grouplike \mathbb{E}_n -space on the pointed space X , the solid vertical maps (induced by composition with the unit $X \rightarrow \Omega^n \Sigma^n X$) in the following diagram of fiber sequences are equivalences:

$$\begin{array}{ccccc}
 L_n & \longrightarrow & \text{Map}_{\mathbb{E}_n}(\Omega^n \Sigma^n X, BGL_1 R_{\downarrow A}) & \longrightarrow & \text{Map}_{\mathbb{E}_n}(\Omega^n \Sigma^n X, BGL_R) \\
 \downarrow & & \downarrow \sim & & \downarrow \sim \\
 L_0 & \longrightarrow & \text{Map}_{\mathbb{E}_0}(X, BGL_1 R_{\downarrow A}) & \longrightarrow & \text{Map}_{\mathbb{E}_0}(X, BGL_1 R).
 \end{array}$$

It follows that the induced map $L_0 \rightarrow L_n$ is an equivalence, too. □

REMARK 3.10. This result also effortlessly extends to more general operads in place of \mathbb{E}_n , which in fact is the version proved by Lewis. We can replace \mathbb{E}_0 as well; for example, the same argument as used above also proves that if f is an m -fold loop map and \bar{f} denotes the universal extension to an n -fold loop for $n \geq m$, then $\text{Ind}_{\mathbb{E}_m}^{\mathbb{E}_n}(Mf) \xrightarrow{\sim} M\bar{f}$.

3.2. \mathbb{E}_n -orientations

The Thom spectrum MG for a topological group $G \rightarrow O$ arises classically in the theory of orientations of vector bundles, representing the universal cohomology theory that orients manifolds with structure group G . This point of view admits a generalization to \mathbb{E}_n -ring spectra, as we briefly summarize now; see [2, 3] for a comparison of the various notions of orientations in the \mathbb{E}_1 and \mathbb{E}_∞ cases, as well as [1].

For the remainder of this section let R be an \mathbb{E}_{n+1} -ring spectrum and A be an \mathbb{E}_{n+1} -ring spectrum under R (see Notation 3.6).

CONVENTION 3.11. For $n = 0$ we make special arrangements: by *grouplike \mathbb{E}_0 -space* we mean a connected pointed space, and by *0-fold loop map* we mean a pointed map with connected domain.

DEFINITION 3.12. Let $B(R, A)$ be the full subgroupoid of $\text{Pic}(R)_{\downarrow A}$ consisting of morphisms of R -modules $h: M \rightarrow A$ such that the adjoint $h^\dagger: \text{Ind}_R^A(M) \rightarrow A$ is an equivalence.

To study the relation between $B(R, A)$ and orientations, we need a lemma about these R -module morphisms:

LEMMA 3.13. Let $h_i: M_i \rightarrow A$ for $i = 1, 2$ be two morphisms of R -modules. Then $(h_1 \otimes_{\downarrow A} h_2)^\dagger$ is an equivalence of A -modules if and only if both h_1^\dagger and h_2^\dagger are (here $\otimes_{\downarrow A}$ denotes the tensor product in $\text{Pic}(R)_{\downarrow A}$).

Proof. Recall that in the monoidal structure of $\text{Pic}(R)_{\downarrow A}$, $h_1 \otimes_{\downarrow A} h_2$ is given by the composite $M \otimes_R N \xrightarrow{h_1 \otimes_R h_2} A \otimes_R A \xrightarrow{\mu_A} A$, where μ_A is the multiplication on A . Since Ind_R^A is \mathbb{E}_n -monoidal, we see that $(h_1 \otimes_{\downarrow A} h_2)^\dagger = h_1^\dagger \otimes_A h_2^\dagger$.

This makes it clear that if both h_i^\dagger are equivalences, then so is $(h_1 \otimes_{\downarrow A} h_2)^\dagger$.

Now assume that $(h_1 \otimes_{\downarrow A} h_2)^\dagger$ is an equivalence with inverse g and let us prove that both h_i^\dagger are equivalences. To lighten the notation, set $M_i^\dagger := \text{Ind}_R^A(M_i)$. Note that by definition of g , the morphism of A -modules given by the composite

$$A \xrightarrow{g} M_1^\dagger \otimes_A M_2^\dagger \xrightarrow{\text{id}_{M_1^\dagger} \otimes_A h_2^\dagger} M_1^\dagger \otimes_A A$$

is a section of h_1^\dagger , which shows that M_1^\dagger splits as $A \oplus F_1$ where $F_1 := \text{fib}(h_1^\dagger)$. Under this splitting, h_1^\dagger corresponds to the projection $A \oplus F_1 \rightarrow A$. Of course there is an analogous splitting of M_2^\dagger as $A \oplus F_2$.

Then we get that $h_1^\dagger \otimes_A h_2^\dagger$ is the projection of $(A \oplus F_1) \otimes_A (A \oplus F_2) = A \oplus F_1 \oplus F_2 \oplus F_1 \otimes F_2$ onto the first summand. Since this map is assumed to be an equivalence, we conclude $F_1 = F_2 = 0$ and thus both h_1^\dagger and h_2^\dagger are equivalences. \square

As a corollary of (one direction) of the lemma, the set of objects in $B(R, A)$ is closed under tensor products in $\text{Pic}(R)_{\downarrow A}$, and thus $B(R, A)$ inherits an \mathbb{E}_n -monoidal structure.

We can now define orientations analogously to the definitions in [2, 3] for the \mathbb{E}_1 and \mathbb{E}_∞ cases.

DEFINITION 3.14. The space of \mathbb{E}_n A -orientations of an \mathbb{E}_n -map $f: X \rightarrow \text{Pic}(R)$ is the space of \mathbb{E}_n -lifts of f indicated below:

$$\begin{array}{ccc} & & B(R, A) \\ & \nearrow & \downarrow \\ X & \longrightarrow & \text{Pic}(R) \end{array}$$

Note that because $B(R, A)$ is an \mathbb{E}_n -subspace of $\text{Pic}(R)_{\downarrow A}$, the universal property (Theorem 3.5) implies that an \mathbb{E}_n A -orientation of f determines a map of $\mathbb{E}_n R$ -algebras $Mf \rightarrow A$. Of course, not every such map of algebras is an orientation; one way to state the requirement of factoring through $B(R, A)$ is to say that for every point $x: * \rightarrow X$, the adjoint θ_x^\dagger of the R -module map $\theta_x: M(f \circ x) \rightarrow Mf \rightarrow A$ is an equivalence. But if the \mathbb{E}_n -space X is grouplike, then this condition is automatically satisfied, as we show next.

LEMMA 3.15. If $f: X \rightarrow \text{Pic}(R)$ is an n -fold loop map, then an $\mathbb{E}_n R$ -algebra morphism $Mf \rightarrow A$ is automatically an orientation and the space of such algebra morphisms is equivalent to the space of \mathbb{E}_n A -orientations of f .

Proof. We just need to show that the lifts of $f: X \rightarrow \text{Pic}(R)$ to $\text{Pic}(R)_{\downarrow A}$ as considered in Theorem 3.5 factor through $B(R, A)$ (we defined $B(R, A)$ as the union of some of the connected components of $\text{Pic}(R)_{\downarrow A}$). Such a lift gives for each point $x \in X$ an R -linear map $\alpha_x: f(x) \rightarrow A$.

If $n > 0$, we have at least one multiplication on X and inverses for it, so the map $\alpha_{x^{-1}}: f(x^{-1}) \rightarrow A$ satisfies that $\alpha_x \otimes \alpha_{x^{-1}} \simeq \alpha_{1_X}: f(1_X) \rightarrow A$ is homotopic to the unit $R \rightarrow A$ and thus is an equivalence when induced up to A . By Lemma 3.13, this implies that α_x is in $B(R, A)$.

If $n = 0$, we do not even need Lemma 3.13: X is connected by Convention 3.11 so we can pick a path from x to 1_X and the lift of f gives a corresponding equivalence $f(x) \simeq f(1_X)$ that commutes up to homotopy with the maps α_x and α_{1_X} to A . \square

As expected \mathbb{E}_n A -orientations give rise to Thom isomorphisms of \mathbb{E}_n algebras.

PROPOSITION 3.16. *An \mathbb{E}_n A -orientation of an \mathbb{E}_n -map $f: X \rightarrow \text{Pic}(R)$ gives rise to a Thom isomorphism $\text{Ind}_R^A(Mf) \cong \text{Ind}_S^A(\Sigma_+^\infty X)$ of $\mathbb{E}_n A$ -algebras (where S is the sphere spectrum). The equivalence of R -modules underlying this Thom isomorphism is a map $A \otimes_R Mf \rightarrow A \otimes \Sigma_+^\infty X$.*

Proof. We can also describe $B(R, A)$ as a pullback of \mathbb{E}_n -spaces:

$$\begin{array}{ccc} B(R, A) & \longrightarrow & B(A, A) \\ \downarrow & & \downarrow \\ \text{Pic}(R) & \xrightarrow{\text{Ind}_R^A} & \text{Pic}(A) \end{array}$$

Note that from the definition, $B(A, A)$ is the space of trivialized A -modules denoted by $A\text{-triv}$ in [2]. Since it is the (entire) slice of $\text{Pic}(A)$ over A , it is contractible. Thus a lift of f to $B(R, A)$ gives a null-homotopy of $\text{Ind}_R^A \circ f$, and consequently, $\text{Ind}_R^A \circ f$ has the same Thom spectrum as the null map $X \rightarrow \text{Pic}(A)$.

Since $\text{Ind}_R^A: \text{Mod}_R \rightarrow \text{Mod}_A$ is a left adjoint, it preserves the colimit computing the Thom spectrum of f , that is, $M(\text{Ind}_R^A \circ f) \cong \text{Ind}_R^A(Mf)$. On the other hand the null map can be expressed as $\text{Ind}_R^S \circ c$ where $c: X \rightarrow \text{Pic}(S)$ is null. The same argument shows that $M(\text{Ind}_S^A \circ c) \cong \text{Ind}_S^A(\text{colim}_X S) = \text{Ind}_S^A(\Sigma_+^\infty X)$.

Finally, the underlying R -module spectrum of $\text{Ind}_R^A(M)$ is given by $Mf \otimes_R A$. □

COROLLARY 3.17. *Let $n > 0$ and $f: X \rightarrow \text{Pic}(R)$ be an n -fold loop map with Thom spectrum Mf . Then f is canonically \mathbb{E}_{n-1} Mf -orientable.*

Proof. Recall that Mf is an \mathbb{E}_n R -algebra, which we can regard as an \mathbb{E}_n -ring spectrum under R (see the second paragraph of Remark 3.7). The identity morphism $Mf \rightarrow Mf$ gives the required orientation. □

The corresponding Thom isomorphism is an equivalence of \mathbb{E}_{n-1} R -algebras, $\text{Ind}_R^{Mf}(Mf) \cong \text{Ind}_S^{Mf}(\Sigma_+^\infty X)$ whose underlying R -module equivalence is a morphism $Mf \otimes_R Mf \cong Mf \otimes \Sigma_+^\infty X$. This Thom isomorphism is due to Mahowald, see [1, Corollary 1.8; 20, Theorem 1.2].

Now we give an alternative proof and mild generalization of the description of \mathbb{E}_n -orientations of ring spectra due to Chadwick and Mandell, see [9, Theorem 3.2]. While their argument uses the general Thom isomorphism, we deduce this result directly from the universal property of Thom ring spectra.

COROLLARY 3.18. *Let $f: X \rightarrow \text{Pic}(R)$ be an n -fold loop map, and let A be an \mathbb{E}_{n+1} -ring spectrum under R . The space of \mathbb{E}_n R -algebra maps from Mf to A (regarded as an \mathbb{E}_n R -algebra) is either empty or equivalent to the space of \mathbb{E}_n R -algebra maps from $R \otimes \Sigma_+^\infty X$ to A , that is*

$$\text{Map}_{\text{Alg}_R^{\mathbb{E}_n}}(Mf, A) \simeq \text{Map}_{\text{Alg}_R^{\mathbb{E}_n}}(R \otimes \Sigma_+^\infty X, A).$$

Proof. In view of the pullback square of \mathbb{E}_n -spaces

$$\begin{array}{ccc} \text{Pic}(R)_{\downarrow A} & \longrightarrow & \text{Pic}(A)_{\downarrow A} \\ \downarrow & & \downarrow \\ \text{Pic}(R) & \xrightarrow{\text{Ind}_R^A} & \text{Pic}(A), \end{array}$$

the space of \mathbb{E}_n -lifts of $f: X \rightarrow \text{Pic}(R)$ to $\text{Pic}(R)_{\downarrow A}$ is equivalent to the space of \mathbb{E}_n -lifts of $\text{Ind}_R^A \circ f$ to $\text{Pic}(A)_{\downarrow A}$. Assume that $\text{Map}_{\text{Alg}_{\mathbb{E}_n}^R}(Mf, A)$ is non-empty. Any \mathbb{E}_n R -algebra map in that space is automatically an orientation by Lemma 3.15 and thus gives a homotopy between $\text{Ind}_R^A \circ f$ and the constant map $c_A: X \rightarrow \text{Pic}(A)$ through \mathbb{E}_n -maps. That homotopy guarantees the spaces of \mathbb{E}_n -lifts of $\text{Ind}_R^A \circ f$ and c_A are equivalent, and the universal property (Theorem 3.5) says these spaces of lifts are the spaces of algebra maps in the statement. \square

A similar result is true for spaces of orientations with a very similar proof.

COROLLARY 3.19. *Let $f: X \rightarrow \text{Pic}(R)$ be an \mathbb{E}_n -map, and let A be an \mathbb{E}_{n+1} -ring spectrum under R . The space of \mathbb{E}_n A -orientations for f is either empty or equivalent to the space of \mathbb{E}_n A -orientations of the constant map $c_R: X \rightarrow \text{Pic}(R)$, namely $\Omega \text{Map}_{\mathbb{E}_n}(X, \text{Pic}(A))$.*

Proof. Arguing just as for the previous corollary using $B(-, A)$ in place of $\text{Pic}(-)_{\downarrow A}$ yields everything but the concrete description of the space of orientations for the constant map. To see that, note that unlike $\text{Pic}(A)_{\downarrow A}$, the space $B(A, A) \simeq \text{Pic}(A)_{/A}$ is contractible. This says first that $B(R, A)$ is the fiber of $\text{Ind}_R^A: \text{Pic}(R) \rightarrow \text{Pic}(A)$ and second that the space of \mathbb{E}_n -lifts of the null map c_R to $B(R, A)$ is equivalent to $\text{Map}_{\mathbb{E}_n}(X, \Omega \text{Pic}(A)) \simeq \Omega \text{Map}_{\mathbb{E}_n}(X, \text{Pic}(A))$. \square

4. Thom spectra and versal \mathbb{E}_n -algebras

4.1. Characteristics of ring spectra

Let $0 \leq n \leq \infty$ be an integer or ∞ and consider an \mathbb{E}_{n+1} -ring spectrum R with associated category Mod_R of left R -module. As explained above, this category admits a monoidal product $\otimes = \otimes_R$ equipping it with the structure of an \mathbb{E}_n -monoidal category, so that there is a category $\text{Alg}_R^{\mathbb{E}_n}$ of \mathbb{E}_n -algebras over R .

We now extend the definition of characteristic introduced in [27, 28] in two ways: First, we allow arbitrary homotopy classes $S^k \rightarrow R$ in non-negative degrees and second we consider \mathbb{E}_n -algebras. For background and further results in the \mathbb{E}_∞ case we refer to Szymik's papers.

DEFINITION 4.1. Given $k \geq 0$ and $\chi \in \pi_k R$ thought of as an R -linear map $\Sigma^k R \rightarrow R$, an algebra $A \in \text{Alg}_R^{\mathbb{E}_n}$ with unit $\eta: R \rightarrow A$ is said to be of *characteristic* χ if $\eta \circ \chi: \Sigma^k R \rightarrow A$ is null-homotopic.

REMARK 4.2. If $n \geq 1$ and $A \in \text{Alg}_R^{\mathbb{E}_n}$ is of characteristic p , then the existence of the multiplication map $\mu: A \otimes_R A \rightarrow A$ shows $p \cdot \text{id}_A = 0$. However, note that for $n = 0$ an algebra being of characteristic p does not imply $p \cdot \text{id}_A$ is null, as the mod 2 Moore spectrum demonstrates.

DEFINITION 4.3. For a given $\chi: \Sigma^k R \rightarrow R$, we define the *versal R -algebra* $R //_{\mathbb{E}_n} \chi$ of *characteristic* χ as the following pushout in $\text{Alg}_R^{\mathbb{E}_n}$:

$$\begin{array}{ccc}
 F_{\mathbb{E}_n} \Sigma^k R & \xrightarrow{\bar{0}} & R \\
 \bar{\chi} \downarrow & & \downarrow \\
 R & \longrightarrow & R //_{\mathbb{E}_n} \chi,
 \end{array}$$

where

- $F_{\mathbb{E}_n} : \text{Mod}_R \rightarrow \text{Alg}_R^{\mathbb{E}_n}$ is the free \mathbb{E}_n R -algebra functor, the left adjoint to the forgetful functor $\text{Alg}_R^{\mathbb{E}_n} \rightarrow \text{Mod}_R$;
- $\bar{g} : F_{\mathbb{E}_n} M \rightarrow N$ denotes the algebra map which is adjoint to the R -linear map $M \rightarrow N$ with $N \in \text{Alg}_R^{\mathbb{E}_n}$.

We use the term *versal* rather than *universal* as these algebras are not initial but only weakly initial in general, see [28, Proposition 3.11]. When there is no risk of confusion, we shall write $R//\chi$ instead of $R//_{\mathbb{E}_n}\chi$.

The versal \mathbb{E}_n R -algebra $R//\chi$ admits a different characterization as the free \mathbb{E}_n R -algebra on the *pointed spectrum* $R \rightarrow R/\chi = \text{cof}(\Sigma^k R \xrightarrow{\chi} R)$, that is, $R//_{\mathbb{E}_n}\chi = \text{Ind}_{\mathbb{E}_0}^{\mathbb{E}_n}(R//_{\mathbb{E}_0}\chi)$. More generally we have:

LEMMA 4.4. *Let $0 \leq m \leq n$, then the following \mathbb{E}_n R -algebras are equivalent:*

- (1) *the versal characteristic χ algebra $R//_{\mathbb{E}_n}\chi$,*
- (2) *the free \mathbb{E}_n R -algebra $\text{Ind}_{\mathbb{E}_m}^{\mathbb{E}_n}(R//_{\mathbb{E}_m}\chi)$ on the \mathbb{E}_m R -algebra $R//_{\mathbb{E}_m}\chi$,*
- (3) *the coequalizer $C_R(\chi) = \text{coeq}(F_{\mathbb{E}_n}\Sigma^k R \rightrightarrows R)$ taken in $\text{Alg}_R^{\mathbb{E}_n}$.*

Proof. We first prove the equivalence of the algebras in (1) and (2). On the one hand, applying the left adjoint $\text{Ind}_{\mathbb{E}_m}^{\mathbb{E}_n}$ to the defining pushout diagram for $R//_{\mathbb{E}_m}\chi$ gives a pushout square

$$\begin{array}{ccc} \text{Ind}_{\mathbb{E}_m}^{\mathbb{E}_n} F_{\mathbb{E}_m} \Sigma^k R & \xrightarrow{\bar{0}} & \text{Ind}_{\mathbb{E}_m}^{\mathbb{E}_n} R \\ \text{Ind}_{\mathbb{E}_m}^{\mathbb{E}_n} \bar{\chi} \downarrow & & \downarrow \\ \text{Ind}_{\mathbb{E}_m}^{\mathbb{E}_n} R & \longrightarrow & \text{Ind}_{\mathbb{E}_m}^{\mathbb{E}_n}(R//_{\mathbb{E}_m}\chi). \end{array}$$

On the other hand, since $\text{Ind}_{\mathbb{E}_m}^{\mathbb{E}_n}$ preserves the initial R -algebra R and $\text{Ind}_{\mathbb{E}_m}^{\mathbb{E}_n} F_{\mathbb{E}_m} \simeq F_{\mathbb{E}_n}$, this diagram is naturally equivalent to the pushout square

$$\begin{array}{ccc} F_{\mathbb{E}_n} \Sigma^k R & \xrightarrow{\bar{0}} & R \\ \bar{\chi} \downarrow & & \downarrow \\ R & \longrightarrow & R//_{\mathbb{E}_n}\chi, \end{array}$$

hence it follows immediately that $R//_{\mathbb{E}_n}\chi$ is equivalent to $\text{Ind}_{\mathbb{E}_m}^{\mathbb{E}_n}(R//_{\mathbb{E}_m}\chi)$.

To prove that $R//_{\mathbb{E}_n}\chi$ is equivalent to the coequalizer $C_R(\chi)$, we observe that both algebras corepresent the same functor on $\text{Alg}_R^{\mathbb{E}_n}$, which maps a given algebra A with unit map η to the space of null-homotopies of the composite $\Sigma^k R \xrightarrow{\chi} R \xrightarrow{\eta} A$. \square

There is a Thom-isomorphism type lemma for algebras of a given characteristic, which is a straightforward generalization of [28, Proposition 3.2].

LEMMA 4.5. *Let $n \geq 1$ and suppose that $\eta : R \rightarrow A$ is a map of \mathbb{E}_{n+1} -ring spectra where A has characteristic χ , then there is a natural equivalence of \mathbb{E}_n A -algebras*

$$\text{Ind}_R^A(F_{\mathbb{E}_n}(\Sigma^{k+1}R)) \xrightarrow{\sim} \text{Ind}_R^A(R//_{\mathbb{E}_n}\chi).$$

Proof. The two algebras in question are constructed as pushouts in $\text{Alg}_{\mathbb{E}_n}^R$

$$\begin{array}{ccc} F_{\mathbb{E}_n} \Sigma^k R & \xrightarrow{\bar{0}} & R \\ \bar{0} \downarrow & & \downarrow \\ R & \longrightarrow & F_{\mathbb{E}_n}(\Sigma^{k+1} R) \end{array} \qquad \begin{array}{ccc} F_{\mathbb{E}_n} \Sigma^k R & \xrightarrow{\bar{0}} & R \\ \bar{\chi} \downarrow & & \downarrow \\ R & \longrightarrow & R // \chi. \end{array}$$

Since $F_{\mathbb{E}_n}(\Sigma^k R) \otimes_R A \xrightarrow{\chi \otimes \text{id}_A} A$ is homotopic to 0 if A has characteristic χ , the two diagrams become equivalent after applying the functor $\text{Ind}_R^A: \text{Alg}_R^{\mathbb{E}_n} \rightarrow \text{Alg}_A^{\mathbb{E}_n}$ (see Notation 3.6), which is given on the level of R -modules by smashing with A . \square

LEMMA 4.6. *Suppose $A \in \text{Alg}_R^{\mathbb{E}_n}$. If A is of characteristic $\chi: \Sigma^k R \rightarrow R$, then*

$$\text{Map}_{\text{Alg}_R^{\mathbb{E}_n}}(R //_{\mathbb{E}_n} \chi, A) = \Omega^{\infty+k+1} A;$$

otherwise the mapping space is empty.

Proof. The proof of [28, Corollary 2.9] generalizes effortlessly: By definition of $R // \chi$ as an \mathbb{E}_n -algebra, there exists a pullback square

$$\begin{array}{ccc} \text{Map}_{\text{Alg}_R^{\mathbb{E}_n}}(R // \chi, A) & \longrightarrow & \text{Map}_{\text{Alg}_R^{\mathbb{E}_n}}(R, A) = * \\ \downarrow & & \downarrow \bar{0}^* \\ * = \text{Map}_{\text{Alg}_R^{\mathbb{E}_n}}(R, A) & \xrightarrow{\bar{\chi}^*} & \text{Map}_{\text{Alg}_R^{\mathbb{E}_n}}(F_{\mathbb{E}_n} \Sigma^k R, A) = \Omega^{\infty+k} A \end{array}$$

showing that $\text{Map}_{\text{Alg}_R^{\mathbb{E}_n}}(R // \chi, A) \xrightarrow{\sim} \Omega^{\infty+k+1} A$ if A is of characteristic χ so that the two points lie in the same component of $\Omega^{\infty} A$, and empty otherwise. \square

In other words, specializing to the case in which χ is the multiplication by p map we see that $\text{Map}_{\text{Alg}_R^{\mathbb{E}_n}}(R // p, A)$ is the space of null-homotopies of the composite map $R \xrightarrow{p} R \xrightarrow{\eta} A$ in R -modules, which is equivalent to the space of homotopies $\eta \sim (1 - p)\eta$ under the map $\text{Hom}_R(R, A) \rightarrow \text{Hom}_R(R, A)$ given by $\alpha \mapsto \alpha + \eta$. This in turn admits an interpretation in terms of Thom spectra, given in the next section.

4.2. Thom spectra as versal characteristic χ algebras

The goal of this section is to identify the Thom spectrum classified by a map $f: S^{k+1} \rightarrow BGL_1 R$ with the versal characteristic χ algebra, where $\chi = \chi(f): \Sigma^k R \rightarrow R$ is the characteristic corresponding to f as defined below.

LEMMA 4.7. *If $k \geq 0$ and A is any unital R -module, that is, an R -module equipped with an R -linear map $\eta: R \rightarrow A$, then there is a natural pullback square of spaces*

$$\begin{array}{ccc} \text{Map}_*(S^{k+1}, BGL_1 R_{\downarrow A}) & \longrightarrow & * \\ \downarrow & & \downarrow \eta \\ \text{Map}_*(S^{k+1}, BGL_1 R) & \longrightarrow & \text{Map}_*(S^k, \Omega^{\infty} A) = \Omega^{\infty+k} A \end{array}$$

where the right vertical map picks out the constant map with value the unit $\eta \in \text{Hom}_R(R, A) = \Omega^{\infty} A$.

Proof. If we regard GL_1R as the space of R -linear self-equivalences of R , it acts on $\text{Hom}_R(R, A)$ by pre-composition. This induces a map $\eta_* : GL_1R \rightarrow \text{Hom}_R(R, A)$ defined by $g \mapsto \eta \circ g$, where $\eta : R \rightarrow A$ is the unit of A . Let $GL_1R_{\downarrow A}$ be the fiber of this map at the point $\eta \in \text{Hom}_R(R, A)$, so that $GL_1R_{\downarrow A}$ can be thought of as the space of automorphisms g of R compatible with η — in the sense that $\eta \circ g$ is homotopic to η . This $GL_1R_{\downarrow A}$ can alternatively be described as the space of endomorphisms of the object η in the ∞ -groupoid $BGL_1R_{\downarrow A}$. In other words, $\Omega_\eta(BGL_1R_{\downarrow A}) = GL_1R_{\downarrow A}$.

This shows that $\text{Map}_*(S^k, GL_1R) = \text{Map}_*(S^{k+1}, BGL_1R)$ and similarly for $GL_1R_{\downarrow A}$. Therefore, applying $\text{Map}_*(S^k, -)$ to the pullback square

$$\begin{array}{ccc} GL_1R_{\downarrow A} & \longrightarrow & * \\ \downarrow & & \downarrow \eta \\ GL_1R & \xrightarrow{\eta_*} & \text{Hom}_R(R, A) \end{array}$$

gives the desired result. □

DEFINITION 4.8. Suppose given a map $f : S^{k+1} \rightarrow BGL_1R$ and let $\tilde{f} : \Sigma^k R \rightarrow R$ be the associated homotopy class. The *characteristic* $\chi(f) : \Sigma^k R \rightarrow R$ associated with f is then defined by

$$\chi(f) = \begin{cases} \tilde{f} - 1 & \text{if } k = 0 \\ \tilde{f} & \text{if } k > 0. \end{cases}$$

PROPOSITION 4.9. If $f : S^{k+1} \rightarrow BGL_1R$ is a based map and $\bar{f} : \Omega^n \Sigma^n S^{k+1} \rightarrow BGL_1R$ is the corresponding n -fold loop map, then for any $A \in \text{Alg}_R^{\mathbb{E}_n}$, there is an equivalence of spaces

$$\text{Map}_{\text{Alg}_R^{\mathbb{E}_n}}(M\bar{f}, A) = \Omega^{\infty+k+1}A,$$

if A has characteristic $\chi(f)$; otherwise, the mapping space is empty.

Proof. By Corollary 3.9, it suffices to prove that $\text{Alg}_R^{\mathbb{E}_0}(Mf, A) = \Omega^{\infty+k+1}A$. To this end, let A be a unital R -module and consider the following commutative diagram of spaces

$$\begin{array}{ccccc} \text{Alg}_R^{\mathbb{E}_0}(Mf, A) & \longrightarrow & \text{Map}_*(S^{k+1}, BGL_1R_{\downarrow A}) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \eta \\ * & \xrightarrow{f} & \text{Map}_*(S^{k+1}, BGL_1R) & \longrightarrow & \text{Map}_*(S^k, A) = \Omega^{\infty+k}A. \\ & & \searrow \eta_* \circ f & & \end{array}$$

The left square is a pullback by the universal property of Thom spectra (Theorem 3.5), while the right square is one due to Lemma 4.7. Hence the outer rectangle is a pullback square.

The right vertical arrow η is induced by $* \rightarrow \Omega^\infty A$ corresponding to the unit η of A , so

$$[\eta] = \begin{cases} \eta \in \pi_0(\Omega^\infty A) & \text{if } k = 0 \\ 0 \in \pi_k(\Omega^\infty A) & \text{if } k > 0. \end{cases}$$

This explains the distinction of the two cases appearing in Definition 4.8: the pullback is non-empty if $k = 0$ and $\eta_* f \simeq \eta$, or if $k > 0$ and $\eta_* f \simeq 0$. □

We thus obtain:

THEOREM 4.10. *For any $n \geq 0$ and $f: S^{k+1} \rightarrow BGL_1 R$ with corresponding n -fold loop map $\bar{f}: \Omega^n \Sigma^n S^{k+1} \rightarrow BGL_1 R$ and with associated characteristic $\chi = \chi(f)$, there is an equivalence $M\bar{f} = R //_{\mathbb{E}_n, \chi}$ of \mathbb{E}_n R -algebras.*

Proof. It follows from Lemma 4.6 that $MF_{\mathbb{E}_n} f$ is of characteristic χ , so the universal property of $R // \chi$ gives a map $R // \chi \rightarrow MF_{\mathbb{E}_n} f$ of \mathbb{E}_n R -algebras. In view of Proposition 4.9 and Lemma 4.6, $MF_{\mathbb{E}_n} f$ and $R // \chi$ corepresent the same functor on $\text{Alg}_R^{\mathbb{E}_n}$, hence this map is an equivalence. \square

REMARK 4.11. An alternative proof of Theorem 4.10 starts with Corollary 3.9 to first reduce the claim to the \mathbb{E}_0 case, and then identifies the Thom spectrum with the versal characteristic χ \mathbb{E}_0 -algebra. The latter statement admits a direct and easy computational proof, as can be found for example in [21, Lemma 3.3] for the case $f = (p - 1)$.

5. Applications

5.1. The Hopkins–Mahowald theorem I: $H\mathbb{F}_p$

In this section we will assume that all spaces and spectra are implicitly p -complete for some prime p . Let $f_p: S^1 \rightarrow BGL_1(S^0)$ be the map corresponding to the element $1 - p \in \mathbb{Z}_p^\times \cong \pi_1 BGL_1(S^0)$ and \bar{f}_p the extension of f_p to a double loop map, making the following diagram commute:

$$\begin{array}{ccc} S^1 & \xrightarrow{f_p} & BGL_1 S^0 \\ \downarrow & \nearrow \bar{f}_p & \\ \Omega^2 \Sigma^2 S^1 & & \end{array}$$

THEOREM 5.1. *If $f_p: S^1 \rightarrow BGL_1(S^0)$ is the map corresponding to the element $1 - p \in \mathbb{Z}_p^\times \cong \pi_1 BGL_1(S^0)$, then the following three spectra are equivalent as \mathbb{E}_2 -algebras:*

- (1) the Thom spectrum $M\bar{f}_p$,
- (2) the versal characteristic p \mathbb{E}_2 -algebra $S^0 //_{\mathbb{E}_2} p$, and
- (3) the Eilenberg–Mac Lane spectrum $H\mathbb{F}_p$ viewed as an \mathbb{E}_2 -algebra.

Proof. The equivalence of $M\bar{f}_p$ and $S^0 // p$ as \mathbb{E}_2 -algebras is a special case of Theorem 4.10. In order to prove that these algebras are also equivalent to $H\mathbb{F}_p$ viewed as an \mathbb{E}_2 -algebra, we observe that there exists a (canonical) map of connective \mathbb{E}_2 -algebras

$$\phi: S^0 // p \longrightarrow H\mathbb{F}_p$$

witnessing the homotopy $p \sim 0$, because $H\mathbb{F}_p$ is of characteristic p and $\pi_1 H\mathbb{F}_p = 0$. Taking $A = H\mathbb{F}_p$ in Lemma 4.5 and using the computation of the homology of free \mathbb{E}_2 -algebras by Araki and Kudo [15, Theorem 7.1] for $p = 2$ and by Dyer and Lashof [12, Theorem 5.2] for $p > 2$ shows that

$$\pi_*(S^0 // p \otimes H\mathbb{F}_p) \cong \pi_*(F_{\mathbb{E}_2} S^1 \otimes H\mathbb{F}_p) \cong H_*(F_{\mathbb{E}_2} S^1, \mathbb{F}_p) \cong \mathcal{A}_p,$$

the dual of the mod p Steenrod algebra. Moreover, the $H\mathbb{F}_p$ \mathbb{E}_2 -algebra morphism

$$F_{\mathbb{E}_2} S^1 \otimes H\mathbb{F}_p \xrightarrow{\sim} S^0 // p \otimes H\mathbb{F}_p \xrightarrow{\phi \otimes H\mathbb{F}_p} H\mathbb{F}_p \otimes H\mathbb{F}_p$$

is adjoint to $S^1 \rightarrow H\mathbb{F}_p \otimes H\mathbb{F}_p$. Unwinding the construction, we see that this map picks out the Bockstein, so the equivalence of (2) and (3) follows from Steinberger's computation of the Dyer–Lashof operations [8, Chapter 3, Theorems 2.2, 2.3] on both sides. \square

REMARK 5.2. The equivalence between the versal \mathbb{E}_2 -algebra of characteristic p and $H\mathbb{F}_p$ has been proven independently in [22].

REMARK 5.3. The reader might wonder why the above argument does not apply to the unique (up to contractible choice) map $\psi: F_{\mathbb{E}_2}S^1 \rightarrow H\mathbb{F}_p$ to imply that $F_{\mathbb{E}_2}S^1 \simeq S^0//p$. This conclusion is false since the two algebras corepresent different functors: $F_{\mathbb{E}_2}S^1$ corepresents $\Omega^{\infty+1}$, while the functor corepresented by $S^0//p$ only agrees with that on \mathbb{E}_2 -algebras of characteristic p by Lemma 4.6.

The reason the argument does not apply is that the algebra map $\psi \otimes \mathbb{F}_p: F_{\mathbb{E}_2}S^1 \otimes H\mathbb{F}_p \rightarrow H\mathbb{F}_p \otimes H\mathbb{F}_p$ does not correspond to a non-zero multiple of the Bockstein, so it is not an equivalence. This also shows that, in general, the equivalence in Lemma 4.5 does not arise from a morphism of algebras $F_{\mathbb{E}_n}(\Sigma^{k+1}R) \rightarrow R//\chi$.

The Hopkins–Mahowald theorem has the following well-known application. Recall that the Morava K -theory spectrum $K(n)$ of height n is an \mathbb{E}_1 -ring spectrum with coefficients $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$, where the degree of v_n is $2p^n - 2$.

COROLLARY 5.4. *There is no \mathbb{E}_2 -refinement of the \mathbb{E}_1 -ring structure on $K(n)$.*

Proof. Assume that $K(n)$ has the structure of an \mathbb{E}_2 -ring spectrum. Since p acts trivially on the coefficients, the universal property of $H\mathbb{F}_p$ induces a map of \mathbb{E}_2 -ring spectra

$$H\mathbb{F}_p \longrightarrow K(n).$$

However, since $H\mathbb{F}_p$ is $K(n)$ -acyclic for all finite n by [25, Theorem 2.1(i)], this map must be null, which yields a contradiction. \square

5.2. The Hopkins–Mahowald theorem II: $H\mathbb{Z}$

The goal of this section is to deduce an integral analogue of Theorem 5.1, using the framework of this paper. Our approach is different from the arguments given in [6, 10, 20], as it does not rely on any further explicit homology computations. Unless otherwise stated, we will implicitly work in the p -complete category of spaces and spectra.

We start with a folklore result of independent interest, which shows that the \mathbb{E}_n -ring structure on Eilenberg–MacLane ring spectra are essentially unique, for all $0 \leq n \leq \infty$. Since we do not know a published reference for this result, we sketch an argument we learned from Tyler Lawson.

LEMMA 5.5. *If R is a commutative ring, then the space of \mathbb{E}_∞ -structures on the Eilenberg–MacLane spectrum HR that induce the given ring structure on $\pi_*HR = R$ is contractible.*

Proof. This follows from the observation that the endomorphism operad of HR in spectra is discrete; indeed, for all $m \geq 0$:

$$\begin{aligned} \mathrm{Map}((HR)^{\otimes m}, HR) &\simeq \mathrm{Map}(\tau_{\leq 0}(HR)^{\otimes m}, HR) \\ &\simeq \mathrm{Map}(H(R^{\otimes m}), HR) \\ &\simeq \mathrm{Hom}(R^{\otimes m}, R). \end{aligned}$$

In order to construct a coherent multiplication on HR , we have to pick out the union of path components corresponding to the iterated multiplication maps. This is a suboperad which is levelwise contractible, and hence we can replace it by an \mathbb{E}_∞ -operad. \square

To construct $H\mathbb{Z}$ as a Thom spectrum, we will combine the construction of $H\mathbb{F}_p$ as a Thom spectrum given in Theorem 5.1 with the idea of intermediate Thom spectra from [5]. Consider the fiber sequence

$$S^3\langle 3 \rangle \longrightarrow S^3 \longrightarrow K(\mathbb{Z}, 3)$$

which realizes the bottom of the Whitehead tower for S^3 . Looping twice gives another fiber sequence

$$\Omega^2(S^3\langle 3 \rangle) \longrightarrow \Omega^2 S^3 \xrightarrow{\pi} S^1 \simeq \Omega^2 K(\mathbb{Z}, 3). \tag{6}$$

As in the previous section, let $f_p: \Omega^2 S^3 \rightarrow BGL_1(S_p^0)$ be the free \mathbb{E}_2 -map classifying the element $(1-p) \in \pi_0 S_p^0$, and let g_p be the composite

$$g_p: \Omega^2(S^3\langle 3 \rangle) \longrightarrow \Omega^2 S^3 \xrightarrow{f_p} BGL_1 S_p^0.$$

For clarity, we will indicate the composite of a map $h: X \rightarrow BGL_1 R$ with the inclusion $BGL_1 R \rightarrow \text{Mod}_R$ by the corresponding capital letter H ; in particular, Mh is the colimit of H .

THEOREM 5.7. *There is an equivalence $Mg_p \simeq H\mathbb{Z}_p$ of \mathbb{E}_2 -ring spectra.*

Proof. Following ideas of Beardsley [5], we will first construct an intermediate Thom spectrum $M\phi_p$ associated to a map $\phi_p: S^1 \rightarrow BGL_1(Mg_p)$ from the base of the fiber sequence (6). The properties of this spectrum allow us then to determine the homotopy groups of Mg_p , from which the claim will follow.

Let Φ_p be the operadic Kan extension of F_p along π with respect to the \mathbb{E}_2 -operad. By [5], Φ_p factors canonically through a map $\phi_p: S^1 \rightarrow BGL_1(Mg_p)$, so we obtain the following (non-commutative) diagram

$$\begin{array}{ccccc} & & g_p & & \\ & \curvearrowright & & \curvearrowleft & \\ \Omega^2(S^3\langle 3 \rangle) & \longrightarrow & \Omega^2 S^3 & \xrightarrow{f_p} & BGL_1 S_p^0 \longrightarrow \text{Sp} \\ & & \downarrow \pi & & \nearrow \\ & & S^1 & \xrightarrow[\phi_p]{} & BGL_1 Mg_p \end{array}$$

We will now identify the Thom spectrum of ϕ_p in two different ways. On the one hand, by [5], there is an equivalence $M\phi_p \simeq Mf_p$ of \mathbb{E}_1 -ring spectra, which in turn is equivalent to $H\mathbb{F}_p$ by Theorem 5.1. On the other hand, Theorem 4.10 shows that $M\phi_p$ is also the versal \mathbb{E}_0 - Mg_p -algebra of characteristic χ_p , where $\chi_p \in \pi_0(Mg_p)^\times$ is the element corresponding to ϕ_p . Combining these two descriptions, we obtain a fiber sequence

$$Mg_p \xrightarrow{1-\chi_p} Mg_p \longrightarrow H\mathbb{F}_p \tag{8}$$

of spectra. By construction, Mg_p is a connected p -complete \mathbb{E}_2 -ring spectrum.

We now claim that Mg_p is of finite type. To this end, first observe that it follows from Lemma 3.15 that Mg_p is $Mf_p \simeq H\mathbb{F}_p$ -oriented. Therefore, we get that

$$H_*(Mg_p; \mathbb{F}_p) \cong H_*(\Omega^2(S^3\langle 3 \rangle); \mathbb{F}_p).$$

In particular, the homology of Mg_p is finitely generated in each degree, so an argument with Serre classes shows that $\pi_*((Mg_p)_p) \cong \pi_*(Mg_p)$ is also finitely generated over \mathbb{Z}_p in each degree. Furthermore, the Hurewicz theorem implies that

$$\mathbb{F}_p \otimes \pi_0 Mg_p \cong H_0(\Omega^2(S^3\langle 3 \rangle); \mathbb{F}_p) \cong \mathbb{F}_p,$$

because $\Omega^2(S^3\langle 3 \rangle)$ is connected. In particular, $\pi_0 Mg_p$ is a cyclic \mathbb{Z}_p -module as it is finitely generated over \mathbb{Z}_p .

Next, consider the long exact sequence of homotopy groups associated to (8), which degenerates to a short exact sequence

$$0 \longrightarrow \pi_0 Mg_p \xrightarrow{1-\chi_p} \pi_0 Mg_p \longrightarrow \mathbb{F}_p \longrightarrow 0$$

as well as isomorphisms $1 - \chi_p: \pi_i Mg_p \xrightarrow{\sim} \pi_i Mg_p$ for $i \geq 1$. Using that Mg_p is of finite type once more, one easily sees that this forces $1 - \chi_p = p$ and

$$\pi_* Mg_p \cong \begin{cases} \mathbb{Z}_p & i = 0 \\ 0 & i \neq 0, \end{cases}$$

hence $Mg_p \simeq H\mathbb{Z}_p$ as spectra. By virtue of Lemma 5.5, we conclude that this equivalence is also one of \mathbb{E}_2 -ring spectra. \square

As in [6, Section 9.3], we may glue these maps together to construct $H\mathbb{Z}$ as a Thom spectrum as well. Here, we work in the category of all spectra, not just p -complete ones.

COROLLARY 5.9. *There is a map $g: \Omega^2 S^3 \rightarrow BGL_1 S^0$ whose associated Thom spectrum is equivalent to $H\mathbb{Z}$ as \mathbb{E}_2 -ring spectra.*

REMARK 5.10. Recent work of Kitchloo [14] gives a description of $H\mathbb{Z}/p^k$ as a Thom spectrum for all primes p and all $k \geq 1$ with the exception of the case $(p, n) = (2, 2)$.

5.3. Topological Hochschild homology and the cotangent complex

Theorem 4.10 allows to transport results proven for Thom spectra to versal algebras and vice versa. We illustrate this idea with two examples, computing the topological Hochschild homology and the cotangent complex of these algebras. These results generalize previous computations of Szymik [28]. In this section, R is an \mathbb{E}_∞ -ring spectrum.

PROPOSITION 5.11. *Let $R//_{\mathbb{E}_n} \chi$ be the versal \mathbb{E}_n -algebra of characteristic χ corresponding to a map $\bar{f}: \Omega^n \Sigma^n S^{k+1} \rightarrow BGL_1 R$. If either $n \geq 3$ or $n \geq 2$ and the versal \mathbb{E}_n -algebra $R//_{\mathbb{E}_n} \chi$ of characteristic χ admits an \mathbb{E}_∞ -refinement, then there is an equivalence*

$$B\Omega^n \Sigma^n S_+^{k+1} \otimes R//_{\mathbb{E}_n} \chi \xrightarrow{\sim} \mathrm{THH}(R//_{\mathbb{E}_n} \chi).$$

Proof. Using Theorem 4.10, it is enough to compute the topological Hochschild homology of $M\bar{f}$. If $n \geq 3$, the claim thus follows from [7, Theorem 3], while the $n \geq 2$ case is covered by [6, 1.6]. \square

The deformation theory of an \mathbb{E}_n R -algebra A is captured by its \mathbb{E}_n -cotangent complex $L_A^{(n)}$, considered as an object in the \mathbb{E}_n -monoidal category $\mathrm{Mod}_A^{\mathbb{E}_n} = \mathrm{Sp}((\mathrm{Alg}_R^{\mathbb{E}_n})/A)$ of \mathbb{E}_n A -modules, see [13, 18].

PROPOSITION 5.12. *If $n \geq 1$ and $R //_{\mathbb{E}_n} \chi$ a the versal \mathbb{E}_n -algebra of characteristic $\chi: \Sigma^k R \rightarrow R$, then there is an equivalence*

$$\Sigma^{k+1} F_{\mathbb{E}_1}(\Sigma^{n-1} R) \otimes R //_{\mathbb{E}_n} \chi \xrightarrow{\sim} L_{R //_{\mathbb{E}_n} \chi}^{(n)}$$

of $\mathbb{E}_n R //_{\mathbb{E}_n} \chi$ -modules.

Proof. First, the cotangent complex of the free \mathbb{E}_n -algebra $F_{\mathbb{E}_n} \Sigma^k R$ can be computed as an $\mathbb{E}_n F_{\mathbb{E}_n} \Sigma^k R$ -module using [13, 2.12, 2.25] as follows:

$$L_{F_{\mathbb{E}_n} \Sigma^k R}^{(n)} \simeq \Sigma^k U_{F_{\mathbb{E}_n} \Sigma^k R} \simeq \Sigma^k F_{\mathbb{E}_1}(\Sigma^{k+n-1} R) \otimes F_{\mathbb{E}_n}(\Sigma^k R),$$

where $U_{F_{\mathbb{E}_n} \Sigma^k R}$ denotes the enveloping algebra of $F_{\mathbb{E}_n} \Sigma^k R$ as in [13]. Second, the natural cofiber sequence for computing the relative cotangent complex [13, 2.11] specializes to give a cofiber sequence

$$L_{F_{\mathbb{E}_n} \Sigma^k R}^{(n)} \otimes_{F_{\mathbb{E}_n} \Sigma^k R} R \longrightarrow L_R^{(n)} \longrightarrow L_{R/F_{\mathbb{E}_n} \Sigma^k R}^{(n)}.$$

Since $L_R^{(n)}$ is contractible, we get

$$\begin{aligned} L_{R/F_{\mathbb{E}_n} \Sigma^k R}^{(n)} &\simeq \Sigma L_{F_{\mathbb{E}_n} \Sigma^k R}^{(n)} \otimes_{F_{\mathbb{E}_n} \Sigma^k R} R \\ &\simeq \Sigma^{k+1} F_{\mathbb{E}_1}(\Sigma^{k+n-1} R) \otimes F_{\mathbb{E}_n} \Sigma^k R \otimes_{F_{\mathbb{E}_n} \Sigma^k R} R \\ &\simeq \Sigma^{k+1} F_{\mathbb{E}_1}(\Sigma^{k+n-1} R). \end{aligned}$$

The base-change formula [18, 7.3.3.7] applied to the defining pushout diagram

$$\begin{array}{ccc} F_{\mathbb{E}_n} \Sigma^k R & \xrightarrow{\bar{0}} & R \\ \bar{\chi} \downarrow & & \downarrow \psi \\ R & \longrightarrow & R // \chi \end{array}$$

now gives the desired equivalence

$$L_{R // \chi}^{(n)} \xrightarrow{\sim} \psi_! L_{R/F_{\mathbb{E}_n} \Sigma^k R}^{(n)} \simeq \Sigma^{k+1} F_{\mathbb{E}_1}(\Sigma^{k+n-1} R) \otimes R // \chi$$

of $\mathbb{E}_n R // \chi$ -modules. □

By Theorem 4.10, we can translate this result immediately into a statement about the \mathbb{E}_n -cotangent complex of certain Thom spectra.

COROLLARY 5.13. *Let $n \geq 1$ and $f: S^{k+1} \rightarrow BGL_1(R)$ with corresponding n -fold loop map $\bar{f}: \Omega^n \Sigma^n S^{k+1} \rightarrow BGL_1(R)$, then there is an equivalence*

$$\Sigma^{k+1} F_{\mathbb{E}_1}(S^{k+n-1}) \otimes M \bar{f} \xrightarrow{\sim} L_{M \bar{f}}^{(n)}$$

of $\mathbb{E}_n M \bar{f}$ -modules.

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