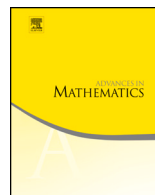




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Real topological Hochschild homology via the norm and Real Witt vectors

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ABSTRACT

We prove that Real topological Hochschild homology can be characterized as the norm from the cyclic group of order 2 to the orthogonal group $O(2)$. From this perspective, we then prove a multiplicative double coset formula for the restriction of this norm to dihedral groups of order $2m$. This informs our new definition of Real Hochschild homology of rings with anti-involution, which we show is the algebraic analogue of Real topological Hochschild homology. Using extra structure on Real Hochschild homology, we define a new theory of p -typical Witt vectors of rings with anti-involution. We end with an explicit computation of the degree zero D_{2m} -Mackey functor homotopy groups of $\mathrm{THR}(\mathbb{Z})$ for m odd. This uses a Tambara reciprocity formula for sums for general finite groups, which may be of independent interest.

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1. Introduction

Real algebraic K-theory, KR , is an invariant of a ring (spectrum) with anti-involution [15]. It is a generalization of Karoubi's Hermitian K-theory [22], and an analogue of Atiyah's topological K-theory with reality [2]. Real topological Hochschild homology, $THR(A)$, is an $O(2)$ -equivariant spectrum that receives a trace map from Real algebraic K-theory [15,6,21]. Just as topological Hochschild homology is essential to the trace method approach to algebraic K-theory, THR is useful for computing Real algebraic K-theory.

In Hill, Hopkins, and Ravenel's solution to the Kervaire invariant one problem [17], they developed the theory of a multiplicative norm functor N_H^G from H -spectra to G -spectra, for $H \leq G$ finite groups. In the case of non-finite compact Lie groups, however, norms are currently only accessible in a few specific cases. In [1], the authors extend the norm construction to the circle group \mathbb{T} , defining the equivariant norm $N_e^{\mathbb{T}}(R)$ for an associative ring spectrum R . They further show that this equivariant norm is a model for topological Hochschild homology. In the present paper, we extend the norm construction to an equivariant norm from D_2 to $O(2)$, using the dihedral bar construction, demonstrating that this norm is a model for Real topological Hochschild homology.

Before defining the norm, we introduce some notation. We write $B_{\bullet}^{\text{di}}(A)$ for the dihedral bar construction on A (cf. Section 2). Let \mathcal{U} denote a complete $O(2)$ -universe, and let \mathcal{V} be the complete D_2 -universe constructed by restricting \mathcal{U} to D_2 . We write $\tilde{\mathcal{V}}$ for the $O(2)$ -universe associated to \mathcal{V} by inflation along the determinant homomorphism. The input for Real topological Hochschild homology is a ring spectrum with anti-involution, (A, ω) . Ring spectra with anti-involution can be alternatively described as algebras in D_2 -spectra over an E_{σ} -operad, where σ is the sign representation of D_2 , the cyclic group of order 2. We refer the reader to [18] for more details.

Definition 1.1. We define the functor

$$N_{D_2}^{O(2)}: \text{Assoc}_{\sigma}(\text{Sp}_{\mathcal{V}}^{D_2}) \longrightarrow \text{Sp}_{\mathcal{U}}^{O(2)}$$

as the composite $\mathcal{I}_{\mathcal{V}}^{\mathcal{U}} | B_{\bullet}^{\text{di}}(-)$. Here $\mathcal{I}_{\mathcal{V}}^{\mathcal{U}}$ denotes the change of universe functor.

We then prove that this functor satisfies one of the fundamental properties of equivariant norms: in the commutative setting it is left adjoint to restriction. Let \mathcal{R} denote the family of subgroups of $O(2)$ which intersect \mathbb{T} trivially, and let $\text{Sp}_{\mathcal{U}}^{O(2), \mathcal{R}}$ denote the \mathcal{R} -model structure on genuine $O(2)$ -spectra [17, Appendix B] (cf. [1, Theorems 2.26, 2.29]).

Theorem 1.2. *The restriction*

$$N_{D_2}^{O(2)}: \text{Comm}(\text{Sp}_{\mathcal{V}}^{D_2}) \rightarrow \text{Comm}(\text{Sp}_{\mathcal{U}}^{O(2), \mathcal{R}})$$

of the norm functor $N_{D_2}^{O(2)}$ to genuine commutative D_2 -ring spectra is left Quillen adjoint to the restriction functor $\iota_{D_2}^*$.

This equivariant norm, $N_{D_2}^{O(2)} A$, is a model for Real topological Hochschild homology, $\text{THR}(A)$.

In [8], the authors prove that for a flat ring spectrum with anti-involution (R, ω) in the sense of [8, Definition 2.6], there is a stable equivalence of D_2 -spectra

$$\iota_{D_2}^* \text{THR}(R) \simeq R \wedge_{N_e^{D_2} \iota_e^* R}^{\mathbb{L}} R.$$

We generalize this result by proving a multiplicative double coset formula for the norm $N_{D_2}^{O(2)}$.

Theorem 1.3 (*Multiplicative Double Coset Formula*). *Let ζ denote the $2m$ -th root of unity $e^{2\pi i/2m}$. When R is a flat E_σ -ring and m is a positive integer, there is a stable equivalence of D_{2m} -spectra*

$$\iota_{D_{2m}}^* N_{D_2}^{O(2)} R \simeq N_{D_2}^{D_{2m}} R \wedge_{N_e^{D_{2m}} \iota_e^* R}^{\mathbb{L}} N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta R.$$

Ordinary topological Hochschild homology is the topological analogue of classical Hochschild homology of algebras. The two theories are related by a linearization map.

$$\pi_k \text{THH}(R) \rightarrow \text{HH}_k(\pi_0 R),$$

which is an isomorphism in degree 0, where R is a (-1) -connected ring spectrum. It is natural to ask, then, what is the algebraic analogue of Real topological Hochschild homology? In this paper, we define such an analogue: a theory of Real Hochschild homology for rings with anti-involution, or more generally for discrete E_σ -rings. A discrete E_σ -ring is a type of D_2 -Mackey functor that arises as the algebraic analogue of E_σ -rings in spectra (cf. Definition 4.7). Indeed, if R is an E_σ -ring in spectra, $\pi_0^{D_2}(R)$ is a discrete E_σ -ring.

Definition 1.4. The *Real D_{2m} -Hochschild homology* of a discrete E_σ -ring \underline{M} is the graded D_{2m} -Mackey functor

$$\underline{\text{HR}}_*^{D_{2m}}(\underline{M}) = H_* \left(\underline{\text{HR}}_\bullet^{D_{2m}}(\underline{M}) \right),$$

where

$$\underline{\text{HR}}_\bullet^{D_{2m}}(\underline{M}) = B_\bullet(N_{D_2}^{D_{2m}} \underline{M}, N_e^{D_{2m}} \iota_e^* \underline{M}, N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta \underline{M}).$$

This theory of Real Hochschild homology is computable using homological algebra for Mackey functors. We prove that Real topological Hochschild homology and Real Hochschild homology are related by a linearization map.

Proposition 1.5. *For any (-1) -connected E_σ -ring A , we have a natural homomorphism*

$$\pi_k^{D_{2m}} \text{THR}(A) \longrightarrow \underline{\text{HR}}_k^{D_{2m}}(\pi_0^{D_2} A),$$

which is an isomorphism when $k = 0$.

This relationship facilitates the computation of Real topological Hochschild homology. As an example, we compute the degree zero D_{2m} -Mackey functor homotopy groups of $\text{THR}(H\mathbb{Z})$, for odd m . When restricted to $\pi_0^{D_2}(\text{THR}(H\mathbb{Z})^{D_{2^k}})$, this computation recovers a computation of [9], proven by different methods.

As part of this computation, we do some of the first calculations to appear in the literature of dihedral norms for Mackey functors. In doing so, we establish a Tambara reciprocity formula for sums for general finite groups.

Theorem 1.6 (*Tambara reciprocity for sums*). *Let G be a finite group and H a subgroup, and let \underline{R} be a G -Tambara functor. For each $F \in \text{Map}^H(G, \{a, b\})$, let K_F be the stabilizer of F . Then for any $a, b \in \underline{R}(G/H)$, we have*

$$\begin{aligned} & N_H^G(a + b) \\ &= \sum_{[F] \in \text{Map}^H(G, \{a, b\})/G} \text{tr}_{K_F}^G \left(\prod_{[\gamma] \in K_F \backslash G/H} N_{K_F \cap (\gamma^{-1}H\gamma)}^{K_F} \left(\gamma \text{res}_{H \cap (\gamma K \gamma^{-1})}^H (F(\gamma^{-1})) \right) \right) \end{aligned}$$

In the case of dihedral groups, this leads to a very explicit formula for Tambara reciprocity for sums (cf. Lemma 6.11), facilitating the computation of dihedral norms.

Our definition of Real Hochschild homology also leads to a definition of Witt vectors for rings with anti-involution. Classically, Witt vectors are closely related to topological Hochschild homology. Indeed, in [14] Hesselholt and Madsen show that for a commutative ring R

$$\pi_0(\text{THH}(R)^{C_{p^n}}) \cong W_{n+1}(R; p),$$

where $W_{n+1}(R; p)$ denotes the length $n + 1$ p -typical Witt vectors of R . Further, in [13], Hesselholt generalized the theory of Witt vectors to non-commutative rings and showed that for any associative ring R there is a relationship between Witt vectors and topological cyclic homology,

$$\text{TC}_{-1}(R; p) \cong W(R; p)_F.$$

Here $W(R)_F$ denotes the coinvariants of the Frobenius endomorphism on the p -typical Witt vectors $W(R; p)$.

In our work, we prove analogous results for rings with anti-involution. We provide a Real cyclotomic structure on Real Hochschild homology, and use this to define Witt

vectors of rings with anti-involution R , denoted $\underline{\mathbb{W}}(R; p)$, (cf. Definition 5.10). As a consequence of this work, we show that $\pi_{-1} \text{TCR}(R)$ can be described purely in terms of equivariant homological algebra.

Theorem 1.7. *Let A be an E_σ -ring and assume $R^1 \lim_k \pi_1^{C_2} \text{THR}(A)^{\mu_{p^k}} = 0$. There is an isomorphism*

$$\pi_{-1} \text{TCR}(A; p) \cong \underline{\mathbb{W}}(\pi_0^{D_2} A; p)_F,$$


where $\underline{\mathbb{W}}(A; p)_F$ is the coinvariants of an operator

$$F: \underline{\mathbb{W}}(\pi_0^{D_2} A; p) \longrightarrow \underline{\mathbb{W}}(\pi_0^{D_2} A; p).$$

1.1. Conventions

Let \mathbf{Top} denote the category of based compactly generated weak Hausdorff spaces. We refer to objects in \mathbf{Top} as spaces and morphisms in \mathbf{Top} as maps of spaces. Let G be a compact Lie group. Then \mathbf{Top}^G denotes the category of based G -spaces and based G -equivariant maps of spaces. For a G -universe \mathcal{U} , let $\mathbf{Sp}_{\mathcal{U}}^G$ be the category of orthogonal G -spectra indexed on \mathcal{U} [27, II. 2.6].

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2. Real topological Hochschild homology via the norm

Recall that the third author, Hopkins and Ravenel [17] define multiplicative norm functors from H -spectra to G -spectra

$$N_H^G : \mathbf{Sp}^H \rightarrow \mathbf{Sp}^G$$

for a finite group G and subgroup H . It is shown in [1] and [5] that one can extend this construction to a norm to the circle group, $N_e^{\mathbb{T}}$, on associative ring orthogonal spectra. The functor

$$N_e^{\mathbb{T}} : \text{Assoc}(\text{Sp}) \rightarrow \text{Sp}_U^{\mathbb{T}}$$

is defined via the cyclic bar construction, $N_e^{\mathbb{T}}(R) = \mathcal{I}_{\mathbb{R}\infty}^U |B_{\wedge}^{cyc} R|$. It follows from this construction of $N_e^{\mathbb{T}}$ that topological Hochschild homology can be viewed as a norm from the trivial group to \mathbb{T} .

In this section, we consider the analogous story for Real topological Hochschild homology. In particular, we define a norm functor $N_{D_2}^{O(2)}$ using the dihedral bar construction and we prove that the norm satisfies the adjointness properties that one would expect from an equivariant norm [17]. As Real topological Hochschild homology can be constructed via the dihedral bar construction [8], this gives a characterization of $\text{THR}(A)$ as an equivariant norm $N_{D_2}^{O(2)}(A)$.

We begin by fixing some conventions. Let $O(2)$ denote the compact Lie group of two-by-two orthogonal matrices. The determinant map determines an extension

$$1 \longrightarrow \mathbb{T} \longrightarrow O(2) \xrightarrow{\det} \{-1, 1\} \longrightarrow 1$$

of groups. We choose a splitting by sending -1 to the matrix $\tau := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and write D_2 for the subgroup of $O(2)$ generated by τ . We then write $\mu_m \subset \mathbb{T}$ for the subgroup of m -th roots of unity generated by $\zeta_m = e^{2\pi i/m}$. Finally, we fix a presentation $D_{2m} = \langle \tau, \zeta_m \mid \tau^2 = \zeta_m^m = (\tau\zeta_m)^2 = 1 \rangle$ for the dihedral group of order D_{2m} regarded as a subgroup of $\mathbb{T} \rtimes D_2 = O(2)$.

The input for Real topological Hochschild homology is a ring spectrum with anti-involution. These can alternatively be described as E_{σ} -rings, algebras in D_2 -spectra over an E_{σ} -operad, where σ is the sign representation (cf. [18] for more details on the theory of E_{σ} -operads). For convenience, we now unpack the definition of an E_{σ} -ring.

Definition 2.1. By an E_1 -ring A , we mean an algebra over the associative operad Assoc in the category of orthogonal spectra Sp . By an E_0 - A -algebra we mean an A -bimodule M equipped with a map $A \rightarrow M$ of A -bimodules (cf. [26, Rem. 2.1.3.10]).

Corollary 2.2. *An E_{σ} -ring R is exactly a D_2 -spectrum R such that*

- (1) *the spectrum $\iota_e^* R$ is an E_1 -ring with anti-involution, denoted $\tau : \iota_e^* R^{\text{op}} \rightarrow \iota_e^* R$, given by the action of the generator of the Weyl group,*
- (2) *the spectrum R is an E_0 - $N_e^{D_2} \iota_e^* R$ -algebra and applying ι_e^* to the E_0 - $N_e^{D_2} \iota_e^* R$ -algebra structure map gives $\iota_e^* R$ the standard E_0 - $\iota_e^* R \wedge \iota_e^* R^{\text{op}}$ -algebra structure.*

Example 2.3. Given an E_1 -ring with anti-involution, regarding R as an object in $\text{Sp}_{\mathbb{V}^2}^{D_2}$ produces an E_{σ} -ring structure on R .

Example 2.4. If R is in $\text{Comm}(\text{Sp}_{\mathbb{V}}^{D_2})$, then R is an E_σ -ring.

For an E_σ -ring A , we now recall the definition of the dihedral bar construction on A .

Definition 2.5. For A an E_σ -ring, the dihedral bar construction $B_\bullet^{\text{di}}(A)$ has k -simplices

$$B_k^{\text{di}}(A) = A \wedge A^{\wedge k}.$$

It has the same simplicial structure maps and cyclic operator as the cyclic bar construction, $B_\bullet^{\text{cy}}(A)$, with the addition of a levelwise involution ω_k acting on the k -simplices. To specify the levelwise involution, let \mathbf{k} be the D_2 -set $\{1, 2, \dots, k\}$ with the generator α of D_2 acting by $\alpha(i) = k - i + 1$ for $1 \leq i \leq k$. Then we define the action of ω_k by the composite

$$\omega_k: A \wedge A^{\wedge \mathbf{k}} \xrightarrow{\text{id} \wedge A^\alpha} A \wedge A^{\wedge \mathbf{k}} \xrightarrow{\tau \wedge \tau \wedge \dots \wedge \tau} M \wedge A^{\wedge \mathbf{k}}$$

where A^α is the automorphism of $A^{\mathbf{k}}$ induced by $\alpha: \mathbf{k} \rightarrow \mathbf{k}$. It is straightforward to check that the structure maps satisfy the usual simplicial and cyclic identities, together with the additional relations

$$\begin{aligned} d_i \omega_k &= \omega_{k-1} d_{k-i} \text{ for } 0 \leq i \leq k \\ s_i \omega_k &= \omega_{k+1} s_{k-i} \text{ for } 0 \leq i \leq k \\ \omega_k t_k &= t_k^{-1} \omega_k \end{aligned}$$

Therefore, the dihedral bar construction is a dihedral object in the sense of [25], and hence by [12, Theorem 5.3] its geometric realization has an action of $O(2)$.

Lemma 2.6. *There is a functor $|B_\bullet^{\text{di}}(-)|: \text{Assoc}_\sigma(\text{Sp}_{\mathbb{V}}^{D_2}) \rightarrow \text{Sp}_{\mathbb{V}}^{O(2)}$.*

Proof. By [12] the realization of a dihedral space has an $O(2)$ -action. Since geometric realization in orthogonal spectra is computed level-wise, we know $|B_\bullet^{\text{di}}(A)|$ has an $O(2)$ -action and this orthogonal $O(2)$ -spectrum is indexed on $\tilde{\mathcal{V}}$. This construction is also clearly functorial. \square

Real topological Hochschild homology can be defined via the dihedral bar construction, as in [8,7].

Definition 2.7. For an E_σ -ring A , the Real topological Hochschild homology of A is

$$\text{THR}(A) = |B_\bullet^{\text{di}}(A)|.$$

By Lemma 2.6, this is an $O(2)$ -spectrum. We now show that it can be viewed as an equivariant norm.

We define the norm from D_2 to $O(2)$ using the dihedral bar construction.

Definition 2.8. Let A be an E_σ -ring in $\mathrm{Sp}_V^{D_2}$. We define

$$N_{D_2}^{O(2)} A = \mathcal{I}_V^{\mathcal{U}} |B_\bullet^{\mathrm{di}}(A)|.$$

This is functorial for $A \in \mathrm{Assoc}_\sigma(\mathrm{Sp}_V^{D_2})$. This defines a functor

$$N_{D_2}^{O(2)} : \mathrm{Assoc}_\sigma(\mathrm{Sp}_V^{D_2}) \rightarrow \mathrm{Sp}_U^{O(2)}.$$

Remark 2.9. In particular, we produce a functor

$$N_{D_2}^{O(2)} : \mathrm{Comm}(\mathrm{Sp}_V^{D_2}) \longrightarrow \mathrm{Comm}(\mathrm{Sp}_U^{O(2)})$$

by restriction to the subcategory $\mathrm{Comm}(\mathrm{Sp}_V^{D_2}) \subset \mathrm{Assoc}_\sigma(\mathrm{Sp}_V^{D_2})$.

Note that the category of commutative monoids in $\mathrm{Sp}_V^{D_2}$ is tensored over the category of D_2 -sets. This follows because $\mathrm{Sp}_V^{D_2}$ is tensored over D_2 -sets and the forgetful functor $\mathrm{Comm}(\mathrm{Sp}_V^{D_2}) \rightarrow \mathrm{Sp}_V^{D_2}$ creates all indexed limits and the category $\mathrm{Comm}(\mathrm{Sp}_V^{D_2})$ contains all Top^{D_2} -enriched equalizers, by a generalization of [28, Lem. 2.8]. We simply write \otimes for this tensoring, viewed as a functor

$$- \otimes - : \mathrm{Comm}(\mathrm{Sp}_V^{D_2}) \times D_2\text{-Set} \longrightarrow \mathrm{Comm}(\mathrm{Sp}_V^{D_2}).$$

Note that this naturally extends to a functor

$$- \otimes - : \mathrm{Comm}(\mathrm{Sp}_V^{D_2}) \times (D_2\text{-Set})^{\Delta^{\mathrm{op}}} \longrightarrow \mathrm{Comm}(\mathrm{Sp}_V^{D_2}).$$

Definition 2.10. Consider the minimal simplicial model

$$D_2 \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D_4 \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D_6 \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} \dots$$

for $O(2)$. Note that $D_{2(\bullet+1)}$ is a dihedral set [25, Lemma 6.3.1]. Therefore $\mathrm{sq}(D_{2(\bullet+1)})$, its Segal-Quillen subdivision (as in [29]), is a simplicial D_2 -set. We define

$$O(2)_\bullet = \mathrm{sq}(D_{2(\bullet+1)})$$

as a simplicial D_2 -set.

Definition 2.11. Let $A \in \text{Comm}(\text{Sp}_V^{D_2})$. Form the coequalizer of the diagram

$$A \otimes D_2 \otimes O(2)_\bullet \begin{array}{c} \xrightarrow{\text{id}_A \otimes \psi} \\ \xrightarrow{N \otimes \text{id}_{O(2)_\bullet}} \end{array} A \otimes O(2)_\bullet$$

where $N: A \otimes D_2 = N_e^{D_2} \iota_e^* A \rightarrow A$ is the E_0 - $N_e^{D_2} \iota_e^* A$ -algebra structure map and $\psi: D_2 \times O(2)_\bullet \rightarrow O(2)_\bullet$ is the left D_2 -action on $O(2)_\bullet$. We define the coequalizer in the category of simplicial objects in $\text{Comm}(\text{Sp}_V^{D_2})$ to be $A \otimes_{D_2} O(2)_\bullet$ and we may consider the geometric realization $|A \otimes_{D_2} O(2)_\bullet|$ as an object in $\text{Sp}_V^{O(2)}$. We therefore make the following definition

$$A \otimes_{D_2} O(2) = \mathcal{I}_V^{\mathcal{U}} |A \otimes_{D_2} O(2)_\bullet|.$$

We now identify the norm (Definition 2.8) with the tensor (Definition 2.11) in the commutative case. Recall that an orthogonal G -spectrum X indexed on a complete universe \mathcal{U} is *well-pointed* if $X(V)$ is well-pointed in Top^G for all finite dimensional orthogonal G -representations V . We say an E_σ -ring in $\text{Sp}_V^{D_2}$ is *very well-pointed* if it is well-pointed and the unit map $S^0 \rightarrow X(\mathbb{R}^0)$ is a Hurewicz cofibration in Top^{D_2} .

Proposition 2.12. *Let A be an object in $\text{Comm}(\text{Sp}_V^{D_2})$ and assume that A is very well-pointed. There is a natural map*

$$N_{D_2}^{O(2)} A \rightarrow A \otimes_{D_2} O(2),$$

in $\text{Comm}(\text{Sp}_V^{O(2)})$, which is a weak equivalence after forgetting to $\text{Sp}_V^{D_2}$.

Proof. We first prove that there is an equivalence

$$\text{sq}(B_\bullet^{\text{di}}(A)) \simeq A \otimes_{D_2} O(2)_\bullet.$$

We consider the k simplices on each side. On the left, the k -simplices, are $A \wedge A^{\wedge 2k+1}$ with $D_{4(k+1)} = \langle \omega, t | t^{2(k+1)} = \omega^2 = t\omega t\omega = 1 \rangle$ action given by letting t cyclically permute the $2k+2$ copies of A , and letting τ act on $\mathbf{k} = \{1, \dots, 2k+1\}$ by $\tau(i) = 2k+1-i+1$. The k -simplices on the right hand side are given by the coequalizer of the diagram

$$A \otimes D_2 \otimes D_{4(k+1)} \begin{array}{c} \xrightarrow{\text{id}_A \otimes \psi} \\ \xrightarrow{N \otimes \text{id}_{D_{4(k+1)}}} \end{array} A \otimes D_{4(k+1)}$$

in the category of $D_{4(k+1)}$ -spectra. This is $A \otimes_{D_2} D_{4(k+1)}$ and therefore the result on k -simplices follows from the $D_{4(k+1)}$ -equivariant map

$$A \otimes_{D_2} D_{4(k+1)} \simeq A \wedge A^{\wedge 2k+1},$$

which is clearly an equivalence on underlying commutative D_2 -spectra. To see that this map is $D_{4(k+1)}$ -equivariant, note that $D_{4(k+1)}/D_2 \cong \mu_{2k+2}$ as $D_{4(k+1)}$ -sets and the right-hand-side can also be considered as a tensoring with the $D_{4(k+1)}$ -set μ_{2k+2} . Since this map is $D_{4(k+1)}$ -equivariant it is compatible with the automorphisms in the dihedral category. It is also easy to check that this is compatible with the face and degeneracy maps. Since A is very well-pointed, both sides are good in the sense of [8, Definition 1.5], this level equivalence induces an equivalence on geometric realizations in the category of D_2 -spectra by [8, Lemma 1.6]. \square

Remark 2.13. Note that in the \mathcal{F} -model structure on $\mathbf{Sp}_{\mathcal{W}}^G$ where G is a compact Lie group, \mathcal{F} is a family of subgroups, and \mathcal{W} is a universe, the \mathcal{F} -equivalences can either be taken to be the maps $X \rightarrow Y$ that induce isomorphisms on homotopy groups $\pi_*(X^H) \rightarrow \pi_*(Y^H)$ for all $H \in \mathcal{F}$ or the maps that induce isomorphisms $\pi_*(\Phi^H X) \rightarrow \pi_*(\Phi^H Y)$, for all $H \in \mathcal{F}$, where Φ^H denotes the H -geometric fixed points.

Recall that \mathcal{R} denotes the family of subgroups of $O(2)$ which intersect \mathbb{T} trivially.

Corollary 2.14. *Let A be an object in $\text{Comm}(\mathbf{Sp}_{\mathbb{V}^2}^{D_2})$ and assume that A is very well-pointed and flat in the sense of [8, Definition 2.6]). Then there is an \mathcal{R} -equivalence*

$$N_{D_2}^{O(2)} A \simeq A \otimes_{D_2} O(2).$$

Proof. Since there is a zig-zag of stable equivalences

$$\Phi^{D_2}(N_{D_2}^{O(2)} A) \simeq \Phi^{D_2}(\text{THR}(A))$$

by [9, Remark 3.6] there is a zig-zag of stable equivalences

$$\Phi^{D_2}(N_{D_2}^{O(2)} A) \simeq \Phi^{D_2}(A) \wedge_{l_{e^*A}^{\mathbb{L}}} \Phi^{D_2}(A)$$

where the right-hand-side is the derived smash product. Since $\Phi^{D_2}(-)$ sends homotopy colimits of $O(2)$ -orthogonal spectra indexed on \mathcal{U} to homotopy colimits of orthogonal spectra,¹ we can identify $\Phi^{D_2}(A \otimes_{D_2} O(2)_{\bullet})$ with the homotopy coequalizer of

$$\Phi^{D_2}(A \otimes_{D_2} O(2)_{\bullet}) \rightrightarrows \Phi^{D_2}(A \otimes O(2)_{\bullet})$$

which is level-wise equivalent to

$$\Phi^{D_2} N_{D_2}^{D_{4(n+1)}}(A) \simeq \Phi^{D_2} \left(\bigwedge_{\gamma} N_{D_2 \cap \gamma D_2 \gamma^{-1}}^{D_2} (l_{D_2 \cap \gamma D_2 \gamma^{-1}}^*(l_{D_2 \cap \gamma D_2 \gamma^{-1}}^* c_{\gamma} A)) \right)$$

¹ Using the Bousfield–Kan formula for homotopy colimits, we can write any homotopy colimit as the geometric realization of a simplicial spectrum and then use the fact that genuine geometric fixed points commute with sifted colimits.

where γ ranges over the representatives of double cosets $D_2 \backslash \gamma / D_2$ in the set of double cosets $D_2 \backslash D_{4(n+1)} / D_2$, which in turn is equivalent to

$$B_n(\Phi^{D_2}(A), \iota_e^*(A), \Phi^{D_2}(A)).$$

It is tedious, but routine, to check that this equivalence is compatible with the face and degeneracy maps. Since both source and target are Reedy cofibrant by our assumptions (cf. [8, Definition 1.5, Lemma 1.6]), this produces a stable equivalence

$$\Phi^{D_2}(A \otimes_{D_2} O(2)) \simeq \Phi^{D_2}(A) \wedge_{\iota_e^* A}^{\mathbb{L}} \Phi^{D_2}(A)$$

of orthogonal spectra on geometric realizations. Since all the groups of order 2 in \mathcal{R} are conjugate in $O(2)$, we have proven the claim. \square

Lemma 2.15. *Given a stable equivalence of very well-pointed E_σ -rings $A \rightarrow A'$ in D_2 -orthogonal spectra indexed on \mathcal{V} , there is an \mathcal{R} -equivalence*

$$N_{D_2}^{O(2)} A \rightarrow N_{D_2}^{O(2)} A'$$

of $O(2)$ -spectra.

Proof. The induced map $N_{D_2}^{O(2)} A \rightarrow N_{D_2}^{O(2)} A'$ is $O(2)$ -equivariant by construction, so it suffices to check that after restricting to D_2 -spectra it is a stable equivalence of D_2 -spectra. Note that this is independent of the choice of subgroup in \mathcal{R} of order 2 because all of the subgroups of order two in \mathcal{R} are conjugate. After restricting to D_2 -spectra, there is a zigzag of stable equivalences of D_2 -spectra

$$\iota_{D_2}^* N_{D_2}^{O(2)} A \simeq \iota_{D_2}^* \text{THR}(A) \xrightarrow{\simeq} \iota_{D_2}^* \text{THR}(A') \simeq \iota_{D_2}^* N_{D_2}^{O(2)} A',$$

by [8, Theorem 2.20] and this agrees with the restriction of the map induced by $A \rightarrow A'$ by naturality of [9, Remark 3.6]. \square

We now show that the norm from D_2 to $O(2)$ (cf. Definition 2.8) satisfies the universal property that one would expect of a norm. For a group G , let All denote the family of all subgroups of G .

Theorem 2.16. *The restriction*

$$N_{D_2}^{O(2)} : \text{Comm}(\text{Sp}_{\mathcal{V}}^{D_2}) \rightarrow \text{Comm}(\text{Sp}_{\mathcal{U}}^{O(2), \mathcal{R}})$$

of the norm functor $N_{D_2}^{O(2)}$ to genuine commutative D_2 -ring spectra is left Quillen adjoint to the restriction functor $\iota_{D_2}^*$ where

$$\text{Comm}(\text{Sp}_{\mathcal{V}}^{D_2}) \text{ and } \text{Comm}(\text{Sp}_{\mathcal{U}}^{O(2), \mathcal{R}})$$

are equipped with the *All*-model structure and the \mathcal{R} -model structure respectively.

Proof. By Corollary 2.14, there is a natural \mathcal{R} -equivalence

$$N_{D_2}^{O(2)}(A) \simeq A \otimes_{D_2} O(2),$$

of $O(2)$ -orthogonal spectra. If $A \rightarrow A'$ is an *All*-equivalence of D_2 -spectra where both source and target are very well-pointed then there is an \mathcal{R} -equivalence

$$N_{D_2}^{O(2)}(A) \simeq N_{D_2}^{O(2)}(A')$$

of $O(2)$ -spectra by Lemma 2.15. In particular, if A and A' are cofibrant in the positive complete stable model structure on $\text{Assoc}_\sigma(\text{Sp}_V^{D_2})$, then they are in particular very well-pointed. This shows that both the functors $N_{D_2}^{O(2)}(-)$ and $(-) \otimes_{D_2} O(2)$ induce well-defined functors between the homotopy categories

$$N_{D_2}^{O(2)}: \text{Ho} \left(\text{Comm}(\text{Sp}_V^{D_2}) \right) \rightarrow \text{Ho} \left(\text{Comm}(\text{Sp}_U^{O(2), \mathcal{R}}) \right)$$

and

$$- \otimes_{D_2} O(2): \text{Ho} \left(\text{Comm}(\text{Sp}_V^{D_2}) \right) \rightarrow \text{Ho} \left(\text{Comm}(\text{Sp}_U^{O(2), \mathcal{R}}) \right)$$

and they are naturally isomorphic on the homotopy categories. It is clear that $- \otimes_{D_2} O(2)$ is left adjoint to the restriction functor

$$\iota_{D_2}^*: \text{Ho} \left(\text{Comm}(\text{Sp}_U^{O(2), \mathcal{R}}) \right) \rightarrow \text{Ho} \left(\text{Comm}(\text{Sp}_V^{D_2}) \right).$$

Moreover, the restriction functor sends cofibrations and weak equivalences to cofibrations and weak equivalences by definition of the stable \mathcal{R} -equivalences and the positive stable \mathcal{R} -cofibrations. Consequently it also preserves all fibrations and acyclic fibrations. \square

3. A multiplicative double coset formula

The multiplicative double coset formula for finite groups gives an explicit formula for the restriction to K of the norm from H to G where H and K are subgroups of G . For compact Lie groups, no such multiplicative double coset formula is known in general. In this section, we present a multiplicative double coset formula for the restriction to D_{2m} of the norm from D_2 to $O(2)$.

Conventions 3.1. When the integer m is understood from context, let

$$\zeta = \zeta_{2m} = e^{2i\pi/2m} \in \mathbb{T} \subset O(2).$$

We consider the element ζ as a lift of the element -1 along

$$D_{2m} \setminus O(2) / D_2 \cong \mu_m \setminus \mathbb{T} \cong \mathbb{T}.$$

We make this choice of homeomorphism simply so that the formula for ζ can be chosen consistently for all m independent of whether m is odd or even. We observe that $\zeta D_2 \zeta^{-1} = \langle \zeta_m \tau \rangle$.

Fix total orders on the D_{2m} -sets D_{2m}/e , D_{2m}/D_2 , and $D_{2m}/\zeta D_2 \zeta^{-1}$. Let

$$\begin{aligned} D_{2m}/e &= \{1 \leq \zeta_m \tau \leq \zeta_m \leq \zeta_m^2 \tau \leq \zeta_m^2 \leq \dots \leq \zeta_m^{m-1} \leq \tau\}, \\ D_{2m}/D_2 &= \{D_2 \leq \zeta_m D_2 \leq \dots \leq \zeta_m^{m-1} D_2\}, \\ D_{2m}/\zeta D_2 \zeta^{-1} &= \{\zeta D_2 \zeta^{-1} \leq \zeta_m \cdot \zeta D_2 \zeta^{-1} \leq \dots \leq \zeta_m^{m-1} \cdot \zeta D_2 \zeta^{-1}\}. \end{aligned}$$

These choices of total orderings on the D_{2m} -sets D_{2m}/e , D_{2m}/D_2 , and $D_{2m}/\zeta D_2 \zeta^{-1}$ also fix group homomorphisms

$$\begin{aligned} \lambda_e &: D_{2m} \rightarrow \Sigma_{2m}, \\ \lambda_{D_2} &: D_{2m} \rightarrow \Sigma_m \wr D_2 \\ \lambda_{\zeta D_2 \zeta^{-1}} &: D_{2m} \rightarrow \Sigma_m \wr \zeta D_2 \zeta^{-1}. \end{aligned}$$

Denote the associated norms by $N_e^{D_{2m}}$, $N_{D_2}^{D_{2m}}$ and $N_{\zeta D_2 \zeta^{-1}}^{D_{2m}}$ respectively.²

Remark 3.2. Any finite subset F of $\mathbb{T} \subset \mathbb{C}$ can be equipped with a total order by considering $1 \leq e^{2i\pi\theta} \leq e^{2i\pi\theta'}$ for all $0 \leq \theta \leq \theta' < 1$. In particular, the subset of m -th roots of unity μ_m in \mathbb{T} can be equipped with a total order.

Lemma 3.3. *There is an isomorphism of totally ordered D_{2m} -sets*

$$f_{m,k}: \mu_{2m(k+1)} \cong D_{2m}/D_2 \amalg D_{2m}^{\amalg k} \amalg D_{2m}/\zeta D_2 \zeta^{-1}$$

where $\zeta = \zeta_{2m}$, the D_{2m} -sets on the right have the total orders from Convention 3.1, and $\mu_{2m(k+1)}$ is equipped with a total order by Remark 3.2.

Proof. Without the total ordering this isomorphism is clear. The total ordering in Convention 3.1 was chosen so that this lemma would be true. \square

Remark 3.4. We will also write $f_{m,k}$ for the underlying map of totally ordered sets from Lemma 3.3 after forgetting the D_{2m} -set structure.

Given an E_σ -ring R , then $\iota_e^* R$ is an E_1 -ring and R is a $N_e^{D_2} \iota_e^* R$ -bimodule with right action

² The choice of ordering does not matter for our norm functors up to canonical natural isomorphism, but remembering the choice of ordering clarifies our constructions later.

$$\bar{\psi}_R: R \wedge N_e^{D_2} \iota_e^* R \longrightarrow R$$

and left action

$$\bar{\psi}_L: N_e^{D_2} \iota_e^* R \wedge R \longrightarrow R.$$

We also note that there is an equivalence of categories

$$c_\zeta: \mathbf{Sp}^{D_2} \longrightarrow \mathbf{Sp}^{\zeta D_2 \zeta^{-1}}$$

which is symmetric monoidal and therefore sends E_σ -rings in \mathbf{Sp}^{D_2} to E_σ -rings in $\mathbf{Sp}^{\zeta D_2 \zeta^{-1}}$. In particular, $c_\zeta R$ is a left $N_e^{\zeta D_2 \zeta^{-1}} \iota_e^* R$ -module with

$$c_\zeta(\bar{\psi}_L): N_e^{\zeta D_2 \zeta^{-1}} \iota_e^* R \wedge c_\zeta R \longrightarrow c_\zeta R.$$

Definition 3.5. Let R be an E_σ -ring. We define a right $N_e^{D_{2m}} \iota_e^* R$ -module structure on $N_{D_2}^{D_{2m}} R$ as the composite

$$\psi_R: N_{D_2}^{D_{2m}} R \wedge N_e^{D_{2m}} \iota_e^* R \xrightarrow{\cong} N_{D_2}^{D_{2m}} (R \wedge N_e^{D_2} \iota_e^* R) \xrightarrow{N_{D_2}^{D_{2m}}(\bar{\psi}_R)} N_{D_2}^{D_{2m}} R.$$

We define a left $N_e^{D_{2m}} \iota_e^* R$ -module structure on $N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta R$

$$\psi_L: N_e^{D_{2m}} \iota_e^* R \wedge N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta R \longrightarrow N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta R$$

as the composite of the isomorphism

$$N_e^{D_{2m}} \iota_e^* R \wedge N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta R \xrightarrow{\cong} N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} \left(N_e^{\zeta D_2 \zeta^{-1}} \iota_e^* R \wedge c_\zeta R \right)$$

with the map

$$N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} \left(N_e^{\zeta D_2 \zeta^{-1}} \iota_e^* R \wedge c_\zeta R \right) \xrightarrow{N_{\zeta D_2 \zeta^{-1}}^{D_{2m}}(c_\zeta(\bar{\psi}_L))} N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta R.$$

Example 3.6. When $m = 2$, we note that $\zeta_4 D_2 \zeta_4^{-1}$ in D_8 can be identified with the diagonal subgroup Δ of D_4 . In this case, Δ and D_2 are conjugate in D_8 even though they are not conjugate in D_4 . We still define a left $N_e^{D_4} \iota_e^* R$ -module structure on $N_\Delta^{D_4} c_\zeta R$ by composing the map

$$N_e^{D_4} \iota_e^* R \wedge N_\Delta^{D_4} c_\zeta R \xrightarrow{\cong} N_\Delta^{D_4} \left(N_e^\Delta \iota_e^* R \wedge c_\zeta R \right)$$

with the map

$$N_{\Delta}^{D_4}(N_e^{\Delta} \iota_e^* R \wedge c_{\zeta} R) \xrightarrow{N_{\Delta}^{D_4}(c_{\zeta}(\bar{\psi}_L))} N_{\Delta}^{D_4} c_{\zeta} R.$$

Remark 3.7. Note that there is an isomorphism of simplicial D_{2m} -sets

$$\mu_{2m(\bullet+1)} \rightarrow D_{2m}/D_2 \amalg D_{2m}^{\amalg \bullet} \amalg D_{2m}/\zeta D_2 \zeta^{-1}$$

which is given by the isomorphism $f_{m,k}$ of totally ordered D_{2m} -sets of Lemma 3.3 on k -simplices. The simplicial maps on the left are given by the simplicial maps in the simplicial set $\text{sd}_{D_{2m}} S_{\bullet}^1$ where S_{\bullet}^1 is the minimal model of S^1 as a simplicial set. On the right, the face maps

$$D_{2m}/D_2 \amalg D_{2m}^{\amalg k} \amalg D_{2m}/\zeta D_2 \zeta^{-1} \rightarrow D_{2m}/D_2 \amalg D_{2m}^{\amalg k-1} \amalg D_{2m}/\zeta D_2 \zeta^{-1}$$

are given by the canonical quotient composed with the fold map

$$D_{2m}/D_2 \amalg D_{2m} \xrightarrow{D_{2m}/D_2 \amalg q} D_{2m}/D_2 \amalg D_{2m}/D_2 \xrightarrow{\nabla} D_{2m}/D_2$$

for the first face map, the fold map

$$D_{2m} \amalg D_{2m} \xrightarrow{\nabla} D_{2m}$$

for the middle maps, and for the last face map it is given by the composite

$$D_{2m} \amalg D_{2m}/\zeta D_2 \zeta^{-1} \xrightarrow{q \amalg 1} D_{2m}/\zeta D_2 \zeta^{-1} \amalg D_{2m}/\zeta D_2 \zeta^{-1} \xrightarrow{\nabla} D_{2m}/\zeta D_2 \zeta^{-1}.$$

The degeneracy maps are given by the canonical inclusions.

Proposition 3.8. *Suppose R is an E_{σ} -ring in D_2 -spectra indexed on the complete universe $\mathcal{V} = \iota_{D_2}^* \mathcal{U}$ where \mathcal{U} is a fixed complete $O(2)$ -universe. More generally, let $\mathcal{V}_n = \iota_{D_{2n}}^* \mathcal{U}$ for $n \geq 1$. There is an isomorphism of simplicial D_{2m} -spectra*

$$\mathcal{I}_{\mathcal{V}}^{\mathcal{V}_m}(\text{sd}_{D_{2m}} B_{\bullet}^{\text{di}}(R)) \cong B_{\bullet} \left(N_{D_2}^{D_{2m}} R, N_e^{D_{2m}} \iota_e^* R, N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_{\zeta} R \right),$$

where we write $\tilde{\mathcal{V}}$ for the D_2 -universe \mathcal{V} regarded as a D_{2m} -universe via inflation along the canonical quotient $D_{2m} \rightarrow D_2$.

Proof. When R is an E_{σ} -ring, we explicitly define the simplicial map

$$\mathcal{I}_{\tilde{\mathcal{V}}}^{\mathcal{V}_m} \text{sd}_{D_{2m}} B_{\bullet}^{\text{di}}(R) \rightarrow B_{\bullet} \left(N_{D_2}^{D_{2m}} R, N_e^{D_{2m}} \iota_e^* R, N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_{\zeta} R \right)$$

on k -simplices. There is an isomorphism given by composition of two maps. The first map

$$f_{m,k}: R^{\wedge \mu_{2m(k+1)}} \longrightarrow R^{\wedge m} \wedge (R^{\wedge m} \wedge (R^{\text{op}})^{\wedge m})^{\wedge k} \wedge (R^{\text{op}})^{\wedge m}$$

is the isomorphism induced by $f_{m,k}$ of Remark 3.4 regarded simply as a map of totally ordered sets. The second map is

$$\begin{array}{c} R^{\wedge m} \wedge ((R \wedge R^{\text{op}})^{\wedge m})^{\wedge k} \wedge (R^{\text{op}})^{\wedge m} \\ \downarrow 1^{\wedge m} \wedge (1 \wedge \tau)^{\wedge mk} \wedge \tau^{\wedge m} \\ N_{D_2}^{D_{2m}} R \wedge (N_e^{D_{2m}} R)^{\wedge k} \wedge N_{\zeta D_{2\zeta^{-1}}}^{D_{2m}} c_{\zeta} R, \end{array}$$

where we use Convention 3.1. This is a D_{2m} -equivariant isomorphism on k -simplices essentially because it comes from the isomorphism of ordered D_{2m} -sets of Lemma 3.3. It follows that the map is compatible with the simplicial structure maps by comparing the structure maps in the isomorphism of simplicial D_{2m} -sets in Remark 3.7 to the structure maps on either side. \square

Consequently, for a flat E_{σ} -ring in the sense of [8, Definition 2.6]), we have the following multiplicative double coset formula. This generalizes the $m = 1$ case appearing in [8].

Theorem 3.9 (*Multiplicative Double Coset Formula*). *When R is a flat E_{σ} -ring and $m > 0$, there is a stable equivalence of D_{2m} -spectra*

$$\iota_{D_{2m}}^* N_{D_2}^{O(2)} R \simeq N_{D_2}^{D_{2m}} R \wedge_{N_e^{D_{2m}} \iota_e^* R}^{\mathbb{L}} N_{\zeta D_{2\zeta^{-1}}}^{D_{2m}} c_{\zeta} R.$$

Proof. By Proposition 3.8, we know there is an equivalence

$$\mathcal{I}_{\mathbb{V}}^{\vee n}(\text{sd}_{D_{2m}} B_{\bullet}^{\text{di}}(R)) \cong B_{\bullet}(N_{D_2}^{D_{2m}} R, N_e^{D_{2m}} \iota_e^* R, N_{\zeta D_{2\zeta^{-1}}}^{D_{2m}} c_{\zeta} R).$$

When R is a flat E_{σ} -algebra in D_2 -spectra in the sense of [8, Definition 2.6]), then $N_{D_2}^{D_{2m}} R$ is a flat $N_e^{D_{2m}} \iota_e^* R$ -module by [30, Theorem 3.4.22-23] and [5] and therefore we may identify

$$N_{D_2}^{D_{2m}} R \wedge_{N_e^{D_{2m}} \iota_e^* R}^{\mathbb{L}} N_{\zeta D_{2\zeta^{-1}}}^{D_{2m}} c_{\zeta} R$$

with the realization of the bar resolution of $N_{D_2}^{D_{2m}} R$ by free $N_e^{D_{2m}} \iota_e^* R$ -modules then smashed with the right $N_e^{D_{2m}} \iota_e^* R$ -module $N_{\zeta D_{2\zeta^{-1}}}^{D_{2m}} c_{\zeta} R$. This is exactly the realization of the simplicial spectrum

$$B_{\bullet}(N_{D_2}^{D_{2m}} R, N_e^{D_{2m}} \iota_e^* R, N_{\zeta D_{2\zeta^{-1}}}^{D_{2m}} c_{\zeta} R). \quad \square$$

4. Real Hochschild homology

In this section we address the question: What is the algebraic analogue of THR? We do this by defining a theory of Real Hochschild homology for discrete E_σ -rings. We then show how this leads to a theory of Witt vectors for rings with anti-involution.

To begin, we recall some basic terminology in the theory of Mackey functors, we define norms in the category of Mackey functors, and E_σ -algebras in D_2 -Mackey functors, which we call discrete E_σ -rings.

4.1. Mackey functors and norms

See [3, §2] for a more thorough review of the theory of Mackey functors. Here we simply recall the constructions and notation we use in the present paper. Let G be a finite group. Write \mathcal{A} for the Burnside category of G . For a finite G -set X , $\underline{\mathcal{A}}_X^G := \mathcal{A}(X, -)$ denotes the representable G -Mackey functor represented by X . This construction forms a co-Mackey functor object in Mackey functors, by viewing it also as a functor in the variable X , so in particular

$$\underline{\mathcal{A}}_{X \amalg Y}^G = \mathcal{A}(X \amalg Y, -) = \mathcal{A}(X, -) \oplus \mathcal{A}(Y, -) = \underline{\mathcal{A}}_X^G \oplus \underline{\mathcal{A}}_Y^G.$$

Write $\underline{\mathcal{A}}^G$ for the Burnside Mackey functor associated to G , which can be identified with $\underline{\mathcal{A}}_*^G$ where $*$ = G/G . This Mackey functor has the property that $\underline{\mathcal{A}}^G(G/H) = A(H)$ where $A(H)$ denotes the Burnside ring for a finite group H . Recall that as an abelian group $A(H)$ is free with basis $\{[H/K]\}$ where K ranges over all conjugacy classes of subgroups $K \leq H$. When $K = H$ we simply write $1 = [H/H]$ and when K is the trivial group we simply write $[H] = [H/\{e\}]$. The transfer and restriction maps in $\underline{\mathcal{A}}^G$ are given by induction and restriction maps on finite sets.

Example 4.1. The Burnside Mackey functor $\underline{\mathcal{A}}^{D_2}$ can be described by the following diagram.

$$\begin{array}{ccccccc} 1 & [D_2] & \underline{M}^{D_2}(D_2/D_2) = \mathbb{Z}\langle 1, [D_2] \rangle & & [D_2] \\ \downarrow & \downarrow & \left(\begin{array}{c} \text{res}_e^{D_2} \downarrow \\ \uparrow \text{tr}_e^{D_2} \end{array} \right) & & \uparrow \\ 1 & 2 & \underline{M}^{D_2}(D_2/*) = \mathbb{Z}\langle 1 \rangle & & 1 \end{array}$$

Given a finite group G , a subgroup $H \leq G$, and a H -set X we write $\text{Map}^H(G, X)$ for the G -set of H -equivariant maps from G to X , which is a functor in the variable X known as coinduction.

Recall that these categories $\text{Sp}_{\mathcal{U}}^G$ of G -spectra indexed on a complete universe \mathcal{U} and Mack_G of G -Mackey functors are both symmetric monoidal, and the symmetric monoidal structures are compatible in the following sense.

Proposition 4.2. [24] *For X and Y cofibrant, (-1) -connected orthogonal G -spectra, there is a natural isomorphism*

$$\pi_0(X \wedge Y) \cong \pi_0 X \square \pi_0 Y.$$

A G -Mackey functor \underline{M} has an associated Eilenberg–MacLane G -spectrum, $H\underline{M}$. The defining property of this spectrum is that

$$\pi_k^G(H\underline{M}) \cong \begin{cases} \underline{M} & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases}$$

It then follows from Proposition 4.2 above that the box product of Mackey functors has a homotopical description:

$$\underline{M} \square \underline{N} \cong \pi_0(H\underline{M} \wedge H\underline{N}).$$

The category \mathbf{Sp}_U^G has an equivariant enrichment of the symmetric monoidal product, a G -symmetric monoidal category structure [17,16]. Such a G -symmetric monoidal structure requires multiplicative norms for all subgroups $H \leq G$. In \mathbf{Sp}_U^G these are given by the Hill–Hopkins–Ravenel norm. The G -symmetric monoidal structure on \mathbf{Sp}_U^G induces such a structure on Mack_G as well. In particular, one can define norms for G -Mackey functors.

Definition 4.3 (cf. [16]). Given a finite group G with subgroup H and an H -Mackey functor \underline{M} , the norm in Mackey functors is defined by

$$N_H^G \underline{M} = \pi_0^G N_H^G H\underline{M}.$$

These Mackey functor norms also appear under a different guise in earlier work of Bouc [4].

The following lemma is immediate.

Lemma 4.4. *The norm in Mackey functors commutes with sifted colimits.*

4.2. Discrete E_σ -rings

In Section 2, we discussed E_σ -rings in D_2 -spectra, which serve as the input for Real topological Hochschild homology. We now define their algebraic analogues, discrete E_σ -rings. These discrete E_σ -rings will be the input for our construction of Real Hochschild homology.

Definition 4.5. Let V be a finite dimensional representation of a finite group G . An E_V -algebra in G -Mackey functors is a \mathcal{P}_V -algebra in G -Mackey functors, where \mathcal{P}_V is the monad

$$\mathcal{P}_V(-) = \bigoplus_{n \geq 0} \pi_0^G(((E_{V,n})_+ \wedge_{\Sigma_n} H(-)^{\wedge n}))$$

and $H(-)$ is the Eilenberg–MacLane functor.

When $V = \sigma$, this monad is particularly simple, since the spaces in the E_σ -operad are homotopy discrete.

Proposition 4.6. *For D_2 -Mackey functors, the monad \mathcal{P}_σ is given by*

$$\mathcal{P}_\sigma(\underline{M}) = T(N_e^{D_2} \iota_e^* \underline{M}) \square (\underline{A} \oplus \underline{M}),$$

where $T(-)$ is the free associative algebra functor.

Proof. Recall that we have an equivariant equivalence

$$E_{\sigma,n} \simeq (D_2 \times \Sigma_n) / \Gamma_n.$$

If n is even, then we have natural isomorphisms

$$\pi_0(((E_{\sigma,n})_+ \wedge_{\Sigma_n} H\underline{M}^{\wedge n})) \cong \pi_0((N_e^{D_2} \iota_e^* H\underline{M})^{\wedge n/2}) \cong (N_e^{D_2} \iota_e^* \underline{M})^{\square n/2}.$$

If n is odd, then since fixed points contribute a box-factor of \underline{M} itself:

$$\pi_0(((E_{\sigma,n})_+ \wedge_{\Sigma_n} H\underline{M}^{\wedge n})) \cong (N_e^{D_2} \iota_e^* \underline{M})^{\square \lfloor n/2 \rfloor} \square \underline{M}.$$

The result follows from grouping the terms according to the number of box-factors involving the norm. \square

Definition 4.7. By a *discrete E_σ -ring*, we mean an algebra over the monad \mathcal{P}_σ in the category of D_2 -Mackey functors.

We can further unpack this structure to describe the monoids.

Lemma 4.8. *A discrete E_σ -ring is the following data:*

- (1) A D_2 -Mackey functor \underline{M} , together with an associative product on $\underline{M}(D_2/e)$ for which the Weyl action is an anti-homomorphism,
- (2) a $N_e^{D_2} \iota_e^* \underline{M}$ -bimodule structure on \underline{M} that restricts to the standard action of $\underline{M}(D_2/e) \otimes \underline{M}(D_2/e)^{\text{op}}$ on $\underline{M}(D_2/e)$.
- (3) an element $1 \in \underline{M}(D_2/D_2)$ that restricts to the element $1 \in \underline{M}(D_2/e)$.

Remark 4.9. These conditions are almost those of a Hermitian Mackey functor in the sense of [10]: the only difference is that a discrete E_σ -ring includes the additional assumption that there is a fixed unit element $1 \in \underline{M}(D_2/D_2)$, or in other words, an $E_0\text{-}N_e^{D_2}\iota_e^*\underline{M}(D_2/e)$ -ring structure on \underline{M} .

Example 4.10. Given an E_σ -ring R it is clear that $\underline{\pi}_0^{D_2}(R)$ is a discrete E_σ -ring. In fact, this does not depend on our choice of E_σ -operad. By Example 2.4, we also conclude that if R is a commutative monoid in $\text{Sp}_V^{D_2}$, then $\underline{\pi}_0^{D_2}R$ is a discrete E_σ -ring.

Example 4.11 (*Rings with anti-involution*). Let R be a discrete ring with anti-involution $\tau: R^{\text{op}} \rightarrow R$, regarded as the action of the generator of D_2 . Then there is an associated Mackey functor \underline{M} with $\underline{M}(D_2/e) = R$ and $\underline{M}(D_2/D_2) = R^{C_2}$. The restriction map $\text{res}_e^{D_2}$ is the inclusion of fixed points, the transfer $\text{tr}_e^{D_2}$ is the map $1 + \tau$, and the Weyl group action of D_2 on $\underline{M}(D_2/e) = R$ is defined on the generator of D_2 by the anti-involution $\tau: R^{\text{op}} \rightarrow R$. Since $R^{\text{op}} \rightarrow R$ is a ring map, there is an element $1 \in R^{C_2}$ that restricts to the multiplicative unit in $1 \in R$. This specifies a discrete E_σ -ring structure on the Mackey functor \underline{M} .

4.3. Real Hochschild homology of discrete E_σ -rings

In this section, we define the Real Hochschild homology $\underline{\text{HR}}_*^{D_{2m}}(\underline{M})$ of a discrete E_σ -ring \underline{M} , which takes values in graded D_{2m} -Mackey functors. We first need to specify a right $N_e^{D_{2m}}\iota_e^*\underline{M}$ -action on $N_{D_2}^{D_{2m}}\underline{M}$, and a left $N_e^{D_{2m}}\iota_e^*\underline{M}$ -action on $N_{\zeta D_2 \zeta^{-1}}^{D_{2m}}c_\zeta \underline{M}$. Here c_ζ is the symmetric monoidal equivalence of categories

$$c_\zeta: \text{Mack}_{D_2} \rightarrow \text{Mack}_{\zeta D_2 \zeta^{-1}}.$$

For \underline{M} a discrete E_σ -ring, $\iota_e^*\underline{M}$ is an (associative unital) ring so $N_e^{D_{2m}}\iota_e^*\underline{M}$ is an associative Green functor. By Lemma 4.8, there is a left action

$$\overline{\psi}_L: N_e^{D_2}\iota_e^*\underline{M} \square \underline{M} \rightarrow \underline{M}$$

and a right action

$$\overline{\psi}_R: \underline{M} \square N_e^{D_2}\iota_e^*\underline{M} \rightarrow \underline{M}.$$

Definition 4.12. We define a right $N_e^{D_{2m}}\iota_e^*\underline{M}$ -module structure on $N_{D_2}^{D_{2m}}\underline{M}$ as the composite

$$\psi_R: N_{D_2}^{D_{2m}}\underline{M} \square N_e^{D_{2m}}\iota_e^*\underline{M} \xrightarrow{\cong} N_{D_2}^{D_{2m}}(\underline{M} \square N_e^{D_2}\iota_e^*\underline{M}) \xrightarrow{N_{D_2}^{D_{2m}}(\overline{\psi}_R)} N_{D_2}^{D_{2m}}\underline{M}.$$

We define a left $N_e^{D_{2m}}\iota_e^*\underline{M}$ -module structure ψ_L on $N_{\zeta D_2 \zeta^{-1}}^{D_{2m}}c_\zeta \underline{M}$ as the composite of the map

$$N_e^{D_{2m}} \iota_e^* \underline{M} \square N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta \underline{M} \xrightarrow{\cong} N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} (N_e^{\zeta D_2 \zeta^{-1}} \iota_e^* \underline{M} \square c_\zeta \underline{M})$$

with the map

$$N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} (N_e^{\zeta D_2 \zeta^{-1}} \iota_e^* \underline{M} \square c_\zeta \underline{M}) \xrightarrow{N_{\zeta D_2 \zeta^{-1}}^{D_{2m}}(c_\zeta(\overline{\psi}_L))} N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta \underline{M},$$

where $c_\zeta(\overline{\psi}_L)$ is the left action of $N_e^{\zeta D_2 \zeta^{-1}} \iota_e^* \underline{M}$ on $c_\zeta \underline{M}$ coming from the fact that c_ζ is symmetric monoidal and therefore sends E_σ -rings in Mack_{D_2} to E_σ -rings in $\text{Mack}_{\zeta D_2 \zeta^{-1}}$.

Definition 4.13. Given Mackey functors \underline{R} , \underline{M} , and \underline{N} where \underline{R} is an associative Green functor, \underline{N} is a right \underline{R} -module and \underline{M} is a left \underline{R} -module, we define the two-sided bar construction $B_\bullet(\underline{M}, \underline{R}, \underline{N})$ with k -simplices

$$B_k(\underline{M}, \underline{R}, \underline{N}) = \underline{M} \square \underline{R}^{\square k} \square \underline{N}$$

and the usual face and degeneracy maps.

Definition 4.14. The *Real D_{2m} -Hochschild homology* of a discrete E_σ -ring \underline{M} is defined to be the graded D_{2m} -Mackey functor

$$\underline{\text{HR}}_*^{D_{2m}}(\underline{M}) = H_*\left(\underline{\text{HR}}_\bullet^{D_{2m}}(\underline{M})\right),$$

where

$$\underline{\text{HR}}_\bullet^{D_{2m}}(\underline{M}) = B_\bullet(N_{D_2}^{D_{2m}} \underline{M}, N_e^{D_{2m}} \iota_e^* \underline{M}, N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta \underline{M}).$$

Remark 4.15. Given a right $N_e^{D_{2m}} \iota_e^* \underline{M}$ -module \underline{N} , we can also define Real Hochschild homology with coefficients

$$\underline{\text{HR}}_0^{D_{2m}}(\underline{M}; \underline{N}) = H_*(B_\bullet(\underline{N}, N_e^{D_{2m}} \iota_e^* \underline{M}, N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta \underline{M})).$$

Recall that the homology of a simplicial Mackey functor is defined to be the homology of the associated normalized dg Mackey functor, as in [3].

Lemma 4.16. *If \underline{M} is a D_2 -Tambara functor, then $\underline{\text{HR}}_0^{D_{2m}}(\underline{M})$ is a D_{2m} -Tambara functor.*

Proof. Reflexive coequalizers in the category of Tambara functors are computed as the reflexive coequalizer of the underlying Mackey functors [31]. \square

Proposition 4.17. *There is an isomorphism of D_{2m} -Mackey functors*

$$\underline{\text{HR}}_0^{D_{2m}}(\underline{M}) \cong N_{D_2}^{D_{2m}} \underline{M} \square_{N_e^{D_{2m}} \iota_e^* \underline{M}} N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta \underline{M}.$$

Proof. Both sides are given by the coequalizer

$$N_{D_2}^{D_{2m}} \underline{M} \square N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta \underline{M} \begin{matrix} \xleftarrow{\text{id} \square \psi_L} \\ \xleftarrow{\psi_R \square \text{id}} \end{matrix} N_{D_2}^{D_{2m}} \underline{M} \square N_e^{D_{2m}} \iota_e^* \underline{M} \square N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta \underline{M}. \quad \square$$

Definition 4.18. We say that a Mackey functor \underline{M} is *flat* if the derived functors of the functor $\underline{M} \square -$ vanish.

Proposition 4.19. *For any discrete E_σ -ring \underline{M} that can be written as a filtered colimit of representable D_2 -Mackey functors, there is an isomorphism*

$$\underline{\text{HR}}_*^{D_{2m}}(\underline{M}) = \underline{\text{Tor}}_*^{N_e^{D_{2m}} \iota_e^* \underline{M}}(N_{D_2}^{D_{2m}} \underline{M}, N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta \underline{M}).$$

Proof. Norms send representable D_2 -Mackey functors to representable D_{2m} -Mackey functors by [3, Proposition 3.7]. Norms also commute with sifted colimits. Filtered colimits of representable D_{2m} -Mackey functors are flat. \square

4.4. *Comparison between Real Hochschild homology and THR*

We now show Real topological Hochschild homology and Real Hochschild homology are related by a linearization map which is an isomorphism in degree 0.

Theorem 4.20. *For any (-1) -connected E_σ -ring A , we have a natural homomorphism*

$$\pi_k^{D_{2m}} \text{THR}(A) \longrightarrow \underline{\text{HR}}_k^{D_{2m}}(\pi_0^{D_2} A),$$

which is an isomorphism when $k = 0$.

Proof. By [11, X.2.9], there is a spectral sequence

$$E_{p,q}^2 = H_p(\pi_q(X_\bullet)) \implies \pi_{p+q}(|X_\bullet|),$$

from filtering by skeleta. The same proof yields an equivariant version of this spectral sequence. By Proposition 3.8, there is a weak equivalence

$$\iota_{D_{2m}}^* \text{THR}(A) \simeq |B_\bullet(N_{D_2}^{D_{2m}} A, N_e^{D_{2m}} \iota_e^* A, N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta A)|,$$

so the spectral sequence in this case will be of the form:

$$E_{p,q}^2 = H_p(\pi_q(N_{D_2}^{D_{2m}} A \wedge N_e^{D_{2m}} \iota_e^* A^{\wedge \bullet} \wedge N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta A)) \implies \pi_{p+q}^{D_{2m}}(\text{THR}(A)).$$

The edge homomorphism of this spectral sequence is a map

$$\pi_p^{D_{2m}}(\text{THR}(A)) \rightarrow H_p(\pi_0^{D_{2m}}(N_{D_2}^{D_{2m}} A \wedge N_e^{D_{2m}} \iota_e^* A^{\wedge \bullet} \wedge N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta A)).$$

We can identify the right hand side as

$$H_p \left(\pi_0^{D_{2m}} N_{D_2}^{D_{2m}} A \square (\pi_0^{D_{2m}} N_e^{D_{2m}} \iota_e^* A) \square \bullet \square \pi_0^{D_{2m}} (N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta A) \right), \tag{1}$$

using the collapse of the Künneth spectral sequence [24] in degree 0. By Definition 4.3, there are isomorphisms of D_{2m} -Mackey functors

$$\begin{aligned} \pi_0^{D_{2m}} (N_e^{D_{2m}} \iota_e^* A) &\cong N_e^{D_{2m}} \iota_e^* \pi_0^{D_2} (A), \\ \pi_0^{D_{2m}} (N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta A) &\cong N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta \pi_0^{D_2} (A), \\ \pi_0^{D_{2m}} (N_{D_2}^{D_{2m}} A) &\cong N_{D_2}^{D_{2m}} \pi_0^{D_2} (A). \end{aligned}$$

We can therefore identify (1) as

$$H_p \left(N_{D_2}^{D_{2m}} \pi_0^{D_2} A \square (N_e^{D_{2m}} \iota_e^* \pi_0^{D_2} A) \square \bullet \square N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta \pi_0^{D_2} A \right),$$

and hence the edge homomorphism gives a linearization map

$$\pi_p^{D_{2m}}(\text{THR}(A)) \longrightarrow \underline{\text{HR}}_p^{D_{2m}}(\pi_0^{D_2} A).$$

To prove the claim that this map is an isomorphism in degree zero, we note that the only contribution to $t + s = 0$ is

$$E_{0,0}^2 \cong N_{D_2}^{D_{2m}} \pi_0^{D_2} A \square_{N_e^{D_{2m}} \iota_e^* \pi_0^{D_2} A} N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta \pi_0^{D_2} A$$

concentrated in degree $s = t = 0$. By Proposition 4.17 this is $\underline{\text{HR}}_0^{D_{2m}}(\pi_0^{D_2} A)$. Since this is a first quadrant spectral sequence, we observe that

$$E_{0,0}^2 \cong E_{0,0}^\infty \cong \pi_0^{D_{2m}} \text{THR}(A). \quad \square$$

Remark 4.21. In [23], Chloe Lewis constructs a Bökstedt spectral sequence for Real topological Hochschild homology, which computes the equivariant homology of $\text{THR}(A)$. The E_2 -term of this spectral sequence is described by Real Hochschild homology, further justifying that $\underline{\text{HR}}$ is the algebraic analogue of THR .

5. Witt vectors of rings with anti-involution

Hesselholt–Madsen [14] proved that for a commutative ring A ,

$$\pi_0^{\mu_{p^n}}(\text{THH}(A))(\mu_{p^n} / \mu_{p^n}) \cong W_{n+1}(A; p).$$

This was extended to associative rings in [13]. Recall that topological Hochschild homology is a cyclotomic spectrum, which yields restriction maps

$$R_n: \text{THH}(A)^{\mu_{p^n}} \rightarrow \text{THH}(A)^{\mu_{p^{n-1}}}.$$

One can then define $\text{TR}(A; p) = \lim_{n, R_n} \text{THH}(A)^{\mu_{p^n}}$, and it follows from the Hesselholt–Madsen result above that

$$\pi_0 \text{TR}(A; p) \cong W(A; p),$$

where $W(A; p)$ denotes the p -typical Witt vectors of A . This was then extended by Hesselholt to non-commutative Witt vectors. From [13, Theorem A], for an associative ring A there is an isomorphism

$$\text{TC}_{-1}(A; p) \cong W(A; p)_F.$$

Here $W(A; p)$ denotes the non-commutative p -typical Witt vectors of A , and

$$W(A; p)_F = \text{coker}(1 - F: W(A; p) \rightarrow W(A; p)),$$

where F is the Frobenius map. Analogously, one would like to have a notion of (non-commutative) Witt vectors for discrete E_σ -rings, such that for an E_σ -ring A , $\pi_0^{D_{2m}} \text{THR}(A)$ is closely related to the Witt vectors of $\pi_0^{D_2} A$. In this section, we define such a notion of Witt vectors.

Real topological Hochschild homology also has restriction maps

$$R_n: \text{THR}(A)^{\mu_{p^n}} \rightarrow \text{THR}(A)^{\mu_{p^{n-1}}}.$$

We want to understand the algebraic analogue of these restriction maps, which requires defining a Real cyclotomic structure on Real Hochschild homology. To do this, we first recall from [3] the definition of geometric fixed points for Mackey functors. Let G be a finite group and let \underline{A} be the Burnside Mackey functor for the group G . For N a normal subgroup of G , let $\mathcal{F}[N]$ denote the family of subgroups of G such that $N \not\subseteq H$.

Definition 5.1. Fix a finite group G and let $N \leq G$ be a normal subgroup. Let $E\mathcal{F}[N](\underline{A})$ be the subMackey functor of the Burnside Mackey functor \underline{A} for G generated by $\underline{A}(G/H)$ for all subgroups H such that H does not contain N . Then define

$$\tilde{E}\mathcal{F}[N](\underline{A}) = \underline{A}/(E\mathcal{F}[N](\underline{A})).$$

If \underline{M} is a G -Mackey functor and N is a normal subgroup of G , then

$$\begin{aligned} E\mathcal{F}[N](\underline{M}) &:= \underline{M} \square E\mathcal{F}[N](\underline{A}), \text{ and} \\ \tilde{E}\mathcal{F}[N](\underline{M}) &:= \underline{M} \square \tilde{E}\mathcal{F}[N](\underline{A}). \end{aligned}$$

More generally, if \underline{M}_\bullet is a dg- G -Mackey functor we define

$$(E\mathcal{F}[N](\underline{M}_\bullet))_n := E\mathcal{F}[N](\underline{M}_n).$$

Note that the G -Mackey functor $\tilde{E}\mathcal{F}[N](\underline{A})$ has the property that

$$\tilde{E}\mathcal{F}[N](\underline{A})(G/H) = \begin{cases} 0 & N \not\subset H \\ \underline{A}((G/N)/(N/H)) & N \subset H \end{cases}$$

which is desired for isotropy separation; i.e. there is an exact sequence

$$E\mathcal{F}[N](\underline{A}) \rightarrow \underline{A} \rightarrow \tilde{E}\mathcal{F}[N](\underline{A})$$

which models the isotropy separation sequence.

We now recall the definition of geometric fixed points for Mackey functors, as in [3]. Let \underline{M} be a D_{2m} -Mackey functor. By [3, Proposition 5.8], we know $\tilde{E}\mathcal{F}[\mu_d](\underline{M})$ is in the image of π_d^* , the pullback functor from D_{2m}/μ_d -Mackey functors to D_{2m} -Mackey functors. Consequently, we may produce a $D_{2m}/\mu_d \cong D_{2m/d}$ -Mackey functor $(\pi_d^*)^{-1}(\tilde{E}\mathcal{F}[\mu_d](\underline{M}))$.

Definition 5.2 (Definition 5.10 [3]). Let \underline{M} be a D_{2m} -Mackey functor. We define the $D_{2m/d}$ -Mackey functor of Φ^{μ_d} -geometric fixed points to be

$$\Phi^{\mu_d}(\underline{M}) := (\pi_d^*)^{-1}(\tilde{E}\mathcal{F}[\mu_d]\underline{M}).$$

We now provide Real Hochschild homology with a Real cyclotomic structure.

Proposition 5.3. Given $D_2 \subset D_{2m} \subset O(2)$, μ_d a normal subgroup of D_{2m} where $d \mid m$ and \underline{M} a discrete E_σ -ring, there is a natural isomorphism

$$\Phi^{\mu_d}(\underline{\mathbb{H}\mathbb{R}}_\bullet^{D_{2m}}(\underline{M})) \cong \underline{\mathbb{H}\mathbb{R}}_\bullet^{D_{2m/d}}(\underline{M})$$

of simplicial $D_{2m/d}$ -Mackey functors and consequently an isomorphism

$$\Phi^{\mu_d}(\underline{\mathbb{H}\mathbb{R}}_*^{D_{2m}}(\underline{M})) \cong \underline{\mathbb{H}\mathbb{R}}_*^{D_{2m/d}}(\underline{M})$$

of $D_{2m/d}$ -Mackey functors.

Proof. We apply Φ^{μ_d} level-wise to the bar construction

$$(\underline{\mathbb{H}\mathbb{R}}^{D_{2m}}(\underline{M}))_\bullet = B_\bullet(N_{D_2}^{D_{2m}}\underline{M}, N_e^{D_{2m}}\iota_e^*\underline{M}, N_{\zeta_{2m}D_2\zeta_{2m}^{-1}}^{D_{2m}}c_{\zeta_{2m}}\underline{M}).$$

By [3, Proposition 5.13], Φ^{μ_d} is strong symmetric monoidal so

$$\begin{aligned} & \Phi^{\mu_d} \left((\underline{\mathbb{H}\mathbb{R}}^{D_{2m}}(\underline{M}))_k \right) \\ & \cong \Phi^{\mu_d} \left(N_{D_2}^{D_{2m}} \underline{M} \right) \square \left(\Phi^{\mu_d} \left(N_e^{D_{2m}} \iota_e^* \underline{M} \right) \right)^{\square k} \square \Phi^{\mu_d} \left(N_{\zeta_{2m} D_2 \zeta_{2m}^{-1}}^{D_{2m}} c_{\zeta_{2m}} \underline{M} \right). \end{aligned} \tag{2}$$

The interaction of the geometric fixed points and the norm is described in [3, Theorem 5.15]. It follows that

$$\Phi^{\mu_d} \left(\underline{\mathbb{H}\mathbb{R}}^{D_{2m}}(\underline{M})_k \right) \cong N_{D_2}^{D_{2m/d}} \underline{M} \square \left(N_e^{D_{2m/d}} \iota_e^* \underline{M} \right)^{\square k} \square N_{\zeta_{2m/d} D_2 \zeta_{2m/d}^{-1}}^{D_{2m/d}} c_{\zeta_{2m/d}} \underline{M} \tag{3}$$

where we use the fact that $D_{2m}/\mu_d \cong D_{2m/d}$ by an isomorphism sending ζ_m to $\zeta_{m/d}$ and τ to τ . Similarly,

$$(\zeta_{2m} D_2 \zeta_{2m} \cdot \mu_d) / \mu_d \cong \zeta_{2m/d} D_2 \zeta_{2m/d},$$

since $\zeta_{2m} D_2 \zeta_{2m}^{-1} = \langle \zeta_m \tau \rangle$ and the isomorphism sends $\zeta_m \tau$ to $\zeta_{m/d} \tau$. It therefore suffices to check that the simplicial structure maps commute with the isomorphisms (2) and (3). Consider the isomorphism

$$\mu_{2m(\bullet+1)} \longrightarrow D_{2m}/D_2 \amalg D_{2m}^{\amalg \bullet} \amalg D_{2m}/\zeta_{2m} D_2 \zeta_{2m}^{-1}$$

from Lemma 3.3. We observe that these maps form an isomorphism of simplicial D_{2m} -sets (cf. Remark 3.7). Applying μ_d -orbits, we have

$$\mu_{2m(\bullet+1)/d} \longrightarrow D_{2m/d}/D_2 \amalg D_{2m/d}^{\amalg \bullet} \amalg D_{2m/d}/\zeta_{2m/d} D_2 \zeta_{2m/d}^{-1}. \tag{4}$$

If \underline{M} is the restriction of a D_{2m} -Mackey functor then the isomorphism is simply induced by tensoring with the isomorphism (4) of simplicial $D_{2m/d}$ -sets. To see this, note that given a finite group G with normal subgroup $N \leq G$, the geometric fixed points Φ^N of the norm N^T of a G -set T is given by taking the norm $N^{T/N}$ of the orbits T/N . The more general statement also holds since we described the isomorphism level-wise and the compatibility with the face and degeneracy maps can be described on each box product factor indexing by the isomorphism of simplicial $D_{2m/d}$ -sets (4) and using the compatibility of that isomorphism with the face and degeneracy maps. \square

Definition 5.4. Given a normal subgroup N in G we define a functor

$$(-)^N: \text{Mack}_G \rightarrow \text{Mack}_{G/N}$$

as the composite $\underline{\pi}_0^{G/N}((H(-))^N)$.

Given a D_{2m} -Mackey functor \underline{M} , the D_2 -Mackey functor \underline{M}^{μ_m} is the data

$$\begin{array}{ccc} & \curvearrowright & \\ \underline{M}(D_{2m}/D_{2m}) & & \underline{M}(D_{2m}/\mu_m) \\ & \curvearrowleft & \end{array}$$

regarded as a D_2 -Mackey functor with the action of the Weyl group $W_{D_2}(e) = D_2$ given by the action of the Weyl group $W_{D_{2m}}(\mu_m) = D_{2m}/\mu_m \cong D_2$.

Remark 5.5. This Mackey “fixed points” functor is the functor denoted q_* in [20], where it was shown to preserve Green and Tambara functors.

These fixed points in Mackey functors relate to categorical fixed points.

Proposition 5.6. For a G -spectrum E and for all integers k , we have

$$\pi_k^{G/N}(E^N) \cong (\pi_k^G(E)).$$

Remark 5.7. This gives an alternate characterization of the geometric fixed points $\Phi^{\mu_d} \underline{M}$ of a D_{2m} -Mackey functor \underline{M} for $d|m$ as

$$\Phi^{\mu_d} \underline{M} \cong (\tilde{E} \mathcal{F}[\mu_d] \underline{M})^{\mu_d}$$

since it is clear in this case that there is a natural isomorphism

$$(\tilde{E} \mathcal{F}[\mu_d](\underline{M}))^{\mu_d} \cong (\pi_{\mu_d}^*)^{-1}(\tilde{E} \mathcal{F}[\mu_d](\underline{M})).$$

Construction 5.8. Given a simplicial D_{2p^k} -Mackey functor \underline{M}_\bullet , there is a natural map

$$\underline{M}_\bullet \rightarrow \tilde{E} \mathcal{F}[\mu_p](\underline{M}_\bullet)$$

and then an induced natural map

$$((\underline{M}_\bullet)^{\mu_p})^{\mu_{p^{k-1}}} \rightarrow \left((\tilde{E} \mathcal{F}[\mu_p](\underline{M}_\bullet))^{\mu_p} \right)^{\mu_{p^{k-1}}}$$

Note that we can identify

$$((\underline{M}_\bullet)^{\mu_p})^{\mu_{p^{k-1}}} = (\underline{M}_\bullet)^{\mu_{p^k}}$$

by unraveling the definition. So, there is a natural transformation

$$R_k: (-)^{\mu_{p^k}} \longrightarrow (\Phi^{\mu_p}(-))^{\mu_{p^{k-1}}}$$

We give this map the name R_k because in our case of interest, where $\underline{M}_\bullet = \underline{\mathbf{HR}}_\bullet^{D_{2p^k}}(\pi_0^{D_2} A)$ for an E_σ -ring A , it is an algebraic analogue of the restriction map R_k on $\text{THR}(A)$.

Construction 5.9. If \underline{M} is a discrete E_σ -ring, it follows from Proposition 5.3 that there is an isomorphism of $D_{2p^{k-1}}$ -Mackey functors

$$\Phi^{\mu_p} \underline{\mathrm{HR}}_n^{D_{2p^k}}(\underline{M}) \cong \underline{\mathrm{HR}}_n^{D_{2p^{k-1}}}(\underline{M}).$$

The above construction therefore produces restriction maps

$$R_k : \left(\underline{\mathrm{HR}}_n^{D_{2p^k}}(\underline{M}) \right)^{\mu_{p^k}} \rightarrow \left(\underline{\mathrm{HR}}_n^{D_{2p^{k-1}}}(\underline{M}) \right)^{\mu_{p^{k-1}}},$$

which are maps of D_2 -Mackey functors. The maps R_k can be described explicitly on Lewis diagrams by

$$\begin{array}{ccc} \underline{\mathrm{HR}}_n^{D_{2p^k}}(\underline{M})(D_{2p^k}/D_{2p^k}) & \xrightarrow{R_k} & \underline{\mathrm{HR}}_n^{D_{2p^{k-1}}}(\underline{M})(D_{2p^{k-1}}/D_{2p^{k-1}}) \\ \left(\begin{array}{c} \uparrow \\ \\ \downarrow \end{array} \right) & & \left(\begin{array}{c} \uparrow \\ \\ \downarrow \end{array} \right) \\ \underline{\mathrm{HR}}_n^{D_{2p^k}}(\underline{M})(D_{2p^k}/\mu_{p^k}) & \xrightarrow{R_k} & \underline{\mathrm{HR}}_n^{D_{2p^{k-1}}}(\underline{M})(D_{2p^{k-1}}/\mu_{p^{k-1}}). \end{array}$$

Definition 5.10. Given a discrete E_σ -ring \underline{M} , we define the *truncated p -typical Real Witt vectors* of \underline{M} by the formula

$$\underline{\mathbb{W}}_{k+1}(\underline{M}; p) = \underline{\mathrm{HR}}_0^{D_{2p^k}}(\underline{M})^{\mu_{p^k}}$$

and the *p -typical Real Witt vectors* of \underline{M} as

$$\underline{\mathbb{W}}(\underline{M}; p) := \lim_{k, R_k} \underline{\mathrm{HR}}_0^{D_{2p^k}}(\underline{M})^{\mu_{p^k}}$$

where the limit is computed in D_2 -Mackey functors. This is functorial in \underline{M} by naturality of the restriction maps R so we produce a functor

$$\underline{\mathbb{W}}(-; p) : \mathrm{Alg}_\sigma(\mathrm{Mack}_{D_2}) \longrightarrow \mathrm{Mack}_{D_2}.$$

Remark 5.11. If \underline{M} is a D_2 -Tambara functor, then by Lemma 4.16 and [20, Prop 5.16], $\underline{\mathrm{HR}}_0^{D_{2p^n}}(\underline{M})^{\mu_{p^n}}$ is a D_2 -Tambara functor and we produce

$$\underline{\mathbb{W}}(-; p) : \mathrm{Tamb}_{D_2} \longrightarrow \mathrm{Tamb}_{D_2}.$$

We now consider how the p -typical Real Witt vectors are related to Real topological Hochschild homology. Let A be an E_σ -ring. Since the family \mathcal{R} does not contain μ_{p^k} and the family $\mathcal{F}[\mu_p]$ does not contain μ_{p^k} for any $k \geq 1$, on μ_{p^k} -fixed points, we do not need to distinguish between these two.

Recall that Real topological restriction homology is defined as

$$\mathrm{TRR}(A; p) = \mathrm{holim}_{k, R_k} \mathrm{THR}(A)^{\mu_{p^k}}$$

in the category $\mathrm{Sp}_V^{D_2}$ [21, Definition 3.6].

Theorem 5.12. *Let A be an E_σ -ring and $\underline{M} = \pi_0^{D_2} A$. There is an isomorphism of D_2 -Mackey functors*

$$\pi_0^{D_2} \text{TRR}(A; p) \cong \underline{\mathbb{W}}(\underline{M}; p)$$

whenever $R^1 \lim_k \pi_1^{D_2} \text{THR}(A)^{\mu_{p^k}} = 0$.

Proof. We note that there is an isomorphism

$$\pi_0^{D_{2p^k}} \text{THR}(A) \cong \underline{\text{HR}}_0^{D_{2p^k}}(\underline{M})$$

of D_{2p^k} -Mackey functors by Theorem 4.20 and consequently a natural isomorphism of D_2 -Mackey functors

$$\pi_0^{D_2}(\text{THR}(A)^{\mu_{p^k}}) \cong (\underline{\text{HR}}_0^{D_{2p^k}}(\underline{M}))^{\mu_{p^k}}. \tag{5}$$

By construction, the diagram

$$\begin{array}{ccc} \pi_0^{D_2} \text{THR}(-)^{\mu_{p^k}} & \xrightarrow{R_k} & \pi_0^{D_2} \text{THR}(-)^{\mu_{p^{k-1}}} \\ \downarrow \cong & & \downarrow \cong \\ (\underline{\text{HR}}_0^{D_{2p^k}}(\pi_0^{D_2}(-)))^{\mu_{p^k}} & \xrightarrow{R_k} & (\underline{\text{HR}}_0^{D_{2p^{k-1}}}(\pi_0^{D_2}(-)))^{\mu_{p^{k-1}}} \end{array}$$

commutes. To see this, we note that the edge homomorphism in the spectral sequence associated to the skeletal filtration of a simplicial spectrum (cf. Proof of 4.20) is compatible with the restriction maps R_k . Consequently,

$$\begin{aligned} \pi_0^{D_2} \text{TRR}(A; p) &\cong \lim_R \pi_0^{D_2} \text{THR}(A)^{\mu_{p^k}} \\ &\cong \lim_R \left(\underline{\text{HR}}_0^{D_{2p^k}}(\pi_0^{D_2} A) \right)^{\mu_{p^k}} \\ &= \underline{\mathbb{W}}(\pi_0^{D_2} A; p) \end{aligned}$$

where the first isomorphism holds by our assumption that

$$R^1 \lim_k \pi_1^{D_2} \text{THR}(A)^{\mu_{p^k}} = 0. \quad \square$$

Construction 5.13. We now define a Frobenius map

$$F: \underline{\mathbb{W}}(\underline{M}; p) \rightarrow \underline{\mathbb{W}}(\underline{M}; p).$$

The restriction maps $\text{res}_{\mu_{p^k}}^{\mu_{p^{k-1}}}$ and $\text{res}_{D_{2p^{k-1}}}^{D_{2p^k}}$ on the Mackey functor $\underline{\text{HR}}_0^{D_{2p^k}}(\underline{M})$ induce a map of D_2 -Mackey functors

$$\begin{array}{ccc}
 \left(\underline{\mathrm{HR}}_0^{D_{2p^k}}(\underline{M}) \right)^{\mu_{p^k}}(D_2/D_2) & \longrightarrow & \left(\underline{\mathrm{HR}}_0^{D_{2p^k}}(\underline{M}) \right)^{\mu_{p^k-1}}(D_{2p}/D_2) \\
 \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) & & \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \\
 \left(\underline{\mathrm{HR}}_0^{D_{2p^k}}(\underline{M}) \right)^{\mu_{p^k}}(D_2/e) & \longrightarrow & \left(\underline{\mathrm{HR}}_0^{D_{2p^k}}(\underline{M}) \right)^{\mu_{p^k-1}}(D_{2p}/e)
 \end{array}$$

that we call F . Observe that the target of this map can be identified with the D_2 -Mackey functor $\left(\underline{\mathrm{HR}}_0^{D_{2p^k-1}}(\underline{M}) \right)^{\mu_{p^k-1}}$, so we simply write

$$F_k : \left(\underline{\mathrm{HR}}_0^{D_{2p^k}}(\underline{M}) \right)^{\mu_{p^k}} \longrightarrow \left(\underline{\mathrm{HR}}_0^{D_{2p^k-1}}(\underline{M}) \right)^{\mu_{p^k-1}}$$

for this map. Note that this is natural so we can apply it to the map

$$\underline{\mathrm{HR}}_0^{D_{2p^k}}(M) \longrightarrow \tilde{E} \mathcal{F}[\mu_p] \left(\underline{\mathrm{HR}}_0^{D_{2p^k}}(M) \right).$$

Therefore, by construction and Proposition 5.3 this map is compatible with the restriction maps in the sense that there are commutative diagrams

$$\begin{array}{ccc}
 \left(\underline{\mathrm{HR}}_0^{D_{2p^k}}(\underline{M}) \right)^{\mu_{p^k}} & \xrightarrow{F_k} & \left(\underline{\mathrm{HR}}_0^{D_{2p^k-1}}(\underline{M}) \right)^{\mu_{p^k-1}} \\
 \downarrow R_k & & \downarrow R_k \\
 \left(\underline{\mathrm{HR}}_0^{D_{2p^k-1}}(\underline{M}) \right)^{\mu_{p^k-1}} & \xrightarrow{F_k} & \left(\underline{\mathrm{HR}}_0^{D_{2p^k-2}}(\underline{M}) \right)^{\mu_{p^k-2}}
 \end{array}$$

Therefore, we have an induced map

$$F_k : \underline{\mathbb{W}}(\underline{M}; p) \longrightarrow \underline{\mathbb{W}}(\underline{M}; p).$$

Remark 5.14. We can also define a Verschiebung operator

$$V : \underline{\mathbb{W}}(\underline{M}; p) \longrightarrow \underline{\mathbb{W}}(\underline{M}; p)$$

in exactly the same way as in Construction 5.13 by replacing the restriction maps in the Mackey functor with the transfer maps in the Mackey functor.

There are also topological analogues of the maps F , V and R on $\mathrm{THR}(A)^{\mu_{p^k}}$ when A is an E_σ -ring, which satisfy certain relations (cf. [21, §3]). In particular, R_k and F_k are compatible in the sense that $R_{k-1} \circ F_k = F_{k-1} \circ R_k$. The cokernel of the map $\mathrm{id}_{\underline{\mathbb{W}}(A;p)} - F$ is defined to be the coinvariants $\underline{\mathbb{W}}(A;p)_F$. As a consequence, we have the following refinement of [13, Theorem A].

Theorem 5.15. *Let A be an E_σ -ring, and suppose that*

$$R^1 \lim_k \pi_1^{D_2} \mathrm{THR}(A)^{\mu_{p^k}} = 0.$$

Then there is an isomorphism

$$\pi_{-1} \text{TCR}(A; p) \cong \mathbb{W}(\pi_0^{D^2}(A); p)_F.$$

Proof. We compute the homotopy fiber of the topological map $\text{id} - F$ by the long exact sequence in homotopy groups. By Theorem 5.12, we can identify $\pi_0 \text{TRR}(A)$. By inspection, the topological map $F: \text{TRR}(A) \rightarrow \text{TRR}(A)$ induces the algebraic map $F: \mathbb{W}(\pi_0 A; p) \rightarrow \mathbb{W}(\pi_0 A; p)$ and therefore $\pi_{-1} \text{TCR}(A; p)$ is the cokernel of the algebraic map $\text{id} - F$. \square

6. Computations

In this section, we use the new algebraic framework from Section 4.3 to do some concrete calculations. In particular, we compute the D_{2m} -Mackey functor $\pi_0^{D_{2m}} \text{THR}(H\mathbb{Z})$ where $m \geq 1$ is an odd integer and \mathbb{Z} is the constant Mackey functor. This computation has essentially already appeared in [9, Theorem 3.15], but we include it to illustrate how our approach can be done entirely in the setting of homological algebra for Mackey functors and is therefore, in a sense, algorithmic. The first step in this computation is a Tambara reciprocity formula for sums, which we present for a general finite group and may be of independent interest.

6.1. The Tambara reciprocity formulae

The most difficult relations in Tambara functors tend to be the interchange describing how to write the norm of a transfer as a transfer of norms of restrictions. These are described by the condition that if

$$\begin{array}{ccc} U & \xleftarrow{h} & T \xleftarrow{f'} U \times_S \prod_g(T) \\ \downarrow g & & \downarrow g' \\ S & \xleftarrow{h'} & \prod_g(T) \end{array}$$

is an exponential diagram, then we have

$$N_g \circ T_h = T_{h'} \circ N_{g'} \circ R_{f'}.$$

The formulae called ‘‘Tambara reciprocity’’ unpack this in two basic cases: $g : G/H \rightarrow G/K$ is a map of orbits and

- (1) $h : G/H \amalg G/H \rightarrow G/H$ is the fold map or
- (2) $h : G/J \rightarrow G/H$ is a map of orbits.

Also by definition, the restriction along $\prod c_\gamma$ is

$$\prod_{[\gamma] \in H \backslash G / K} \underline{M}(G / (H \cap \gamma K \gamma^{-1})) \xrightarrow{\prod_{[\gamma] \in K \backslash G / H} \gamma} \prod_{[\gamma] \in K \backslash G / H} \underline{M}(G / (K \cap \gamma^{-1} H \gamma))$$

for a Mackey or Tambara functor \underline{M} , where γ here is the Weyl action. These give all the tools needed to understand the Tambara reciprocity formulae. We spell out the formula for a norm of a sum in general; we will not need the formula for the norm of a transfer here.

Theorem 6.2. *Let G be a finite group and H a subgroup, and let \underline{R} be a G -Tambara functor. For each $F \in \text{Map}^H(G, \{a, b\})$, let K_F be the stabilizer of F . Then for any $a, b \in \underline{R}(G/H)$, we have*

$$\begin{aligned} & N_H^G(a + b) \\ &= \sum_{[F] \in \text{Map}^H(G, \{a, b\}) / G} \text{tr}_{K_F}^G \left(\prod_{[\gamma] \in K_F \backslash G / H} N_{K_F \cap (\gamma^{-1} H \gamma)}^{K_F} \left(\gamma \text{res}_{H \cap (\gamma K \gamma^{-1})}^H (F(\gamma^{-1})) \right) \right) \end{aligned}$$

Proof. This follows immediately from the proceeding discussion. The only step to check is the identification of the restriction. This follows from the identification of the map f' with the evaluation map. In the case $T_0 = \{a, b\}$, where here we blur the distinction between a and b as elements of $\underline{R}(G/H)$ and as dummy variables, the map

$$G / (H \cap (\gamma K \gamma^{-1})) \rightarrow G / H \times \{a, b\}$$

coincides with the canonical quotient onto the summand specified by evaluating F at γ^{-1} . \square

Here we only need the Tambara reciprocity formulae for dihedral groups, so we now restrict attention to these cases.

6.2. Formulae for dihedral groups

We use the following two lemmas to describe the coinductions needed for the Tambara reciprocity formulae for

$$N_{D_{2m}}^{D_{2n}}(a + b) \text{ and } N_{D_{2m}}^{D_{2n}} \text{tr}_{\mu_m}^{D_{2m}}(a),$$

where n/m is an odd prime.

Lemma 6.3. *Let H, K be subgroups of a finite group G and T be a G -set. Then there is a natural bijection*

$$\text{Map}^H(G, T)^K \cong \prod_{K\gamma H \in K \backslash G/H} \text{Map}^H(K\gamma H, T)^K \cong \prod_{K\gamma H \in K \backslash G/H} T^{(\gamma^{-1}K\gamma \cap H)}.$$

When T has trivial action this simplifies to

$$\text{Map}^H(G, T)^K \cong \prod_{K\gamma H \in K \backslash G/H} T.$$

Proof. Regarding G as a $K \times H^{\text{op}}$ -space, there is an isomorphism

$$L_{K \times H^{\text{op}}}^* G \cong \coprod_{\gamma} K\gamma H$$

where γ ranges over representatives for double cosets $K\gamma H$ in $K \backslash G/H$. The K fixed points of coinduction up from H to G are the same as the $K \times H^{\text{op}}$ -fixed points of just the set of maps out of G . This gives a natural (in T) bijection

$$\text{Map}^H(G, T)^K \cong \text{Map}^H \left(\prod_{K \backslash G/H} K\gamma H, T \right)^K \cong \prod_{K \backslash G/H} \text{Map}^H(K\gamma H, T)^K$$

as desired.

To understand each individual factor, we use the quotient map $\pi_K: K \times H \rightarrow H$ to rewrite the fixed points:

$$\left(\text{Map}^H(K\gamma H, T) \right)^K \cong \text{Map}^{K \times H}(K\gamma H, \pi_K^* T).$$

Since by definition of the pullback, K acts trivially on T , the map factors through the orbits $K \backslash K\gamma H$, which is an H -orbit. The classical double coset formula identifies this with $H/(H \cap \gamma^{-1}K\gamma)$, and the result follows.

The simplification follows from the action being trivial, and hence all points being fixed. \square

Lemma 6.4. *Let p be an odd prime. There are isomorphisms of D_{2p} -sets*

$$\text{Map}^{D_2}(D_{2p}, \{a, b\}) \cong * \amalg * \amalg \prod_{i=1}^{2^{(p+1)/2}-2} D_{2p}/D_2 \amalg \prod_{i=1}^{((2^p-1)/p)+1-2^{(p-1)/2}} D_{2p}.$$

Proof. We observe that the only fixed points with respect to μ_p and any subgroup of D_{2p} containing μ_p are the constant maps f_a and f_b sending D_{2p} to a or D_{2p} to b , respectively. This gives the first two summands. The D_2 -fixed points are given by a product of $(p+1)/2 = |D_2 \backslash D_{2p}/D_2|$ copies of $\{a, b\}$ with an additional copy corresponding to the constant maps. Combining this information, we have $2^{(p+1)/2} - 2$ copies of D_{2p}/D_2 each contributing one D_2 -fixed point. The remaining summands must be given by copies

of the D_{2p} -set D_{2p} and examining the cardinality the number of copies of D_{2p} must be $((2^{p-1} - 1)/p) + 1 - 2^{(p-1)/2}$. \square

Remark 6.5. There is a geometric interpretation of this. The D_{2p} -set $\text{Map}^{D_2}(D_{2p}, \{a, b\})$ can be thought of as the ways to label the vertices of the regular p -gon with labels a or b . The stabilizer of a function is the collection of those rigid motions which preserve the labeling. Those with stabilizer D_2 are the ones that are symmetric with respect to the reflection through some fixed vertex and passing through the center. These then depend only on the label at the chosen vertex, and then $\frac{p-1}{2}$ labels for the next vertices, moving either clockwise or counterclockwise from that vertex. Of these, there are two that are special: the two where everything has a fixed label.

Notation 6.6. For an odd prime p , let

$$c_p = 2^{\frac{p-1}{2}} - 1 \text{ and } d_p = \frac{2^{p-1} - 1}{p} - c_p.$$

Lemma 6.7. *Let p be an odd prime. We have an isomorphism of D_{2p} -sets*

$$\text{Map}^{D_2}(D_{2p}, D_2) \cong D_{2p}/\mu_p \amalg \coprod^{d_p+c_p} D_{2p}.$$

Proof. Since D_2 is a free D_2 -set, there are no D_{2p} -fixed points, by the universal property of coinduction. On the other hand, D_2 as a D_2 -set is actually in the image of the restriction from D_{2p} -sets via the quotient map $D_{2p} \rightarrow D_2$, and this is compatible with the inclusion of D_2 into D_{2p} . This allows us to rewrite our D_{2p} -set as

$$\text{Map}^{D_2}(D_{2p}, D_2) \cong \text{Map}(D_{2p}/D_2, D_{2p}/\mu_p).$$

Since μ_p acts trivially in the target, the μ_p -fixed points are

$$\text{Map}(D_{2p}/D_2, D_{2p}/\mu_p)^{\mu_p} = \text{Map}(D_{2p}/\mu_p D_2, D_{2p}/\mu_p) \cong D_{2p}/\mu_p.$$

Finally, since D_2 -acts freely in the target, there are no D_2 -fixed points (and hence for any of the conjugates). Counting gives the desired answer. \square

We need two much more general versions of these identifications, both of which follow from the preceding lemmas.

Lemma 6.8. *If $N \subset H \subset G$ with N a normal subgroup of G , then for any H -set T with $T = T^N$, we have a natural bijection of G -sets*

$$\text{Map}^H(G, T) \cong \text{Map}^{H/N}(G/N, T).$$

Proof. Since N is a normal subgroup of G , for any $g \in G$, $n \in N$, and $f \in \text{Map}^H(G, T)$, we have

$$f(gn) = f(c_g(n)g) = c_g(n)f(g) = f(g),$$

where the first equality is by normality of N , the second is by H -equivariance of f , and the third is by the condition that $T^N = T$. \square

Corollary 6.9. *Let p be an odd prime. For any $m \geq 1$, there are isomorphisms of D_{2pm} -sets*

$$\text{Map}^{D_{2m}}(D_{2pm}, \{a, b\}) \cong \{f_a, f_b\} \amalg \prod_{i=1}^{2c_p} D_{2pm}/D_{2m} \amalg \prod_{i=1}^{d_p} D_{2pm}/\mu_m,$$

and

$$\text{Map}^{D_{2m}}(D_{2pm}, D_{2m}/\mu_m) \cong D_{2pm}/\mu_{pm} \amalg \prod_{i=1}^{d_p+c_p} D_{2pm}/\mu_m.$$

Proof. This follows from Lemma 6.8. The subgroup $N = \mu_m$. The quotient D_{2m}/μ_m is D_2 ; the quotient D_{2pm}/μ_m is D_{2p} , and the result follows from the previous lemmas. \square

We will now produce a Tambara reciprocity formula for sums for the group D_{2p} when p is an odd prime.

Notation 6.10. Let

$$X = \left(\text{Map}^{D_2}(D_{2p}, \{a, b\}) \right)^{D_2} - \{f_a, f_b\}$$

be a set of representatives for the D_2 -fixed points. For a point $\underline{x} \in X$, let $x_i = \underline{x}(\zeta_p^i)$. Let

$$Y = \left(\text{Map}^{D_2}(D_{2p}, \{a, b\}) - D_{2p} \cdot X \right) / D_{2p}$$

be the set of free orbits in $\text{Map}^{D_2}(D_{2p}, \{a, b\})$, and for an equivalence class $[\underline{y}] \in Y$, let $y_i = \underline{y}(\zeta_p^i)$.

Lemma 6.11 *(Tambara reciprocity for sums for dihedral groups).* *Let D_{2p} be the dihedral group where p is an odd prime, with a generator τ of order 2 and ζ_p of order p , and let D_2 be the cyclic subgroup generated by τ . Let \underline{S} be a D_{2p} -Tambara functor. Then for all a and b in $\underline{S}(D_{2p}/D_2)$*

$$N_{D_2}^{D_{2p}}(a_{D_2} + b_{D_2}) = N_{D_2}^{D_{2p}}(a) + N_{D_2}^{D_{2p}}(b)$$

$$\begin{aligned}
 &+ \sum_{\underline{x} \in X} \operatorname{tr}_{D_2}^{D_{2p}} \left(x_0 \prod_{i=1}^{(p-1)/2} N_e^{D_2}(\zeta_p^i \operatorname{res}_e^{D_2}(x_i)) \right) \\
 &+ \sum_{[\underline{y}] \in Y} \operatorname{tr}_e^{D_{2p}} \left(\prod_{i=1}^p \zeta_p^i \operatorname{res}_e^{D_2}(y_i) \right)
 \end{aligned}$$

where X and Y are as in Notation 6.10.

Proof. This follows from Theorem 6.2, using the identification of coinduction given by Lemma 6.4. \square

Example 6.12. Explicitly, in the case of $p = 3$, we have the formula

$$\begin{aligned}
 N_{D_2}^{D_6}(a + b) &= N_{D_2}^{D_6}(a_{D_2}) + N_{D_2}^{D_6}(b_{D_2}) \\
 &\quad + \operatorname{tr}_{D_2}^{D_6}(b_{D_2} \cdot N_e^{D_2}(\zeta_5 \cdot \operatorname{res}_e^{D_2}(a_{D_2}))) \\
 &\quad + \operatorname{tr}_{D_2}^{D_6}(a_{D_2} \cdot N_e^{D_2}(\zeta_5 \cdot \operatorname{res}_e^{D_2}(b_{D_2}))),
 \end{aligned}$$

because in this case the set Y is empty. When $p = 7$, abbreviating $a_e = \operatorname{res}_e^{D_2} a$ and $b_e = \operatorname{res}_e^{D_2} b$ there is a summand

$$\operatorname{tr}_e^{D_{14}}(\xi_7 b_e \cdot \xi_7^2 b_e \cdot \xi_7^3 a_e \cdot \xi_7^4 b_e \cdot \xi_7^5 a_e \cdot \xi_7^6 a_e \cdot \xi_7^7 a_e).$$

6.3. Truncated p -typical Real Witt vectors of \mathbb{Z}

For p an odd prime, we compute the D_{2p^k} -Tambara functor $\pi_0^{D_{2p^k}} \operatorname{THR}(H\mathbb{Z})$, using the formula

$$\pi_0^{D_{2p^k}} \operatorname{THR}(H\mathbb{Z}) \cong N_{D_2}^{D_{2p^k}} \mathbb{Z} \square_{N_e^{D_{2p^k}} \iota_e^* \mathbb{Z}} N_{\zeta D_2 \zeta^{-1}}^{D_{2p^k}} c_\zeta \mathbb{Z}$$

from Theorem 4.20 and Proposition 4.17. We therefore begin by computing the Mackey functor norm, $N_{D_2}^{D_{2p^k}} \mathbb{Z}$.

Since we will be working both with dihedral groups as groups and with them as representatives of isomorphism classes of D_{2m} -sets in the corresponding Burnside ring, we will use distinct notation to keep track.

Notation 6.13. If T is a finite G -set, then let $[T]$ denote the isomorphism class of T as an element of the Burnside ring. When $T = G/G$, we will also simply write this as 1.

We also need some notation for generation of a Mackey functor, especially representable ones.

Notation 6.14. If T is a finite G -set, let $\underline{A}_T \cdot f$ be \underline{A}_T , with the canonical element $T \xleftarrow{=} T \xrightarrow{=} T$ named f .

Lemma 6.15. *Let p be an odd prime. There is an isomorphism of D_{2p} -Tambara functors*

$$N_{D_2}^{D_{2p}}(\mathbb{Z}) \cong \underline{A}^{D_{2p}} / (2 - [D_{2p}/\mu_p]).$$

Proof. The constant Mackey functor \mathbb{Z} for D_2 is the quotient of \underline{A}^{D_2} by the element $2 - [D_2] \in \underline{A}^{D_2}(D_2/D_2)$ using the conventions of Section 4.1. Equivalently, we can rewrite this as a coequalizer of maps, both of which are represented by multiplication by a fixed D_2 -set:

$$\underline{A}^{D_2} \begin{array}{c} \xrightarrow{D_2} \\ \xrightarrow{2} \end{array} \underline{A}^{D_2}.$$

Extend this to a reflexive coequalizer by formally putting in the zeroth degeneracy. This represents \mathbb{Z} as a sifted colimit of free Mackey functors:

$$\underline{A}^{D_2} \cdot a \oplus \underline{A}^{D_2} \cdot b \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{d_0} \end{array} \begin{array}{c} \xrightarrow{d_1} \\ \xleftarrow{s_0} \end{array} \underline{A}^{D_2} \longrightarrow \mathbb{Z},$$

where $s_0(1) = b$, and where

$$d_0(b) = d_1(b) = 1 \text{ and } d_i(a) = \begin{cases} [D_2] & i = 0 \\ 2 & i = 1. \end{cases}$$

The norm commutes with sifted colimits, so we deduce that we have a reflexive coequalizer diagram

$$N_{D_2}^{D_{2p}}(\underline{A}^{D_2} \cdot a \oplus \underline{A}^{D_2} \cdot b) \begin{array}{c} \xrightarrow{N(d_0)} \\ \xleftarrow{N(s_0)} \\ \xrightarrow{N(d_1)} \end{array} N_{D_2}^{D_{2p}}(\underline{A}^{D_2}) \longrightarrow N_{D_2}^{D_{2p}}(\mathbb{Z}).$$

The norm is defined by the left Kan extension of coinduction, so we have a canonical isomorphism for representable functors:

$$N_{D_2}^{D_{2p}}(\underline{A}^{D_2} \oplus \underline{A}^{D_2}) \cong N_{D_2}^{D_{2p}}(\underline{A}_{\{a,b\}}^{D_2}) \cong \underline{A}_{\text{Map}^{D_2}(D_{2p}, \{a,b\})},$$

and the norm of the Burnside Mackey functor for D_2 is the Burnside Mackey functor for D_{2p} . Here and from now on we simply write \underline{A}_T for $\underline{A}_T^{D_{2p}}$. Lemma 6.4 determines the D_{2p} -set we see here:

$$\text{Map}^{D_2}(D_{2p}, \{a,b\}) = \{f_a\} \amalg \{f_b\} \amalg \coprod_{\underline{x} \in X} D_{2p}/D_2 \cdot \underline{x} \amalg \coprod_{[y] \in Y} D_{2p} \cdot [y].$$

This decomposition gives a decomposition of the representable:

$$\underline{A}_{\text{Map}^{D_2}(D_{2p},\{a,b\})} \cong \underline{A} \cdot f_a \oplus \underline{A} \cdot f_b \oplus \bigoplus_{x \in X} \underline{A}_{D_{2p}/D_2} \cdot x \oplus \bigoplus_{[y] \in Y} \underline{A}_{D_{2p}} \cdot y.$$

Since the direct sum is the coproduct in Mackey functors, we can view each summand in the coequalizer as independently introducing a relation on \underline{A} . We can therefore work one summand at a time, keeping track of the added relations. By the Yoneda Lemma, maps from a representable Mackey functor $\underline{A}_T \cdot f$ to \underline{A} are in bijective correspondence with elements of $\underline{A}(T)$, and the bijection is given by evaluating a map of Mackey functors on the canonical element f . To determine these, we work directly, using the definition of the representables. For a general summand parameterized by the orbit of a function $D_{2p}/D_2 \rightarrow \{a,b\}$, the value of the corresponding face map is built out of the functions values at the points of D_{2p}/D_2 . The slogan here is that this is simply a “decategorification” of the Tambara reciprocity formula we already described.

The first case is the constant functions. Here, we have

$$d_i(f_*) = N_{D_2}^{D_{2p}}(d_i(*)),$$

for $* = a, b$. Both d_0 and d_1 agree on b with value 1, so the summand $\underline{A} \cdot f_b$ contributes no relation. For the summand $\underline{A} \cdot f_a$, we use that the norms in the Burnside Tambara functor are given by coinduction:

$$\begin{aligned} d_0(f_a) &= N_{D_2}^{D_{2p}}([D_2]) = [D_{2p}/\mu_p] + (d_p + c_p)[D_{2p}] \text{ and} \\ d_1(f_a) &= N_{D_2}^{D_{2p}}(2) = 2 + 2c_p[D_{2p}/D_2] + d_p[D_{2p}], \end{aligned}$$

where

$$c_p = 2^{\frac{p-1}{2}} - 1 \text{ and } d_p = \frac{2^{p-1} - 1}{p} - c_p$$

are as defined in Notation 6.6. Coequalizing these two maps introduces a relation

$$2 + 2c_p[D_{2p}/D_2] + d_p[D_{2p}] - ([D_{2p}/\mu_p] + (d_p + c_p)[D_{2p}]),$$

which simplifies to

$$(2 - [D_{2p}/\mu_p]) + c_p(2[D_{2p}/D_2] - [D_{2p}])$$

in $\underline{A}(D_{2p}/D_{2p})$.

The second case we consider is the easiest one: the summands parameterized by Y . Maps from $\underline{A}_{D_{2p}} \cdot y$ to \underline{A} are in bijection with elements of $\underline{A}(D_{2p}/e) = \underline{\mathbb{Z}}$. The explicit value is the corresponding summand from the Tambara reciprocity formula:

$$d_i(y) = \prod_{j=0}^p \text{res}_e^{D_2} (d_i(y_j)),$$

since the Weyl action on the underlying abelian group in the Burnside Mackey functor is trivial. Since $d_0(b) = d_1(b) = 1$ and since

$$\text{res}_e^{D_2} (d_0(a)) = \text{res}_e^{D_2} (d_1(a)) = 2,$$

both face maps always agree on these summands, with value given by

$$d_i(y) = 2^{|y^{-1}(a)|}.$$

Finally, the trickiest summands are the ones parameterized by X . Since we are mapping out of

$$\underline{A}_{D_{2p}/D_2} \cong \text{Ind}_{D_2}^{D_{2p}} \underline{A}^{D_2},$$

by the induction-restriction adjunction, it suffices to understand instead the restriction to D_2 of the target. Here we use the multiplicative double coset formula: for a general D_2 -Mackey functor \underline{M} , we have

$$i_{D_2}^* N_{D_2}^{D_{2p}} \underline{M} \cong \underline{M} \square_{D_2 \setminus D_{2p} / D_2 - D_2 e D_2} \square N_e^{D_2} i_e^* \underline{M}.$$

For the Burnside Mackey functor, there is a confusing collision: every Mackey functor in this expression is the Burnside Mackey functor, so we cannot distinguish between \underline{A}^{D_2} as itself or as $N_e^{D_2} \mathbb{Z}$. Writing things in terms of the actual norms of a generic Mackey functor, helps disambiguate. To a function \underline{x} with stabilizer D_2 , we have the corresponding summand from the Tambara reciprocity formula:

$$d_i(\underline{x}) = d_i(x_0) \cdot \prod_{j=1}^{(p-1)/2} N_e^{D_2} (\text{res}_e^* d_i(x_j))$$

where again, the triviality of the Weyl group allows us to ignore it. Note also that with the exception of x_0 , we actually only see the restriction of $d_i(x_j)$. As we saw in the second case, the two face maps here always agree, with value 1 if $x_j = b$ and with value 2 if $x_j = a$.

If $x_0 = b$, then $d_0(\underline{x}) = d_1(\underline{x})$, since the product factors always agreed.

If $x_0 = a$, then we have

$$d_i(f) = d_i(a) \cdot \prod_{j=1}^{(p-1)/2} N_e^{D_2} \text{res}_e^* d_i(x_j) = d_j(a)(2 + [D_2])^k,$$

where k is the number of j between 1 and $(p - 1)/2$ such that $x_j = 1$. The coequalizer therefore induces the relation

$$(2 - [D_2]) \cdot (2 + [D_2])^k.$$

Since these are multiples of the case $i = 0$, so we deduce that all of these summands contribute exactly one relation:

$$2 - [D_2] \in \underline{A}(D_{2p}/D_2).$$

Summarizing, we have that the norm $N_{D_2}^{D_{2p}} \underline{\mathbb{Z}}$ is

$$\underline{A} / \left((2[D_{2p}/D_{2p}] - [D_{2p}/\mu_p]) + c_p(2[D_{2p}/D_2] - [D_{2p}]), (2[D_2/D_2] - [D_2]) \right),$$

where to help the reader keep track of where in the Mackey functor the relations are born, we replace $1 \in \underline{A}(G/H)$ with H/H .

This simplifies in several ways, however. Since transfers in the Burnside Mackey functor are given by induction,

$$c_p(2[D_{2p}/D_2] - [D_{2p}]) = c_p \operatorname{tr}_{D_2}^{D_{2p}}(2[D_2/D_2] - [D_2]),$$

so we can remove this from the first relations with impunity, giving

$$\underline{A} / \left(2[D_{2p}/D_{2p}] - [D_{2p}/\mu_p], 2[D_2/D_2] - [D_2] \right).$$

We also have

$$\operatorname{res}_{D_2}^{D_{2p}}(2[D_{2p}/D_{2p}] - [D_{2p}/\mu_p]) = 2[D_2/D_2] - [D_2],$$

so we can now drop the second relation. This yields

$$N_{D_2}^{D_{2p}} \underline{\mathbb{Z}} \cong \underline{A} / (2 - [D_{2p}/\mu_p]).$$

Since $\underline{\mathbb{Z}}$ is a Tambara functor, $N_{D_2}^{D_{2p}} \underline{\mathbb{Z}}$ is, and as a quotient of \underline{A} , it has a unique Tambara functor structure. \square

This has a somewhat surprising consequence: the form of the norm is the same as what we started with, in that we are coequalizing two maps represented by G -sets of cardinality 2. Induction gives the following generalization.

Theorem 6.16. *For any odd integer $m \geq 1$, we have an isomorphism of D_{2m} -Tambara functors*

$$N_{D_2}^{D_{2m}} \underline{\mathbb{Z}} \cong \underline{A}^{D_{2m}} / (2 - [D_{2m}/\mu_m]).$$

6.4. Unpacking the norm

We pause here to unpack this definition some, since the quotient of Mackey functors by a congruence relation might be less familiar than the abelian group case.

Definition 6.17. For any odd natural number m , let

$$\underline{R}_m = N_{D_2}^{D_{2m}} \underline{\mathbb{Z}}.$$

Lemma 6.18. For any k dividing m , we have

$$i_{D_{2k}}^* \underline{R}_m \cong \underline{R}_k,$$

and

$$i_{\mu_k}^* \underline{R}_m \cong \underline{A}^{\mu_k}.$$

Proof. Theorem 6.16 writes the norm \underline{R}_m as the coequalizer of

$$\begin{array}{ccc} \underline{A}^{D_{2m}} & \xrightarrow{2} & \underline{A}^{D_{2m}} \\ & \xrightarrow{[D_{2m}/\mu_m]} & \end{array}$$

Since the restriction functor on Mackey functors is exact, for any subgroup H , the restriction of the norm is the coequalizer of 2 and

$$\text{res}_H^{D_{2m}} [D_{2m}/\mu_m] = [i_H^* D_{2m}/\mu_m].$$

When $H = D_{2k}$, we have

$$i_{D_{2k}}^* D_{2m}/\mu_m = D_{2k}/\mu_k,$$

since there is a single double coset and $D_{2k} \cap \mu_m = \mu_k$. This gives the first part.

When $H = \mu_m$, we have

$$i_{\mu_m}^* D_{2m}/\mu_m = 2\mu_m/\mu_m,$$

since μ_m is normal. This implies that the restriction to μ_m is \underline{A}^{μ_m} , and hence the restriction to μ_k is \underline{A}^{μ_k} . \square

Since we are coequalizing two maps from the Burnside Mackey functor to itself, the value at D_{2m}/D_{2m} can also be readily computed. We need a small lemma about the products of certain D_{2m} -orbits.

Proposition 6.19. *Let m be an odd natural number. Let $H \subset D_{2m}$ be a subgroup, and let $\ell = \gcd(m, |H|)$. Then we have*

$$D_{2m}/H \times D_{2m}/\mu_m \cong \begin{cases} D_{2m}/\mu_\ell & |H| \text{ even,} \\ D_{2m}/H \amalg D_{2m}/H & |H| \text{ odd.} \end{cases}$$

Proof. Since m is odd, any subgroup H is conjugate to either D_{2k} or μ_k , for k dividing m , and the two cases are distinguished by the parity of the cardinality. Hence D_{2m}/H is isomorphic to either D_{2m}/D_{2k} or to D_{2m}/μ_k in exactly the two cases in the statement. The result follows from the isomorphism

$$D_{2m}/H \times D_{2m}/\mu_m \cong D_{2m} \times_H^* D_{2m}/\mu_m,$$

and our earlier analysis of the restrictions. \square

Corollary 6.20. *For any odd m , $\underline{R}_m(*)$ is a free abelian group:*

$$\underline{R}_m(*) \cong \mathbb{Z}\{[D_{2m}/D_{2k}] \mid k|m\}.$$

The image of $[D_{2m}/D_{2k}] \in \underline{A}^{D_{2m}}()$ is $[D_{2m}/D_{2k}]$, while the image of $[D_{2m}/\mu_k] \in \underline{A}^{D_{2m}}(*)$ is $2[D_{2m}/D_{2k}]$.*

Proof. Proposition 6.19 describes the effect of the two maps on the standard basis for the Burnside ring. We see that

$$(2 - [D_{2m}/\mu_m]) \cdot [D_{2m}/\mu_k] = 0,$$

while

$$(2 - [D_{2m}/\mu_m]) \cdot [D_{2m}/D_{2k}] = 2[D_{2m}/D_{2k}] - [D_{2m}/\mu_k].$$

This gives both the additive result and the images. \square

Lemma 6.18 shows then that the same statement is essentially true for the values at dihedral subgroups.

Corollary 6.21. *For any odd m and any k dividing m , we have an isomorphism*

$$\underline{R}_m(D_{2m}/D_{2k}) \cong \mathbb{Z}\{[D_{2k}/D_{2j}] \mid j|k\}.$$

We can also spell out the restriction and transfer maps here. The restriction and transfer to the odd order cyclic subgroups is easier, since there is a unique maximal one.

Proposition 6.22. *The restriction map*

$$\underline{R}_m(*) \rightarrow \underline{R}_m(D_{2m}/\mu_m) \cong \underline{A}^{\mu_m}(\mu_m/\mu_m)$$

is given by

$$[D_{2m}/D_{2k}] \mapsto [\mu_m/\mu_k].$$

The transfer map is given by

$$[\mu_m/\mu_k] \mapsto 2[D_{2m}/D_{2k}].$$

Proof. These follow from the restriction and induction in D_{2m} -sets, together with the relation

$$[D_{2m}/\mu_k] = 2[D_{2m}/D_{2k}]$$

in \underline{R}_m . \square

For the restrictions and transfers to dihedral subgroups, we consider a maximal proper divisor. Let p be a prime dividing m , and let $k = m/p$.

Proposition 6.23. *The restriction map*

$$\underline{R}_m(*) \rightarrow \underline{R}_k(*)$$

is given by

$$[D_{2m}/D_{2j}] \mapsto \frac{p^\ell}{j} [D_{2k}/D_{2\ell}],$$

where $\ell = \gcd(k, j)$. The transfer maps are given by

$$[D_{2k}/D_{2j}] \mapsto [D_{2m}/D_{2j}].$$

Proof. The transfer maps are immediate. For the restriction, since m is odd, the normalizer of any dihedral subgroup is itself. The intersection of D_{2k} with D_{2j} is the dihedral group $D_{2\ell}$, while the intersection of D_{2k} with any conjugate of D_{2j} is just the intersection $\mu_j \cap \mu_k = \mu_\ell$. This means that it suffices to count cardinalities. This gives

$$i_{D_{2k}}^* D_{2m}/D_{2j} = D_{2k}/D_{2\ell} \amalg \coprod^a D_{2k}/\mu_\ell,$$

where $a = \frac{p^\ell - j}{2j}$. Since $[D_{2k}/\mu_\ell] = 2[D_{2k}/D_{2\ell}]$, the result follows. \square

Theorem 6.24. *Let $m \geq 1$ be an odd integer. There is an isomorphism of D_{2m} -Mackey functors*

$$\pi_0^{D_{2m}} \text{THR}(H\mathbb{Z}) \cong \underline{A}^{D_{2m}} / (2 - [D_{2m}/\mu_m]),$$

where $(2 - [D_{2m}/\mu_m])$ is the ideal generated by $2 - [D_{2m}/\mu_m]$ in the Tambara functor $\underline{A}^{D_{2m}}$.

Proof. By Proposition 4.17 and Theorem 4.20, it suffices to compute the coequalizer

$$N_{D_2}^{D_{2m}} \mathbb{Z} \square N_e^{D_{2m}} \mathbb{Z} \square N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta \mathbb{Z} \rightrightarrows N_{D_2}^{D_{2m}} \mathbb{Z} \square N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta \mathbb{Z}.$$

For any G , $N_e^G \mathbb{Z}$ is the Burnside Mackey functor, the symmetric monoidal unit. The E_0 -structure map here is just the unit

$$\underline{A}^{D_{2m}} \rightarrow N_{D_2}^{D_{2m}} \mathbb{Z}.$$

By Theorem 6.16, this is surjective, so $N_{D_2}^{D_{2m}} \mathbb{Z}$ is a “solid” Green functor in the sense that the multiplication map is an isomorphism. Finally, note that the argument we gave to identify $N_{D_2}^{D_{2m}} \mathbb{Z}$ did not depend on the choice of D_2 inside D_{2m} , so we have an isomorphism

$$N_{D_2}^{D_{2m}} \mathbb{Z} \cong N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} \mathbb{Z}$$

of Tambara functors. We deduce that all pieces in the coequalizer diagram are just $N_{D_2}^{D_{2m}} \mathbb{Z}$. \square

When restricted to $\pi_0^{D_2}(\text{THR}(H\mathbb{Z})^{D_{2p^k}})$, our computation recovers the computation in [9].

Corollary 6.25. *Let p be an odd prime. Then there are isomorphisms of abelian groups*

$$\pi_0^{D_{2p^k}}(\text{THR}(H\mathbb{Z}))(D_{2p^k}/D_{2p^k}) \cong \pi_0^{D_{2p^k}}(\text{THR}(H\mathbb{Z}))(D_{2p^k}/\mu_{p^k}) \cong \mathbb{W}_{k+1}(\mathbb{Z}; p).$$

Since we computed the restriction and transfer maps, we have the following computation.

Corollary 6.26. *There is an isomorphism of Mackey functors*

$$\mathbb{W}_k(\mathbb{Z}; p) = \underline{\mathbb{W}}_k(\mathbb{Z}; p)$$

for odd primes p .

Declaration of competing interest

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