WHAT ARE CYCLOTOMIC SPECTRA AND WHY DO WE NEED THEM?

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Contents

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1. Introduction

This paper is an expository account of cyclotomic spectra. They are spectra (in the sense of homotopy theory) with additional structure that includes an action of the circle group, which we will denote by $\mathbb{T}$, for torus. Such objects come up in algebraic K-theory and its close relatives topological Hochschild homology THH and topological cyclic homology TC. They figure prominently in the recent disproof of the telescope conjecture for chromatic heights greater than 1 by Robert Burklund, Jeremy Hahn, Ishan Levy and Tomer Schlank (hereafter referred to collectively as BHLS) [BHLS23]. Those authors show that for each $n \geqslant 1$ and each prime p, there is a p-local ring spectrum X of chromatic height n such that $L_{K(n+1)}\mathrm{TC}(X)$ and $L_{T(n+1)}\mathrm{TC}(X)$ (see Definition 1.12) are distinct.

The present work is part of my attempt to understand theirs. It includes some historical narrative based partly on this old mathematician's personal recollections, such as the comments in §4.5.2, and partly on things he has learned only recently, such as most of Sections 2 and 3, after steering clear of algebraic K-theory for decades.

More precisely, for each prime p and each integer $n > 0$ they consider a form of the truncated Brown-Peterson spectrum $BP\langle n\rangle$, originally defined by Dave Johnson and Steve Wilson in [JW73], and whose algebraic K-theory is the subject of a recent paper by Dylan Wilson and Hahn [HW22]. In [BHLS23, §5] the authors define an action of the additive group of integers $\mathbb{Z}$, and hence of its subgroups $p^k\mathbb{Z}$, via Adams operations. They prove that the "K-theoretic coassembly map" (see Definition 5.6)

$$L_{T(n+1)}\mathrm{K}\left(BP\langle n\rangle^{hp^k\mathbb{Z}}\right) \to \left(L_{T(n+1)}\mathrm{K}(BP\langle n\rangle)\right)^{hp^k\mathbb{Z}} \tag{1.1}$$

is not an equivalence, but becomes one after $K(n+1)$-localization. ([BHLS23, Theorem A] says the spectrum on the left is not $K(n+1)$-local for all $k \geqslant 0$.) In other words, the algebraic K-theory functor on ring spectra with $\mathbb{Z}$-action does does not commute with passage to homotopy fixed points, even $T(n+1)$-locally, but its does so commute $K(n+1)$-locally. Thus, the telescope conjecture, which equates localizations with respect to $T(n+1)$ and $K(n+1)$, fails.

Remark 1.2. *We will follow the font convention of* [BHLS23] *and denote the algebraic K-theory of a ring spectrum R by $\mathrm{K}(R)$ to avoid confusion with its nth Morava K-theory $K(n)_*(R)$.*

The reader of [BHLS23] should note, as its authors indicate in a footnote on page 6, that when they speak of "cyclotomic hyperdescent," "chromatic cyclotomic extensions" and "cyclotomic redshift" as in the title of [BMCSY25], they are using the word "cyclotomic" differently from its use in the title of this paper. Their use has to do with adjoining roots of unity, or higher chromatic analogs thereof. In order to avoid confusion, we will sometimes refer to that construction as "discrete cyclotomy," and the subject of the present work as "smooth cyclotomy" since it involves an action of the circle group $\mathbb{T}$.

Then one has the discrete cyclotomic extensions of the $T(n)$ and $K(n)$-local sphere spectra studied in various papers of Schlank, Shay Ben-Moshe, Shachar Carmeli and Lior Yanovski. The Carmeli-Schlank-Yanovski use of the word has deeper historical roots (by over a century) than that of Bökstedt-Hsiang-Madsen. We will review discrete cyclotomy in the companion paper [Rav26].

Returning to smooth cyclotomy, the term "cyclotomic spectrum" was first introduced by Ib Madsen in [Mad94b, Definition 2.6], and further studied by him and Lars Hesselholt in [HM97, Definition 2.2]. The related notion of the cyclotomic trace was first introduced in 1993 by Madsen, Marcel Bökstedt and Wu-Chung Hsiang in [BHM93]. The construction involves an action of $\mathbb{T}$ where one is interested in the fixed point sets of its finite subgroups. It is related to Alain Connes' notion of cyclic sets (see §2.5), which are simplicial sets with additional structure.

BHLS take their discrete cyclotomic extensions and apply functors such as algebraic K-theory, TC and THH, which are defined in terms of smooth cyclotomy. Their "cyclotomic completion" has to do with discrete cyclotomy, while cyclotomic boundedness (in relation to the Antieau-Nikolaus t-structure of [AN21], which we discuss in §5.11) has to do with smooth cyclotomy. Most of their proof takes place in the ∞-category CycSp of smoothly cyclotomic spectra as in Definition 5.24.

1.1. **Roadmap.** Here is a description of how the contents of this paper and [Rav26] relate to that of [BHLS23]. In [BHLS23, §2] they review cyclotomic spectra and the functors TC, topological cyclic homology and TR, topological restriction homology.

We give two definitions of cyclotomic spectra: Definition 4.51 describes them as orthogonal $\mathbb{T}$-spectra with some additional structure, and Definition 5.23 describes them as objects in a suitable ∞-category. The most important source of examples is topological Hochschild homology, which we introduce at length in Sections 2 and 3. It is defined from the two perspectives in Definitions 3.9 and 5.21. TC is defined in Definitions 3.30 and 5.54. TR is defined in Definition 5.80.

The material of [BHLS23, §3] is partially treated here in the similarly titled §6. [BHLS23, §4] concerns locally unipotent (see [Rav26, Definition 5.1]) actions of integers on certain spectra and the Lichtenbaum-Quillen property of [Rav26, Definition 2.7]. The subject of [BHLS23, §5] is Adams operations on $BP\langle n\rangle$, which we will treat in [Rav26].

In the climactic [BHLS23, §6] they study two coassembly maps as in Definition 5.6:

$$L_{T(n+1)}\mathrm{K}(BP\langle n\rangle^{hp^k\mathbb{Z}}) \to L_{T(n+1)}\mathrm{K}(BP\langle n\rangle)^{hp^k\mathbb{Z}}$$

and

$$L_{K(n+1)}\mathrm{K}(BP\langle n\rangle^{hp^k\mathbb{Z}}) \to L_{K(n+1)}\mathrm{K}(BP\langle n\rangle)^{hp^k\mathbb{Z}}.$$

They show that the first one is not an equivalence, but the second one is, thereby disproving the telescope conjecture. In their words,

> We do this by looking at the coassembly map from two highly divergent perspectives, which are connected via trace theorems:
>
> (1) From the perspective of locally unipotent $\mathbb{Z}$-actions on ring spectra, the results of [BHLS23, §4] tell us that the coassembly map cannot be an isomorphism.
>
> (2) From the perspective of cyclotomic redshift of [BMCSY25], the map
>
> $$L_{T(n)}BP\langle n\rangle^{hp^k\mathbb{Z}} \longrightarrow L_{T(n)}BP\langle n\rangle$$
>
> splits after base change to the maximal abelian extension of the $K(n)$-local sphere, and therefore the coassembly map is a $K(n+1)$-local isomorphism.

We will discuss the maximal abelian extension of the $K(n)$-local sphere in [Rav26]. Like the maximal abelian extension of the p-adic numbers, it is discretely cyclotomic.

1.2. **Actions of the circle group.**

Definition 1.3. The rth root functor ρ_r^*. [BM12, Notation 4.1] *For an integer $r \geqslant 1$, let*

$$\rho_r : \mathbb{T} \to \mathbb{T}/\mathrm{C}_r$$

denote the map which sends $\omega \in \mathbb{T}$ to the image of $\omega^{1/r}$ in $\mathbb{T}/\mathrm{C}_r$. (This isomorphism is not to be confused with the projection of $\mathbb{T}$ to its quotient, which has a kernel of order r, nor with the regular representation ϱ_r of C_r of Theorem 3.12.) For a

$\mathbb{T}/\mathrm{C}_r$-space X (meaning a $\mathbb{T}$-space on which $\mathrm{C}_r \subseteq \mathbb{T}$ acts trivially), let $\rho_r^* X$ denote the $\mathbb{T}$-space induced by the isomorphism ρ_r. Similarly for a representation V of $\mathbb{T}/\mathrm{C}_r$, $\rho_r^* V$ denotes the representation of $\mathbb{T}$ induced by the same isomorphism.

Definition 1.4. *A $\mathbb{T}$-space X is* **smoothly cyclotomic** *if for each finite subgroup $\mathrm{C}_r \subseteq \mathbb{T}$, there is a $\mathbb{T}$-equivariant equivalence*

$$\varphi_r : \rho_r^* X^{\mathrm{C}_r} \to X,$$

the **cyclotomic structure map***, where $\rho_r^* X^{\mathrm{C}_r}$ denotes the fixed point space X^{C_r} with the residual action of $\mathbb{T}/\mathrm{C}_r$, which is isomorphic to $\mathbb{T}$.*

This definition should be compared with (4.40) and (4.41) below.

Example 1.5. The free loop space. *For any space X, the free loop space $\mathcal{L}X$, the space of maps of $\mathbb{T}$ into X, is smoothly cyclotomic. Here we are regarding $\mathbb{T}$ as the unit circle in the complex numbers $\mathbb{C}$, and it acts on $\mathcal{L}X$ by rotation of loops. For each $r > 1$, a loop is fixed by $C_r \subseteq \mathbb{T}$ precisely when it repeats itself r times, meaning that it factors through the r-fold covering of $\mathbb{T}$. Such a loop determines another (possibly nonrepeating) loop by restriction to the subspace*

$$\left\{ e^{2\pi t\sqrt{-1}} : 0 \leqslant t \leqslant 1/r \right\} \subseteq \mathbb{T}.$$

(We do not denote $\sqrt{-1}$ by i since we often use that symbol as an index.) This defines a $\mathbb{T}$-equivariant homeomorphism

$$\varphi_r : \rho_r^* (\mathcal{L}X)^{C_r} \to \mathcal{L}X.$$

Proposition 1.6. Based loops and free loops. *Let*

$$\mathrm{eval} : \mathcal{L}X \to X$$

be the map given by evaluation at a given point on the circle. Then for each point $x \in X$, $\mathrm{eval}^{-1}(x)$ is the space of based loops at x. The map eval *has a section sending each $x \in X$ to the constant loop at that point.*

Under suitable hypotheses on X, including path connectivity, these fibers are all equivalent and a base point can be chosen so that the homotopy fiber of eval is ΩX.

1.3. **Topological Hochschild homology and related notions.** A more important example of a smoothly cyclotomic space for our purposes is the topological Hochschild homology $\mathrm{THH}(R)$ of a topological ring R, defined below in (2.33) and shown to be smoothly cyclotomic in Proposition 2.46.

Making similar definitions (of THH and cyclotomic objects) for spectra is more delicate. Like the definition of spectra themselves, those of THH and cyclotomic spectra have undergone several upheavals in the past 3 decades. For the original definition we refer the reader to Bökstedt's remarkable preprint [Bök85a], the work of Bökstedt, Hsiang and Madsen [BHM93], and Madsen's subsequent expositions of it [Mad94a, Mad94b]. All were written in the days before we knew how to define a smash product in the category of spectra that is strictly associative. This required them to tread very carefully. Their key idea is the use of a **Bökstedt functor** as in Definition 3.1, which they call a **functor with smash product** or FSP. Such a functor $\mathfrak{B}$ determines a spectrum $\mathfrak{B}(\mathbb{S})$ with appropriate multiplicative structure. We will review this material in §3.

In order to do this in the category of spectra, we need to define a symmetric monoidal structure in that category. We know how to do this now, but did not when THH was first considered.

The first category of spectra with a symmetric monoidal structure was that of *S-modules* constructed by Tony Elmendorf, Igor Kriz, Mike Mandell and Peter May in [EKMM97], where THH is treated briefly in Chapter IX.

This was followed shortly by the simpler definition of *symmetric spectra* by Mark Hovey, Brooke Shipley and Jeff Smith [HSS00]. THH in this setting is the subject of Shipley's paper [Shi00]. In it she explains how Bökstedt anticipated the definition of symmetric spectra. She also has to deal with technical problems created by the unfortunate fact that an equivalance of symmetric spectra need not induce an isomorphism of stable homotopy groups.

These difficulties are not present in the category of *orthogonal spectra*, the subject of the book [MM02] by Mandell and May. The study of cyclotomic orthogonal spectra is taken up by Andrew Blumberg and Mandell in [BM12, §4], [BM15] and [BM24]. We will discuss it in §4.

An ∞-categorical treatment of cyclotomic spectra and related matters is given by Thomas Nikolaus and Peter Scholze in their seminal paper [NS18, Chapter II], which we will discuss in §5.

In all three approaches (those of Bökstedt-Hsiang-Madsen, Blumberg-Mandell and Nikolaus-Scholze) there is both a global (in the sense of number theory) definition involving the circle group $\mathbb{T}$ and a p-adic definition involving the Prüffer group $\mathrm{C}_{p^\infty} \subseteq \mathbb{T}$. Only the latter is relevant to [BHLS23] since all spectra there are assumed to be p-adically complete.

1.4. **Telescopic and chromatic localization.** Recall that the subject of the telescope conjecture is the relation between Bousfield localizations with respect to $K(n)$, the nth Morava K-theory, and "the" telescope $T(n)$. The quotation marks will be explained shortly. First we need

Definition 1.7. Two notions of chromatic height.

(i) [Rav92, Definition 1.5.3.] *A p-local finite spectrum Y* **has type** *n if $K(n)_*X \neq 0$ and $K(m)_*X = 0$ for all $m < n$.*
(ii) [MR99, §3] *A p-complete bounded below spectrum Y* **has** fp**-type** *n if it has finite type and for each finite spectrum U of type $n+1$, $U \otimes Y$ is π-finite, meaning that is has only finitely many nontrivial homotopy groups, each of which is finite.*

It is known [Rav84, Theorem 2.11] that for finite spectra X, $K(m)_*X = 0$ implies $K(m-1)_*X = 0$, but this is far from true for infinite CW-spectra Y. For example we have

$$K(m)_*BP\langle n\rangle = \begin{cases} K(m)_* \otimes H_*(BP\langle n\rangle; \mathbb{Z}/p) & \text{for } m \leqslant n \\ 0 & \text{for } m > n, \end{cases}$$

and this spectrum has fp-type n. A spectrum need not have an fp-type even if it is connective and of finite type.

Theorem 1.8. Hopkins-Smith periodicity. [HS98, Theorem 9] *Each type n finite spectrum V admits a map $v : V \to \Sigma^{-d}V$ for some $d > 0$ (when $n > 0$) inducing an isomorphism in $K(n)$ homology and a nilpotent map in ordinary mod*

p homology. The former condition implies that the cofiber of v has type $n+1$. We denote by $T(n)$ the filtered colimit obtained by iterating v, the **height n telescope**.

For $n = 0$, we need a map inducing an isomorphism in rational homology and a nilpotent map in mod p homology. The degree p map fits this description.

For a given prime p and height n, neither the finite spectrum V, the map v, nor the telescope $T(n)$ is unique, hence the phrase "the" telescope above. However for a given finite V of type n, the homotopy type of $T(n)$ is known to be independent of the choice of v. Better still, the Bousfield localization functor $L_{T(n)}$ associated with $T(n)$ is known to be independent of the choice of V as well, hence the notation. It depends only on the height n and the implicit prime p.

The original conjecture of [Rav84] was that the functors $L_{T(n)}$ (telescopic localization, also known as L_n^f or L_n^{Fin}) and $L_{K(n)}$ (chromatic localization) are the same. This was known at the time to be true for $n = 0$ and $n = 1$. A few years later it became apparent that the statement was likely to be false for $n > 1$. *The authors of* [BHLS23] *use algebraic K-theory to construct counterexamples,* as their title indicates.

Example 1.9. *The Johnson-Wilson spectrum $BP\langle n\rangle$ has* fp*-type n. This includes $BP\langle 0\rangle := H\mathbb{Z}_{(p)}$ and $BP\langle -1\rangle := H\mathbb{Z}/p$.*

Example 1.10. *For each prime p and each height $n > 0$, there are p-local Thom spectra $y(n)$ introduced by Mahowald in* [Mah79] *with*

$$H_*(y(n);\mathbb{Z}/p) \cong \begin{cases} P(\xi_i : 1 \leqslant i \leqslant n) & \text{for } p = 2 \\ E(\tau_i : 0 \leqslant i \leqslant n-1) \otimes P(\xi_i : 1 \leqslant i \leqslant n) & \text{for } p > 2 \end{cases}$$

as comodules over the dual Steenrod algebra. Each is an associative (i.e., $\mathbb{E}_1$) ring spectrum. They were studied extensively in [MRS01b] *by Paul Shick, Mahowald and the author in hopes of disproving the telescope conjecture.*

It is known that $K(m)_*(y(n)) = 0$ iff $m < n$. In other words $y(n)$ behaves as if it has type n even though it is not a finite complex. It does not have an fp*-type. It is also known that there is a self-map $\Sigma^{|v_n|}y(n) \to y(n)$ inducing multiplication by v_n in $K(n)_*(-)$, as Theorem 1.8 would lead us to expect.*

Remark 1.11. *The condition on $U \otimes X$ above implies that $K(m)_*(U \otimes X) = 0$ for all $m \geqslant 0$. In the language of* [Rav84, Definition 4.1], *$U \otimes X$ is* **dissonant**. *This condition is weaker than π-finiteness. An infinite wedge of suspended mod p Eilenberg-MacLane spectra, such as* $\mathrm{THH}(\mathbb{Z}/p)$ *(see Theorem 3.18), is dissonant but not π-finite.*

If X has fp*-type n, then $K(m)_*X = 0$ for all $m > n$. On the other hand, a type n spectrum Y has $K(m)_*(Y) \neq 0$ for all $m \geqslant n$.*

In order to describe the counterexamples to the telescope conjecture, the following notation is convenient.

Definition 1.12. Telescopic and chromatic localizations of K-theory and TC**.** *For an $\mathbb{E}_1$-ring spectrum R and $n \geqslant 0$,*

$$\begin{aligned} \mathrm{K}_{T(n)}(R) &:= L_{T(n)}\mathrm{K}(R), & \mathrm{K}_{K(n)}(R) &:= L_{K(n)}\mathrm{K}(R), \\ \mathrm{TC}_{T(n)}(R) &:= L_{T(n)}\mathrm{TC}(R), & \mathrm{TC}_{K(n)}(R) &:= L_{K(n)}\mathrm{TC}(R). \end{aligned}$$

The reader not familiar with the operads $\mathbb{E}_n$ may find a brief introduction to them (with references to other such works) in [Rav26, Appendix B].

1.5. **Outline.** We now describe the rest of the paper in more detail.

In §2.1 we describe some classical algebra starting with some definitions in Gerhard Hochschild's 1945 paper "On the cohomology groups of an associative algebra" [Hoc45] and their generalizations due to Cartan-Eilenberg [CE56]. Hochschild homology is the subject of Definition 2.6.

The simplicial category $\mathbf{\Delta}$, simplicial sets, and related notions are introduced in §2.4. Hochschild's chain complex is reinterpreted as that of a simplicial abelian group in (2.32).

Connes' cyclic category $\mathbf{\Lambda}$ and cyclic sets are the subject of §2.5. The cyclic circle $\mathbf{\Lambda}^0$ is the subject of Corollary 2.43. $\mathbf{\Lambda}$ has the same objects as $\mathbf{\Delta}$, the finite ordered sets $[n]$. In both cases one has Yoneda functors represented by $[n]$, the standard n-simplex Definition 2.21 and the standard n-cyclex of Definition 2.59. In §2.6 we introduce the paracyclic and r-cyclic categories in Definition 2.49 along with edgewise subdivision in Definition 2.55. We summarize these indexing categories in §2.8.

In §2.9 we describe two double complexes associated with cyclic objects in an abelian category. They fit into two short exact sequences, (2.75) and (2.83), that we call **Tate sequences**. There is another related to cyclic spectra, (4.55), which leads to a long exact sequence in homology.

In §3 we describe the groundbreaking work of Bökstedt, Hsiang and Madsen [BHM93]. Their main tool is the Bökstedt functor of Definition 3.1, which Bökstedt himself calls a "functor with smash product." We list some common examples in Example 3.8. To each such functor $\mathfrak{B}$ we associate spaces $\mathrm{THH}(\mathfrak{B})$ in Definition 3.9 and $\mathrm{K}(\mathfrak{B})$ in Definition 3.19. They are related by the Dennis trace of Definition 3.26.

This brings us to the cyclotomic trace. The space $\mathrm{THH}(\mathfrak{B})$, which is defined to be the geometric realization of a certain simplicial set $\mathrm{THH}_\bullet(\mathfrak{B})$, comes equipped with an action of the circle group $\mathbb{T}$ hence of each of its finite subgroups C_r. Replacing $\mathrm{THH}_\bullet(\mathfrak{B})$ with its rth edgewise subdivision (which does not alter its topology) makes this action of C_r simplicial as expained in Theorem 3.12. Topological cyclic homology $\mathrm{TC}(\mathfrak{B})$ and the cyclotomic trace, a map it receives from $\mathrm{K}(\mathfrak{B})$, are the subject of Definition 3.30.

We take up the ordinary (meaning without ∞-categories) theory of spectra in §4. Let $\mathcal{T}$ denote the category of pointed topological spaces. Initially, around 1960, a spectrum X was defined to be a sequence of pointed spaces X_n for $n \geqslant 0$ with structure maps $\epsilon_n^X : \Sigma X_n \to X_{n+1}$. One could require each of the spaces to have an action of a compact Lie group G so that the sructure maps are equivariant as in Definition 4.1.

We call such spectra **sequential**. They can be reinterpreted as enriched $\mathcal{T}$-valued functors on a certain $\mathcal{T}$-enriched indexing category $\mathscr{J}^{\mathbf{N}}$ having as objects the natural numbers. It turns out that $\mathscr{J}^{\mathbf{N}}$ lacks a symmetric monoidal structure, and in hind sight this is the reason for the lack of a workable smash product on the original category of spectra. This was a major headache for a generation. In the late 90s it was found that $\mathscr{J}^{\mathbf{N}}$ could be fattened up into an indexing category that *is* symmetric monoidal. Two instances of this are spelled out in (4.4), and they lead to the categories of symmetric and orthogonal spectra of Definition 4.5.

In §4.1 we study the Mandell-May category of Definition 4.7, the appropriate indexing category for orthogonal G-spectra, the subject of §4.2. Such spectra come equipped with two different kinds of fixed points, categorical and geometric, spelled out in Definition 4.20.

In §4.3 we pay a brief visit to the functor of Jean-Louis Loday, which is a method of tensoring a simplicial or cyclic set X with a commutative ring spectrum A. When X is the simplicial circle it yields $\mathrm{THH}(A)$.

In §4.4 we describe the Greenlees-May diagram, usually called the Tate diagram. The latter name is used because the construction imitates (in the category of spectra) the group cohomology of (4.32) originally defined by John Tate (1925-2019) in [Tat52].

It is an essential tool in equivariant stable homotopy theory. For a G-spectrum for finite G, it leads to a cofiber sequence (4.35) relating the homotopy orbit spectrum X_{hG}, the homotopy fixed point spectrum X^{hG}, and a third spectrum X^{tG}, the Tate construction of X (4.26). When the group is compact but not finite, X_{hG} needs to be suspended by the adjoint representation, whose degree is the dimension of underlying manifold of G.

In §5 we discuss ∞-categories and the work of Nikolaus and Scholze. Jacob Lurie has written thousands of pages on ∞-categories, and we do not expect the reader to be familiar with all of it. We give specific references to this material when needed. As in [Rav23], we will write ∞-categories which are not ordinary categories in the color cyan.

Elementary ∞-categorical notions are discussed in §5.1, limits and colimits in §5.2, and some addtional structures in §5.3. THH is defined in ∞-categorical terms in §5.4. The ∞-category of cyclotomic spectra is defined in §5.5. Polygonic spectra are recalled in §5.6, and epicyclic spaces and spectra are the subject of §5.7.

Topological cyclic homology TC and the related functors TC^{-} and TP are reviewed in §5.8. Nikolaus-Scholze's simplified way of computing TC is the subject of Theorem 5.57. It is applied to the mod p Eilenberg-MacLane spectrum in §5.9.1, and the integer version due to [BM94] is discussed briefly in §5.9.2. The latter is earliest instance of **chromatic redshift**.

In §5.10 we introduce t-structures, which are systems of full subcategories of stable ∞-categories. The standard example is the Postnikov t-structure on the ∞-category of spectra, which has to do with connectivity and coconnectivity. In §5.11 we describe the Antieau-Nikolaus t-structure on the ∞-category of cyclotomic spectra with its surprising definition of coconnectivity.

Every t-structure determines a subcategory known as the heart, which has an abelian homotopy category. In the Postnikov case the latter is the derived category of abelian groups. In the Antieau-Nikolaus case they call its objects p-typical Cartier modules (Definition 5.72), which are abelian p-groups equipped with natural endomorphisms $\mathbf{F}$ and $\mathbf{V}$, the Frobenius and Verschiebung maps. They are related to similar maps (see Definition 2.16) in the theory of Witt vectors, which we review in §2.3.

Topological restriction homology TR (Definition 5.80) is the subject of §5.12. It is a functor that converts a cyclotomic spectrum to one with an additional structure called a Frobenius lift as in Definition 5.23(i). It is known to be a fully faithful right adjoint of the corresponding forgetful functor.

In §6 we indicate how the machinery of the previous sections can be brought to bear on the telescope conjecture. Given a p-complete $\mathbb{E}_1$-ring spectrum R with an action of the integers $\mathbb{Z}$, we get a diagram of cyclotomic spectra

$$\mathrm{THH}(R^{h\mathbb{Z}}) \to \mathrm{THH}(R^{h(p\mathbb{Z})}) \to \mathrm{THH}(R^{h(p^2\mathbb{Z})}) \to \cdots \to \mathrm{THH}(R).$$

and of ordinary spectra with THH replaced by TC, K or their localizations with respect to $K(n+1)$ or $T(n+1)$. Whenever one applies a functor to a limit, such as homotopy fixed points, one has a coassembly map ϵ of Definition 5.6 from the value of the functor on the limit to the limiting value of the functor.

Thus when R is the sphere spectrum $\mathbb{S}$ with trivial $\mathbb{Z}$-action, we have

$$\mathbb{S}^{h(p^i\mathbb{Z})} \simeq \mathbb{S}^{B(p^i\mathbb{Z})_+} \simeq \mathbb{S} \vee \Sigma^{-1}\mathbb{S},$$

also known as the dual circle $\mathbb{D}S^1$. The structure of $\mathrm{THH}(\mathbb{D}S^1)$ is the subject of Theorem 6.2, which is due to Cary Malkiewich. Its underlying spectrum has a single cell in dimension -1, but infinitely many in dimension 0. On the other hand, $\mathrm{THH}(\mathbb{S}) = \mathbb{S}$, so $\mathrm{THH}(\mathbb{S})^{B\mathbb{Z}}$ just has a single cell in dimensions 0 and -1. Hence the coassembly map

$$\epsilon : \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) \to \mathrm{THH}(\mathbb{S})^{B\mathbb{Z}}$$

is very far from being an equivalence.

Theorem 6.7 says that the same is true if we replace $\mathbb{S}$ by a $T(n)$-local ring spectrum R with trivial Z-action on which $\mathrm{K}_{T(n+1)}$ is nontrivial, then the coassembly map

$$\epsilon : \mathrm{K}_{T(n+1)}(R^{B\mathbb{Z}}) \to \mathrm{K}_{T(n+1)}(R)^{B\mathbb{Z}},$$

is not an equivalence.

If we knew that the $K(n+1)$-local analog of the coassembly map of Theorem 6.7 was an equivalence for $n \geqslant 1$, we would know that the telescope conjecture is false. What we do know is two steps removed from this. There is a *particular* R, namely $L_{T(n)}BP\langle n\rangle$, with a *nontrivial* action of $\mathbb{Z}$ for which the coassembly map is a $K(n+1)$-local but not a $T(n+1)$-local equivalence. This will be discussed in [Rav26], where we will see that a crucial ingredient is [BMCSY25, Theorem C].

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2. Algebraic and space level constructions

2.1. **Some classical algebra.** We begin by recalling the relevant algebra. For more background on this material, we recommend Chuck Weibel's book [Wei94, Chapter 9].

Let A be an associative algebra over a field k and let M be a two-sided A-module. In Gerhard Hochschild's 1945 paper [Hoc45], he considered a cochain complex $C^\bullet(A;M)$ in which

$$C^n(A;M) := \mathrm{Hom}_k(A^{\otimes(n+1)}, M) \tag{2.1}$$

(where the tensor products are over k) with coboundary operator δ defined as follows for $f \in C^n(A;M)$ and $a_i \in A$ for $0 \leqslant i \leqslant n+1$.

$$\begin{aligned}(\delta f)(a_0, \cdots, a_{n+1}) &:= a_0 f(a_1, \cdots, a_{n+1}) \\ &\quad + \sum_{1\leqslant i\leqslant n} (-1)^i f(a_0, \cdots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \cdots, a_{n+1}) \\ &\quad + (-1)^n f(a_0, \cdots, a_n) a_{n+1}.\end{aligned} \tag{2.2}$$

Note here that f is M-valued, and the first and last terms above make use of the left and right A-module structures on M. Note also that in no term on the right has the order the a_is changed. We will see such a change in (2.5) below.

Following Henri Cartan and Sammy Eilenberg in [CE56, Chapter IX] (where k was no longer assumed to be a field, but a commutative ring over which A is projective; in more recent literature only flatness over k is needed), define the *enveloping algebra* A^e of A by

$$A^e := A \otimes_k A^{\mathrm{op}}, \tag{2.3}$$

where A^{op} denotes A with the opposite multiplication. When A is a commutative k-algebra, $A^e \cong A \otimes_k A$.

In any case a two-sided A-module M becomes a left A^e-module by the formula

$$(a \otimes b^*)m := amb \qquad \text{for } a \in A,\ m \in M \text{ and } b^* \in A^{\mathrm{op}}.$$

It is also a right A^e-module by the formula

$$m(a \otimes b^*) := bma.$$

In particular A itself is a two-sided A^e-module, leading to an augmentation $\epsilon : A^e \to A$ defined by $\epsilon(a \otimes b^*) = ab$.

Cartan-Eilenberg [CE56, §IX.4] then defined the homology of a right A^e-module M by

$$H_n(A;M) := \mathrm{Tor}_n^{A^e}(M, A) \tag{2.4}$$

and the cohomology of a left A^e-module M by

$$H^n(A;M) := \mathrm{Ext}^n_{A^e}(A, M).$$

They showed that the latter coincides with the cohomology of Hochschild's complex of (2.2). This will be generalized topologically in Definition 5.32.

In both cases we need a projective A^e-resolution of A as a left A^e-module. To define one, let

$$S_n A := A^{\otimes(n+2)} \qquad \text{and} \qquad \tilde{S}_n(A) := A^{\otimes n},$$

where the tensor products are over k. Make $S_n A$ a two-sided A-module, i.e., a left A^e-module, by

$$(a \otimes b^*)(a_0 \otimes \cdots \otimes a_{n+1}) := (aa_0) \otimes a_1 \otimes \cdots \otimes a_n \otimes (a_{n+1} b).$$

We define $\partial_n : S_n(A) \to S_{n-1}(A)$ by

$$\begin{aligned}&\partial_n(a_0 \otimes \cdots \otimes a_{n+1}) \\ &\qquad := \sum_{0\leqslant i\leqslant n} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes (a_i a_{i+1}) \otimes a_{i+2} \otimes \cdots \otimes a_{n+1}.\end{aligned}$$

We have

$$S_n(A) = A \otimes_k \tilde{S}_n(A) \otimes_k A = A^e \otimes_k \tilde{S}_n(A)$$

and

$$M \otimes_{A^e} S_n(A) = M \otimes_{A^e} A^e \otimes_k \tilde{S}_n(A) = M \otimes_k \tilde{S}_n(A).$$

It follows that $H_*(A; M)$ as in (2.4) is the homology of the complex $M \otimes_k \tilde{S}(A)$ in which

$$\begin{aligned} \partial_n(m \otimes a_1 \otimes \cdots \otimes a_n) := {} & ma_1 \otimes a_2 \otimes \cdots \otimes a_n \\ & + \sum_{0<i<n} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ & + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned} \tag{2.5}$$

Note here that in the last term on the right, the order of the a_is is cyclically permuted, unlike in (2.2).

Definition 2.6. Hochschild homology. *The* **cyclic bar construction** *or* **Hochschild complex** $C^{\text{Hoch}}(A; M/k)$ *is the chain complex of (2.5). When k is $\mathbb{Z}$, or it is understood from the context, we drop it from the notation. When M is A itself, we denote it by $C^{\text{Hoch}}(A/k)$ and we have*

$$\begin{aligned} \partial_n(a_0 \otimes \cdots \otimes a_n) := {} & \sum_{0 \leqslant i<n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ & + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

The homology in this case is **the Hochschild homology of** A*, denoted by* $\text{HH}_*(A/k)$.

The **acyclic Hochschild complex** $C^{\text{acyc}}(A/k)$ *has the same chain groups with boundary operator ∂'_n given by*

$$\partial'_n(a_0 \otimes \cdots \otimes a_n) := \sum_{0 \leqslant i<n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n,$$

in which the last term of $\partial_n(a_0 \otimes \cdots \otimes a_n)$ is missing.

The complex $C^{\text{Hoch}}(A)$ is denoted by ZA by Tom Goodwillie in [Goo85], and by $A\natural$ by Alain Connes in [Con83].

Example 2.7. Some easy cases.

(i) For $M = A = k$, we find that the $C^{\text{Hoch}}(A/k)$ has the form

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ k \xleftarrow{0} & k \xleftarrow{1} & k \xleftarrow{0} & k \xleftarrow{1} & k \xleftarrow{0} & k \longleftarrow \cdots, \end{array}$$

leading to

$$\text{HH}_i(k/k) = \begin{cases} k & \text{for } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(ii) For $M = A$ with A commutative, it begins with

$$\begin{array}{ccccc} 0 & & 1 & & 2 \\ A & \longleftarrow & A \otimes_k A & \longleftarrow & A \otimes_k A \otimes_k A \longleftarrow \cdots \\ a_0a_1 - a_1a_0 = 0 & \longleftarrow\!\shortmid & a_0 \otimes a_1 & & \\ & & a_0a_1 \otimes a_2 - a_0 \otimes a_1a_2 + a_2a_0 \otimes a_1 & \longleftarrow\!\shortmid & a_0 \otimes a_1 \otimes a_2 \end{array}$$

leading to $\text{HH}_0(A/k) = A$ and $\text{HH}_1(A/k)$ being a certain quotient of $A \otimes A$, namely the A-module of Kähler differentials $\Omega^1_{R/k}$. This is the A-module *generated by symbols* $\mathrm{d}x$ *for* $x \in A$*, subject to the rules*

$$\begin{aligned} \mathrm{d}c &= 0 && \text{for } c \in k, \\ \mathrm{d}(x+y) &= \mathrm{d}x + \mathrm{d}y && \text{for } x, y \in A, \\ \text{and} \qquad \mathrm{d}(xy) &= y\mathrm{d}x + x\mathrm{d}y. \end{aligned}$$

(iii) For $M = A$ *with* A *noncommutative, the above shows that*

$$\mathrm{HH}_0(A/k) \cong A/[A, A] \cong A \otimes_{A^e} A.$$

It turns out that for a finitely generated projective A*-module* P*, the trace of its identity map has its value in this quotient. The group* $\mathrm{K}_0(A)$ *is the Grothendieck completion of the monoid of isomorphism classes of such* P*. The resulting map* $K_0(A) \to \mathrm{HH}_0(A)$ *is the* **Hattori-Stallings trace** [Hat65, Sta65]*. We will see in §3.2 that it refines to the topological Dennis trace*

$$\mathrm{Tr} : K(A) \to \mathrm{THH}(A).$$

We will define $\mathrm{K}(A)$ *in Definition 3.19 and* THH *in (2.33). The Dennis trace is is the subject of §3.2.*

For a commutative k-algebra A, there is a ring structure on $\mathrm{HH}_*(A/k)$ that arises from the fact that the chain complex $C^{\mathrm{Hoch}}(A)$ is that of a simplicial ring. This is proved by Achim Krause and Thomas Nikolaus in [KN21a, Lemma 2.3]; the construction involves shuffle maps. We can also build a differential graded algebra out of $\Omega^1_{A/k}$ by forming the free exterior algebra $\Omega^*_{A/k} := \Lambda_A \Omega^1_{A/k}$, also known as the *de Rham complex of* A *over* k. When A is a polynomial algebra over k on n variables x_i, $\Omega^*_{A/k}$ is a graded exterior algebra on the n variables $\mathrm{d}x_i$. This multiplicative structure leads to a map

$$\Omega^*_{A/k} \to \mathrm{HH}_*(R/k).$$

The theorem of Hochschild, Bertram Kostant and Alexander Rosenberg [HKR62] says it is an isomorphism when A satisfies a certain smoothness condition, such as being a polynomial algebra over k.

The Hochschild complex is reinterpreted in (2.32) as the chain complex of a simplicial abelian group. The complex $C^{\mathrm{acyc}}(A)$ is acyclic because there is a chain homotopy $u : C^{\mathrm{acyc}}(A)_n \to C^{\mathrm{acyc}}(A)_{n+1}$ given by

$$u(a_0 \otimes \cdots \otimes a_n) := 1 \otimes a_0 \otimes \cdots \otimes a_n. \tag{2.8}$$

We will see in (2.76) that this acyclicity holds in any chain complex associated with a *cyclic (in the categorical sense) abelian group* as in Definition 2.40.

Definition 2.9. *Let* E *be a two-sided* Γ*-space, where* Γ *is a grouplike topological monoid, meaning one for which* $\pi_0\Gamma$ *is a group. The* **topological cyclic bar construction** $N^{\mathrm{cyc}}_\bullet(E; \Gamma)$ *is defined by formulas similar to those of (2.5). It is a simplicial space whose nth component is* $E \times \Gamma^n$*. When* E *is* Γ *itself, we denote it by* $N^{\mathrm{cyc}}_\bullet(\Gamma)$*, and we have*

$$\tau_n(g_0, \ldots, g_n) = (g_n, g_0, \ldots, g_{n-1}) \qquad \text{for } g_i \in \Gamma.$$

2.2. **Morita equivalence.** The following is originally due to Kiiti Morita [Mor58].

Definition 2.10. *Two unital k-algebras R and S are* **Morita equivalent** *if there is a bimodule ${}_RP_S$ (meaning a left R-module and a right S-module), a bimodule ${}_SQ_R$, an isomorphism of R-bimodules $u : P \otimes_S Q \cong R$ and an isomorphism of S-bimodules $v : Q \otimes_R P \cong S$.*

Example 2.11. Morita equivalence of matrix rings. *Let $\mathcal{M}_m(A)$ denote the ring of $m \times m$ matrices over A. For $R = A$ and $S = \mathcal{M}_m(A)$, take $P = A^m$ (row vectors of rank m) and $Q = A_m$ (column vectors).*

Theorem 2.12. [Lod92, Theorem 1.2.7] *If R and S are Morita equivalent k-algebras and M is an R-bimodule, then there is a natural isomorphism*

$$H_*(R; M) \cong H_*(S; Q \otimes_R M \otimes_R P),$$

for $H_*(R; M)$ as in (2.4).

The following is discussed in more detail by Loday in [Lod92, 1.2] and by Weibel in [Wei94, 9.5]. We have homomorphisms

$$\mathrm{Inc} : A \to \mathcal{M}_m(A) \qquad \text{and} \qquad \mathrm{Trace} : \mathcal{M}_m(A) \to A,$$

where the former sends $a \in A$ to the square matrix with a in the upper left corner and zeroes elsewhere, and the latter sends a matrix to the sum of its diagonal entries.

Loday [Lod92, Definition 1.2.1] defines a *generalized trace map*

$$\mathrm{Trace} : \mathcal{M}_m(M) \otimes \mathcal{M}_m(A)^{\otimes n} \to M \otimes A^{\otimes n},$$

by

$$\mathrm{Trace}(\omega \otimes \alpha \otimes \beta \otimes \cdots \otimes \alpha^{(n)}) := \sum \omega_{i_0,i_1} \otimes \alpha_{i_1,i_2} \otimes \beta_{i_2,i_3} \otimes \cdots \otimes \alpha^{(n)}_{i_n,i_0},$$

where $\alpha^{(n)}$ is the nth letter of the Greek alphabet, and the sum is over all possible indices $(i_0, \dots, i_n)$.

Theorem 2.13. Morita equivalence for matrix tensor products. [Lod92, Theorem 1.2.4]. *The maps* Inc *and* Trace *above induce inverse isomorphisms between $H_*(A; M)$ and $H_*(\mathcal{M}_m(A); \mathcal{M}_m(M))$. In particular $\mathrm{HH}_*(\mathcal{M}_m(A))$ is naturally isomorphic to $\mathrm{HH}_*(A)$.*

2.3. **Witt vectors.** This review follows the treatment of Jean-Pierre Serre in [Ser79, §II.6]. For a prime p one has **Witt polynomials**

$$w_n(x_0, x_1, \dots, x_n) \in \mathbb{Z}[x_0, x_1, \dots, x_n]$$

defined by

$$w_n(x) := \sum_{i=0}^{n} p^i x_i^{p^{n-i}} = x_0^{p^n} + p x_1^{p^{n-1}} + \cdots + p^n x_n \qquad \text{for } n \geqslant 0.$$

Theorem 2.14. [Ser79, Theorem II.6] *Given a second series $(y_0, y_1, \dots)$ of indeterminates, for each*

$$\Phi \in \mathbb{Z}[X, Y]$$

there exists a unique sequence of polynomials

$$\varphi_n \in \mathbb{Z}[x_0, \dots x_n; y_0, \dots, y_n] \qquad \textit{for } n \geqslant 0$$

such that

$$w_n(\varphi_0, \dots, \varphi_n) = \Phi(w_n(x), w_n(y)).$$

In particular we have polynomials $S_0, S_1, \dots$ *and* $P_0, P_1, \dots$ *associated with* $\Phi(X,Y) = X + Y$ *and* $\Phi(X,Y) = XY$ *respectively.*

If A is an arbitrary commutative ring with

$$a = (a_0, \dots, a_n, \dots), \quad b = (b_0, \dots, b_n, \dots) \in A^{\mathbb{N}},$$

set

$$a \boxplus b := \big(S_0(a,b), \dots, S_n(a,b), \dots\big) \qquad \text{and} \qquad a \boxtimes b := \big(P_0(a,b), \dots, P_n(a,b), \dots\big).$$

For example

$$\begin{aligned} S_0(a,b) &= a_0 + b_0 & S_1(a,b) &= a_1 + b_1 + \frac{a_0^p + b_0^p - (a_0+b_0)^p}{p} \\ P_0(a,b) &= a_0 b_0 & P_1(a,b) &= a_0^p b_1 + a_1 b_0^p + p a_1 b_1. \end{aligned}$$

Theorem 2.15. [Ser79, Theorem II.7] *The laws of composition defined above make* $A^{\mathbb{N}}$ *into a commutative unitary ring called the* **ring of Witt vectors with coefficients in** A *and denoted by* $W(A)$.

When $A = \mathbb{F}_p$, $W(A)$ is the p-adic integers $\mathbb{Z}_p$. For $A = \mathbb{F}_{p^k}$, $W(A)$ is the degree k extension of $\mathbb{Z}_p$ obtained by adjoining $(p^k - 1)$th roots of unity, the integer lift of the extension $\mathbb{F}_{p^k}$ of $\mathbb{F}_p$.

Definition 2.16. The maps W_*, V, r, and F. *For a Witt vector* $a = (a_0, a_1, \dots)$, *let*

$$W_*(a) := (w_0(a), w_1(a), \dots) = (a_0, a_0^p + p a_1, \dots)$$

and let its **Verscheibung** *or* **shift** *vector be*

$$\mathbf{V}a := (0, a_0, a_1, \dots).$$

For $x \in A$, *let*

$$r(x) := (x, 0, 0, \dots) \in W(A).$$

When A *has characteristic* p, *define the* **Frobenius** $\mathbf{F} : W(A) \to W(A)$ *by*

$$\mathbf{F}(a_0, a_1, \dots) := (a_0^p, a_1^p, \dots).$$

Then we find that

$$\begin{aligned} r(xy) &= r(x) \boxtimes r(y) = (xy, 0, 0, \dots), \\ (a_0, a_1, \dots) &= r(a_0) \boxplus \mathbf{V}r(a_1) \boxplus \mathbf{V}^2 r(a_2) \boxplus \cdots \\ &= \sum_{n \geqslant 0} \mathbf{V}^n r(a_n), \\ r(x) \boxtimes (a_0, a_1, \dots) &= (x a_0, x^p a_1, \dots, x^{p^n} a_n, \dots), \\ \text{and} \qquad \mathbf{VF} = \mathbf{FV} &= p. \end{aligned}$$

2.4. **The simplicial category and simplicial objects.** Simplicial sets were originally defined by Eilenberg and Joseph Zilber in [EZ53]. Here we use the indexing conventions of Paul Goerss and Rick Jardine [GJ99, I.1].

Definition 2.17. *The* **simplical category** $\mathbf{\Delta}$ *is that of finite ordered sets* $[n] = \{0, 1, \ldots, n\}$ *for* $n \geqslant 0$*, and order preserving maps. Such maps include*

$$\begin{aligned} d^i &: [n-1] \to [n], \text{ the injective map not having } i \text{ in its image} \\ \text{and } s^i &: [n+1] \to [n], \text{ the surjection sending both } i \text{ and } i+1 \text{ to } i, \end{aligned} \tag{2.18}$$

(both for $0 \leqslant i \leqslant n$*) known as coface and codegeneracy maps. All morphisms in* $\mathbf{\Delta}$ *are composites of them. These satisfy the following* **cosimplicial identities***:*

(i) $d^j d^i = d^i d^{j-1}$ *for* $i < j$
(ii) $s^i d^j = d^i s^{j-1}$ *for* $i < j$
(iii) $s^j d^i = \mathrm{Id}$ *for* $i = j$ *and for* $i = j+1$
(iv) $s^j d^i = d^{i-1} s^j$ *for* $i > j+1$
(v) $s^j s^i = s^i s^{j+1}$ *for* $i \leqslant j$.

Warning. *The symbol* $\mathbf{\Delta}$ *is not to be confused with* Δ*, which we sometimes use to denote a diagonal map.*

Definition 2.19. *A* **simplicial object** X *in a category* $\mathcal{C}$ *(sometimes denoted by* $X_\bullet$*) is a* $\mathcal{C}$*-valued functor on* $\mathbf{\Delta}^{\mathrm{op}}$*, in which we denote the image of* $[n]$ *by* X_n*. Any such functor comes equipped with face maps* $d_i : X_n \to X_{n-1}$ *and degeneracy maps* $s_i : X_n \to X_{n+1}$ *induced by the morphisms* d^i *and* s^i *in* $\mathbf{\Delta}$*. We denote the category of such functors by* $\mathcal{C}_{\mathbf{\Delta}}$*.*

When $\mathcal{C} = \mathrm{Set}$*, the category of sets, an element in the set* X_n *is called an* n**-simplex***. It is* **degenerate** *if it is in the image of a degeneracy map. Otherwise it is* nondegenerate.

Similarly a **cosimplicial object** $X^\bullet$ *is a* $\mathcal{C}$*-valued functor on* $\mathbf{\Delta}$*, in which we denote the image of* $[n]$ *by* X^n*.*

The corresponding **simplicial identities** are

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{for } i < j \\ d_i s_j &= \begin{cases} s_{j-1} d_i & \text{for } i < j \\ \mathrm{Id} & \text{for } i = j,\ j+1 \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases} \\ s_i s_j &= s_{j+1} s_i && \text{for } i \leqslant j. \end{aligned} \tag{2.20}$$

Definition 2.21. Some simplicial sets. *The simplicial set* $\mathbf{\Delta}^n$*, the* **standard** n**-simplex***, is defined by*

$$(\mathbf{\Delta}^n)_k := \mathbf{\Delta}([k], [n]) = \mathbf{\Delta}^{\mathrm{op}}([n], [k]).$$

This is the Yoneda functor represented by $[n]$*, so we could denote it by* $よ^{[n]}$*. The symbol* よ *is the Japanese hiragana character "yo," the first syllable of Yoneda's name.*

In its **boundary** $\partial\mathbf{\Delta}^n$*, the set of* k*-simplices is the set of such morphisms in* $\mathbf{\Delta}$ *which are not surjective.*

In its i**th face***, the set of* k*-simplices is the set of such morphisms whose image does not contain* i*.*

In the i**th horn** $\partial\mathbf{\Delta}^n_i \subseteq \partial\mathbf{\Delta}^n$ *for* $0 \leqslant i \leqslant n$*, the set of* k*-simplices is the set of nonsurjective morphisms whose image does contain* i*.*

The ith horn is usually denoted (for example in [GJ99] and [Lur09]) by Λ_i^n, but we will use that symbol differently in Definition 2.59.

The following is an exercise for the reader.

Proposition 2.22. *The cardinality of* $(\mathbf{\Delta}^n)_k$ *is* $\binom{n+k+1}{k+1}$, *and the number of non-degenerate* k*-simplices in* $\mathbf{\Delta}^n$ *is* $\binom{n+1}{k+1}$.

Let Top denote the category of compactly generated weak Hausdorff spaces. The *topological* n*-simplex* is the space

$$\Delta^n_{\text{top}} := \left\{ (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_i \geqslant 0 \text{ and } \sum x_i = 1 \right\}. \tag{2.23}$$

One can check that Δ^2_{top} is an equilateral triangle and Δ^3_{top} is a regular tetrahedron.

Definition 2.24. *The* **geometric realization** $|X|$ *of a simplicial set* X *is the colimit of the* Top*-valued functor*

$$[n] \mapsto X_n \times \Delta^n_{\text{top}}.$$

The geometric realization of a simplicial space (as in Definition 2.19) is similarly defined. More generally if X *is a simplicial object in a cocomplete category* $\mathcal{C}$ *that is tensored over* Top, *then* $|X|$ *is also an object in* $\mathcal{C}$.

This space turns out to be a quotient of the disjoint union of geometric realizations of the *nondegenerate* topological simplices of X, meaning ones not in the image of any degeneracy map. The data given by the face maps determine how they are glued together. In particular, $|\mathbf{\Delta}^n| = \Delta^n_{\text{top}} \approx D^n$, $|\partial \mathbf{\Delta}^n| \approx S^{n-1}$, and the geometric realizations of both the the ith face and ith horn of $\mathbf{\Delta}^n$ are homeomorphic to D^{n-1}.

Definition 2.25. The nerve and classifying space of a small category. *For a small category* $\mathcal{C}$, *the* **nerve** $N_\bullet(\mathcal{C})$ *is the simplicial set given by*

$$N_n(\mathcal{C}) := \mathcal{C}at([n], \mathcal{C})$$

where $\mathcal{C}at(-,-)$ *denotes the set of functors from one small catagory to another and* $[n]$ *here denotes the linearly ordered set* $\{0, \dots, n\}$ *regarded as a category. The* **classifying space** $B\mathcal{C}$ *is the geometric realization of the nerve,* $|N(\mathcal{C})|$. *When the category* $\mathcal{C}$ *is topological, we get a simplicial space.*

For a topological monoid Γ, *we denote by* $\mathcal{B}\Gamma$ *the topological category with one object and a morphism for each point in* Γ. *Composition of morphisms is determined by the monoid structure of* Γ.

When $\mathcal{C} = \mathcal{B}\Gamma$, *we denote* $N_\bullet(\mathcal{C})$ *by* $N_\bullet(\Gamma)$, *leading to* $N_n(\Gamma) = \Gamma^n$.

In other words, $N_n(\mathcal{C})$ and $N_n^{\text{cyc}}(\mathcal{C})$ are the sets of diagrams in $\mathcal{C}$ of the form

$$c_0 \to c_1 \to \cdots \to c_{n-1} \to c_n$$

and

$$c_0 \to c_1 \to \cdots \to c_{n-1} \to c_n, \quad (c_n \to c_0)$$

where for each i, the $(n+1)$-fold composite morphism $c_i \to c_i$ is *not* required to be the identity. Of the $n+1$ face maps $N_n(\mathcal{C}) \to N_{n-1}(\mathcal{C})$, $n-1$ are obtained by composing each of the $n-1$ pairs of adjacent arrows above, and the other two are obtained by ignoring the maps from c_0 and to c_n. All face maps

$$N_n^{\text{cyc}}(\mathcal{C}) \to N_{n-1}^{\text{cyc}}(\mathcal{C})$$

are obtained by such composition. In both cases the $n+1$ degeneracy maps are obtained by inserting the identity map on c_i for each i.

The inclusion functors $[n] \to [n]^{\mathrm{cyc}}$ induce maps of simplicial sets

$$N_\bullet^{\mathrm{cyc}}(\mathcal{C}) \to N_\bullet(\mathcal{C})$$

and their geometric realizations. Composing with the action of $\mathbb{T}$ on the cyclic space $|N_\bullet^{\mathrm{cyc}}(\mathcal{C})|$, we get maps

$$\begin{array}{c} \mathbb{T} \times |N_\bullet^{\mathrm{cyc}}(\mathcal{C})| \xrightarrow{\ \mu\ } |N_\bullet^{\mathrm{cyc}}(\mathcal{C})| \xrightarrow{\ \pi\ } |N_\bullet(\mathcal{C})| \\ |N_\bullet^{\mathrm{cyc}}(\mathcal{C})| \xrightarrow{\qquad f \qquad} \mathcal{L}|N_\bullet(\mathcal{C})| \end{array} \tag{2.26}$$

where f is corresponds to $\pi\mu$ under the topological adjunction

$$\mathrm{Map}(\mathbb{T} \times X,\ Y) \ \cong\ \mathrm{Map}(X,\ \mathcal{L}Y).$$

Theorem 2.27. A $\mathbb{T}$-equivariant equivalence. [Lod15, Theorem 7.3.11]. *When $\mathcal{C}$ is the one object category associated with a topological or simplicial group G, the map f of (2.26) is a $\mathbb{T}$-equivariant equivalence.*

Definition 2.28. The chain complex of a simplicial abelian group. *Let C be a simplicial abelian group or more generally a simplicial object in an abelian category. The chain complex* $\mathrm{Ch}(C)$*, in which the nth chain group (or abelian object) is C_n and the nth boundary operator $\partial_n : C_n \to C_{n-1}$ for $n > 0$ is given by*

$$\partial_n := \sum_{0 \leqslant i \leqslant n} (-1)^i d_i. \tag{2.29}$$

When C is the free abelian group $\mathbb{Z}X$ on a simplicial set X, the chain complex $\mathrm{Ch}(\mathbb{Z}X)$ *is called the* **Moore complex** *of X.*

The **normalized chain complex** $N\mathrm{Ch}(C)$ *has as its nth chain group*

$$N\mathrm{Ch}(C)_n := \bigcap_{i=0}^{n-1} \ker(d_i) \subseteq C_n,$$

and nth boundary operator is d_n.

The geometric realization $|\mathrm{Ch}(C)|$ of the underlying simplicial set is known to be a generalized Eilenberg-MacLane space with

$$\pi_*|C| = H_*C. \tag{2.30}$$

This is the *Dold-Kan correspondence*, which is nicely explained by Akhil Mathew in [Mat11].

Definition 2.31. *For a topological space X, the* **singular simplicial set** $\mathrm{Sing}_\bullet X$ *is defined by*

$$\mathrm{Sing}_n X := \mathrm{Map}(\Delta^n_{\mathrm{top}}, X),$$

the set of continuous maps $\Delta^n_{\mathrm{top}} \to X$. Face and degeneracy maps are defined in terms of maps among the Δ^n_{top}s.

The functor Sing is the right adjoint of geometric realization. This was stated without proof by Dan Kan in [Kan58]. The singular chain complex of X is by definition the Moore complex of $\mathrm{Sing}(X)$ as in Definition 2.28.

The Hochschild chain complex $C^{\mathrm{Hoch}}(A)$ of Definition 2.6 is $\mathrm{Ch}(\mathbf{HH}_\bullet(A))$, where the simplicial abelian group $\mathbf{HH}_\bullet(A)$ (which Connes denotes by $A^\natural$ in [Con83, §3]) is defined by $\mathbf{HH}_n(A) = A^{\otimes(n+1)}$ with face and degeneracy maps

$$\begin{aligned} d_i(a_0 \otimes \cdots \otimes a_n) &:= \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & \text{for } i < n \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & \text{for } i = n \end{cases} \\ \text{and} \qquad s_i(a_0 \otimes \cdots \otimes a_n) &:= \begin{cases} 1 \otimes a_0 \otimes \cdots \otimes a_n & \text{for } i = 0 \\ a_0 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_n & \\ & \text{for } i > 0. \end{cases} \end{aligned} \tag{2.32}$$

Suppose we have a symmetric monoidal category $(\mathcal{C}, \otimes)$ with a monoid object R, i.e., an object equipped with a map $R \otimes R \to R$ with suitable properties that include strict associativity. Then we could define a cyclic (and hence simplicial) object $\mathbf{HH}_\bullet(R)$ in $\mathcal{C}$ using the formulas of (2.32) and (2.48). Suppose in addition that $\mathcal{C}$ is tensored over the category of topological spaces, meaning that for an object C in $\mathcal{C}$ and a space X, we can make sense of $C \times X$ as another object in $\mathcal{C}$. Suppose further that $\mathcal{C}$ is cocomplete, meaning closed under colimits. Then we can make sense of the geometric realization of a simplicial or cyclic object in $\mathcal{C}$ and thus define

$$\mathrm{THH}(R) := |\mathbf{HH}_\bullet(R)|, \tag{2.33}$$

an object in $\mathcal{C}$, the **topological Hochschild homology** of the monoid object R. This term was invented by Bökstedt. For reasons explained §2.5.2, it comes equipped with an action of the group $\mathbb{T}$.

It follows from (2.30) that $|\mathbf{HH}_\bullet(A)|$ is a generalized Eilenberg-MacLane space with

$$\pi_*|\mathbf{HH}_\bullet(A)| = \mathrm{HH}_*(A) \qquad \text{as in Definition 2.6.} \tag{2.34}$$

Definition 2.35. *For a discrete set X, $E_\bullet X$ is the simplicial set with*

$$E_n X := \mathrm{Map}([n], X) = X^{n+1},$$

the set of maps to X from a set with $n+1$ elements.

Proposition 2.36. Contractibility and freeness.

(i) The geometric realization $|EX|$ is contractible.
(ii) If X is a discrete (abelian) group G, then $E_\bullet G$ is a simplicial (abelian) group, and $|EG|$ is a contractible free G-space with orbit space

$$|EG|_G = |N(\mathcal{B}G)| \qquad \textit{as in Definition 2.25.}$$

2.5. **Connes' cyclic category and cyclic objects.** In [Con83] Connes defines the *cyclic category* $\mathbf{\Lambda}$, which has the same objects as $\mathbf{\Delta}$, but more morphisms. A formal description can be found in Loday's book [Lod92, 6.1], where it is denoted by ΔC. His category $\mathbf{\Lambda}$ is isomorphic to that of Definition 2.49 below. Connes regards $[n]$ (a set with $n+1$ elements) as the set of $(n+1)$th roots of unity sitting in the unit circle of the complex numbers $\mathbb{C}$; see [Con94, III.A.β]. His morphisms are homotopy classes of orientation preserving self-maps of degree 1 of that circle that preserve these subsets. One such map is $\tau_n : [n] \to [n]$, which denotes counterclockwise rotation by $2\pi/(n+1)$, so $\tau_n^{n+1} = 1_{[n]}$. We denote the corresponding morphism in

$\mathbf{\Lambda}^{\mathrm{op}}$ by t_n. There are some obvious identities involving t_n with the morphisms d_i and s_i of (2.20) spelled out by Loday in [Lod92, 6.1.2], namely

$$\begin{aligned} d_i t_n &= \begin{cases} d_n & \text{for } i = 0 \\ t_{n-1} d_{i-1} & \text{for } 1 \leqslant i \leqslant n \end{cases} \\ s_i t_n &= \begin{cases} t_{n+1}^2 s_n & \text{for } i = 0 \\ t_{n+1} s_{i-1} & \text{for } 1 \leqslant i \leqslant n. \end{cases} \end{aligned} \tag{2.37}$$

Proposition 2.38. [Lod92, Theorem 6.1.3] *Any morphism $[n] \to [k]$ in $\mathbf{\Lambda}$ can be written uniquely as the composite of some iterate of τ_n with a morphism $[n] \to [k]$ in $\mathbf{\Delta}$.*

In particular there are $n+1$ distinct morphisms $[n] \to [0]$ in $\mathbf{\Lambda}$ even though there is just one map between the underlying sets. Let

$$\zeta = e^{2\pi\sqrt{-1}/(n+1)} \in \mathbb{T}.$$

Then for $0 \leqslant i \leqslant n$ there is a map sending the arc joining ζ^i and ζ^{i+1} to the full unit circle for $[0]$, and sending its complement to the point $1 \in \mathbb{T}$. Hence the forgetful functor from $\mathbf{\Delta}$ to the category of sets extends to $\mathbf{\Lambda}$, *but not faithfully*.

Next consider morphisms $[n+1] \to [n]$ in $\mathbf{\Lambda}$ for which the underlying map of sets is onto. If the underlying map fixes 0, it is obtained by collapsing one of the $n+2$ arcs (adjoining adjacent roots unity) in the circle for $[n+1]$ to a single point. Let $\omega = e^{2\pi\sqrt{-1}/(n+2)} \in \mathbb{T}$. If the arc being collapsed is not the one from ω^{-1} to 1, the *last arc*, then the map corresponds to one of the the simplicial degeneracy operators s^i of (2.18) for $0 \leqslant i \leqslant n$. Thus we denote the last arc collapse map by s^{n+1}, which we will also refer to as a degeneracy operator.

The following should be compared with Definition 2.25.

Definition 2.39. *For a small category $\mathcal{C}$, the* **cyclic nerve** $N_\bullet^{\mathrm{cyc}}(\mathcal{C})$ *is the simplicial set given by*

$$N_n^{\mathrm{cyc}}(\mathcal{C}) := \mathcal{C}at([n]^{\mathrm{cyc}}, \mathcal{C})$$

where $[n]^{\mathrm{cyc}}$ is the category $[n]$ equipped with an extra morphism from n to 0. We denote its geometric realization by $B^{\mathrm{cyc}}\mathcal{C}$.

When $\mathcal{C}$ is the one object category $\mathcal{B}\Gamma$ associated with a topological monoid Γ, we denote $N_\bullet^{\mathrm{cyc}}(\mathcal{C})$ by $N_\bullet^{\mathrm{cyc}}(\Gamma)$, leading to $N_n^{\mathrm{cyc}}(\Gamma) = \Gamma^{n+1}$.

2.5.1. *The work of Bill Dwyer, Mike Hopkins and Dan Kan.* In [DHK85], the authors take a slightly different but equivalent approach to the passage from $\mathbf{\Delta}$ to $\mathbf{\Lambda}$. They define an additional degeneracy map $s^{n+1} : [n+1] \to [n]$ (our last arc collapse map) and stipulate that $(s^{n+1}d^0)^{n+1} = 1_{[n]}$. Note that s^{n+1} is related to the other degeneracy maps of (2.18) via the action of C_{n+2} on the set $[n+1]$.

They give a geometric proof of Proposition 2.61 for $r = 1$. The key point in their argument is that the simplicial set $U\mathbf{\Lambda}^n$ has $n+1$ nondegenerate $(n+1)$-simplices corresponding to the composites of the last arc map $s^{n+1} : [n+1] \to [n]$ with powers of the rotation τ_n.

More explicitly, note that the $(n+1)$-*dimensional prism* $I \times |\mathbf{\Delta}^n|$ is the space

$$\left\{ (t; x_0, x_1, \ldots, x_n) \in I \times \mathbb{R}^{n+1} : 0 \leqslant t \leqslant 1,\, x_i \geqslant 0,\, \sum_{0 \leqslant i \leqslant n} x_i = 1 \right\}.$$

For each j with $0 \leqslant j \leqslant n$, define a subspace $P_j \subseteq I \times |\mathbf{\Delta}^n|$ by

$$P_j = \left\{ (t; x_0, x_1, \dots, x_n) : \sum_{0 \leqslant i \leqslant j-1} x_i \leqslant t \leqslant \sum_{0 \leqslant i \leqslant j} x_i \right\}.$$

The union of these subspaces is the entire prism, and the topological $(n+1)$-simplex P_j is the convex hull of the set of $n+2$ points

$$\{(0; \delta_k) : 0 \leqslant k \leqslant j\} \cup \{(1; \delta_k) : j \leqslant k \leqslant n\},$$

where $\delta_k \in \mathbb{R}^{n+1}$ for $0 \leqslant k \leqslant n$ is the vector whose kth coordinate is 1 and all others are 0. The geometric realization of $\mathbf{\Lambda}^n$ is the quotient of the prism obtained by identifying $(0; x_0, x_1, \dots, x_n)$ with $(1; x_0, x_1, \dots, x_n)$, thereby converting it to $\mathbb{T} \times |\mathbf{\Delta}^n|$.

2.5.2. *Connes' cyclic objects.*

Definition 2.40. *A* **cyclic object** X *in a category* $\mathcal{C}$ *is a* $\mathcal{C}$*-valued functor on* $\mathbf{\Lambda}^{\mathrm{op}}$ *and therefore a simplicial object (by restriction of the functor to* $\mathbf{\Delta}^{\mathrm{op}}$*) with some additional structure, which includes an action of the cyclic group* C_{n+1} *on* X_n *for each* n*. We denote the category of such functors by* $\mathcal{C}_{\mathbf{\Lambda}}$*, and the forgetful functor* $\mathcal{C}_{\mathbf{\Lambda}} \to \mathcal{C}_{\mathbf{\Delta}}$ *by* U*. The* **geometric realization** $|X|$ **of a cyclic set** X *is that of the underlying simplicial set* UX*, which we will often denote abusively by* X*.*

The cyclic set $\mathbf{\Lambda}^n$ *the* **standard** n**-cyclex***, is defined by*

$$(\mathbf{\Lambda}^n)_k := \mathbf{\Lambda}([k], [n]) = \mathbf{\Lambda}^{\mathrm{op}}([n], [k]). \tag{2.41}$$

The term **cyclex** (plural **cyclices**) is new and is meant to be the cyclic analog of **simplex**. Like the standard n-simplex of Definition 2.21, the standard n-cyclex is a Yoneda functor.

The above will be generalized in Definition 2.59, for which it is the case $r = 1$.

See [Lod92, Chapter 7] for a formal treatment of cyclic spaces.

This object was denoted by $\mathbf{C}_\bullet$ (note the Roman font) in [Lod92, Proposition 6.1.9]. For $k = 1$, the group C_2 interchanges the degenerate and nondegenerate edges.

Note that the cardinality of the free C_{k+1}-set $(\mathbf{\Lambda}^n)_k$ exceeds that of $(\mathbf{\Delta}^n)_k$ (see Proposition 2.22) by a factor of $k+1$ by Proposition 2.38.

Proposition 2.42. The geometric realization of the standard n**-cyclex** $\mathbf{\Lambda}^n$ *is homeomorphic to* $\mathbb{T} \times \mathbf{\Delta}^n_{\mathrm{top}}$*, where* $\mathbf{\Delta}^n_{\mathrm{top}} = |\mathbf{\Delta}^n|$ *is the standard topological* n*-simplex of (2.23).*

For the proof, see [DHK85, §2.9].

Corollary 2.43. The cyclic circle or standard 0-cyclex. *The cyclic set* $\mathbf{\Lambda}^0$ *of (2.60) is isomorphic as a simplicial set to the* **circle** $\mathbf{\Delta}^1/\partial\mathbf{\Delta}^1$*. Its geometric realization is the circle* $\mathbb{T}$*.* $U\mathbf{\Lambda}^0$ *is isomorphic to the simplicial set with a single vertex* x_0 *and a single nondegenerate edge* x_1*. Thus we have*

$$(U\mathbf{\Lambda}_0)_k = \begin{cases} \{x_0\} & \text{for } k = 0 \\ \{x_1,\ s_0(x_0)\} & \text{for } k = 1 \\ \left\{s_i(s_0)^{k-2}(x_1) : 0 \leqslant i \leqslant k-1\right\} \cup \left\{(s_0)^k(x_0)\right\} & \\ & \text{for } k \geqslant 2, \end{cases} \tag{2.44}$$

making a total of $(k+1)$ k-simplices for each $k \geqslant 0$. The cyclic group C_{k+1} (which is the automorphism group $\mathrm{Aut}_{\mathbf{\Lambda}^{\mathrm{op}}}([k])$) acts freely on this set, so there is single orbit for each k.

Theorem 2.45. The action of the circle group $\mathbb{T}$. [Lod92, Theorem 7.1.4] *Let X be a cyclic space (e.g., a cyclic set) and let $|X|$ be the geometric realization of its underlying simplicial space. Then*

(i) $|X|$ is endowed with a canonical action of the circle $\mathbb{T}$, and
(ii) $X \mapsto |X|$ is a functor from cyclic spaces to $\mathbb{T}$-spaces.

Proposition 2.46. The smooth cyclotomy of $B^{\mathrm{cyc}}(\Gamma)$. *For a topological monoid Γ, the fixed point set $|B^{\mathrm{cyc}}(\Gamma)|^{\mathrm{C}_r}$ (as in Definition 2.39) is $\mathbb{T}$-equivariantly homeomorphic to $|B^{\mathrm{cyc}}(\Gamma)|$ for each integer $r \geqslant 1$.*

We sketch the proof here. The space $|B^{\mathrm{cyc}}(\Gamma)|$ is a quotient of

$$\coprod_{n\geqslant 0} \Gamma^{n+1} \times |\mathbf{\Lambda}^n| = \coprod_{n\geqslant 0} \Gamma^{n+1} \times |\mathbf{\Delta}^n| \times \mathbb{T}.$$

When r divides $n+1$, let $m = (n+1-r)/r$. The actions of C_r on Γ^{n+1} (by permutation of coordinates) and $|\mathbf{\Delta}^n|$ (by permutation of vertices) fix subspaces homeomorphic to Γ^{m+1} and $|\mathbf{\Delta}^m|$ respectively. It follows that $|B^{\mathrm{cyc}}(\Gamma)|^{\mathrm{C}_r}$ is a quotient of

$$\coprod_{m\geqslant 0} \Gamma^{m+1} \times |\mathbf{\Lambda}^m| = \coprod_{m\geqslant 0} \Gamma^{m+1} \times |\mathbf{\Delta}^m| \times \mathbb{T},$$

making it $\mathbb{T}$-equivariantly homeomorphic to $|B^{\mathrm{cyc}}(\Gamma)|$ as claimed.

Definition 2.47. *The* **cyclic geometric realization of a cyclic space** X *is*

$$|X|^{\mathrm{cyc}} := E\mathbb{T} \times_{\mathbb{T}} |X|.$$

In particular for $X = *$, we have $|*|^{\mathrm{cyc}} = B\mathbb{T}$.

Loday [Lod92, 7.1.5] defined a functor $F : \mathrm{Set}_{\mathbf{\Delta}} \to \mathrm{Set}_{\mathbf{\Lambda}}$ (with $F\mathbf{\Delta}^n = \mathbf{\Lambda}^n$) that is left adjoint to the forgetful functor $U : \mathrm{Set}_{\mathbf{\Lambda}} \to \mathrm{Set}_{\mathbf{\Delta}}$. For a simplicial set X, $|FX| \approx \mathbb{T} \times |X|$ with free $\mathbb{T}$-action by [Lod92, Lemma 7.1.8], even though UFX is not a product as a simplicial set. For a simplicial set X and a cyclic set Y, we have an adjunction isomorphism

$$\mathrm{Set}_{\mathbf{\Delta}}(X, UY) \cong \mathrm{Set}_{\mathbf{\Lambda}}(FX, Y).$$

When $X = UY$, this reads

$$\mathrm{Set}_{\mathbf{\Delta}}(UY, UY) \cong \mathrm{Set}_{\mathbf{\Lambda}}(FUY, Y).$$

The morphism set on the left has a distinguished element, the identity map on UY, and the corresponding morphism on the right is $\epsilon_Y : FUY \to Y$, the counit of the adjunction. On geometric realizations we have

$$|FUY| \approx \mathbb{T} \times |UY| := \mathbb{T} \times |Y| \xrightarrow{\quad |\epsilon_Y| \quad} |Y|,$$

making $|Y|$ a $\mathbb{T}$-space.

We can extend the simplicial structure on $\mathbf{HH}_{\bullet}(A)$ as in (2.32) to a cyclic structure by defining the last arc degeneracy map as

$$s^{n+1}(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_n \otimes 1. \tag{2.48}$$

Hence $\mathbf{HH}_\bullet(A)$ is a cyclic (in the sense of Connes) k-module, and the space $|\mathbf{HH}_\bullet(A)|$ of (2.34) is a $\mathbb{T}$-space.

2.6. **The paracyclic and r-cyclic categories, and edgewise subdivision.** The following is taken from [NS18, page 380].

Definition 2.49. *The* **paracyclic category** $\mathbf{\Lambda}_\infty$ *has as objects the linearly ordered sets*

$$[\![n]\!]_{\mathbf{\Lambda}_\infty} := (1/n)\mathbb{Z}$$

on which $\mathbb{Z}$ acts by addition. A morphism is a map of sets $f : [\![m]\!]_{\mathbf{\Lambda}_\infty} \to [\![n]\!]_{\mathbf{\Lambda}_\infty}$ satisfying

$$f(x) \leqslant f(y) \text{ when } x \leqslant y \qquad \text{and} \qquad f(x+1) = f(x) + 1. \tag{2.50}$$

Hence each morphism set has an action of $\mathbb{Z}$ by pointwise addition.

For an integer $r \geqslant 1$, the r-**cyclic category** $\mathbf{\Lambda}_r$ *(denoted simply by $\mathbf{\Lambda}$ when $r = 1$) has the same objects, now denoted by $[\![n]\!]_{\mathbf{\Lambda}_r}$ (or simply $[\![n]\!]_{\mathbf{\Lambda}}$ when $r = 1$), and each morphism set is the quotient of that in $\mathbf{\Lambda}_\infty$ by the action of $r\mathbb{Z}$.*

We will often drop the subscript on $[\![n]\!]$.

The conditions of (2.50) mean that f is determined by its behavior on any interval of length 1.

The category $\mathbf{\Lambda} = \mathbf{\Lambda}_1$ coincides with Connes' cyclic category $\mathbf{\Lambda}$ of Definition 2.40.

Remark 2.51. The symbols $\mathbf{\Lambda}$, $[n]$ and $[\![n]\!]$. *The symbol $\mathbf{\Lambda}$ is not to be confused with $\Lambda(-)$, which we sometimes use to denote an exterior algebra, or $\wedge$ as in Definition 4.21.*

Our notation differs from that of [NS18, page 380] *and* [McC24, Example 2.1.5], *in which our $[\![n]\!]_{\mathbf{\Lambda}_r}$ is denoted by $[n]_{\mathbf{\Lambda}_r}$. In this paper (starting in §2.4) and many others, $[n]$ denotes the finite ordered set $\{0, 1, \ldots, n\}$, which has $n+1$ elements, while $[\![n]\!]_{\mathbf{\Lambda}}$ behaves like a set with n elements. Thus we write $[n] \times \mathbb{Z} \simeq [\![n+1]\!]_{\mathbf{\Lambda}_\infty}$ for the isomorphism written in* [McC24, Example 2.1.6] *as $[n] \times \mathbb{Z} \simeq [n+1]_\Lambda$.*

Thus we have projection functors

$$\begin{array}{ccccc} \mathbf{\Lambda}_\infty & \xrightarrow{\mathrm{Proj}_{\infty,r}} & \mathbf{\Lambda}_r & \xrightarrow{\mathrm{Proj}_{r,r/d}} & \mathbf{\Lambda}_{r/d} \\ \mathrm{Map}_{\mathbf{\Lambda}_\infty}([\![m]\!],[\![n]\!]) & \longmapsto & \mathrm{Map}_{\mathbf{\Lambda}_r}([\![m]\!],[\![n]\!]) & \longmapsto & \mathrm{Map}_{\mathbf{\Lambda}_{r/d}}([\![m]\!],[\![n]\!]) \\ & & \| & & \| \\ & & \mathrm{Map}_{\mathbf{\Lambda}_\infty}([\![m]\!],[\![n]\!])/r\mathbb{Z} & & \mathrm{Map}_{\mathbf{\Lambda}_r}([\![m]\!],[\![n]\!])/(\mathbb{Z}/d) \end{array} \tag{2.52}$$

for each $r \geqslant 1$ and each divisor d of r. The groups $r\mathbb{Z} \subseteq \mathbb{Z}$ and $\mathbb{Z}/d \subseteq \mathbb{Z}/r$ act freely on the morphism sets in question.

Equivalently $\mathbf{\Lambda}_r$ for $1 \leqslant r \leqslant \infty$ is the category which contains $\mathbf{\Delta}$ and additional morphisms $\tau_n : [n] \to [n]$ subject to the relations of (2.37) and the relation $\tau_n^{r(n+1)} = 1_{[n]}$ when $r < \infty$.

The objects of $\mathbf{\Lambda}$ behave like finite sets and there is an inclusion functor

$$\begin{aligned} V : \mathbf{\Lambda} &\to \mathcal{F}in \\ [\![n]\!]_{\mathbf{\Lambda}} &\mapsto \langle n \rangle \end{aligned}$$

to the category of finite sets of Definition D.1 sending $[\![n]\!]_{\mathbf{\Lambda}}$, which has n elements, to the unpointed set with the same cardinality. We know by [NS18, Proposition B.1] that it lifts to a similarly named functor

$$V : \mathbf{\Lambda} \to \mathrm{Assoc}^{\otimes}_{\mathrm{act}},$$

where the codomain is the associative operad of (D.6). Precomposing with the isomorphism between $\mathbf{\Lambda}$ and its opposite gives a functor

$$V^{\mathrm{op}} : \mathbf{\Lambda}^{\mathrm{op}} \to \mathrm{Assoc}^{\otimes}_{\mathrm{act}}. \tag{2.53}$$

The self-duality of $\mathbf{\Lambda}_\infty$, which implies that of each $\mathbf{\Lambda}_r$, is spelled out in [NS18, page 381].

For each r, there is an embedding

$$\iota_r : \mathbf{\Delta} \to \mathbf{\Lambda}_r \qquad [n] \mapsto [\![n+1]\!]_{\mathbf{\Lambda}_r} \tag{2.54}$$

Simplicial and r-cyclic (paracyclic for $r = \infty$) sets are contravariant Set-valued functors on $\mathbf{\Delta}$ and $\mathbf{\Lambda}_r$ respectively. See Definition 2.59 below. Both have geometric realizations, and that of an r-cyclic set for finite r has a natural action of the circle $\mathbb{T}$. That of a paracyclic set has natural action of $\mathbb{R}$.

We will want to look at the fixed point sets of finite subgroups of $\mathbb{T}$. In order to describe these simplicially, the following notion of subdivision is helpful.

Definition 2.55. *For each positive integer r, the* **r-fold edgewise subdivision functor** $\mathrm{sd}_r : \mathbf{\Delta} \to \mathbf{\Delta}$ *sends $[n-1]$ to $[rn-1]$ for $n \geqslant 1$, and each morphism to its rth iterated disjoint union. (Face and degeneracy maps can also be written as r-fold compositions as in (2.57).) Let $d_{(r)} : \Delta^{n-1}_{\mathrm{top}} \to \Delta^{rn-1}_{\mathrm{top}}$ be the map of topological simplices induced by the diagonal embedding*

$$(x_0, \dots, x_{n-1}) \mapsto \frac{(x_0, \dots, x_{n-1}, x_0, \dots, x_{n-1}, \dots, x_0, \dots, x_{n-1})}{r}$$

where the coefficients in the codomain are repeated r times with

$$x_i \geqslant 0 \qquad \text{and} \qquad \sum_{0 \leqslant i < n} x_i = 1.$$

For a simplicial set X, we define the **rth subdivided simplicial set** $\mathrm{sd}_r^* X$ *to be the composite functor*

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{sd}_r} \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X} \mathrm{Set}, \tag{2.56}$$

so that $(\mathrm{sd}_r^* X)_{n-1} = X_{rn-1}$. Its ith face map d_i (for $0 \leqslant i \leqslant n-1$) is the composite of r face maps in X,

$$\begin{array}{ccccccc} (\mathrm{sd}_r^* X)_{n-1} & & \xrightarrow{\quad d_i \quad} & & & & (\mathrm{sd}_r^* X)_{n-2} \\ \Vert & & & & & & \Vert \\ X_{rn-1} & \xrightarrow{d_{i+(r-1)n}} & \cdots & \xrightarrow{d_{i+n}} & X_{r(n-1)} & \xrightarrow{d_i} & X_{r(n-1)-1} \end{array} \tag{2.57}$$

and its degeneracy maps are defined similarly.

Lemma 2.58. Edgewise subdivision induces a homeomorphism. [BHM93, Lemma 1.1] *The map $D_r : |\mathrm{sd}_r^* X| \to |X|$ induced by*

$$1 \times d_{(r)} : (\mathrm{sd}_r^* X)_{n-1} \times \mathbf{\Delta}^{n-1} \to X_{rn-1} \times \mathbf{\Delta}^{rn-1}$$

is a homeomorphism.

Definition 2.59. [BHM93, Definition 1.5] *An* r**-cyclic (or paracyclic) object** *in a category* $\mathcal{C}$ *is a contravariant* $\mathcal{C}$*-valued functor on* $\mathbf{\Lambda}_r$ *for finite* r *(or* $\mathbf{\Lambda}_\infty$*), and we denote the category of such functors by* $\mathcal{C}_{\mathbf{\Lambda}_r}$*. The value of such a functor* X *on* $[n]$ *is denoted by* X_n*. The* **geometric realization** $|X|$ *of an* r*-cyclic set* X *is that of the underlying (via the embedding of (2.54)) simplicial set.*

The r*-cyclic set* $\mathbf{\Lambda}_r^n$ *the* **standard** n**-cyclex of degree** r*, is defined by*

$$(\mathbf{\Lambda}_r^n)_k := \mathbf{\Lambda}_r([k],[n]) = \mathbf{\Lambda}_r^{\mathrm{op}}([n],[k]). \tag{2.60}$$

We will omit the subscript when it is 1.

Proposition 2.61. [BHM93, Lemma 1.6] **The geometric realization of** $\mathbf{\Lambda}_r^n$ *is homeomorphic to* $\mathbb{R}/r\mathbb{Z} \times \mathbf{\Delta}_{\mathrm{top}}^n$*. The action of* τ_n *on* $|\mathbf{\Lambda}_r^n|$ *given by*

$$\tau_n(\theta; x_0, \ldots, x_n) := (\theta - x_0; x_1, \ldots, x_n, x_0), \tag{2.62}$$

where $\theta \in \mathbb{R}/r\mathbb{Z}$*.*

(2.62) implies that

$$\tau_n^{n+1}(\theta; x_0, \ldots, x_n) = (\theta - 1; x_0, x_1, \ldots, x_n),$$

making $\tau_n^{(n+1)r}$ the identity map for finite r. The duals of the relations of (2.37) imply that τ_n^{n+1} commutes with d^i and s^i in that for $0 \leqslant i \leqslant n$,

$$\tau_n^{n+1} d^i = d^i \tau_{n-1}^n \qquad \text{and} \qquad \tau_n^{n+1} s^i = s^i \tau_{n+1}^{n+2}.$$

From this it follows that an r-cyclic space or set has an action of the cyclic group C_r.

For $r = 1$, Proposition 2.61 was first proved by Dwyer, Hopkins and Kan in [DHK85, §2.9]. We describe their proof in §2.5.1.

2.7. The epicyclic category.

The ideas of this subsection were first published in [BFG94], whose authors attribute them to [Goo87]. A more contemporary reference is [McC24], which lists several other papers on the topic.

We will define analogs $\widetilde{\mathbf{\Lambda}}_r$ of the categories of Definition 2.49 having the same objects but more morphisms. Recall that Connes regards the finite ordered set $[n]$ as the set of $(n+1)$th roots of unity in the unit circle. Morphisms in $\mathbf{\Lambda}$ are homotopy classes of degree one maps of the circle that preserve the subset. The categories $\mathbf{\Lambda}_r$ and $\mathbf{\Lambda}_\infty$ can be described in similar terms with the unit circle replaced by its r-fold and universal covers. Maps in the former case have degree 1 and on the real line they are equivariant with respect to the action of the integers by addition. *In the epicyclic version we drop the degree one requirement.* We allow maps of arbitrary positive degree k in $\widetilde{\mathbf{\Lambda}}_r$, and a map f in $\widetilde{\mathbf{\Lambda}}_\infty$ must satisfy $f(x+1) = f(x) + k$. The resulting morphism sets still have an action of $\mathbb{Z}$ by pointwise addition.

Definition 2.63. [BFG94, Definition 1.1]. *The* **epicyclic category** $\widetilde{\mathbf{\Lambda}}_\infty$ *has as objects the linearly ordered sets*

$$[\![n]\!]_{\widetilde{\mathbf{\Lambda}}_\infty} := (1/n)\mathbb{Z}$$

on which $\mathbb{Z}$ *acts by addition, as in Definition 2.49. A morphism is a map of sets* $f : [\![m]\!] \to [\![n]\!]$ *satisfying*

$$f(x) \leqslant f(y) \text{ when } x \leqslant y \qquad \text{and} \qquad f(x+1) = f(x) + k, \tag{2.64}$$

for a positive integer k, the **degree** *of f. For $k = 1$, this is the same as (2.50). Hence each morphism set has an action of $\mathbb{Z}$ by pointwise addition.*

For an integer $r \geqslant 1$, the **r-epicyclic category** $\widetilde{\mathbf{\Lambda}}_r$ *(denoted simply by $\widetilde{\mathbf{\Lambda}}$ when $r = 1$) has the same objects, and each morphism set is the quotient of that in $\widetilde{\mathbf{\Lambda}}_\infty$ by the action of $r\mathbb{Z}$.*

Equivalently (see [BFG94, Definition 1.1]), $\widetilde{\mathbf{\Lambda}}$ has the same objects and includes the morphisms of $\mathbf{\Lambda}$ as in (2.37) with additional morphisms

$$\pi_n^k : [k(n+1)-1] \to [n], \qquad k, n \in \mathbb{N}, \quad k \geqslant 1 \tag{2.65}$$

defined by $a(n+1)+b \mapsto b$ for $0 \leqslant b \leqslant n$ and $0 \leqslant a < k$. These are subject to the relations

(i) $\pi_n^1 = 1_{[n]}$, $\pi_n^\ell \pi_{\ell(n+1)-1}^k = \pi_n^{k\ell}$
(ii) $\alpha\pi_m^k = \pi_n^k \mathrm{sd}_k(\alpha)$ for $\alpha \in \mathbf{\Delta}([m],[n])$ for sd_k as in Definition 2.55
(iii) $\tau_n \pi_n^k = \pi_n^k \mathrm{sd}_k(\tau_n)$ for τ_n as in §2.5.

There is an analog of Proposition 2.38 that says any morphism $[m] \to [n]$ in $\widetilde{\mathbf{\Lambda}}$ can be written uniquely as a composite

$$[m] \xrightarrow{(\tau_m)^i} [m] \xrightarrow{\alpha} [k(n+1)-1] \xrightarrow{\pi_n^k} [n] \tag{2.66}$$

for some $k > 0$ with α being a morphism in $\mathbf{\Delta}$.

A more conceptual definition of $\widetilde{\mathbf{\Lambda}}$ due to Nikolaus is [McC24, Definition 2.1.9].

Definition 2.67. The categories $[\![n]\!]_{\mathbf{\Lambda}_r}$. *The objects $[\![n]\!]_{\mathbf{\Lambda}_r} = [\![n]\!]_{\widetilde{\mathbf{\Lambda}}_r}$ of Definitions 2.49 and 2.63 for $1 \leqslant r \leqslant \infty$ are themselves categories. For $r = \infty$ it is a poset, which is a category whose objects are the elements in the set and in which there is a unique morphism from each element to each larger one. The sets of morphisms and objects have an action of the integers by addition. For each $r < \infty$ we get a new category by passage to orbit sets. Thus $[\![n]\!]_{\mathbf{\Lambda}_r}$ has nr objects and the geometric realization of its nerve is a circle. That of $[\![n]\!]_{\mathbf{\Lambda}_\infty}$ is the real line.*

Definition 2.68. *An* **epicyclic object** *X in a category $\mathcal{C}$ is a $\mathcal{C}$-valued functor on $\widetilde{\mathbf{\Lambda}}^{\mathrm{op}}$ and therefore a simplicial object (by restriction of the functor to $\mathbf{\Delta}^{\mathrm{op}}$) with some additional structure, which includes an action of the cyclic group C_{n+1} on X_n for each n. We denote the category of such functors by $\mathcal{C}_{\widetilde{\mathbf{\Lambda}}}$, and the forgetful functor $\mathcal{C}_{\widetilde{\mathbf{\Lambda}}} \to \mathcal{C}_{\mathbf{\Delta}}$ by $\widetilde{U}$. The* **geometric realization $|X|$ of an epicyclic set** *X is that of the underlying simplicial set $\widetilde{U}X$, which we will often denote abusively by X.*

The epicyclic set $\widetilde{\mathbf{\Lambda}}^n$, the **standard n-epicyclex**, *is defined by*

$$(\widetilde{\mathbf{\Lambda}}^n)_k := \widetilde{\mathbf{\Lambda}}([k],[n]) = \widetilde{\mathbf{\Lambda}}^{\mathrm{op}}([n],[k]). \tag{2.69}$$

Since $\mathbf{\Lambda}$ is a wide subcategory (consisting of morphisms of degree 1) of $\widetilde{\mathbf{\Lambda}}$, an epicyclic object is also a cyclic object. We know by Theorem 2.45 that the geometric realization of a cyclic comes equipped with an action of the circle group $\mathbb{T}$. The corresponding structure in the epicyclic case involves the following.

Definition 2.70. [BFG94, (0.1)]. *The* **rotation-power monoid** $\mathscr{M} := \mathbb{T} \rtimes \mathbb{N}^\times$, *where $\mathbb{N}^\times$ denotes the multiplicative monoid of positive integers, is the monoid with multiplication given by*

$$(z_1, k_1)(z_2, k_2) = (z_1 z_2^{k_1}, k_1 k_2) \qquad \text{for } z_i \in \mathbb{T} \text{ and } k_i \in \mathbb{N}^\times.$$

For a prime number p, the p**-typical rotation-power monoid** *is*

$$\mathscr{M}_p := \mathrm{C}_{p^\infty} \rtimes \mu_{p^{\mathbb{N}}},$$

where $\mathrm{C}_{p^\infty} \subseteq \mathbb{T}$ *is the Prüffer group and*

$$\mu_{p^{\mathbb{N}}} := \left\{1, p, p^2, \dots\right\}.$$

Jonas McCandless [McC24] calls $\mathscr{M}$ the Witt monoid and denotes it by $\mathbf{W}$. It is studied but not named in [BFG94].

$\mathscr{M}$ acts on the free loop space $\mathcal{L}X$ with $\mathbb{T}$ acting by rotation of loops and $\mathbb{N}^\times$ acting by power maps, hence the name. This means that for a point $f : \mathbb{T} \to X$ in $\mathcal{L}X$ and $(z, k) \in \mathscr{M}$, we have

$$(z, k)(f)(u) : f(zu^k) \in X \qquad \text{for each } u \in \mathbb{T}. \tag{2.71}$$

Theorem 2.72. [BFG94, Theorems A and B].

(i) The geometric realization of an epicyclic space has a canonical, right action of the rotation-power monoid $\mathscr{M}$ of Definition 2.70.
(ii) The classifying space $B\mathscr{M}$ is homotopy equivalent to $|N(\widetilde{\mathbf{\Lambda}})|$.
(iii) The fundamental group of $B\mathscr{M}$ is isomorphic to the multiplicative group of positive rational numbers. The universal covering of $B\mathscr{M}$ has the homotopy type of the Eilenberg-MacLane space $K(\mathbb{Q}, 2)$ with π_1 acting by multiplication of rational numbers.

2.8. **Summary of our indexing categories.** We have introduced the categories

$$\begin{array}{ccccccccc}
[n-1] \underset{s^i}{\overset{d^i}{\rightleftarrows}} [n] & \mathbf{\Delta} & & & & & & \\
 & \subseteq\downarrow & & & & & & \\
[n] \xrightarrow{\tau_n} [n] & \mathbf{\Lambda} & \xleftarrow{\mathrm{Proj}_{r/d,1}} & \mathbf{\Lambda}_{r/d} & \xleftarrow{\mathrm{Proj}_{r,r/d}} & \mathbf{\Lambda}_r & \xleftarrow{\mathrm{Proj}_{\infty,r}} & \mathbf{\Lambda}_\infty & \\
 & \subseteq\downarrow & & & & \downarrow\subseteq & & \downarrow\subseteq & \\
[kn+k-1] \xrightarrow{\pi_n^k} [n] & \widetilde{\mathbf{\Lambda}} & & \xleftarrow{\mathrm{Proj}_{r,1}} & & \widetilde{\mathbf{\Lambda}}_r & \xleftarrow{\mathrm{Proj}_{\infty,r}} & \widetilde{\mathbf{\Lambda}}_\infty & [\![n]\!]_{\widetilde{\mathbf{\Lambda}}_\infty} := (1/n)\mathbb{Z}
\end{array}$$

with the indicated objects and morphisms in Definitions 2.17, 2.49 and 2.63. These are respectively the simplicial, cyclic/paracyclic and epicyclic categories. The coface maps d^i and the codegeneracy maps s^i are given in Definition 2.17, τ_n is the automorphism of order $r(n+1)$ given in the first paragraph of §2.5, and π_n^k is given in (2.65). In each case we are interested in contravariant functors to the category of sets or spaces.

2.9. **The double complexes of Tsygan-Goodwillie and Connes.** The first quadrant version of the following appeared first in less general form in a paper of Tsygan [Tsy83]. It was later studied in [LQ84] and [Lod92, 2.1.2]. The treatment here is taken from Goodwillie's paper [Goo85, II.2] and is also in Weibel's book [Wei94, 9.6]. To our knowledge, Goodwillie's paper is the first to treat the case of a general cyclic object in an abelian category rather than the specific case $\mathbf{HH}_\bullet(A)$ as in (2.32).

Definition 2.73. Tsygan and Goodwillie's first double complex. *Let X be a cyclic object in an abelian category. Then $C^{\mathrm{per}}_{**}(X)$, the* **periodic double**

complex of X, *is the upper half plane double complex with a contracting chain homotopy u in its oddly indexed columns*

$$\begin{array}{ccccccccc}
 & & \downarrow b & & u\uparrow\downarrow -b' & & \downarrow b & & \\
\xleftarrow{N} & & C^{\mathrm{per}}_{2i,j+1}(X) & \xleftarrow{\epsilon} & C^{\mathrm{per}}_{2i+1,j+1}(X) & \xleftarrow{N} & C^{\mathrm{per}}_{2i+2,j+1}(X) & \xleftarrow{\epsilon} & \\
 & & \downarrow b & & u\uparrow\downarrow -b' & & \downarrow b & & \\
\xleftarrow{N} & & C^{\mathrm{per}}_{2i,j}(X) & \xleftarrow{\epsilon} & C^{\mathrm{per}}_{2i+1,j}(X) & \xleftarrow{N} & C^{\mathrm{per}}_{2i+2,j}(X) & \xleftarrow{\epsilon} & \\
 & & \downarrow b & & u\uparrow\downarrow -b' & & \downarrow b & &
\end{array} \tag{2.74}$$

where

$$\begin{aligned}
C^{\mathrm{per}}_{i,j}(X) &:= X_j && \text{for } i \in \mathbb{Z} \text{ and } j \geqslant 0\\
b &:= \sum_{0\leqslant k\leqslant j} (-1)^k d_k, && \text{the Hochschild boundary,}\\
b' &:= \sum_{0\leqslant k< j} (-1)^k d_k, && \text{the modified Hochschild boundary,}\\
u &:= (-1)^j s_j, && \text{the contracting chain homotopy,}\\
\epsilon &:= 1-(-1)^j t_j && \text{for } t_j \text{ as in (2.37),}\\
\text{and}\qquad N &:= \sum_{0\leqslant k\leqslant j} \left((-1)^j t_j\right)^k .
\end{aligned}$$

The definition of each arrow in the picture depends on the vertical coordinate, which we suppress from the notation to avoid clutter, and the parity of the horizontal coordinate. We denote the evenly and oddy indexed columns by $C^{\mathrm{Hoch}}_*(X)$, the Hochschild complex of X, *and $C^{\mathrm{acyc}}_*(X)$, the* acyclic complex of X, *respectively. We denote the homology of the former by $\mathrm{HH}_*(X)$. We will denote the corresponding first ($i \geqslant 0$) and third ($i < 0$) quadrant double complexes by $C_{**}(X)$ and $C^-_{**}(X)$, so there is a short exact sequence of double complexes*

$$0 \to C^-_{**}(X) \to C^{\mathrm{per}}_{**}(X) \to C_{**}(X) \to 0. \tag{2.75}$$

When X is $\mathbf{HH}_\bullet(A)$ as in (2.32), we will denote $C^{\mathrm{per}}_{*,*}(X)$ by $C^{\mathrm{per}}_{*,*}(A)$, and similarly for the related complexes C^{Hoch}_*, C^{acyc}_*, $C^-_{*,*}$ and $C_{*,*}$.

We will see analogs (2.75) below in (2.83) and (4.55). The latter two have to do with $\mathbb{T}$-spectra rather than cyclic objects in an abelian category. We call them **Tate sequences**. See Definition 4.54.

The acyclic complex is so named because the map

$$u = (-1)^j s_j : X_j \to X_{j+1} \tag{2.76}$$

satisfies $b'u + ub' = 1_{X_j}$, making it a contracting chain homotopy generalizing (2.8).

Remark 2.77. The curious nature of the Tsygan double complex. *The Hochschild complex $C^{\mathrm{Hoch}}(X)$ is the complex $\mathrm{Ch}(X)$ of Definition 2.28. The acyclic complex $C^{\mathrm{acyc}}(X)$, which has the same chain objects as $C^{\mathrm{Hoch}}(X)$, is also defined for any simplicial abelian object X. No use of the cyclic structure is made in defining either. Only the horizontal arrows in (2.74) are defined in terms of it.*

However in the simplicial setting there is no obvious map between the two chain complexes, so the existence of boundary operator on $\mathrm{Ch}(X)$ *that renders it acyclic appears to be an idle curiosity.*

Let d^h and d^v denote the horizontal and vertical arrows in (2.74). Goodwillie shows that that $d^h d^v + d^v d^h$ vanishes on each $C_{i,j}(X)$, generalizing a similar argument in [LQ84, Lemma 1.1].

Definition 2.78. Cyclic homology. *With notation as above,* $\mathrm{HC}_*(X)$*, the* **cyclic homology of** X*, is the homology of the first quadrant total complex* $\mathrm{Tot}^{\oplus}(C_{**}(X))$*. This is the chain complex defined by*

$$\mathrm{Tot}^{\oplus}(C_{**}(X))_n := \bigoplus_{\substack{i+j=n\\ i,j\geqslant 0}} C_{i,j}(X)$$

with boundary operator $d^h + d^v$*.*

Following [Goo85, page 191], the double complex map $s : C_{i,j}(X) \to C_{i-2,j}(X)$, which is an isomorphism for $i \geqslant 2$, leads to a short exact sequence

$$0 \to \ker(s) \to \mathrm{Tot}^{\oplus}(C_{**}(X))_* \to \mathrm{Tot}^{\oplus}(C_{**}(X))_{*-2} \to 0$$

and a long exact sequence

$$\cdots \to \mathrm{HH}_*(X) \to \mathrm{HC}_*(X) \to \mathrm{HC}_{*-2}(X) \to \mathrm{HH}_{*-1}(X) \to \cdots \tag{2.79}$$

due in cohomological form to Connes [Con83] and [Con94, III.1.γ]; see [Lod92, Theorem 2.2.1]. Weibel refers to this as the SBI sequence [Wei94, Proposition 9.6.11]. It follows that a map of cyclic objects inducing an isomorphism in HH_* also induces one in HC_*.

Composing the map $\mathrm{HH}_*(X) \to \mathrm{HC}_*(X)$ in (2.79) with the reindexed

$$\mathrm{HC}_*(X) \to \mathrm{HH}_{*+1}(X)$$

gives the **Connes operator**

$$B : \mathrm{HH}_j(X) \to \mathrm{HH}_{j+1}(X). \tag{2.80}$$

It is induced by the composite

$$X_j = C_{2i+2,j}(X) \xrightarrow{\epsilon u N} C_{2i,j+1}(X) = X_{j+1}$$

in (2.74), which we will also denote by B. Note that

$$BB = (\epsilon u N)(\epsilon u N) = \epsilon u (N\epsilon) u N = 0,$$

and the cochain complex $(\mathrm{HH}_*(X), B)$ is the **de Rham complex of** X.

Definition 2.81. *The* **second double complex of a cyclic abelian object** X *is obtained from that of (2.74) by removing the acyclic oddly indexed columns of (2.74) and suitably regrading, thereby obtaining the half plane complex* $\mathcal{B}^{\mathrm{per}}_{**}(X)$ *in which* $\mathcal{B}^{\mathrm{per}}_{i,j}(X) = X_{j-i}$ *for* $j \geqslant i$ *and vanishes for* $j < i$*. In the following picture,* *the horizontal and vertical coordinates are i and j.*

(2.82)
$$\begin{array}{cccccccccc}
 & & & -1 & & 0 & & 1 & & 2 \\
 & & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & & \downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle b} \\
2 & \cdots & \xleftarrow{B} & X_3 & \xleftarrow{B} & X_2 & \xleftarrow{B} & X_1 & \xleftarrow{B} & X_0 \\
 & & & \downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b} & & \\
1 & \cdots & \xleftarrow{B} & X_2 & \xleftarrow{B} & X_1 & \xleftarrow{B} & X_0 & & \\
 & & & \downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle b} & & & & \\
0 & \cdots & \xleftarrow{B} & X_1 & \xleftarrow{B} & X_0 & & & & \\
 & & & \downarrow{\scriptstyle -b} & & & & & & \\
-1 & \cdots & \xleftarrow{B} & X_0, & & & & & &
\end{array}$$

where on X_j,

$$b = \sum_{0\leqslant k\leqslant j} (-1)^k d_k$$

$$\text{and} \qquad B = \epsilon u N = ((-1)^j + t_j)s_j \sum_{0\leqslant k\leqslant j} ((-1)^j t_j)^k.$$

The horizontal and vertical differentials are B and $(-1)^i b$ respectively.

There is a subcomplex $\mathcal{B}^-_{*,*}(X)$ obtained by replacing the groups with $i \geqslant 0$ by 0, which is concentrated in the second quadrant plus half of the third one. Its quotient $\mathcal{B}_{**}(X)$ is concentrated in the first quadrant.

Thus this complex is concentrated in the northwest half plane defined by $j \geqslant i$. It is both a chain complex and a cochain complex, and is called a *mixed complex* by Kassel in [Kas87]. See [Wei94, 9.8] for further discussion. It should be compared with the bicomplex $\mathcal{B}(A)$ of [Lod92, 2.1.7]. A cohomological variant of it is studied by Connes in [Con94, 3.1.γ].

The corresponding $\mathrm{Tot}^{\oplus}$s, in all three versions, are known to be quasi-isomorphic (up to regrading) to those of (2.74). We have

$$\mathrm{Tot}^{\oplus}(\mathcal{B}_{**}(X))_n = \bigoplus_{0\leqslant i\leqslant n/2} X_{n-2i},$$

$$\mathrm{Tot}^{\oplus}(\mathcal{B}^{\mathrm{per}}_{**}(X))_n = \begin{cases} \bigoplus_{i\geqslant 0} B_{2i} & \text{for } n \text{ even} \\ \bigoplus_{i\geqslant 0} B_{2i+1} & \text{for } n \text{ odd,} \end{cases}$$

$$\text{and} \qquad \mathrm{Tot}^{\oplus}(\mathcal{B}^{-}_{**}(X))_n = \begin{cases} \bigoplus_{i\geqslant \max((n+2)/2,0)} B_{2i} & \text{for } n \text{ even} \\ \bigoplus_{i\geqslant \max((n+1)/2,0)} B_{2i+1} & \text{for } n \text{ odd.} \end{cases}$$

It follows that we have a short exact sequence of double complexes similar to (2.75),

$$0 \to \mathcal{B}^-_{**}(X) \to \mathcal{B}^{\mathrm{per}}_{**}(X) \to \mathcal{B}_{**}(X) \to 0 \tag{2.83}$$

and similarly for their $\mathrm{Tot}^{\oplus}$s. We denote their homology groups by $\mathrm{HC}^-(X)$, $\mathrm{H}^{\mathrm{per}}(X)$, and $\mathrm{HC}(X)$ respectively. In each case one can filter the double complex increasingly by its columns and obtain a *Connes spectral sequence* converging to its homology with input $\mathrm{HH}_*(X)$ in which the first differential is the Connes operator B.

2.10. **The free loop space of the circle.** We recall properties of the free loop space $\mathcal{L}S^1$ for future reference. The circle group $\mathbb{T}$ acts (as it does on any free loop space) on by rotation of loops. Each point in it has a winding number or degree associated with it, and we denote by $\mathcal{L}_r S^1$ the space of loops with degree r. The rotation-power monoid $\mathscr{M}$ of Definition 2.70 acts on $\mathcal{L}S^1$ as explained in (2.71).

Let

$$p : \mathbb{R} \to S^1 \subseteq \mathbb{C} \qquad \text{with} \qquad p(t) := e^{2\pi t\sqrt{-1}},$$

where $S^1 \subseteq \mathbb{C}$ is the unit circle. Then a loop of degree r lifts via p to a path in $\mathbb{R}$ between two points $|r|$ units apart, for example to a closed path when $r = 0$. The space of such loops is equivalent to S^1 with $\mathbb{T}$ acting via the rth power map. It follows that we have a $\mathbb{T}$-equivariant equivalence

$$\mathcal{L}S^1 \simeq \coprod_{r\in\mathbb{Z}} \mathbb{T}/\mathrm{C}_{|r|}. \tag{2.84}$$

Here "$\mathbb{T}/\mathrm{C}_0$" is understood to be the circle with trivial $\mathbb{T}$-action, there being no such group as C_0. The underlying space is equivalent to $\mathbb{Z} \times S^1$, and $\mathbb{T}$ acts on the rth component via the rth power map. This result is the $G = \mathbb{Z}$ case of the following.

Lemma 2.85. [KSS09, Lemma 9.1] *Let G be any topological group of CW type. Then there is a fibrewise homotopy equivalence*

$$\mathcal{L}BG \simeq EG \times_G G^{\mathrm{ad}},$$

where G^{ad} denotes G with the conjugation action, of fibrewise H-spaces over BG. In particular if G is abelian,

$$\mathcal{L}BG \simeq BG \times G.$$

For $G = \mathbb{Z}$, $BG \simeq S^1$, and this is our description of the space underlying $\mathcal{L}S^1$. Similarly for $G = \mathbb{Z}_p$, we have

$$\mathcal{L}B\mathbb{Z}_p \simeq B\mathbb{Z}_p \times \mathbb{Z}_p.$$

Like any free loop space, this is a $\mathbb{T}$-space. In order to describe it as such, note that as topological spaces,

$$\mathbb{Z}_p \cong \{0\} \amalg \coprod_{j\geqslant 0} p^j\mathbb{Z}_p^\times.$$

This is the stratification of $\mathbb{Z}_p$ by p-adic valuation. Each subspace on the right other than $\{0\}$ is open, and all of them are closed.

Since $\mathbb{Z}_p$ is abelian, its classifying space is also an abelian group which we denote by $\mathbb{T}_p$. The homomorphism $\mathbb{Z} \to \mathbb{Z}_p$ induces $\mathbb{T} \to \mathbb{T}_p$. For each $j \geqslant 0$ we have

$$\mathrm{C}_{p^j} \subseteq \mathbb{T} \subseteq \mathbb{T}_p$$

with $\mathbb{T}_p/\mathrm{C}_{p^j} \cong \mathbb{T}_p$ as underlying spaces, since $\mathbb{Z}_p$ and $p^j\mathbb{Z}_p$ are isomorphic as groups. Then we find that as $\mathbb{T}$-spaces

$$\mathcal{L}B\mathbb{Z}_p \simeq B\mathbb{Z}_p \amalg \coprod_{j\geqslant 0} \left(p^j\mathbb{Z}_p^\times \times \mathbb{T}_p/\mathrm{C}_{p^j}\right), \tag{2.86}$$

where $\mathbb{T}$ acts trivially on the first summand on the right. This is a special case of [Lod92, Theorem 7.3.11].

3. The work of Bökstedt, Hsiang and Madsen

Now we quote some results from [BHM93, §1] indicating the action of $\mathbb{T}$ on spaces related to THH. Madsen's expository accounts [Mad94a, Mad94b] of this work are very helpful.

3.1. THH via Bökstedt functors.

In most papers on the subject, what we are calling a Bökstedt functor is referred to as a **functor with smash products** or **FSP** and is usually denoted by the letter F.

Definition 3.1. [Bök85a, Definition 1.1] *A* **Bökstedt functor** *is an endofunctor* $\mathfrak{B}$ *on the category of pointed topological spaces equipped with natural transformations*

$$\mu : \mathfrak{B}(-) \wedge \mathfrak{B}(-) \implies \mathfrak{B}(- \wedge -) \qquad \text{and} \qquad \iota : \mathrm{Id} \implies \mathfrak{B}$$

(which we will denote by $\mu(\mathfrak{B})$ *and* $\iota(\mathfrak{B})$ *when there is more than one such functor in play) inducing morphisms*

$$\mu_{X,Y} : \mathfrak{B}(X) \wedge \mathfrak{B}(Y) \to \mathfrak{B}(X \wedge Y) \qquad \text{and} \qquad \iota_X : X \to \mathfrak{B}(X)$$

for which the following three diagrams commute:

$$\begin{array}{ccc} & & \mathfrak{B}(X) \wedge \mathfrak{B}(Y) \\ & \overset{\iota_X \wedge \iota_Y}{\nearrow} & \downarrow{\scriptstyle \mu_{X,Y}} \\ X \wedge Y & \underset{\iota_{X \wedge Y}}{\searrow} & \mathfrak{B}(X \wedge Y), \end{array} \tag{3.2}$$

$$\begin{array}{ccc} \mathfrak{B}(X) \wedge \mathfrak{B}(Y) \wedge \mathfrak{B}(Z) & \xrightarrow{\mu_{X,Y} \wedge \mathfrak{B}(Z)} & \mathfrak{B}(X \wedge Y) \wedge \mathfrak{B}(Z) \\ {\scriptstyle \mathfrak{B}(X) \wedge \mu_{Y,Z}} \downarrow & & \downarrow {\scriptstyle \mu_{X \wedge Y, Z}} \\ \mathfrak{B}(X) \wedge \mathfrak{B}(Y \wedge Z) & \xrightarrow{\mu_{X, Y \wedge Z}} & \mathfrak{B}(X \wedge Y \wedge Z), \end{array} \tag{3.3}$$

and

$$\begin{array}{ccccc} X \wedge \mathfrak{B}(Y) & \xrightarrow{\iota_X \wedge \mathfrak{B}(Y)} & \mathfrak{B}(X) \wedge \mathfrak{B}(Y) & \xrightarrow{\mu_{X,Y}} & \mathfrak{B}(X \wedge Y) \\ {\scriptstyle t} \downarrow & & & & \downarrow {\scriptstyle \mathfrak{B}(t)} \\ \mathfrak{B}(Y) \wedge X & \xrightarrow{\mathfrak{B}(Y) \wedge \iota_X} & \mathfrak{B}(Y) \wedge \mathfrak{B}(X) & \xrightarrow{\mu_{Y,X}} & \mathfrak{B}(Y \wedge X), \end{array} \tag{3.4}$$

where t is transposition of smash product factors. $\mathfrak{B}$ *is* **commutative** *if the following diagram also commutes.*

$$\begin{array}{ccc} \mathfrak{B}(X) \wedge \mathfrak{B}(Y) & \xrightarrow{\mu_{X,Y}} & \mathfrak{B}(X \wedge Y) \\ {\scriptstyle t} \downarrow & & \downarrow {\scriptstyle \mathfrak{B}(t)} \\ \mathfrak{B}(Y) \wedge \mathfrak{B}(X) & \xrightarrow{\mu_{Y,X}} & \mathfrak{B}(Y \wedge X), \end{array} \tag{3.5}$$

The adjoint **stabilization and costabilization maps** σ_X *and* η_X *are defined by*

$$\begin{array}{ccccc} S^1 \wedge \mathfrak{B}(X) & \xrightarrow{\iota_{S^1}\wedge \mathfrak{B}(X)} & \mathfrak{B}(S^1) \wedge \mathfrak{B}(X) & \xrightarrow{\mu_{S^1,X}} & \mathfrak{B}(S^1 \wedge X) \\ \| & & & & \| \\ \Sigma\mathfrak{B}(X) & & \xrightarrow{\sigma_X} & & \mathfrak{B}(\Sigma X) \\ & & \perp & & \\ \mathfrak{B}(X) & & \xrightarrow[\eta_X]{} & & \Omega\mathfrak{B}(\Sigma X) \end{array} \tag{3.6}$$

We require in addition that $\mathfrak{B}$ *preserves connectivity and that* $\pi_{n+i}\mathfrak{B}(\Sigma^n X)$ *is independent of* n *for* $n \gg i$.

Definition 3.7. *A Bökstedt functor* $\mathfrak{B}$ *is* **cofibrant** *if for each CW-complex* X, $\mathfrak{B}(X)$ *is again a CW-complex, and the stabilization map* σ_X *is a cofibration.*

Nothing like this last definition is found in [BHM93], where is no discussion of model structures. They are only considered in subsequent work on cyclotomic spectra as orthogonal spectra, starting with that of Blumberg and Mandell [BM12]. The condition of Definition 3.7 insures that the spectrum $\mathfrak{B}(\mathbb{S})$ is cofibrant in the stable projective model structure on the categories of sequential, symmetric and orthogonal spectra of Definitions 4.1 and 4.5. The cofibrancy condition is the subject of [HHR21, Corollary 7.1.37], and we will need it in Theorem 3.12. There is a concise review of the homotopy theory of (meaning model structures in) the categories of associative and commutative ring spectra in [ABG⁺18, §2.3] by Blumberg, Mandell, Vigleik Angeltveit, Teena Gerhardt, Mike Hill and Tyler Lawson.

Example 3.8. Some Bökstedt functors.

(i) *For a group-like topological monoid* Γ, *let* $\mathfrak{B}_\Gamma := (- \wedge \Gamma_+)$, *making* $\mathfrak{B}_\Gamma(\mathbb{S})$ *the suspension spectrum for* Γ_+. *It is denoted by* $\underline{\underline{\Gamma}}$ *in* [BHM93, Example 3.2 (i)]. *When* Γ *is a single point,* $\mathfrak{B}_\Gamma$ *is the identity functor* Id. *For a pointed space* X, *the Moore loop space* $\Omega^M X$ *(see* [CM95, 5.1]*), which we will denote simply by* ΩX, *is such a monoid.* $\mathfrak{B}_\Gamma$ *is commutative as when* Γ *is. We will say that a Bökstedt functor of this form is* **monoidal**. *It is cofibrant as in Definition 3.7 if* Γ *is a CW-complex.*

(ii) *For a discrete ring* R, *let* $\mathfrak{B}_R$ *be the Bökstedt functor defined by*

$$\mathfrak{B}_R(X) := |R\mathrm{Sing}_\bullet(X)/R\mathrm{Sing}_\bullet(\mathrm{pt})|,$$

where $\mathrm{Sing}_\bullet$ *is as in Definition 2.31 and* $R\mathrm{Sing}_\bullet(-)$ *denotes the free simplicial* R*-module on the indicated simplicial set. It is denoted by* $\underline{\underline{R}}$ *in* [BHM93, Example 3.2 (iii)]. *This is a generalized Eilenberg-MacLane space with*

$$\pi_*\mathfrak{B}_R(X) \cong \overline{H}_*(X; R).$$

This means the spectrum $\mathfrak{B}_R(\mathbb{S})$ *is the Eilenberg-MacLane spectrum for* R. $\mathfrak{B}_R$ *is commutative as in (3.5) when* R *is commutative. It is cofibrant for any* R.

(iii) *For a Bökstedt functor* $\mathfrak{B}$ *and integer* $m > 0$, *we can form the corresponding* **matrix Bökstedt functor**

$$\mathcal{M}_m(\mathfrak{B})(X) := \mathrm{Map}(\langle m\rangle, \langle m\rangle \wedge \mathfrak{B}(X)),$$

where $\langle m\rangle := \{1,\dots,m\}$. A point in this mapping space is an (m^2)-tuple of points in $\mathfrak{B}(X)$.

We need to specify the natural transformations $\mu(\mathcal{M}_m(\mathfrak{B}))$ and $\iota(\mathcal{M}_m(\mathfrak{B}))$. For $\alpha_1 \in \mathcal{M}_m(\mathfrak{B})(X)$ and $\alpha_2 \in \mathcal{M}_m(\mathfrak{B})(Y)$, the composite

$$\begin{array}{ccc} \langle m\rangle \xrightarrow{\alpha_2} \langle m\rangle \wedge \mathfrak{B}(Y) & & \\ \downarrow{\scriptstyle \alpha_1\wedge\mathfrak{B}(Y)} & & \\ \langle m\rangle \wedge \mathfrak{B}(X)\wedge\mathfrak{B}(Y) & \xrightarrow{\langle m\rangle\wedge\mu_{X,Y}} & \langle m\rangle\wedge\mathfrak{B}(X\wedge Y), \end{array}$$

which lives in $\mathcal{M}_m(\mathfrak{B})(X\wedge Y)$, behaves like matrix multiplication. This can be reinterpreted as a map

$$\mu(\mathcal{M}_m(\mathfrak{B}))_{X,Y} : \mathcal{M}_m(\mathfrak{B})(X)\wedge\mathcal{M}_m(\mathfrak{B})(Y) \to \mathcal{M}_m(\mathfrak{B})(X\wedge Y)$$

as in Definition 3.1, with $\iota(\mathcal{M}_m(\mathfrak{B}))_X : X \to \mathcal{M}_m(\mathfrak{B})(X)$ given by

$$(\iota(\mathcal{M}_m(\mathfrak{B}))_X(x))(i) := i\wedge\iota(\mathfrak{B})_X(x) \qquad \text{for } x\in X \text{ and } i\in\langle m\rangle.$$

It is cofibrant when $\mathfrak{B}$ is.

Definition 3.9. [BHM93, (3.4)]

(i) *The* **topological Hochschild homology** $\mathrm{THH}(\mathfrak{B})$ **of a Bökstedt functor** $\mathfrak{B}$ *is the geometric realization of the simplicial space $\mathrm{THH}_\bullet(\mathfrak{B})$ defined by*

$$\mathrm{THH}_n(\mathfrak{B}) := \operatorname*{hocolim}_k \Omega^{|k|}\left(\mathfrak{B}(S^{k_0})\wedge\cdots\wedge\mathfrak{B}(S^{k_n})\right), \tag{3.10}$$

where the colimit runs over all $(n+1)$-tuples of nonnegative integers

$$k = (k_0,\dots,k_n) \qquad \text{with} \qquad |k| := k_0+\cdots+k_n,$$

and the maps are induced by the costabilization map of (3.6). The face and degeneracy maps are defined as in (2.32), with the unit 1 being replaced by S^0.

(ii) *For a pointed space X, we define $\mathrm{THH}^X_\bullet(\mathfrak{B})$ similarly by*

$$\mathrm{THH}^X_n(\mathfrak{B}) := \operatorname*{hocolim}_k \Omega^{|k|}\left(\mathfrak{B}(S^{k_0})\wedge\cdots\wedge\mathfrak{B}(S^{k_n})\wedge X\right).$$

(iii) *The simplicial space $\mathrm{THH}_\bullet(\mathfrak{B})$ is a simplicial infinite loop space, and we denote the corresponding spectrum by $\mathrm{tHH}(\mathfrak{B})$. The spectrum associated with $\mathrm{THH}^X_\bullet(\mathfrak{B})$ is $\mathrm{tHH}(\mathfrak{B})\wedge X$.*

(iv) *When $\mathfrak{B} = \mathfrak{B}_R$ for a discrete ring R as in Example 3.8(ii), we denote the spectrum $\mathrm{tHH}(\mathfrak{B}_R)$ by $\mathrm{THH}(R)$. It is denoted by $T(R)$ in* [BM94, §4].

Here we are replacing the associative ring A of §2.1 by the associative ring spectrum $\mathfrak{B}(\mathbb{S})$. There is a spectral sequence converging to $H_*\mathrm{tHH}(\mathfrak{B})\wedge X$ (with field coefficients) whose input is $\mathrm{HH}_*(H_*\mathfrak{B}(\mathbb{S})\wedge X)$.

Proposition 3.11. THH **and the free loop space.**[BHM93, Proposition 3.7 for $r=1$] *For $\mathfrak{B}_{\Omega X}$ as in Example 3.8(i), we have*

$$\mathrm{tHH}(\mathfrak{B}_{\Omega X}) \simeq \Sigma^\infty_+(\mathcal{L}X).$$

$\mathrm{THH}_\bullet(\mathfrak{B})$ is also a cyclic space as in Definition 2.40, which means that the space $\mathrm{THH}(\mathfrak{B})$ and the spectrum $\mathrm{tHH}(\mathfrak{B})$ each have a $\mathbb{T}$-action.

The following is [BHM93, Proposition 3.7] in case of monoidal $\mathfrak{B}$ (as in Example 3.8(i)) and can be deduced from [ABG+16] for $\mathfrak{B}$ cofibrant as in Definition 3.7.

Theorem 3.12. The action of C_r on THH. *For each monoidal or cofibrant Bökstedt functor $\mathfrak{B}$ and each integer $r \geqslant 1$, the simplicial C_r-space $\mathrm{sd}_r^*\mathrm{THH}_\bullet^X(\mathfrak{B})$ is*

$$\mathrm{sd}_r^*\mathrm{THH}_n^X(\mathfrak{B}) = \operatorname*{hocolim}_k \Omega^{|k|\varrho_r}\left(\mathfrak{B}(S^{k_0})^{\wedge r} \wedge \cdots \wedge \mathfrak{B}(S^{k_n})^{\wedge r} \wedge X\right), \tag{3.13}$$

where ϱ_r denotes the real regular representation of the cyclic group C_r, $\Omega^{k\varrho_r}(-)$ denotes the twisted loop space of Definition A.15, and C_r permutes the factors of each of the the r-fold smash powers of the spaces $\mathfrak{B}(S^{k_i})$.

The following is not proved in [BHM93], so we will sketch a proof here.

Proposition 3.14. Fixed point sets. *For a monoidal or cofibrant Bökstedt functor $\mathfrak{B}$ and each subgroup $\mathrm{C}_q \subseteq \mathrm{C}_r$, the fixed point set $(\mathrm{sd}_r^*\mathrm{THH}^X(\mathfrak{B}))^{\mathrm{C}_q}$ has a residual action (Definition A.1) of $\mathrm{C}_{r/q}$ and is equivariantly homeomorphic to $\mathrm{sd}_{r/q}^*\mathrm{THH}^X(\mathfrak{B})$ with the action of Theorem 3.12.*

Proof. We will do this for $q = r$, leaving the general case to the reader. The space of (3.13) is a filtered colimit of C_r-spaces. Filtered colimits commute with finite limits, so it suffices to determine the colimit of fixed point sets. For each k we have a space of nonequivariant maps between C_r-spaces on which C_r acts by conjugation. Its fixed point set is the corresponding space of equivariant maps. This is the image of the kth space of (3.10) under the norm map that sends a pointed space X to its r-fold smash product with C_r permuting the factors. It follows that

$$(\mathrm{sd}_r^*\mathrm{THH}_n^X(\mathfrak{B}))^{\mathrm{C}_r} = \mathrm{THH}_n^X(\mathfrak{B}),$$

and similarly

$$(\mathrm{sd}_r^*\mathrm{THH}_n^X(\mathfrak{B}))^{\mathrm{C}_q} = \mathrm{sd}_{r/q}^*\mathrm{THH}_n^X(\mathfrak{B}). \qquad \square$$

Corollary 3.15. Smooth cyclotomy of THH. *For each monoidal or cofibrant Bökstedt functor $\mathfrak{B}$, $\mathrm{THH}(\mathfrak{B})$ is smoothly cyclotomic as in Definition 1.4.*

Example 3.16. THH of the identity functor. [Mad94b, Example 2.8] *It follows from Definition 3.9 that $\mathrm{Id}(\mathbb{S}) = \mathfrak{B}_{\mathrm{pt}}(\mathbb{S}) \simeq \mathbb{S}$. The identification of its fixed point sets under finite cyclic p-groups is more interesting. We have*

$$\mathrm{THH}(\mathrm{Id})^{\mathrm{C}_{p^n}} \simeq \prod_{i=0}^{n} Q_+(B\mathrm{C}_{p^i}),$$

where

$$Q_+(X) := \operatorname*{hocolim}_j \Omega^j\Sigma^j(X_+). \tag{3.17}$$

The maps F and R of Definition A.1 are given by

$$\begin{aligned} &\mathrm{F}(x_0,\ldots,x_n) = (x_0 + T(x_1), T(x_2),\ldots,T(x_n)) \\ \textit{and}\qquad &\mathrm{R}(x_0,\ldots,x_n) = (x_0,\ldots,x_{n-1}) \end{aligned}$$

where $T : Q_+(B\mathrm{C}_{p^i}) \to Q_+(B\mathrm{C}_{p^{i-1}})$ is the transfer mapping associated to the degree p covering $B\mathrm{C}_{p^{i-1}} \to B\mathrm{C}_{p^i}$.

It follows that the spectrum $\mathrm{tHH}(\mathrm{Id})$ is the sphere spectrum $\mathbb{S}$.

Theorem 3.18. **THH of $\mathbb{Z}$ and $\mathbb{Z}/p$.** [Bök85b, Theorem 1.1] *For a discrete ring R, let $\mathfrak{B}_R$ be as in Example 3.8(ii).*

(i) *For each prime p,*

$$\mathrm{THH}(\mathfrak{B}_{\mathbb{Z}/p}) \cong \prod_{i=0}^{\infty}{}' K(\mathbb{Z}/p, 2i)$$
$$\text{and } \pi_*\mathrm{THH}(\mathfrak{B}_{\mathbb{Z}/p}) = \mathbb{Z}/p[b] \text{ with } b \in \pi_2,$$

where $\prod'$ *denotes the restricted product, meaning the colimit of finite products.*

(ii)

$$\mathrm{THH}(\mathfrak{B}_{\mathbb{Z}}) \cong \mathbb{Z} \times \prod_{i=1}^{\infty}{}' K(\mathbb{Z}/i, 2i-1),$$

(iii) [BCS10, Theorems 1.3 and 1.4] $\mathrm{THH}(\mathbb{Z}/p)$ *and* $\mathrm{THH}(\mathbb{Z})$ *(as in Definition 3.9(iv)) are the corresponding Eilenberg-MacLane spectra with*

$$\mathrm{THH}(\mathbb{Z}/p) \simeq H\mathbb{Z}/p \otimes \Omega S^3_+$$
$$\text{and} \qquad \mathrm{THH}(\mathbb{Z}) \simeq H\mathbb{Z} \otimes \tau_{\geqslant 3}\Omega S^3_+.$$

Here $\tau_{\geqslant 3}\Omega S^3$ *(denoted by* $\Omega(S^3\langle 3\rangle)$ *in* [BCS10]*) denotes the 3-connective cover of* ΩS^3*, which is the fiber of the evident map to* $K(\mathbb{Z}, 2)$*. A classical calculation involving the Serre spectral sequence for the fiber sequence*

$$S^1 \to \tau_{\geqslant 3}\Omega S^3 \to \Omega S^3$$

(see [Rav86, Lemma 1.2.3] *for a closely related computation) shows that its integer homology coincides with the homotopy groups of (ii).*

(iv) *The map between the these spaces induced by* $\mathbb{Z} \to \mathbb{Z}/p$ *preserves the product decomposition, is the obvious reduction for* $i = 0$*, is trivial for* i *not divisible by* p*, and is the Bockstein* $K(\mathbb{Z}/pj, 2pj-1) \to K(\mathbb{Z}/p, 2pj)$ *for* $i = pj$ *with* $j > 0$*.*

(v) *In the decomposition of (i), let* $\iota_{2i} \in H^{2i}(K(\mathbb{Z}/p, 2i); \mathbb{Z}/p)$ *be the fundamental class. Then the coproduct in cohomology is given by*

$$\iota_{2i} \mapsto \sum_{0 \leqslant k \leqslant i} \iota_{2k} \otimes \iota_{2i-2k},$$

which is dual to the multiplication of (i).

(vi) *In the decomposition of (ii), let*

$$\iota_{2pj-1} \in H^{2pj-1}(K(\mathbb{Z}/pj, 2pj-1); \mathbb{Z}/p)$$

be the fundamental class. Then the coproduct in cohomology is given by

$$\iota_{2pj-1} \mapsto \iota_{2pj-1} \otimes 1 + 1 \otimes \iota_{2pj-1} + \sum_{0<k<j} \beta(\iota_{2pk-1} \otimes \iota_{2p(j-k)-1}),$$

where $\beta : H^{2pj-2} \to H^{2pj-1}$ *denotes the mod p Bockstein.*

In §5.9 we will describe the cyclotomic structure (Definition 5.23) of the spectrum $\mathrm{THH}(\mathbb{Z}/p)$. The above is merely a description of its underlying homotopy type. We will see that the action of $\mathbb{T}$ on it is nontrivial.

Here is a sketch of a proof of Theorem 3.18 due to Blumberg, Ralph Cohen and Christian Schlichtkrull [BCS10], which we learned from Tomer Schlank. It is also treated by Krause and Nikolaus in [KN21a, §4], who in addition give a description of $\mathrm{THH}(\mathbb{Z}/p^j)$ for all $j > 0$, and by Joseph Hlavinka in [HYn24, §3].

For any space X, one has

$$\mathcal{L}\Omega X \simeq \Omega^2 X \times \Omega X.$$

Thus for $X = S^3$, we have $\mathcal{L}\Omega S^3 \simeq \Omega^2 S^3 \times \Omega S^3$. A theorem of Mahowald [Mah79] says there is a map $\Omega^2 S^3 \to BO$ for which the Thom spectrum is $H\mathbb{F}_2$. It is the double loop functor applied to a map $S^3 \to B^3 O$. Applying the functor $\mathcal{L}\Omega$ gives a map

$$\mathcal{L}\Omega S^3 \longrightarrow \mathcal{L}\Omega B^3 O \simeq BO \times BBO \xrightarrow{p_1} BO.$$

An odd primary analog due to Mike Hopkins can be found in [MRS01a, Lemma 3.3].

Let $Y = \tau_{\geqslant 4} S^3$ (sometimes denoted by $S^3\langle 3\rangle$) be the 4-connective (or 3-connected) cover of S^3, the fiber of the map $S^3 \to K(\mathbb{Z}, 3)$. We know that the Thom spectrum associated with $\Omega^2 Y$ is $H\mathbb{Z}$. By studying the Serre spectral sequence for the fiber sequence

$$S^1 \longrightarrow \Omega Y \longrightarrow \Omega S^3,$$

one can show

$$H_i(\Omega Y; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ \mathbb{Z}/m & \text{for } i = 2m-1 \text{ and } m > 1 \\ 0 & \text{otherwise,} \end{cases}$$

and this leads to Theorem 3.18(ii).

Definition 3.19. *The* **algebraic K-theory of a Bökstedt functor $\mathfrak{B}$.** [Bök85a, Definition 2.3]. *Let $\mathrm{GL}_m(\mathfrak{B})$ be the union of the invertible components of*

$$\operatorname*{hocolim}_k \Omega^k \mathcal{M}_m(\mathfrak{B})(S^k),$$

where $\mathcal{M}_m(\mathfrak{B})$ is the matrix Bökstedt functor of Example 3.8(iii). It is a group-like topological monoid, and

$$\mathrm{K}(\mathfrak{B}) := \Omega_0 B\left(\coprod_m B\mathrm{GL}_m(\mathfrak{B})\right) = \Omega_0 B\left(\coprod_m |N_\bullet \mathrm{GL}_m(\mathfrak{B})|\right),$$

where Ω_0 on the right indicates the degree zero component of the indicated loop space, in which π_0 is $\mathbb{Z}$.

As in Quillen's original definition of algebraic K-theory, the space on the right is equivalent to $B\mathrm{GL}(\mathfrak{B})^+$, the plus construction on the filtered homotopy colimit of the spaces $B\mathrm{GL}_m(\mathfrak{B})$ for $m \geqslant 0$, which is described by Weibel in [Wei13, IV.1].

Example 3.20. The K-theories of Quillen and Waldhausen.

(i) For $\mathfrak{B}_R$ as in Example 3.8(ii), $\pi_*\mathrm{K}(\mathfrak{B}_R) = \mathrm{K}_*(R)$ as defined by Quillen in [Qui73]. *This is discussed in* [Mad94a, §2.6].

(ii) For $\mathfrak{B}_{\Omega X}$ as in Example 3.8(i), $\mathrm{K}(\mathfrak{B}_{\Omega X})$ is Waldhausen's space $A(X)$ as in [Wal85]. *This is discussed in* [Mad94b, §6].

Definition 3.21. *The* **cyclic K-theory of a Bökstedt functor $\mathfrak{B}$.** [BHM93, (5.7)].

$$\mathrm{K}^{\mathrm{cyc}}(\mathfrak{B}) := \Omega_0 B\left(\coprod_m |N_\bullet^{\mathrm{cyc}} \mathrm{GL}_m(\mathfrak{B})|\right),$$

where Ω_0 on the right indicates the degree zero component of the indicated loop space.

3.2. **The Dennis trace.** The topological Dennis trace $\mathrm{Tr} : \mathrm{K}(\mathfrak{B}) \to \mathrm{THH}(\mathfrak{B})$, was first defined in algebraic form by Keith Dennis in an unpublished paper in 1976. For a topological monoid Γ, we have simplicial sets $N_\bullet(\Gamma)$ and $N_\bullet^{\mathrm{cyc}}(\Gamma)$ as in Definitions 2.25 and 2.39. There is a map

$$I_\bullet : N_\bullet(\Gamma) \to N_\bullet^{\mathrm{cyc}}(\Gamma) \quad \text{with } I(\gamma_1, \dots, \gamma_n) = \left((\gamma_1, \cdots, \gamma_n)^{-1}, \gamma_1, \dots, \gamma_n\right) \tag{3.22}$$

whose geometric realization is homotopic to the inclusion of $B\Gamma$ into the free loop space $\mathcal{L}B\Gamma$ as the constant loops.

For $\Gamma = \mathrm{GL}_m(\mathfrak{B})$ as in Definition 3.19, we denote the map of (3.22) by $I_\bullet^{(m)}$. Consider the composite

$$N_\bullet \mathrm{GL}_m(\mathfrak{B}) \xrightarrow{I_\bullet^{(m)}} N_\bullet^{\mathrm{cyc}} \mathrm{GL}_m(\mathfrak{B}) \xrightarrow{S_\bullet^{(m)}} \mathrm{THH}_\bullet(\mathcal{M}_m(\mathfrak{B}))$$

where

$$S_n^{(m)}(f_0, \dots, f_n) = f_0 \wedge \cdots \wedge f_n \qquad \text{for } f_i : S^{k_i} \to \mathcal{M}_m(\mathfrak{B})(S^{k_i}). \tag{3.23}$$

Passing to geometric realizations gives a map

$$S^{(m)} I^{(m)} : B\mathrm{GL}_m(\mathfrak{B}) \to \mathrm{THH}(\mathcal{M}_m(\mathfrak{B})). \tag{3.24}$$

The following analog of Theorem 2.13 was proved in [Bök85a].

Theorem 3.25. Morita invariance of Bökstedt functors. *For each Bökstedt functor $\mathfrak{B}$, there is an equivalence*

$$\mathrm{Trace} : \Omega B \left(\coprod_{m \geqslant 0} \mathrm{THH}(M_m(\mathfrak{B})) \right) \to \mathrm{THH}(\mathfrak{B}) \times \mathbb{Z}.$$

Definition 3.26. The Dennis trace of a Bökstedt functor $\mathfrak{B}$ *is the following composite.*

$$\begin{array}{ccc} \Omega_0 B \left(\coprod_m B\mathrm{GL}_m(\mathfrak{B})\right) & \xrightarrow{\Omega_0 B\left(\coprod_m S^{(m)} I^{(m)}\right)} & \Omega_0 B \left(\coprod_m \mathrm{THH}(\mathcal{M}_m(\mathfrak{B}))\right) \\ \| & & \simeq \downarrow \mathrm{Trace} \\ \mathrm{K}(\mathfrak{B}) & & \mathrm{THH}(\mathfrak{B}), \end{array}$$

where each of the coproducts is over $m \geqslant 0$, and the equivalence on the right is that of Theorem 3.25.

3.3. **The cyclotomic trace.** Now we need to consider the action of $\mathbb{T}$ and its finite subgroups C_r for $r \geqslant 1$ on the cyclic space $\mathrm{THH}(\mathfrak{B})$ of Definition 3.9.

First we need some elementary definitions.

We will now make use of the edgewise subdivision of Definition 2.55. Recall that the geometric realization of r-fold edgewise subdivision of a simplicial set X is homeomorphic to $|X|$ by Lemma 2.58. Thus we get a residual action map as in Definition A.1(i),

$$\mathrm{R}_p := \mathrm{R}_{\mathrm{C}_p}^{\mathrm{C}_{p^n}} : (\mathrm{sd}_{p^n}^* \mathrm{THH}(\mathfrak{B}))^{\mathrm{C}_{p^n}} \to ((\mathrm{sd}_{p^n}^* \mathrm{THH}(\mathfrak{B}))^{\mathrm{C}_p})^{\mathrm{C}_{p^n/p}}.$$

Recall that

$$(\mathrm{sd}_{p^n}^* \mathrm{THH}(\mathfrak{B}))^{\mathrm{C}_p} \simeq \mathrm{sd}_{p^{n-1}}^* \mathrm{THH}(\mathfrak{B})$$

by Proposition 3.14. The map $\mathrm{R}_p := \mathrm{R}_p^{p^n}$ figures in the original definition of $\mathrm{TC}(\mathfrak{B}, p)$, [BHM93, Definition 5.12 (i)], where it is denoted by Φ_p.

Since $\mathrm{sd}_{p^n}^* \mathrm{THH}(\mathfrak{B}) \approx \mathrm{THH}(\mathfrak{B})$, we have

$$\mathrm{R}_p : \mathrm{THH}(\mathfrak{B})^{\mathrm{C}_{p^n}} \to \mathrm{THH}(\mathfrak{B})^{\mathrm{C}_{p^n/p}}.$$

We can also regard $\mathrm{C}_{p^n/p}$ as a subgroup of C_{p^n}, yielding a restriction map as in Definition A.1(ii).

$$\mathrm{F}_p := \mathrm{F}_{\mathrm{C}_{p^n/p}}^{\mathrm{C}_{p^n}} : \mathrm{THH}(\mathfrak{B})^{\mathrm{C}_{p^n}} \to \mathrm{THH}(\mathfrak{B})^{\mathrm{C}_{p^n/p}},$$

which is denoted by D in [BHM93, (5.10)].

Thus we get maps

$$\mathrm{R}, \mathrm{F} : \mathrm{THH}(\mathfrak{B})^{\mathrm{C}_{p^n}} \to \mathrm{THH}(\mathfrak{B})^{\mathrm{C}_{p^{n-1}}} \tag{3.27}$$

as in Definition A.1 for which the diagram

$$\begin{array}{ccc} \mathrm{THH}(\mathfrak{B})^{\mathrm{C}_{p^{n+1}}} & \xrightarrow[\cong]{\mathrm{R}} & \mathrm{THH}(\mathfrak{B})^{\mathrm{C}_{p^n}} \\ {\scriptstyle \mathrm{F}}\downarrow & & \downarrow{\scriptstyle \mathrm{F}} \\ \mathrm{THH}(\mathfrak{B})^{\mathrm{C}_{p^n}} & \xrightarrow[\cong]{\mathrm{R}} & \mathrm{THH}(\mathfrak{B})^{\mathrm{C}_{p^{n-1}}} \end{array} \tag{3.28}$$

commutes as a special case of (A.3).

These induce maps of filtered homotopy limits

$$\begin{array}{c} \operatorname{holim}_{\mathrm{R}} \mathrm{THH}(\mathfrak{B})^{\mathrm{C}_{p^n}} \underset{1}{\overset{\mathrm{F}}{\rightrightarrows}} \operatorname{holim}_{\mathrm{R}} \mathrm{THH}(\mathfrak{B})^{\mathrm{C}_{p^n}} \\ \operatorname{holim}_{\mathrm{F}} \mathrm{THH}(\mathfrak{B})^{\mathrm{C}_{p^n}} \underset{1}{\overset{\mathrm{R}}{\rightrightarrows}} \operatorname{holim}_{\mathrm{F}} \mathrm{THH}(\mathfrak{B})^{\mathrm{C}_{p^n}}, \end{array} \tag{3.29}$$

and these two diagrams have equivalent equalizers.

Definition 3.30. [BHM93, Definition 5.12]

(i) The **topological cyclic homology at** p **of the Bökstedt functor** $\mathfrak{B}$*,* $\mathrm{TC}(\mathfrak{B}, p)$*, is the equalizer of either diagram of (3.29). Equivalently it is the homotopy limit of the diagram*

$$\cdots \underset{\mathrm{F}}{\overset{\mathrm{R}}{\rightrightarrows}} \mathrm{THH}(\mathfrak{B})^{\mathrm{C}_{p^2}} \underset{\mathrm{F}}{\overset{\mathrm{R}}{\rightrightarrows}} \mathrm{THH}(\mathfrak{B})^{\mathrm{C}_p} \underset{\mathrm{F}}{\overset{\mathrm{R}}{\rightrightarrows}} \mathrm{THH}(\mathfrak{B}).$$

(ii) The **cyclotomic trace at** p *is the infinite loop map*

$$\mathrm{K}(\mathfrak{B}) \xrightarrow{I} (\operatorname{holim}_{\mathrm{F}} \mathrm{K}^{\mathrm{cyc}}(\mathfrak{B})^{\mathrm{C}_{p^n}})^{h\mathrm{R}} \xrightarrow{S} \mathrm{TC}(\mathfrak{B}, p), \qquad \mathrm{Trc} = S \circ I$$

where $\mathrm{K}^{\mathrm{cyc}}(\mathfrak{B})$ *is as in Definition 3.21, and the maps* I *and* S *are as in (3.22) and (3.23). The middle object above is the homotopy limit of the diagram*

$$\cdots \underset{\mathrm{F}}{\overset{\mathrm{R}}{\rightrightarrows}} \mathrm{K}^{\mathrm{cyc}}(\mathfrak{B})^{\mathrm{C}_{p^2}} \underset{\mathrm{F}}{\overset{\mathrm{R}}{\rightrightarrows}} \mathrm{K}^{\mathrm{cyc}}(\mathfrak{B})^{\mathrm{C}_p} \underset{\mathrm{F}}{\overset{\mathrm{R}}{\rightrightarrows}} \mathrm{K}^{\mathrm{cyc}}(\mathfrak{B}).$$

This is the first of several definitions of TC. In Definition 5.54 it is defined as a mapping object in the ∞-category of cyclotomic spectra; see Remark 5.55. A new formula for computing it is given in Theorem 5.57. A formula relating it to TR is given in Theorem 5.88.

Remark 3.31. TC **of a free loop space.** *We have homeomorphisms*

$$\varphi_p^{-1} : \mathcal{L}X^{C_{p^n}} \to \rho_p^* \mathcal{L}X^{C_{p^{n+1}}}$$

for φ_p as in Example 1.5, induced by the pth power map on $\mathbb{T}$. If we replace $\mathrm{THH}(\mathfrak{B})$ *by $\mathcal{L}X$ in (3.28), the horizontal maps become homeomorphisms and*

$$\operatorname*{holim}_{\mathrm{R}} \mathcal{L}X^{C_{p^n}} = \lim_{\mathrm{R}} \mathcal{L}X^{C_{p^n}} \cong \mathcal{L}X.$$

Then the first equalizer of the analog of (3.29) would consist only of constant loops.

The other limit,

$$\lim_{\mathrm{F}} \mathcal{L}X^{C_{p^n}},$$

is the space constant loops, on which R *is the identity. Thus the free loop space analog of* TC *appears to be the space of constant loops, so* TC seems to undo the free loop functor $\mathcal{L}$.

4. Spectra and equivariant stable homotopy theory

A **spectrum** X was originally defined to be a sequence of pointed spaces X_n for $n \geqslant 0$ with **structure maps** $\epsilon_n^X : \Sigma X_n \to X_{n+1}$, or equivalently **costructure maps** $\eta_n^X : X_n \to \Omega X_{n+1}$. Since this is only one of several ways to define spectra these days, we will call such objects **sequential spectra**.

A sequential spectrum in which each structure map is a weak equivalence is called a **suspension spectrum**, and is often denoted by $\Sigma^\infty X_0$. One in which each costructure map is a weak equivalence is called an **Ω-spectrum**. The objects of Lurie's ∞-category of spectra (see [Rav23, §9] and [Lur17, Definition 1.4.3.1]) are sequential Ω-spectra, so we will sometimes refer to them as **Lurie spectra**.

If each X_n has a G-action for some compact Lie group G, with each structure map being equivariant (where ΣX_n and ΩX_n are understood to be the ordinary suspension and loop space of the pointed G-space X_n), the resulting object is called a *sequential spectrum with G-action*. This means that for each closed subgroup $H \subseteq G$ one gets a spectrum X^H in which the nth space is the fixed point set X_n^H. This notion is different from that of a **G-spectrum**, which we will discuss below.

A **map of sequential spectra (with G-action)** $f : X \to Y$ is a collection of (G-equivariant) pointed maps $f_n : X_n \to Y_n$ compatible with the structure maps for X and Y. Such a map is said to be a (G-equivariant) **stable equivalence** if it induces an isomorphism of stable homotopy groups (of fixed point sets for each closed subgroup $H \subseteq G$) as in (4.2). This condition is weaker than requiring each map f_n to be a (G-equivariant) equivalence.

Definition 4.1. *We denote the category of sequential spectra by* $\mathrm{Sp}_{\mathrm{seq}}$*, and the category of those with G-action by* $\mathrm{Sp}_{\mathrm{seq}}^{BG}$.

A sequential spectrum has stable homotopy (or simply homotopy) and reduced homology groups defined by

$$\begin{aligned} \pi_i X &:= \operatorname*{colim}_n \pi_i \Omega^n X_n = \operatorname*{colim}_n \pi_{n+i} X_n \\ \text{and} \qquad H_i X &:= \operatorname*{colim}_n H_{n+i} X_n. \end{aligned} \tag{4.2}$$

When X has a G-action, we can make similar definitions for each closed subgroup $H \subseteq G$ in terms of fixed point sets, the definitions above being those for the trivial subgroup. The definition does not require X_n to be $(n-1)$-connected, so the groups of (4.2) could be nonzero for $i < 0$.

This was our only definition until the late 80's. It suffered from the lack of a convenient way to define the smash product of two spectra.

A sequential spectrum can be reinterpreted as a $\mathcal{T}$-valued functor on a certain $\mathcal{T}$-enriched indexing category $\mathscr{J}^{\mathbf{N}}$ with one object for each finite set $\mathbf{n} = \{1, 2, \ldots n\}$ for $n \geqslant 0$ with $\mathbf{0}$ being the empty set $\varnothing$. Its morphism spaces are

$$\mathscr{J}^{\mathbf{N}}(\mathbf{m}, \mathbf{n}) = \begin{cases} S^{n-m} & \text{for } 0 \leqslant m \leqslant n \\ \mathrm{pt} & \text{for } m > n \geqslant 0 \end{cases} \tag{4.3}$$

For more on this point of view, see [HHR21, Chapter 7], specifically [HHR21, Definition 7.2.4]. From this perspective, the difficulty with the smash product is related to the absence of a symmetric monoidal structure on $\mathscr{J}^{\mathbf{N}}$. See [HHR21, Remark 7.2.11] for more on this lack of symmetry.

One solution is to define a new indexing category which *is* symmetric monoidal. Two such categories are $\mathscr{J}^{\Sigma}$ and $\mathscr{J}^{O}$; we will often omit the superscript in the latter. They have the same objects as $\mathscr{J}^{\mathbf{N}}$ with morphism spaces

$$\begin{aligned} \mathscr{J}^{\Sigma}(\mathbf{m}, \mathbf{n}) &= \begin{cases} \Sigma_{n+} \underset{\Sigma_{n-m}}{\wedge} S^{n-m} & \text{for } 0 \leqslant m \leqslant n \\ \mathrm{pt} & \text{for } m > n \geqslant 0 \end{cases} \\ \mathscr{J}^{O}(\mathbf{m}, \mathbf{n}) &= \begin{cases} O(n)_{+} \underset{O_{n-m}}{\wedge} S^{n-m} & \text{for } 0 \leqslant m \leqslant n \\ \mathrm{pt} & \text{for } m > n \geqslant 0, \end{cases} \end{aligned} \tag{4.4}$$

where Σ_{n+} and $O(n)_{+}$ denote the symmetric and orthgonal groups with disjoint basepoint.

Definition 4.5. Symmetric and orthogonal spectra. *The categories of $\mathcal{T}$-valued functors on the indexing categories of (4.4) are those of* **symmetric spectra** *of* [HSS00]*, which we denote by* Sp^{Σ}*, and* **orthogonal spectra** *of* [MM02]*, which we denote by* Sp^{O}*, respectively.*

Proposition 4.6. *There are forgetful functors* $\mathrm{Sp}^{O} \to \mathrm{Sp}^{\Sigma} \to \mathrm{Sp}_{\mathrm{seq}}$ *induced by the evident functors* $\mathscr{J}^{O} \leftarrow \mathscr{J}^{\Sigma} \leftarrow \mathscr{J}^{\mathbf{N}}$*.*

4.1. **The Mandell-May category.** Now we define the Mandell-May category $\mathscr{J}_G$ for a compact Lie group G, which is enriched over $\mathcal{T}_G$. The case of trivial G is the category $\mathscr{J}^{O}$ of (4.4).

Definition 4.7. The Mandell-May category $\mathscr{J}_G$ *has finite dimensional orthogonal representations V of G as objects. These are actual representations, not virtual ones. For two such representations V and W we have a pointed morphism G-space $\mathscr{J}_G(V, W)$ defined as follows. Let $O(V, W)$ be the space of orthogonal embeddings $t : V \subseteq W$, which need not be equivariant. It is empty when $|W| < |V|$. Otherwise each such embedding gives us an orthogonal complement $t(V)^{\perp} \subseteq W$. This enables us to define a vector bundle over $O(V, W)$ sitting inside the trivial bundle $O(V, W) \times W$.* **The morphism space $\mathscr{J}_G(V, W)$ is its Thom space, which is a pointed G-space.** *A point in it other than the base point is a pair (t, x) where $t : V \to W$ is an embedding as above and $x \in t(V)^{\perp}$. Composition of* *morphisms $U \to V \to W$ leads to a map*

$$\mathscr{J}_G(V,W) \wedge \mathscr{J}_G(U,V) \to \mathscr{J}_G(U,W)$$

induced by composition of orthogonal embeddings.

The following is immediate.

Proposition 4.8. Properties of $\mathscr{J}_G$.

(i) $\mathscr{J}_G(V,W)$ *is a point when* $|W| < |V|$,
(ii) $\mathscr{J}_G(0,W) = S^W$.
(iii) $\mathscr{J}_G(V,V) = O(V,V)_+$, *the orthogonal group* $O(V)$ *of* V *with disjoint base point.*
(iv) The inclusion of V *into* $V \oplus W$ *induces a map*

$$S^V = \mathscr{J}_G(0,V) \to \mathscr{J}_G(W, V \oplus V).$$

(v) $\mathscr{J}_G$ *has a symmetric monoidal structure (which is not closed) related to direct sum of representations.*

The following is part of [MM02, Definition V.4.1].

Definition 4.9. The relative Mandell-May category. *Let* $N \subseteq G$ *be a normal subgroup.*

(i) The $\mathcal{T}_{G/N}$ *enriched category* $\mathscr{J}_{G,N}$ *has as objects the representations* V *of* G, *and the morphism space* $\mathscr{J}_{G,N}(V,W)$ *is the fixed point space* $\mathscr{J}_G(V,W)^N$ *with its residual action of* G/N. *A nonbase point in this space is a pair* (t,x), *where* $t : V \to W$ *is an* N*-equivariant orthogonal embedding and* $x \in (t(V)^\perp)^N$.
(ii) We have a functor

$$\begin{array}{ccc} \mathscr{J}_{G,N} & \xrightarrow{\phi^{G,N}} & \mathscr{J}_{G/N} \\ V & \longmapsto & V^N \end{array}$$

$$(t,x) \in \mathscr{J}_{G,N}(V,W) = \mathscr{J}_G(V,W)^N \longmapsto (t^N, x) \in \mathscr{J}_{G/N}(V^N, W^N)$$

where t^N *denotes the restriction of* t *to* V^N, *and since* $x \in t(V)^\perp$ *is fixed by* N, *it lies* W^N *and hence in the orthogonal complement of* $t(V^N)$ *in* W^N. *Thus for each* V *and* W, *we have a map*

$$\phi^{G,N}_{V,W} : \mathscr{J}_{G,N}(V,W) \to \mathscr{J}_{G/N}(V^N, W^N). \tag{4.10}$$

(iii) The functor $\nu_{G,N} : \mathscr{J}_{G/N} \to \mathscr{J}_{G,N}$ *(the right adjoint of* $\phi^{G,N}$*) sends a representation* W *of* G/N *to the representation of* G *obtained by precomposition with the homomorphism* $G \to G/N$.

Note here that $O(V,W)^N$ is the space of N-equivariant orthogonal embeddings $V \to W$.

We will be interested in the case where G is the circle group $\mathbb{T}$ and N is a finite (and hence cyclic) subgroup.

Remark 4.11. Non-normal subgroups. *We could replace the pair (G, N) for a normal subgroup $N \subseteq G$ by $(N(H), W(H))$ (normalizer and Weyl group) for arbitrary subgroup $H \subseteq G$. We leave these details to the reader. The same goes for the discussion of fixed point spectra in Definition 4.20.*

4.2. **Defining orthogonal G-spectra.** The relative form of the following is [MM02, Definition V.4.2] with different terminology.

Definition 4.12. *An* **orthogonal G-spectrum** *is a $\mathcal{T}_G$-enriched $\mathcal{T}_G$-valued functor on the Mandell-May category $\mathscr{J}_G$ of Definition 4.7. We denote by Sp_G (Sp^G) the category of such functors (and equivariant maps). For such a functor X we denote by X_V its value on the object V in $\mathscr{J}_G$. For $V = \mathbb{R}^n$ with trivial G-action, we denote this space by X_n.*

We say that an orthogonal G-spectrum X **has trivial G-action** *if the same holds for X_n for each $n \geqslant 0$. (In this case the action on X_V is necessarily nontrivial when the action on V is.)*

More generally an **orthogonal (G, N)-spectrum** *is a $\mathcal{T}_{G/N}$-enriched $\mathcal{T}_{G/N}$-valued functor on the relative Mandell-May category $\mathscr{J}_{G,N}$ of Definition 4.9(i).*

A morphism $f : X \to Y$ of orthogonal G-spectra is a natural transformation of functors. This consists of a collection of suitably compatible maps $f_V : X_V \to Y_V$, which are equivariant if we are in the category Sp^G, but need not be if we are in Sp_G.

Hence an orthogonal (G, N)-spectrum X consists of a collection $\{X_V\}$ of pointed G/N-spaces (for each representation V of G) along with structure maps

$$\epsilon^X_{V,W} : \mathscr{J}_{G,N}(V, W) \wedge X_V \to X_W \tag{4.13}$$

with suitable properties that are spelled out in the references cited above. In particular each X_V comes equipped with an action of the orthogonal group $O(V)$. An orthogonal (G, e)-spectrum is an orthogonal G-spectrum. The sphere spectrum $\mathbb{S}$ is defined by $\mathbb{S}_V = S^V$.

It follows that the morphism space $\mathrm{Sp}^G(X, Y)$ is a certain subspace of the product (over all V) of the spaces $\mathcal{T}^G(X_V, Y_V)$. It can be described categorically as an enriched end, and similarly for $\mathrm{Sp}_G(X, Y)$. See [HHR21, §9.1.C].

Remark 4.14. Genuine and naive G-spectra. *It is known (*[MM02, Lemma V.1.1]*, reproved as* [HHR21, Lemma 9.1.8]*) that the space X_V depends only on the dimension $|V|$ of V, although the group actions on the spaces X_V and $X_{|V|}$ will differ. For example if G acts trivially on $X_{|V|}$ but nontrivially on V, then it will act nontivially on*

$$X_V = O(|V|, V)_+ \wedge_{O(|V|)} X_{|V|}.$$

For this reason Schwede in [Sch18]*, which is the reference for orthogonal G-spectra used in* [NS18]*, defines such objects X solely in terms of the spaces X_n. He discusses the relation between the two definitions in* [Sch18, Remark 2.7]*.*

It follows that Sp^G as we have defined it is equivalent to the category of $\mathcal{T}^G$-valued functors on $\mathscr{J}$, the Mandell-May category for the trivial group. Such objects are often called naive G-spectra*, while $\mathcal{T}^G$-valued functors on $\mathscr{J}_G$ are called* **genuine G-spectra***. We will denote the former category by $\mathrm{Sp}^G_{\mathrm{naive}}$. While Sp^G and $\mathrm{Sp}^G_{\mathrm{naive}}$ are equivalent as categories, they come with different homotopical structures, meaning different classes of stable equivalances. See* [HHR21, Theorem 9.3.10 and

Example 9.3.11] *for more discussion. It turns out that* $\mathrm{Sp}^G_{\mathrm{naive}}$ *has more genuine weak equivalences than naive ones.*

Naive G-spectra are sometimes called **Borel G-spectra**, and the category of such is sometimes denoted by Sp^{hG}. Recall that for a G-space X, the homotopy fixed point set X^{hG} is the mapping map $\mathrm{Top}^G(EG, X)$. When G acts trivially on X, this is the same as $\mathrm{Top}(BG, X)$. Hence if $\mathcal{C}$ is an ∞-category on which G acts trivially, it would be reasonable to denote $\mathrm{Fun}(\mathcal{B}G, \mathcal{C}) = \mathrm{Fun}(BG, \mathcal{C})$ by $\mathcal{C}^{hG}$. For $\mathcal{C} = \mathrm{Sp}$, this the ∞-category of Lurie spectra with G-action. See [AMGR17, page 3] (where naive G-spectra are called **homotopy G-spectra**) for more discussion.

The categories Sp_G and Sp^G are both tensored and cotensored over $\mathcal{T}_G$ and $\mathcal{T}^G$ respectively. This means that for a pointed G-space K and an orthogonal G-spectrum X, we can make sense of both the tensor product $X \wedge K$ and the cotensor product X^K. These spectra are defined by

$$(4.15)\qquad \begin{aligned} (X \wedge K)_V &:= X_V \wedge K \\ \text{and} \qquad \left(X^K\right)_V &:= \mathcal{T}_G(K, X_V) \text{ or } \mathcal{T}^G(K, X_V). \end{aligned}$$

Definition 4.16. *For a representation V of G, the V***th Yoneda spectrum** $よ^V$ *defined by*

$$(よ^V)_W = \mathscr{J}_G(V, W).$$

Note that the space on the right is a point when $|W| < |V|$, and that $よ^0 = \mathbb{S}$. As in Definition 4.12, we denote $よ^{\mathbb{R}^n}$ by $よ^n$.

$よ^V$ is a twisted desuspension of the sphere spectrum $\mathbb{S}$, and $\Sigma^V よ^V \simeq \mathbb{S}$.

As noted before, the symbol よ is the Japanese hiragana character "yo," the first syllable of Yoneda's name. This spectrum is denoted by S^{-V} in [HHR16] and [HHR21] (where G is assumed to be finite), while the "shift desuspension functor" $(よ^V \wedge -)$, in which the variable may be either an orthogonal G-spectrum or a pointed G-space, is denoted by F_V in both [MM02, page 11] and [MMSS01, Definition 1.3].

Both Sp_G and Sp^G are known to have a closed symmetric monoidal structure defined in terms of the Day convolution; see [HHR21, §3.3]. This means that the morphism space $\mathrm{Sp}_G(V, W)$ is the 0th space of a morphism or function spectrum $F_G(X, Y)$ defined by

$$F_G(X, Y)_V = \mathrm{Sp}_G(よ^V \wedge X, Y).$$

Again we refer the reader to [HHR21, §9.1] for details. Even though it was written with only finite groups G in mind, most of it holds as well for G a compact Lie group. One feature of the finite group case *not* present in the compact Lie group case is the existence of a finite dimensional representation (the regular one) containing every irreducible one as a summand. The group $\mathbb{T}$ has infinitely many distinct irreducible representations.

Definition 4.17. Equivariant stable homotopy groups and stable equivalences. *An orthogonal G-spectrum has* **stable homotopy groups** $\pi^H_\star(-)$ *graded over $RO(G)$ for each subgroup $H \subseteq G$. For finite dimensional orthogonal representations V and V' of G,*

$$(4.18)\qquad \pi^H_{V-V'}X := \operatorname*{colim}_W \pi^H_V \Omega^W X_{W+V'},$$

where π_V^H and Ω^W on the right are as in Definition A.15, and the colimit is over the category of finite dimensional orthogonal representations W of G and equivariant inclusions.

We also have **geometric homotopy groups**

$$\pi^{\Phi H}_{V-V'}X := \operatorname*{colim}_W \pi_V \Omega^{W^H} X^H_{W+V'}. \tag{4.19}$$

An equivariant map $f : X \to Y$ is a **stable equivalence** *if it induces isomorphisms in $\pi^H_\star(-)$ for all closed subgroups $H \subseteq G$.*

For finite G, let ϱ_G denote the **regular representation of** G, meaning its group ring over $\mathbb{R}$ (a vector space of dimension $|G|$) on which G acts by left multiplication. Then we know that every finite dimensional representation of G is a summand of some finite multiple of ϱ_G. It follows that the colimits of (4.18) and (4.19) can be simplified to the sequential colimits

$$\begin{aligned} \pi^H_{V-V'}X &= \operatorname*{colim}_n \pi^H_V \Omega^{n\varrho_G} X_{n\varrho_G+V'} \\ \text{and} \qquad \pi^{\Phi H}_{V-V'}X &= \operatorname*{colim}_n \pi_V \Omega^{n\varrho_G^H} X^H_{n\varrho_G+V'}. \end{aligned}$$

Such a simplification is *not* available for a compact but infinite Lie group such as $\mathbb{T}$.

A G-spectrum has two flavors of fixed points for each normal subgroup $N \subseteq G$.

Definition 4.20. Fixed point spectra. *Let X be an orthogonal G-spectrum X and let $N \subseteq G$ be a normal subgroup.*

(i) The G/N-spectrum X^N, its **categorical fixed point spectrum**, *is defined by*

$$(X^N)_W = (X_{q(W)})^N$$

for each representation W of G/N, where $q = q_{G/N} : \mathscr{J}_{G/N} \to \mathscr{J}_{G,N}$ is the right adjoint of Definition 4.9(iii).

(ii) The (G, N)-spectrum $\mathrm{Fix}^N X$ is defined by

$$(\mathrm{Fix}^N X)_V = (X_V)^N$$

for each representation V of G. The G/N-spectrum $\Phi^N X$, its **geometric fixed point spectrum**, *is the left Kan extension of $\mathrm{Fix}^N X$ along the left adjoint functor*

$$\phi_{G,N} : \mathscr{J}_{G,N} \to \mathscr{J}_{G/N}$$

of Definition 4.9.

4.3. **The Loday functor.** In this subsection we use the symbol $\underline{\otimes}$ for the categorical tensor product, reserving $\otimes$ for its usual meaning as algebraic tensor product or smash product.

A cocomplete category $\mathcal{C}$ is tensored over sets. For an object W in $\mathcal{C}$ and a finite set F with cardinality f, $W\underline{\otimes}F$ is the f-fold coproduct of W. For a simplicial set X of finite type, $W\underline{\otimes}X$ is a simplicial object in $\mathcal{C}$ with $(W\underline{\otimes}X)_n = W\underline{\otimes}X_n$, *provided that we can define the relevant maps among the various finite coproducts of* W. When in addition $\mathcal{C}$ is tensored over topological spaces, we get a geometric realization $|W\underline{\otimes}X|$ in $\mathcal{C}$. Its homotopy type (assuming this makes sense in $\mathcal{C}$) for a given W depends only on the homotopy type of $|X|$, meaning that weakly equivalent simplicial sets give weakly equivalent objects in this construction.

Of particular interest is the case where X is the simplicial circle or standard 0-cyclex $\mathbf{\Lambda}^0 = \mathbf{\Delta}^1/\partial\mathbf{\Delta}^1$ of Corollary 2.43. Recall that its nth component has $n+1$ elements that are permuted by the cyclic group C_{n+1}.

In the category of commutative k-algebras for a discrete commutative ring k, the coproduct is tensor product over k, denoted as usual by $\otimes_k$. The algebra multiplication gives us the maps we need between finite coproducts to give us the desired simplicial structure. Thus for a commutative algebra A and a simplicial set X we get a simplicial k-algebra which we denote by $A\underline{\otimes}_k X$, which is also a simplicial abelian group, omitting the subscript when $k = \mathbb{Z}$. We denote the associated chain complex as in (2.29) by $\mathrm{Ch}(A\underline{\otimes}_k X)$. For example the chain complex $\mathrm{Ch}(A\underline{\otimes}_k \mathbf{\Delta}^0)$ is

$$\begin{array}{ccccccccccc} 0 & & 1 & & 2 & & 3 & & 4 & & \\ A & \xleftarrow{0} & A & \xleftarrow{1} & A & \xleftarrow{0} & A & \xleftarrow{1} & A & \longleftarrow & \cdots, \end{array}$$

so its homology is A concentrated in degree 0. More interestingly,

$$A\underline{\otimes}_k \mathbf{\Lambda}^0 = \mathbf{HH}_\bullet(A),$$

the simplicial abelian group of (2.32).

In the category of commutative ring spectra which are algebras over a fixed commutative ring spectrum R, the coproduct is the smash product over R, denoted by $\otimes_R$. When A is such an algebra, we denote its categorical tensor product with a simplicial set X by $A\underline{\otimes}_R X$. When $R = \mathbb{S}$, we omit it.

The following notation is due to Brun, Carlsson and Dundas [BCD10], who study the Loday functor in detail.

Definition 4.21. *For a commutative ring spectrum A and a simplicial set X, the* **Loday functor** $\wedge_X A$ *is the geometric realization of the simplicial ring spectrum $A\underline{\otimes} X$. More generally when A is a commutative algebra over a commutative ring spectrum R, the* **Loday functor relative to** R *is*

$$\wedge_X A/R := |A\underline{\otimes}_R X|.$$

Our $\wedge_X A$ is denoted by $\Lambda_X A$ and called "smash X of A" in [BCD10], and by $\mathcal{L}_X(A)$ in [HHL+18]. In the latter it is generalized (in a different direction from the above) to $\wedge_{X,Y}(A, B; C)$, for $Y \subseteq X$ and commutative rings $A \to B \to C$. This relative version places C over the basepoint of X, B over all points of Y that are not the basepoint and A over the complement of Y in X.

Remark 4.22. The category of associative R-algebras. *When A is an associative (but not commutative) R-algebra and $X = \mathbf{\Lambda}^0$, we still have a simplicial ring spectrum in which the nth component is*

$$A \otimes_R A \otimes_R A \otimes_R \cdots \otimes_R A \qquad \text{with } n+1 \text{ factors}$$

whose geometric realization is $\mathrm{THH}(A/R)$*. However the $(n+1)$-fold coproduct in the category of* associative *R-algebras is not the $(n+1)$-fold relative smash product, but something far more complicated. This is analogous to the fact that while the coproduct in the category of abelian groups is the direct product, that in the category of arbitary groups is the far more complicated free product. Hence we cannot identify* $\mathrm{THH}(A/R)$ *with* $|A\underline{\otimes}_R \mathbf{\Lambda}^0|$.

Note that $\wedge_X A$ is covariantly functorial in both A and X. Jim McClure, Roland Schwänzel and RainerVogt [MSV97, Theorem B] show that when A is an $\mathbb{E}_\infty$-ring spectrum and $X = \mathbf{\Lambda}^0$ (which they denote simply by S^1), $\wedge_X A = \mathrm{THH}(A)$. In [MSV97, Theorem G] they show that any map $A \to B$ to an $\mathbb{E}_\infty$-ring spectrum with $\mathbb{T}$-action extends uniquely to $\mathrm{THH}(A)$.

Angeltveit and Rognes [AR05, §3] show that it has a Hopf algebebra structure. Brun, Fiedorowicz and Vogt [BFV07] show that for an $\mathbb{E}_n$-ring spectrum A with $n \geqslant 2$, $\mathrm{THH}(A)$ can be defined in the same way and is an $\mathbb{E}_{n-1}$-ring spectrum.

In [BCS10], Andrew Blumberg, Cohen and Schlichtkrull determine the THH of certain Thom spectra. Let BF be the classifying space for stable spherical fibrations and suppose we have a map $f : X \to BF$ which is the loop map associated to a map of connected based spaces $Bf : BX \to B^2F$. Then the resulting Thom spectrum $T(f)$ is an $\mathbb{E}_1$-ring. They identify $\mathrm{THH}(T(f))$ as a certain Thom spectrum associated with the free loop space $\mathcal{L}(BX)$. When f is a triple loop map, [BCS10, Theorem 3] says

$$\mathrm{THH}(T(f)) \simeq T(f) \wedge BX_+.$$

For example,

$$\mathrm{THH}(MU) \simeq MU \wedge SU_+.$$

In [HW22] the authors study the relative version $\mathrm{THH}(BP\langle n\rangle/MU)$. MU is a commuatative ring spectrum, but the $\mathbb{E}_3$-ring spectrum $BP\langle n\rangle$ is known not to admit an $\mathbb{E}_\infty$-structure by work of Lawson [Law18] for $p = 2$ and Andrew Senger [Sen24] for odd primes.

4.4. **The Greenlees-May or Tate diagram.** The standard reference for the material in this subsection is John Greenlees and Peter May's classic book [GM95]. As noted above and illustrated in Example A.6, each compact Lie group G has a contractible free G-space EG. Let EG_+ denote EG with a disjoint base point, and consider the base point preserving map $j : EG_+ \to S^0$ that sends all of EG to the nonbase point of S^0. The resulting cofiber sequence

$$EG_+ \xrightarrow{\ j\ } S^0 \xrightarrow{\ k\ } \widetilde{E}G, \tag{4.23}$$

is useful to us. Since EG is contractible, the first map is underlain by a homotopy equivalence, making the underlying space of $\widetilde{E}G$ contractible. On the other hand for any nontrivial subgroup $H \subseteq G$, the fixed point set $(EG_+)^H$ is a point since H acts freely away from the basepoint. It follows that

$$(EG_+)^H \simeq \begin{cases} S^0 & \text{for } H = e \\ \mathrm{pt} & \text{otherwise} \end{cases} \qquad \text{while} \qquad \widetilde{E}G^H \simeq \begin{cases} \mathrm{pt} & \text{for } H = e \\ S^0 & \text{otherwise.} \end{cases}$$

The cofiber sequence of (4.23) is not to be confused with the **isotropy separation sequence** (see [HHR21, §9.11A])

$$E\mathcal{P}_+ \longrightarrow S^0 \longrightarrow \widetilde{E}\mathcal{P},$$

where $\mathcal{P}$ denotes the family of proper subgroups of G, and $E\mathcal{P}$ is a G-space for which $E\mathcal{P}^G$ is empty but $E\mathcal{P}^H$ is weakly contractible for each proper subgroup $H \subsetneq G$. The two cofiber sequences coincide when $G = \mathrm{C}_p$.

For an orthogonal G-spectrum X, cotensoring (as in (4.15)) with the first map of (4.23) gives a map

$$X = X^{S^0} \xrightarrow{\ \epsilon_X := X^j\ } X^{EG_+} \tag{4.24}$$

known as **Borel completion**.

We now smash (4.23) with (4.24) and get the **Greenlees-May diagram** of [GM95, Diagram (D)],

$$\begin{array}{ccccc} X \wedge EG_+ & \xrightarrow{X\wedge j} & X & \xrightarrow{X\wedge k} & X \wedge \widetilde{E}G \\ \downarrow{\scriptstyle \epsilon_X \wedge EG_+} & & \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \epsilon_X \wedge \widetilde{E}G} \\ X^{EG_+} \wedge EG_+ & \xrightarrow{X^{EG_+}\wedge j} & X^{EG_+} & \xrightarrow{X^{EG_+}\wedge k} & X^{EG_+} \wedge \widetilde{E}G \\ \| & & \| & & \| \\ f_G(X) & & c_G(X) & & t_G(X) \end{array} \tag{4.25}$$

in which each row is a cofiber sequence and the spectra in the bottom row are defined by the indicated equalities. Here the letters f, c and t stand for "free," "complete," and "Tate." The left vertical map is an equivalence by [GM95, Proposition 1.2]. This means that *the square on the right is a homotopy pullback diagram.*

Greenlees and May [GM95, Introduction] call $t_G(X)$ (which they denote by $t(X)$ without the subscript) the **Tate G-spectrum associated to** X, and its fixed point set

$$X^{tG} := t_G(X)^G \tag{4.26}$$

the Tate construction of X. It is surprisingly interesting even when the action of G on X is trivial as in Definition 4.12.

When X is a naive G-spectrum, [GM95, Proposition 2.1] says that the fixed point sets of $f_G(X)$ and $c_G(X)$ can by be identified with $(\Sigma^{\mathrm{Ad}(G)}X)_{hG}$, the homotopy orbit spectrum of the twisted suspension (Definition A.15) of X_{hG} of (A.13), and X^{hG}, the homotopy fixed point set of (A.9). Here $\mathrm{Ad}(G)$ denotes the **adjoint representation of** G, whose degree is the dimension of G. In particular it is trivial when G is finite. It is defined in terms of the action of G on its Lie algebra, and that action is trivial when G is abelian. Thus for the circle group $\mathbb{T}$ acting on a spectrum X, we have

$$(f_{\mathbb{T}}X)^{\mathbb{T}} \simeq (\Sigma^{\mathrm{Ad}(\mathbb{T})}X)_{h\mathbb{T}} \simeq \Sigma X_{h\mathbb{T}}. \tag{4.27}$$

The map

$$\mathrm{Nm}_G : (\Sigma^{\mathrm{Ad}(G)}X)_{hG} \simeq f_G(X)^G \to c_G(X)^G \simeq X^{hG} \tag{4.28}$$

is known as **the norm** and has the X^{tG} as its cofiber. For finite G it can be described as follows.

For a module M over the integral group ring of G there is a map

$$\mathrm{Nm}_G : M_G \to M^G,$$

originally due to John Tate, which we describe in §4.4.1. There is no comparable map $X_G \to X^G$ for a G-space X, but there is one for a G-spectrum X that we describe in §4.4.2.

4.4.1. *The norm map in algebra.* The original reference for this paragraph is [CE56, Chapter XII], whose authors attributed the ideas to Tate. Let $\mathbb{Z}G$ denote the integer group ring of a finite group G. For each $\gamma \in G$, we denote by $[\gamma]$ the corresponding basis element of $\mathbb{Z}G$. We have the augmentation map

$$\nabla : \mathbb{Z}G \to \mathbb{Z} \quad \text{given by} \quad [\gamma] \mapsto 1 \quad \text{for each } \gamma \in G, \tag{4.29}$$

and we denote its kernel, the augmentation ideal, by I. A $\mathbb{Z}G$-module M is an abelian group with an action of G. As such it has an orbit module

$$M_G = M/IM = M \otimes_{\mathbb{Z}G} \mathbb{Z}$$

and a fixed point set

$$M^G = \{m \in M : [\gamma]m = m \ \forall \gamma \in G\} = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \subseteq M.$$

Now consider the **Tate norm map** $\mathrm{Nm}_G : M \to M$ given by

$$m \mapsto \sum_{\gamma \in G} [\gamma] m.$$

It sends each element $m \in M$ to the sum of the elements in its G-orbit. Hence two elements in an orbit have the same image under it, so it factors through the quotient M_G. Its image is fixed by G and thus contained in M^G, and we have

$$\mathrm{Nm}_G : M_G \to M^G.$$

When $M = \mathbb{Z}G$, it is an isomorphism between $\mathbb{Z}G/I$ and the subgroup of $\mathbb{Z}G$ generated by the norm of the unit element. Hence its kernel and cokernel both vanish. When $M = \mathbb{Z}$ with trivial G-action, this map is multiplication by $|G|$, the order of G. We will describe its kernel and cokernel in (4.31).

Recall that

$$H^i(G; M) := \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, M),$$
$$H_i(G; M) := \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, M),$$

and both can be computed using a free (or projective) $\mathbb{Z}G$-resolution of $\mathbb{Z}$ of the form

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 = \mathbb{Z}G \xrightarrow{\nabla} \mathbb{Z} \longrightarrow 0, \tag{4.30}$$

where ∇ is as in (4.29).

Then $H^*(G; M)$ is the cohomology of the cochain complex

$$\cdots \leftarrow \mathrm{Hom}_{\mathbb{Z}G}(P_2, M) \leftarrow \mathrm{Hom}_{\mathbb{Z}G}(P_1, M) \leftarrow \mathrm{Hom}_{\mathbb{Z}G}(P_0, M) = M$$

and $H_*(G; M)$ is the homology of the chain complex

$$\cdots \to P_2 \otimes_{\mathbb{Z}G} M \to P_1 \otimes_{\mathbb{Z}G} M \to P_0 \otimes_{\mathbb{Z}G} M = M.$$

The augmentation $\nabla : \mathbb{Z}G \to \mathbb{Z}$ of (4.29) induces maps $H_0(G; M) \to M_G$ and $M^G \to H^0(G; M)$. These lead to a 4-term exact sequence

$$0 \to H_0(G; M) \to M_G \xrightarrow{\mathrm{Nm}_G} M^G \to H^0(G; M) \to 0. \tag{4.31}$$

Tate cohomology $\hat{H}^*$ is defined by

$$\hat{H}^i(G; M) = \begin{cases} H^i(G; M) & \text{for } i > 0 \\ H_{-1-i}(G; M) & \text{for } i \leqslant 0, \end{cases} \tag{4.32}$$

so we can write the 4-term exact sequence as

$$0 \to \hat{H}^{-1}(G; M) \to M_G \xrightarrow{\mathrm{Nm}_G} M^G \to \hat{H}^0(G; M) \to 0.$$

When M is a ring, there is a multiplication in $\hat{H}^*(G; M)$ that makes it a module over $H^*(G; M)$, its nonnegatively graded part.

Example 4.33. The case $G = \mathrm{C}_p$ and $M = \mathbb{Z}/p$. *We find that*

$$\widehat{H}^i(\mathrm{C}_p; \mathbb{Z}/p) = \mathbb{Z}/p \qquad \text{for all } \mathbb{Z}.$$

Given generators $a \in \widehat{H}^1$ and $x \in \widehat{H}^2$, $\widehat{H}^{2i+\epsilon}$ is generated by $a^\epsilon x^i$ for $\epsilon = 0, 1$ and all $i \in \mathbb{Z}$. As graded rings we have (for $p > 2$)

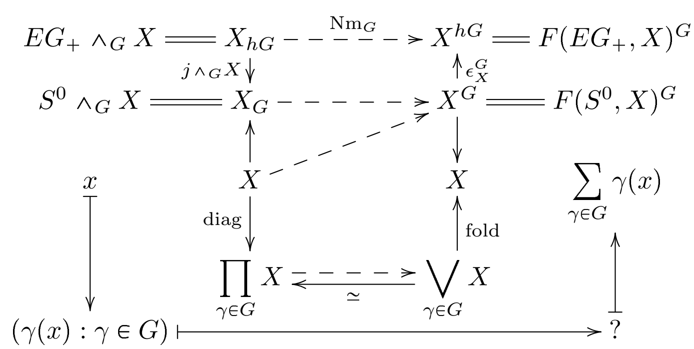
$$\begin{aligned} & H^*(\mathrm{C}_p; \mathbb{Z}/p) \cong \Lambda(a) \otimes \mathbb{Z}/p[x] \\ \text{and} \qquad & \widehat{H}^*(\mathrm{C}_p; \mathbb{Z}/p) \cong \Lambda(a) \otimes \mathbb{Z}/p[x, x^{-1}], \end{aligned}$$

and for $p = 2$ there is a similar description with $x = a^2$.

The case $p = 2$ is discussed further in §4.5.1.

4.4.2. *The norm map in equivariant stable homotopy theory.* For a finite group G acting on a space X (pointed or not) we do *not* get a similar map $X_G \to X^G$ because in general there is no way to sum the points in a G-orbit of X. However we can do this in the stable world as follows. For a naive G-spectrum X there are G-maps

(4.34)



where G acts as usual on X, and on the product and coproduct by permuting factors and summands. The diagonal map is twisted as indicated, and the fold map is the usual one. The map from the G-indexed coproduct to the G-indexed product is functorial, and it is an equivalence because we are in the stable category, and therefore it has an (admittedly mysterious) inverse whose composite with the fold map is the indicated addition up to homotopy. The sum in the lower right is fixed by G, so the composite lifts to the fixed point spectrum X^G as indicated. This lifting factors through the orbit spectrum X_G since G acts trivially on X^G. The resulting composite map $X_{hG} \to X^{hG}$ is the norm.

This norm is *not* that of [HHR16], which is a functor that converts an H-spectrum to a G-spectrum for G finite and $H \subseteq G$. A generalization of it to the case $e \subseteq \mathbb{T}$ is used in [ABG+18] to study the cyclotomic structure on $\mathrm{THH}(R)$.

As noted in [NS18, Definition I.1.13], (4.34) makes sense in any stable ∞-category in which limits and colimits indexed by G are defined. The stability condition guarantees the equivalance of the product and coproduct there.

Thus the fixed point diagram for (4.25), the **Greenlees-May fixed point diagram**, is

$$\begin{array}{ccccc} X_{hG} & \longrightarrow & X^G & \longrightarrow & (X \wedge \widetilde{E}G)^G \\ \| & & \downarrow{\scriptstyle \epsilon_X^G} & \lrcorner & \downarrow \\ X_{hG} & \xrightarrow{\mathrm{Nm}_G^X} & X^{hG} & \xrightarrow{s_X^G} & X^{tG}. \end{array} \tag{4.35}$$

We remind the reader that G is assumed to be finite. We will omit the index X in the three maps when it is clear from the context.

Definition 4.36. *When G acts trivially on X (making $X^G = X$), the* **Tate map**

$$\theta_X^G : X \to X^{tG}$$

is the composite $s_X^G \epsilon_X^G$ in (4.35). We will often drop the subscript X.

Definition 4.37. The functors TC^- and TP. *For a $\mathbb{T}$-spectrum X,*

$$\mathrm{TC}^-(X) := X^{h\mathbb{T}} \qquad \text{and} \qquad \mathrm{TP}(X) := X^{t\mathbb{T}}.$$

For a C_{p^∞}-spectrum X,

$$\mathrm{TC}^-(X,p) := X^{h\mathrm{C}_{p^\infty}} \qquad \text{and} \qquad \mathrm{TP}(X,p) := X^{t\mathrm{C}_{p^\infty}}.$$

For a cyclotomic spectrum X (see Definition 4.51), we have a diagram of cofiber sequences

$$\begin{array}{ccccc} & & \mathrm{TC}^-(X) & & \mathrm{TP}(X) \\ & & \| & & \| \\ \Sigma X_{h\mathbb{T}} & \xrightarrow{\mathrm{Nm}_{\mathbb{T}}} & X^{h\mathbb{T}} & \xrightarrow{s_X^{\mathbb{T}}} & X^{t\mathbb{T}} \\ \downarrow & & \downarrow & & \downarrow \\ X_{h\mathrm{C}_{p^j}} & \xrightarrow{\mathrm{Nm}_{\mathrm{C}_{p^j}}} & X^{h\mathrm{C}_{p^j}} & \xrightarrow{s_X^{\mathrm{C}_{p^j}}} & X^{t\mathrm{C}_{p^j}}, \end{array} \tag{4.38}$$

where the equalities are those of Definition 4.37, and the norm maps are those of (4.28) and (4.35). The one for $\mathbb{T}$ was constructed using different methods by John Klein in [Kle01, §3] and is defined for any $\mathbb{T}$-spectrum X; the cyclotomic structure is not needed for it. It is known [BM94, Lemma 2.18] that after p-adic completion $X^{h\mathbb{T}}$ is the homotopy limit of the $X^{h\mathrm{C}_{p^j}}$ when X has finite type.

The left vertical map in (4.38) is constructed as follows. Let

$$H := \mathrm{C}_{p^j} \subseteq \mathbb{T} =: G.$$

The spaces in (4.25) for $G = \mathbb{T}$ are defined in terms of a contractible from G-space EG, which is also a contractible free H-space. Hence (4.25) is also a Greenlees-May diagram for H. Consider the inclusion map

$$f_G(X)^G \to f_G(X)^H \simeq f_H(X)^H.$$

These spectra are identified by [GM95, Proposition 2.1] as

$$\Sigma X_{h\mathbb{T}} = (\Sigma^{\mathrm{Ad}(G)} X)_{hG} \to (\Sigma^{\mathrm{Ad}(H)} X)_{hH} = X_{h\overline{\mathrm{C}}_{p^j}}.$$

4.5. **The case $G = \mathrm{C}_r$.** When $G = \mathrm{C}_r$ we denote the map s_X^G of (4.35) by s_X^r, and we have

$$(X \wedge \widetilde{E}\mathrm{C}_r)^{\mathrm{C}_r} = \Phi^{\mathrm{C}_r} X,$$

the geometric fixed point spectrum of Definition 4.20(ii).

The right square in (4.35) is the pullback diagram

(4.39)
$$\begin{array}{ccc} X^{\mathrm{C}_r} & \longrightarrow & \Phi^{\mathrm{C}_r} X \\ {\scriptstyle \epsilon_X^{\mathrm{C}_r}}\downarrow & \lrcorner & \downarrow \\ X^{h\mathrm{C}_r} & \xrightarrow{s_X^r} & X^{t\mathrm{C}_r}. \end{array}$$

Blumberg and Mandell (see Definition 4.51) define a cylotomic spectrum to be a $\mathbb{T}$-spectrum equipped with an equivalence

(4.40)
$$\Phi_p : \rho_p^* \Phi^{\mathrm{C}_p} X \to X$$

for each prime p, where $\Phi^{\mathrm{C}_p} X$ is the geometric fixed point spectrum of Definition 4.20(ii). There is a residual action of $\overline{\mathbb{T}} := \mathbb{T}/\mathrm{C}_p \cong \mathbb{T}$ on $\Phi^{\mathrm{C}_p} X$, and ρ_p^* is as in Definition 1.3. The map is required to be $\mathbb{T}$-equivariant.

In Definition 5.23 Nikolaus and Scholze define it in terms of a $\mathbb{T}$-equivariant map

(4.41)
$$\varphi_p : X \to \rho_p^* X^{t\mathrm{C}_p}.$$

Nikolaus and Scholze show that the two definitions agree when X is bounded below. Both (4.40) and (4.41) should be compared with Definition 1.4. We will discuss this further in §5.

4.5.1. *The group* C_2 *and stunted projective spaces.* When X has trivial group action and $p = 2$, we have

(4.42)
$$X^{t\mathrm{C}_2} \simeq \operatorname*{holim}_{i \to \infty} (\mathbb{R}\mathrm{P}_{-i} \wedge \Sigma X),$$

where $\mathbb{R}\mathrm{P}_{-i}$ denotes the Thom spectrum for the $(-i)$-fold Whitney sum of the canonical real real bundle over $\mathbb{R}P^\infty$. There is an analogous statement for the group C_p for an odd prime p. These are proved as [GM95, Theorem 16.1].

To explain (4.42), we need the following notation. For $0 \leqslant i < j$, let

$$\mathbb{R}\mathrm{P}_i^j := \Sigma^\infty \mathbb{R}\mathrm{P}^j / \mathbb{R}\mathrm{P}^{i-1},$$

the suspension spectrum of the **stunted real projective space**, which is known to have the following properties.

- It has a CW-structure with a single cell in each dimension ranging from i to j.
- For $i < j < k$ there is a cofiber sequence

(4.43)
$$\mathbb{R}\mathrm{P}_i^j \longrightarrow \mathbb{R}\mathrm{P}_i^k \longrightarrow \mathbb{R}\mathrm{P}_{j+1}^k$$

- $\mathbb{R}\mathrm{P}_i^j$ is the suspension spectrum of the Thom space for the i-fold Whitney sum of the canonical line bundle over $\mathbb{R}\mathrm{P}^{j-i}$.

- There is an integer ℓ depending only on $j-i$ such that
$$\mathbb{R}\mathrm{P}_{i+2^\ell}^{j+2^\ell} \simeq \Sigma^{2^\ell}\mathbb{R}\mathrm{P}_i^j$$
This was first proved by James in [Jam59] and is known as **James periodicity**. (The value of ℓ is known and is roughly $(j-i)/2$, but we do not need it here.) It means that for each finite $k>0$, the homotopy type of $\Sigma^{-i}\mathbb{R}\mathrm{P}_i^{i+k}$ varies periodically with i.

We can use James periodicity to make sense of $\mathbb{R}\mathrm{P}_i^j$ for *any integers* i and j with $i \leqslant j$, and define

$$\begin{aligned} \mathbb{R}\mathrm{P}_i = \mathbb{R}\mathrm{P}_i^\infty &:= \operatorname*{hocolim}_{j\to\infty} \mathbb{R}\mathrm{P}_i^j, \\ \mathbb{R}\mathrm{P}_{-\infty}^j &:= \operatorname*{holim}_{i\to\infty} \mathbb{R}\mathrm{P}_{-i}^j, \\ \text{and} \qquad \mathbb{R}\mathrm{P}_{-\infty}^\infty &:= \operatorname*{holim}_{i\to\infty} \mathbb{R}\mathrm{P}_{-i}^\infty. \end{aligned} \tag{4.44}$$

Note that when G acts trivially on a space X, its homotopy orbit space is $BG\times X$, and its homotopy fixed point set is $\mathrm{Map}(BG,X)$. Hence for a C_2-spectrum X with trivial group with trivial action we have

$$\begin{aligned} X_{h\mathrm{C}_2} &\simeq \mathbb{R}\mathrm{P}_0 \wedge X \\ \text{and} \qquad X^{h\mathrm{C}_2} &\simeq \mathrm{map}(\mathbb{R}\mathrm{P}_0, X) \simeq \mathrm{map}(\operatorname*{hocolim}_{m\to\infty} \mathbb{R}\mathrm{P}_0^{m-1}, X) \\ &\simeq \operatorname*{holim}_{m\to\infty} \mathrm{map}(\mathbb{R}\mathrm{P}_0^{m-1}, X) \\ &\simeq \operatorname*{holim}_{m\to\infty} (\mathrm{map}(\mathbb{R}\mathrm{P}_0^{m-1}, \mathbb{S}) \wedge X) \end{aligned}$$

Now we claim that the Spanier-Whitehead dual (see [HHR21, §8.0B]) of $\mathbb{R}\mathrm{P}_0^{m-1}$ is

$$\mathbb{D}\mathbb{R}\mathrm{P}_0^{m-1} := \mathrm{map}(\mathbb{R}\mathrm{P}_0^{m-1}, \mathbb{S}) \simeq \Sigma\mathbb{R}\mathrm{P}_{-m}^{-1}; \tag{4.45}$$

see (5.11). Assuming this and that X is finite, we have

$$\begin{aligned} X^{h\mathrm{C}_2} &\simeq (\operatorname*{holim}_{m\to\infty} \Sigma\mathbb{R}\mathrm{P}_{-m}^{-1} \wedge X) \\ &\simeq \Sigma\mathbb{R}\mathrm{P}_{-\infty}^{-1} \wedge X. \end{aligned}$$

It follows that for finite X and $G=\mathrm{C}_2$, the bottom row of (4.35) is the smash product of X with

$$\begin{array}{ccccccc} \mathbb{R}\mathrm{P}_0^\infty & \longrightarrow & \Sigma\mathbb{R}\mathrm{P}_{-\infty}^{-1} & \longrightarrow & \Sigma\mathbb{R}\mathrm{P}_{-\infty}^{\infty} & \longrightarrow & \Sigma\mathbb{R}\mathrm{P}_0^\infty \\ \| & & \| & & \| & & \| \\ \mathbb{S}_{h\mathrm{C}_2} & \longrightarrow & \mathbb{S}^{h\mathrm{C}_2} & \longrightarrow & \mathbb{S}^{t\mathrm{C}_2} & \longrightarrow & \Sigma\mathbb{S}_{h\mathrm{C}_2}, \end{array} \tag{4.46}$$

where the last map in the top row is the extension of the cofiber sequence to the right. It turns out that the last two maps of the top row form the suspended limiting/colimiting case of the cofiber sequence of (4.43) with $i=-\infty$, $j=-1$ and $k=\infty$. This gives us (4.42) once we have proved (4.45).

Atiyah Duality [Ati61] tells us that the Spanier-Whitehead dual of the suspension spectrum of a closed manifold M (such as $\mathbb{R}\mathrm{P}^{m-1}$) is up to suspension the Thom spectrum M^ν of its normal bundle. (See [HHR21, §8.0B, especially Theorem 8.0.12] for more discussion.) We know that the tangent bundle of $\mathbb{R}\mathrm{P}^{m-1}$ is stably equivalent to the m-fold Whitney sum of the canonical line bundle; see [MS74, Theorem 4.5]. This means its normal Thom spectrum is $\mathbb{RP}^{-1}_{-m}$. The suspension on the righthand side of (4.45) is needed to insure that the top cell is in dimension 0.

We also know that the top cell of the normal Thom spectrum is spherical. This means there is a map $\mathbb{S} \to \Sigma\mathbb{RP}^{-1}_{-m}$ that is nontrivial on mod 2 homology. It is compatible with the maps in the limit, leading to a diagram

$$\begin{array}{ccccc}
\mathbb{S} & \longrightarrow & \Sigma\mathbb{RP}^{-1}_{-m} & \longrightarrow & \Sigma\mathbb{RP}^{\infty}_{-m} & & \\
 & \searrow & \uparrow & & \uparrow & & \\
 & & \Sigma\mathbb{RP}^{-1}_{-\infty} & \longrightarrow & \Sigma\mathbb{RP}^{\infty}_{-\infty} & = & \mathbb{S}^{tC_2}
\end{array}$$

The Segal Conjecture says that the composite map $\mathbb{S} \to \mathbb{S}^{tC_2}$ is 2-adic completion.

4.5.2. *The Segal conjecture.* For more information on this topic, we recommend the recent account by Kaif Hilman [Hil20]. The conjecture was first stated by Graeme Segal in [Seg71] and later proved by Gunnar Carlsson in [Car84]. In its simplest form it describes the 0th stable cohomotopy group of the classifying space BG of a finite group G with an isomorphism

$$\lim_k \pi^0_S\left(BG^{(k)}_+\right) \cong \widehat{A}(G). \tag{4.47}$$

The homotopy limit of (4.42) was first examined by Adams in [Ada74]. It was the subject of extensive computations by Mahowald in the 70s and 80s, some of which were eventually reported in [MR93]. That paper and [MS88] were the first discussions of *blueshift phenomena* in chromatic homotopy theory (not to be confused with the redshift of §5.9.2), although the term "blueshift" was not coined until later. As Mahowald and Shick said in their opening paragraph,

> Calculations of various sorts lead to the following slogan:
> The root invariant of v_n-periodic homotopy is v_n-torsion.

Mahowald's computations also paved the way to W. H. Lin's proof of the Segal Conjecture for the group C_2 in [Lin79] and [LDMA80], and of Gunawardena's proof of it for groups of odd prime order [Gun80].

For a pointed space X, $\pi^0_S(X)$ denotes the group of homotopy classes of maps from the suspension spectrum of X, $\Sigma^\infty X$, to the sphere spectrum, $\mathbb{S} = \Sigma^\infty S^0$. Like ordinary cohomology, stable cohomotopy has a ring structure (cup product) induced by the diagonal inclusion $X \to X \times X$.

The space $BG^{(k)}_+$ is the k-skeleton of BG_+, the classifying space of G with a disjoint basepoint. The functor $\pi^0_S(-)$ is contravariant, so it converts the sequential colimit associated with the skeletal filtration of BG_+ to a limit of abelian groups. While differing choices of a contractible free G-space EG (with orbit space BG) can lead to differing skeleta $BG^{(k)}_+$, it is known that the limit in question is independent of such choices.

$A(G)$ denotes the *Burnside ring of* G, originally defined by William Burnside in [Bur11]. In modern language it is the Grothendieck group of the abelian monoid (under disjoint union) of isomorphism classes of finite G-sets under disjoint union, with multiplication induced by Cartesian product. Additively it is the free abelian group on the set of conjugacy classes of finite subgroups $H \subseteq G$. It admits an *augmentation* map $\epsilon : A(G) \to \mathbb{Z}$, defined by sending a finite G-set to its cardinality, with kernel I, the *augmentation ideal*. $\widehat{A}(G)$ denotes the I-adic completion of $A(G)$.

There is a stronger form of the conjecture, of which (4.47) is just the tip of the iceberg. It has to do with the function spectrum $F(\Sigma^\infty BG_+, \mathbb{S})$, or equivalently the function G-spectrum $F^G(\Sigma^\infty EG_+, \mathbb{S})$ with G acting trivially on $\mathbb{S}$. It can also be thought of as the Spanier-Whitehead dual of $\Sigma^\infty BG_+$. The group of (4.47) is its 0th homotopy group. See [Hil20, §4] for details.

Theorem 4.48. Properties of the Tate construction.

(i) [Ada82] *The Segal Conjecture implies that for finite X with trivial C_p-action, $X^{t\mathrm{C}_p}$ is the p-adic completion of X.*

(ii) Let H/p be the mod p Eilenberg-MacLane spectrum. Then

$$(H/p)^{t\mathrm{C}_p} = \operatorname*{holim}_{k \leqslant 0} \bigvee_{i \geqslant k} \Sigma^i H/p.$$

For $p = 2$ we can write this as

$$\operatorname*{holim}_{i} (\mathbb{R}\mathrm{P}_{-i} \wedge \Sigma H/2),$$

which is wildly different from

$$\operatorname*{holim}_{i} (\mathbb{R}\mathrm{P}_{-i}) \wedge \Sigma H/2 = H/2.$$

(iii) [AMS98] *Let $E(n)$ be the nth Johnson-Wilson spectrum for the prime p and $n > 0$. Then $E(n)^{t\mathrm{C}_p}$ is a certain completion of the coproduct of all even supensions of $E(n-1)$. This is another instance of chromatic blueshift.*

(ii) above is a spectacular example of the failure of the smash product to preserve limits.

4.5.3. *More about $G = \mathbb{T}$.*

Definition 4.49. Real orthogonal representations of $\mathbb{T}$. *Let λ denote the complex numbers with the usual multiplication by elements of $\mathbb{T}$, regarded as a 2-dimensional real vector space. More generally for each integer $r \geqslant 0$, let λ^r denote complex numbers with $\omega \in \mathbb{T}$ acting by multiplication by ω^r, again regarded as a 2-dimensional real vector space. Using the notation of Definition A.15, let*

$$\mathbb{T}(r) := S(\lambda^r), \qquad \mathbb{S}[\mathbb{T}/\mathrm{C}_r] := \Sigma^\infty_+ \mathbb{T}(r) \qquad \text{and} \qquad \mathbb{S}^{(r)} := \Sigma^\infty S^{\lambda^r}.$$

The map $a_{\lambda^r} : S^0 \to S^{\lambda^r}$ of Definition A.15 leads to a map $a_{(r)} : \mathbb{S}^{-(r)} \to \mathbb{S}$, the **Euler class**.

The notation $\mathbb{S}[\mathbb{T}/\mathrm{C}_r]$ (but not $\mathbb{T}(r)$) is abusive for $r = 0$ since there is no group C_0.

Proposition 4.50. The action of the Prüffer group. *For $r > 0$, let $r = sp^j$ with s not divisible by p. Then the space $\mathbb{T}/\mathrm{C}_r$ is C_{p^∞}-equivariantly isomorphic to $\mathbb{T}/\mathrm{C}_{p^j}$.*

In the action of $\mathbb{T}$ on λ^r, the subgroup $\mathrm{C}_r \subseteq \mathbb{T}$ acts trivially, so the action factors through $\mathbb{T}/\mathrm{C}_r$. The underlying spectrum of $\mathbb{S}^{(r)}$ is $\Sigma^2\mathbb{S} = \Sigma^\infty S^2$, and it is tensor invertible. The representation λ^r is denoted in [BM12, Notation 4.1] by $\mathbb{C}(r)$. We are mostly using the notation of [BHLS23, §2.1.1].

For each positive integer r, the circle group $\mathbb{T}$ has a subgroup isomorphic to the cyclic group C_r. The group $\mathbb{T}$ is isomorphic to its quotient by any finite subgroup, a property not enjoyed by any nontrivial finite group. For a $\mathbb{T}$-space X, we have a residual action (Definition A.1) of $\mathbb{T}/\mathrm{C}_r \cong \mathbb{T}$ on the fixed point set X^{C_r}.

The following is part of [BM12, Definition 4.2].

Definition 4.51. *A* **cyclotomic spectrum** *is an orthogonal $\mathbb{T}$-spectrum X along with $\mathbb{T}$-equivariant maps*

$$r_{p,V} : \rho_p^*\left(X_V^{\mathrm{C}_p}\right) \to X_{\rho_p^*(V^{\mathrm{C}_p})}.$$

for each prime $p > 0$ and each representation V of $\mathbb{T}$ (where ρ_p^* is as in Definition 1.3) satisfying certain conditions implying that there is an isomorphism

$$\rho_p^*\Phi^{\mathrm{C}_p}X \to X,$$

where $\Phi^{\mathrm{C}_p}X$ is the geometric fixed point spectrum of Definition 4.20(ii).

4.6. **Tate resolutions.** Applying the functor $\mathrm{Hom}_{\mathbb{Z}G}(-,\mathbb{Z})$ to (4.30), we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}G}(P_0,\mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}G}(P_1,\mathbb{Z}) & \longrightarrow & \cdots \\ & & \| & & \| & & & & \\ & & \mathbb{Z}, & \xrightarrow{\quad\Delta\quad} & \mathbb{Z}G & & & & \end{array} \tag{4.52}$$

where Δ is the diagonal or coaugmentation

$$1 \mapsto \sum_{\gamma\in G}[\gamma].$$

We define $P_{-i} := \mathrm{Hom}_{\mathbb{Z}G}(P_{i-1},\mathbb{Z})$ for $i > 0$. Then we can splice (4.30) with (4.52) and get

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & P_{-1} & \longrightarrow & P_{-2} & \longrightarrow & P_{-3} \longrightarrow \cdots \\ & & & & & & \| & & \| & & & & \\ & & & & & & \mathbb{Z}G & \xrightarrow{\Delta\nabla} & \mathbb{Z}G & & & & \end{array} \tag{4.53}$$

where the augmentation $\nabla : \mathbb{Z}G \to \mathbb{Z}$ is as in (4.30).

Definition 4.54. *A* **Tate resolution** P **of** $\mathbb{Z}$ **over a finite group** G *is a long exact sequence of the form (4.53). $P^- \subseteq P$ is the subsequence obtained by replacing P_i by 0 for $i \geqslant 0$, and $P^+ := P/P^-$, the quotient obtained by killing P_i for $i < 0$. For a G-CW spectrum X with cellular chain complex (of $\mathbb{Z}G$-modules) $C(X)$, let*

$$P(X) := P \otimes_{\mathbb{Z}G} C(X),$$

with $P^-(X)$ and $P^+(X)$ defined similarly. The **Tate sequence for** X *is following short exact sequence of chain complexes over $\mathbb{Z}G$.*

$$0 \to P^-(X) \to P(X) \to P^+(X) \to 0 \tag{4.55}$$

5. ∞-categories and the work of Nikolaus-Scholze

Most of [NS18] is written in the language of ∞-categories. For a very brief introduction to them we refer the reader to [Rav23], where other references can be found. As in that paper, we will write ∞-categories which are not ordinary categories in the color cyan.

5.1. **Elementary ∞-categorical notions.** Stable ∞-categories, of which the ∞-category of spectra Sp is the marquee example, are defined in Definition 5.8(iv), [Rav23, §9] and in [Lur17, Definition 1.1.1.9].

In a pointed ∞-category $\mathcal{C}$, the suspension ΣX is the pushout of the diagram $0 \leftarrow X \rightarrow 0$, while the loop object ΩY is the pullback of the diagram $0 \rightarrow Y \leftarrow 0$. In a stable ∞-category (see Definition 5.8(iv)), a commutative square diagram is a pushout iff it is a pullback, so $X \simeq \Omega\Sigma X$ and $\Sigma\Omega Y \simeq Y$. This makes the functors Σ and Ω both invertible in the homotopy category of a stable ∞-category $\mathcal{C}$.

Definition 5.1. Some familiar ∞-categories.

- (i) *$\mathcal{S}$ is Lurie's ∞-category of spaces (meaning Kan complexes, aka ∞-groupoids)* [Lur09, Definition 1.2.16.1]*, which is described illustratively in* [Rav23, §5]*. It is sometimes called the category of* **anima***. This term is used in writings on condensed mathematics, for which the original references appear to be* [CS20a] *and* [CS20b]*, even though the term is not used in the former.*
- (ii) *For a space (or Kan complex) K, K denotes the ∞-category in which objects are points (0-simplices) in K, 1-morphisms are paths (1-simplices), 2-morphisms are homotopies between paths (2-simplices), and so on. The existence inverse paths makes K an* ∞-groupoid*, meaning that its homotopy category is a groupoid. See* [Lur09, §1.1.1 and Proposition 1.2.5.1].
- (iii) *For objects X and Y in an ∞-category $\mathcal{C}$, we denote the space (or Kan complex) of morphisms $X \rightarrow Y$ by $\mathrm{Map}_{\mathcal{C}}(X, Y)$.*
- (iv) *Sp denotes Lurie's ∞-category of spectra as in* [Lur17, Definition 1.4.3.1] *and in* [Rav23, §9]*. It is the homotopy limit of the diagram obtained by iterating the loop functor on ∞-category of pointed spaces $\mathcal{S}_*$. We will give a different definition of the ∞-category of orthogonal G-spectra in Definition B.6.*
- (v) *For ∞-categories $\mathcal{C}$ and $\mathcal{D}$, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denotes the ∞-category of functors $\mathcal{C} \rightarrow \mathcal{D}$; see* [Lur09, Proposition 1.2.7.3].
- (vi) *When $\mathcal{C} = K$ for a space or Kan complex K, we will write $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ as $\mathcal{D}^K$. In particular, for a group G with classifiying space BG (or equivalently the nerve of the one object category $\mathcal{B}G$), the ∞-category $\mathcal{C}^{BG}$ is that of* **objects in $\mathcal{C}$ with G-action** [NS18, Definition I.1.2].
- (vii) *Replacing $\mathcal{C}$ by the nerve $N(\mathcal{C}^{\mathrm{op}})$ for on ordinary category $\mathcal{C}$, the ∞-category of functors becomes*

$$\mathrm{Fun}(N(\mathcal{C}^{\mathrm{op}}), \mathcal{D}) =: \mathcal{P}_{\mathcal{D}}(\mathcal{C}),$$

the **category of $\mathcal{D}$-valued presheaves (or contravariant functors) on** $\mathcal{C}$*. Note that for a group G the one object category $\mathcal{B}G$ is isomorphic to its opposite (since each morphisms is invertible), so $\mathcal{C}^{BG} \cong \mathcal{P}_{\mathcal{D}}(\mathcal{C})$.*
- (viii) *An ∞-category is* **perfect** *if it is small, idempotent-complete, and stable. $\mathrm{Cat}_{\infty}^{\mathrm{perf}}$ is the* **∞-category of perfect ∞-categories***.*

Recall that in classical category theory an adjunction for a pair of functors

$$\mathcal{C} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{D}$$

is a natural isomorphism between the morphism sets $\mathcal{C}(X, UY)$ and $\mathcal{D}(FX, Y)$ for objects X in $\mathcal{C}$ and Y in $\mathcal{D}$. For ∞-categories $\mathcal{C}$ and $\mathcal{D}$ we are looking instead for a natural equivalance between morphism spaces.

Definition 5.2. Adjoint functors of ∞-categories. [CSY22, Definition 2.1.1] *and* [Lur09, §5.2, specifically Definition 5.2.2.1]. *For a functor $f : \mathcal{C} \to \mathcal{D}$,*

(i) a **left adjoint of** *f is a pair $(f_!, \eta)$ where where $f_! : \mathcal{D} \to \mathcal{C}$ is a functor and $\eta : \mathrm{Id}_{\mathcal{D}} \implies f \circ f_!$ is a unit natural transformation in the sense of* [Lur09, Definition 5.2.2.7]*, and*

(ii) a **right adjoint of** *f is a pair (f_*, ϵ) where where $f_* : \mathcal{D} \to \mathcal{C}$ is a functor and $\epsilon : f_* \circ f \implies \mathrm{Id}_{\mathcal{C}}$ is a counit natural transformation in the sense of* [Lur09, the dual of Definition 5.2.2.7].

Adjoints of composite functors are suitably defined composites of adjoints as in [CSY22, Definition 2.1.3].

Definition 5.3. The stabilization of an ∞-category. *As noted above, a sequential Ω-spectrum X is a sequence of pointed spaces X_n for $n \geqslant 0$ and pointed weak equivalences $X_n \to \Omega X_{n+1}$. In a pointed ∞-category $\mathcal{C}$ with finite limits, the loop of an object Y is the pullback of the diagram $0 \to Y \leftarrow 0$, so we can define a* **spectrum object in** *$\mathcal{C}$ in the same way we do it in the pointed ∞-category of spaces $\mathcal{S}_*$. In* [Lur17, §1.4] *Lurie denotes the category of such objects by $\mathrm{Sp}(\mathcal{C})$. In* [BGT13] *it is denoted by $\mathrm{Stab}(\mathcal{C})$, the* stabilization of *$\mathcal{C}$. In any case there is a functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ sending a spectrum X object to the object X_0 in $\mathcal{C}$. When $\mathcal{C}$ is presentable (see* [Lur09, Chapter 5]*), this functor admits a left adjoint* Σ^∞ [Lur17, Proposition 1.4.4.4].

Remark 5.4. Spectra with G-action and G-spectra. *The ordinary categories of spectra with G-action (Sp^{BG} as in Definition 4.1), sometimes called* **naive G-spectra***, and of orthogonal G-spectra as in Definition 4.12, are different. The same goes for the ∞-categories Sp^{BG} of Definition 5.1(vi) and Sp^G of Definition B.6.*

5.2. **Limits and colimits.** In Definition 5.1(v), let $f : K \to L$ be a map of Kan complexes. It leads to a pullback functor $f^* : \mathcal{C}^L \to \mathcal{C}^K$ between functor categories induced by precomposition. For suitable $\mathcal{C}$ and f, this functor has left and right adjoints denoted by $f_!$ and f_* [Lur17, Notation 6.1.6.1]. Equivalently they are the functors sending each functor $X : K \to \mathcal{C}$ (a K-shaped diagram in $\mathcal{C}$) to its left and right Kan extensions along f.

When L is a point, these functors from $\mathcal{C}^K$ to $\mathcal{C}$ send X to its colimit and limit. One advantage of working with ∞-categories is that homotopy limits and homotopy colimits, the subject of Bousfield and Kan's "yellow monster" [BK72], are the same as ordinary limits and colimits. The standard reference for this is [Lur09, 4.2.4]. See [Rav23, §9] for a simple illustration.

The following is originally due to Joyal [Joy02, Definition 4.5] and is quoted by Lurie as [Lur09, Definition 1.2.13.4].

Definition 5.5. Limits and colimits in an ∞-category. *For a simplicial set K and an ∞-category $\mathcal{C}$, let $f : K \to \mathcal{C}$ be a simplicial map. This is an object in $\mathcal{C}^K$. When K is a Kan complex, this is a $\mathcal{C}$-valued functor on the ∞-category K of Definition 5.1(ii). An* **object X in $\mathcal{C}$ over** *f is one equipped with compatible maps from X to the images under f of all vertices in K. Equivalently it is a natural transformation from the X-valued constant functor on K to f. The collection of* *all such objects and suitable morphisms (and higher morphisms) between them is an* ∞*-category* $\mathcal{C}_{/f}$ [Lur09, Proposition 1.2.9.3]*, the* **over category of** f*. A* **limit of** f*,* $\lim_K f$*, is a terminal object in it (i.e., an object for which the mapping space* $\mathrm{Map}_{\mathcal{C}_{/f}}(Y, \lim_K f)$ *is contractible for all* Y*), regarded as an object in* $\mathcal{C}$*. Dually, one can construct an* **under category** $\mathcal{C}_{f/}$ *of objects under* f*, and define a colimit of* f *to be an initial object in it regarded as an object in* $\mathcal{C}$*. Such terminal and initial objects may or may not exist depending on* $\mathcal{C}$*.*

In [Lur09, §4.2.4] Lurie shows that Definition 5.5 agrees with the classical theory of homotopy (co)limits when we specialize to the case where $\mathcal{C}$ is the nerve of a topological category.

Definition 5.6. Induced maps to/from a limit/colimit. *Now suppose that in addition to the data of Definition 5.5, we have an* ∞*-functor* $F : \mathcal{C} \to \mathcal{D}$ *and that* $\mathcal{D}$ *has the needed limits or colimits. Then* F *induces a functor*

$$F_{/f} : \mathcal{C}_{/f} \to \mathcal{D}_{/Ff}.$$

Then we have objects $F(\lim_K f)$*, the functor of the limit, and* $\lim_K (Ff)$*, the limiting value of the functor, in* $\mathcal{D}$*. Since the latter is a terminal object, we have a unique (up to homotopy) map* $\epsilon : F(\lim_K f) \to \lim_K (Ff)$*, the* **coassembly map of** F **and** f*. If it is an equivalence, we say that* F **preserves** *the limit of* f*. Dually we have the* **assembly map***,* $\eta : \operatorname{colim}_K (Ff) \to F(\operatorname{colim}_K f)$*.*

Lurie discusses this situation in [Lur09, page 48] but does not give names to the two maps.

Definition 5.7. Assembly and coassembly maps for G**-spectra.** *For compact Lie group* G*, let* $\mathcal{B}G$ *be the one object topological category of Definition 2.25. Let* $K = N(\mathcal{B}G)$ *(so* $|K| = BG$*, the classifying space of* G*) and let* $\mathcal{C}$ *be the* ∞*-category of spaces or spectra. Then* $f : BG \to \mathcal{C}$ *defines a* G*-action on an object* X *in* $\mathcal{C}$*. The limit and colimit are* X^{hG} *and* X_{hG}*, the homotopy fixed points (A.9) and homotopy orbits (A.13). Thus we have the unit map* $\eta : (FX)_{hG} \to F(X_{hG})$ *and the counit map* $\epsilon : F(X^{hG}) \to (FX)^{hG}$ *of* F *and* X*, the assembly and coassembly maps. The map of (1.1) is an example of a coassembly map.*

The norm map. We can ask for a norm map $\mathrm{Nm}_G : X_{hG} \to X^{hG}$ as in (4.35) when G is finite. In [Lur17, 6.1.6] Lurie shows that for any finite group G, the norm map Nm_G exists and has a cofiber whenever $\mathcal{C}$ is a stable ∞-category with countable limits and colimits. This cofiber is the **categorical Tate construction** X^{tG}. In [Rav26] we will consider situations in which it is defined for spaces that are π-finite (meaning all homotopy groups are finite and only finitely many of them are nontrivial), such as BG, and certain ∞-categories (such as that of $K(n)$-local spectra) for which it is always an equivalence, rendering the Tate construction contractible.

5.3. **Additional structures on** ∞**-categories.** The first four parts of the following are similar to [NS18, Definition I.1.1].

Definition 5.8. Some properties of ∞**-categories.** *Let* $\mathcal{C}$ *be a* ∞*-category.*

(i) [Lur17, Definition 1.1.1.1] $\mathcal{C}$ *is* **pointed** *if it has an object, usually denoted by 0, that is both initial and final.*

(ii) [Lur17, Definition 6.1.6.13] $\mathcal{C}$ *is* **semiadditive** *(called* preadditive *by Nikolaus and Scholze) if it is pointed, has finite products and coproducts, and for any two objects* $X, Y \in \mathcal{C}$*, the map*

$$\begin{pmatrix} 1_X & 0 \\ 0 & 1_Y \end{pmatrix} : X \sqcup Y \to X \times Y$$

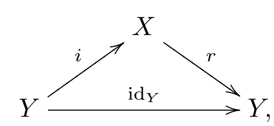
is an equivalence. Here, $0 \in \mathrm{Map}_{\mathcal{C}}(X, Y)$ *denotes the composition* $X \to 0 \to Y$ *for any zero object* $0 \in \mathcal{C}$*. In this case we write* $X \oplus Y$ *for this object, and note that* $\pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y)$ *has a natural commutative monoid structure.*

(iii) $\mathcal{C}$ *is* **additive** *if it is semiadditive and* $\pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y)$ *has a natural abelian group structure.*

(iv) [Lur17, Definition 1.1.1.9] $\mathcal{C}$ *is* **stable** *if it is additive and the loop functor* $\Omega : \mathcal{C} \to \mathcal{C}$ *sending* X *to* $0 \times_X 0$ *is an equivalence. A functor between stable* ∞*-categories is* **exact** *if it preserves finite colimits, or equivalently, finite limits,* [Lur17, Proposition 1.1.4.1].

(v) [Mat16, Definition 2.1] *A* **stable homotopy theory** *is a stable* ∞*-category which is also a symmetric monoidal* ∞*-category* $(\mathcal{C}, \otimes, \mathbf{1})$ *(Definition D.14) in which the tensor product commutes with all colimits.*

(vi) [Lur09, 4.4.5] *An object* Y *in* $\mathcal{C}$ *is a* **retract** *of an object* X *if there exists a 2-simplex* $\mathbf{\Delta}^2 \to \mathcal{C}$ *corresponding to a diagram*



or equivalently if Y *is a retract of* X *in the classical sense in the homotopy category* $h\mathcal{C}$*. It follows that the endomorphism* $i \circ r : X \to X$ *is idempotent and that the object* Y *is both the equalizer and coequalizer of the pair* $(\mathrm{id}_X, i \circ r)$*.* $\mathcal{C}$ *is* **idempotent complete** *if every idempotent endomorphism (that is, every* $\mathcal{C}$*-valued functor on the one object category equipped with an idempotent map) arises in this way, meaning that said equalizer/coequalizer always exists. An* **idempotent completion of** $\mathcal{C}$ *is a fully faithful functor* $f : \mathcal{C} \to \mathcal{D}$ *where* $\mathcal{D} =: \mathrm{Ind}(\mathcal{C})$ *is idempotent complete and each of its objects is a retract of one in the image of* f*. See Remark 5.12.*

(vii) $\mathcal{C}$ *is* **exact** *if it is small and stable. The* ∞*-category of exact* ∞*-categories and exact functors is denoted by* $\mathrm{Cat}_\infty^{\mathrm{ex}}$*. The* ∞*-category of idempotent complete exact* ∞*-categories and exact functors is denoted by* $\mathrm{Cat}_\infty^{\mathrm{perf}}$*. The relevant example of such is the category of* **compact left (or right) modules over a ring spectrum** R*, which we denote by* $\mathrm{Mod}_R(\mathrm{Sp})$ *or simply* Mod_R*. See* [Lur17, §4.2 and §7.1.1] *for more discussion. An exact functor* $F : \mathcal{A} \to \mathcal{B}$ *between small stable* ∞*-categories is a* **Morita equivalence** *if it induces an equivalence their idempotent completions.*

Definition 5.9. *A functor*

$$E \colon \mathrm{Cat}_\infty^{\mathrm{perf}} \longrightarrow \mathcal{D},$$

valued in a stable presentable ∞*-category* $\mathcal{D}$*, is a* **localizing invariant** *if it satisfies the following conditions:*

(i) ***Morita invariance.*** *If* $F\colon \mathcal{C} \to \mathcal{C}$ *is an exact functor that induces an equivalence on idempotent completions (see Definition 5.8(vi))*

$$\mathrm{Ind}(\mathcal{C}) \xrightarrow{\simeq} \mathrm{Ind}(\mathcal{D}),$$

then $E(F)$ *is an equivalence.*

(ii) ***Exactness.*** *For every exact sequence of small stable* ∞*-categories*

$$\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C},$$

the induced sequence

$$E(\mathcal{A}) \longrightarrow E(\mathcal{B}) \longrightarrow E(\mathcal{C})$$

is a fiber (equivalently, cofiber) sequence in $\mathcal{D}$*.*

(iii) ***Filtered colimits.*** *The functor* E *preserves filtered colimits:*

$$E\Big(\operatorname*{colim}_i \mathcal{C}_i\Big) \simeq \operatorname*{colim}_i E(\mathcal{C}_i).$$

We know (see [Lur09, §5.1.3]) that for any ∞-category $\mathcal{C}$ there is a mapping space functor

$$\mathrm{Map}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S},$$

meaning that the set of morphisms $X \to Y$ comes equipped with a topology. When $\mathcal{C}$ is pointed, we get a pointed morphism space. When $\mathcal{C}$ is stable as in (iv), this functor lifts to

$$\mathrm{map}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp} \qquad \text{with} \qquad \Omega^{\infty}\mathrm{map}_{\mathcal{C}}(X,Y) = \mathrm{Map}_{\mathcal{C}}(X,Y). \tag{5.10}$$

In other words, in addition to a space one has a **spectrum** $\mathrm{map}_{\mathcal{C}}(X,Y)$ (sometimes called the **function spectrum**) of maps between objects in a stable ∞-category. The 0th space of this mapping spectrum is the previously defined mapping space. In the category Sp itself we have

$$\mathrm{map}_{\mathrm{Sp}}(\mathbb{S}, X) \simeq X,$$

but this need not be the case in a stable subcategory such as CycSp or CycSp_p as in Definition 5.24. We also define the Spanier-Whitehead dual of a spectrum X by

$$\mathbb{D}X := \mathrm{map}_{\mathrm{Sp}}(X, \mathbb{S}). \tag{5.11}$$

For a space X, we will write $\mathbb{D}\Sigma^{\infty}X$ abusively as $\mathbb{D}X$.

Remark 5.12. Idempotent completion. *Lurie shows in* [Lur09, Proposition 5.1.4.2] *that the idempotent completion of (vi) always exists, and in* [Lur17, Corollary 1.1.3.7] *that it is stable when* $\mathcal{C}$ *is.* [Lur09, Propositions 5.1.4.9] *implies the inclusion functor* $\mathrm{Cat}_{\infty}^{\mathrm{perf}} \to \mathrm{Cat}_{\infty}^{\mathrm{ex}}$ *has a left adjoint which is the idempotent completion functor* Ind *denoted in* [BGT13] *by* Idem*.*

Definition 5.13. R**-linearity.** *Let* R *be an* $\mathbb{E}_n$*-ring spectrum for* $n \geqslant 1$*. A stable* ∞*-category* $\mathcal{C}$ *is* R**-linear** *if:*

- *It is enriched over* Mod_R*, the* ∞*-category of* R*-module spectra as in Definition 5.8(vii), meaning that for objects* $X, Y \in \mathcal{C}$*, the mapping spectrum* $\mathrm{map}_{\mathcal{C}}(X,Y)$ *is an* R*-module.*
- *Tensoring with* R*-modules acts on* $\mathcal{C}$*, and this action is compatible with the stable structure.*

$\mathrm{Cat}_{R,\infty}^{\mathrm{perf}}$ *denotes the* ∞*-category small, idempotent-complete* R*-linear stable* ∞*-categories and* R*-linear functors between them.*

Definition 5.14. [NS18, Definition II.1.4] *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be* ∞*-categories with functors* $F_0, F_1 : \mathcal{C} \to \mathcal{D}$*. The* **lax equalizer** $\mathrm{LEq}(F_0, F_1)$ *of* F_0 *and* F_1*, also written as*

$$\mathrm{LEq}\left(\mathcal{C} \overset{F_0}{\underset{F_1}{\rightrightarrows}} \mathcal{D}\right),$$

is the pullback in the following diagram of simplicial sets.

$$\begin{array}{ccc} \mathrm{LEq}(F_0, F_1) & \longrightarrow & \mathcal{D}^{\mathbf{\Delta}^1} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathcal{C} & \xrightarrow{(F_0, F_1)} & \mathcal{D} \times \mathcal{D} \end{array}$$

In particular, an object of $\mathrm{LEq}(F_0, F_1)$ *is a pair* (c, f)*, where* c *is an object in* $\mathcal{C}$ *and* $f : F_0(c) \to F_1(c)$ *is a morphism in* $\mathcal{D}$*.*

Similarly the **equalizer** $\mathrm{Eq}(F_0, F_1)$ *is the pullback of the diagram above with the right column replaced by the diagonal functor* $\Delta : \mathcal{D} \to \mathcal{D} \times \mathcal{D}$*. An object in it is an object* c *in* $\mathcal{C}$ *on which the two functors agree.*

The word "lax" above refers to the fact that the two images of c in $\mathcal{D}$ are related by a morphism rather than equality.

Lemma 5.15. The Tate orbit and fixed point lemmas. [NS18, Lemmas I.2.1 and I.2.2] *Let* X *be a spectrum with an action of* C_{p^2}*. Recall that the quotient group* $\overline{\mathrm{C}}_p := \mathrm{C}_{p^2}/\mathrm{C}_p$ *acts residually on* $X_{h\mathrm{C}_p}$ *and* $X^{h\mathrm{C}_p}$*.*

(i) If X *is bounded below, meaning that* $\pi_i X = 0$ *for* $i \ll 0$*, then*

$$(X_{h\mathrm{C}_p})^{t\overline{\mathrm{C}}_p} \simeq \mathrm{pt}.$$

(ii) If X *is bounded above, meaning that* $\pi_i X = 0$ *for* $i \gg 0$*, then*

$$(X^{h\mathrm{C}_p})^{t\overline{\mathrm{C}}_p} \simeq \mathrm{pt}.$$

5.4. **The** ∞**-categorical construction of** THH**.** Here we recall the results of [NS18, §III.1], referring the reader to that paper for the proofs.

Proposition 5.16. Exactness of T_p**.** [NS18, Proposition III.1.1] *The functor*

$$T_p : \mathrm{Sp} \to \mathrm{Sp} \qquad X \mapsto (X^{\otimes p})^{t\mathrm{C}_p},$$

where C_p *acts on* $X^{\otimes p}$ *by permuting its factors, is exact as in Definition 5.8(iv).*

This functor is studied by Lunøe-Nielsen and Rognes in [LNR12], where they call it the **topological Singer construction**.

Proposition 5.17. [NS18, Proposition III.1.2]. *Let* $\mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Sp}, \mathrm{Sp})$ *be the category of exact endofunctors of* Sp*, and let* $\mathrm{Id}_{\mathrm{Sp}} \in \mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Sp}, \mathrm{Sp})$ *be the identity functor. For any* $F \in \mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Sp}, \mathrm{Sp})$*, evaluation at the sphere spectrum* $\mathbb{S}$ *induces an equivalence*

$$\mathrm{Map}_{\mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Sp},\mathrm{Sp})}(\mathrm{Id}_{\mathrm{Sp}}, F) \xrightarrow{\simeq} \Omega^\infty F(\mathbb{S}).$$

Definition 5.18. [NS18, Definition III.1.4] **The Tate diagonal** *is the natural transformation*

$$\begin{aligned} \Delta_p : \mathrm{Id}_{\mathrm{Sp}} &\to T_p \\ X &\mapsto (X^{\otimes p})^{t\mathrm{C}_p} \end{aligned}$$

of endofunctors of Sp *which, under the equivalence of Proposition 5.17, corresponds to the map*

$$\mathbb{S} \to \mathbb{S}^{h\mathrm{C}_p} \to \mathbb{S}^{t\mathrm{C}_p}.$$

Recall that the Segal Conjecture for the group C_p implies that the map above is p-adic completion.

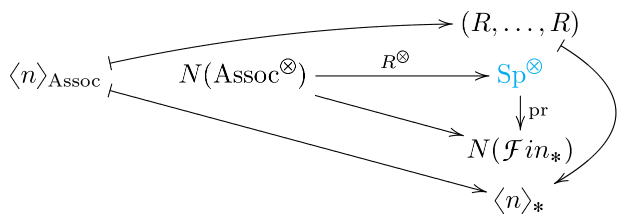
Theorem 5.19. The generalized Segal Conjecture. [NS18, Theorem III.1.7] *For any bounded below spectrum X, the Tate diagonal map*

$$\Delta_p : X \to (X^{\otimes p})^{t\mathrm{C}_p}$$

is p-adic completion.

The following makes use of the definition of a symmetric monoidal ∞-category and related notions in Appendix D.

Definition 5.20. [NS18, Definition III.2.1] *The* **∞-category** $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$ **of $\mathbb{E}_1$-ring spectra** *is the ∞-category of functors $R^{\otimes}$ from $N(\mathrm{Assoc}^{\otimes})$ to $\mathrm{Sp}^{\otimes}$ over $N(\mathcal{F}in_*)$ as shown below, where the sequence $(R, \ldots, R)$ has n coordinates and the triangle commutes.*



Definition 5.21. [NS18, Definition III.2.3] **The ∞-categorical form of** THH *sends an $\mathbb{E}_1$-ring spectrum $R^{\otimes}$ as above to the geometric realization of the cyclic spectrum*

$$\begin{array}{ccccccc} N(\mathbf{\Lambda}^{\mathrm{op}}) & \xrightarrow{N(V^{\mathrm{op}})} & N(\mathrm{Assoc}^{\otimes}_{\mathrm{act}}) & \xrightarrow{R^{\otimes}} & \mathrm{Sp}^{\otimes} & \xrightarrow{\otimes} & \mathrm{Sp} \\ [n]_{\mathbf{\Lambda}} & \longmapsto & & & & & R^{\otimes(n+1)}, \end{array}$$

where V^{op} is as in (2.53).

The following is needed by Hahn and Wilson [HW22] in their study of the relative spectrum $\mathrm{THH}(BP\langle n\rangle/MU)$.

Definition 5.22. Relative THH. *Let A be an $\mathbb{E}_\infty$-ring spectrum, and let Mod_A be the ∞-category of A-module spectra. It is a special case of the category* $\mathrm{LMod}_A(\mathrm{Sp})$ *of* [Lur17, Notation 7.1.1.1]. *(Lurie only requires A to be associative, so he has to distinguish between left and right modules. Since our A is commutative, modules over it are two sided.)* Mod_A *is symmetric monoidal under the relative smash product $X \otimes_A Y$, which is defined to be the coequalizer of the two maps*

$$X \otimes A \otimes Y \rightrightarrows X \otimes Y$$

induced by the right A-module structure on X and the left one on Y.

For an $\mathbb{E}_1$-algebra $R^{\otimes}$ in Mod_A, $\mathrm{THH}(R/A)$ the geometric realization of the cyclic spectrumin the category of A-modules,

$$\begin{array}{ccccccc} N(\mathbf{\Lambda}^{\mathrm{op}}) & \xrightarrow{N(V^{\mathrm{op}})} & N(\mathrm{Assoc}^{\otimes}_{\mathrm{act}}) & \xrightarrow{R^{\otimes_A}} & \mathrm{Mod}_A^{\otimes} & \xrightarrow{\otimes} & \mathrm{Mod}_A \\ [n]_{\mathbf{\Lambda}} & \longmapsto & & & & & R^{\otimes_A(n+1)}. \end{array}$$

5.5. **The ∞-category of cyclotomic spectra.** Defining the ∞-category Sp^G of orthogonal G-spectra involves some technicalities that would distract us here, so we postpone that discussion to Appendix B, specifically Definition B.6. The definition of a symmetric monoidal structure on an ∞-category $\mathcal{C}$ is also complicated and is the subject of Appendix D, specifically Definition D.14.

Definition 5.23. [NS18, Definition II.1.1]

(i) *A* **cyclotomic spectrum** *X is an object in $\mathrm{Sp}^{\mathbb{T}}$ (see Definition B.6) with a $\mathbb{T}$-equivariant* **cyclotomic structure map** *(which Nikolaus and Scholze call the* Frobenius map*)*

$$\varphi_p : X \to X^{t\mathrm{C}_p},$$

where $X^{t\mathrm{C}_p}$ is as in (4.35), for each prime p. We will write this as $(X,(\varphi_p)_{p\in\mathbb{P}})$, where $\mathbb{P}$ denotes the set of primes.

(ii) *A p-***cyclotomic spectrum** *X is an object in $\mathrm{Sp}^{\mathrm{C}_{p^\infty}}$ (where $\mathrm{C}_{p^\infty} \subset \mathbb{T}$ is the Prüffer group consisting of the p^ith roots of unity for all $i \geqslant 0$) with a C_{p^∞}-equivariant map φ_p as above.*

(iii) *A p-***cyclotomic spectrum with Frobenius lift** *X is an object in $\mathrm{Sp}^{\mathrm{C}_{p^\infty}}$ with a C_{p^∞}-equivariant map $F : X \to X^{h\mathrm{C}_p}$ such that $\varphi_p = s_X^{\mathrm{C}_p} F$ for $s_X^{\mathrm{C}_p}$ as in (4.35). This map F, which is a lifting of φ_p, is not to be confused with the restriction map F_H^G of Definition A.1(ii).*

The word "Frobenius" (as a map) has several meanings in the literature. It has long been used for the pth power map in an algebra in characteristic p. Nikolaus-Scholze use it for the cyclotomic structure map while McCandless uses it for a lifting of the latter to $X^{h\mathrm{C}_p}$ as in (iii) above. McCandless also has a more general notion of Frobenius lift in an ∞-category in [McC24, Definition 2.1.2] that involves the rotation-power monoid $\mathscr{M} := \mathbb{T} \rtimes \mathbb{N}^{\times}$ of Definition 2.70; see Definition 5.48.

Definition 5.24. [NS18, Definition II.1.6] *and* [BHLS23, Definition 2.3].

(i) *The ∞-***category** CycSp **of cyclotomic spectra** *is the lax equalizer (as in Definition 5.14) in which $\mathcal{C}$ and $\mathcal{D}$ are each $\mathrm{Sp}^{\mathbb{T}}$, and the two functors are the identity and the one induced by $\prod_p \varphi_p$.*

(ii) *That of p-***cyclotomic spectra***, CycSp_p, is the lax equalizer in which $\mathcal{C}$ and $\mathcal{D}$ are each $\mathrm{Sp}^{\mathrm{C}_{p^\infty}}$, and the two functors are the identity and the one induced by φ_p.*

(iii) *That of p-***cyclotomic spectra with Frobenius lifts***, $\mathrm{CycSp}_p^{\mathrm{Fr}}$, is the lax equalizer in which $\mathcal{C}$ and $\mathcal{D}$ are each $\mathrm{Sp}^{\mathrm{C}_{p^\infty}}$, and the two functors are the identity and the one induced by F as in Definition 5.23(iii).*

Remark 5.25. *All three of the above are presentable stable ∞-categories whose objects are spectra with additional structure. In each case the forgetful functor to Sp is exact and preserves equivalences and small colimits. Hence all three are* *enriched over spectra as in (5.10). In each case function spectrum* $\mathrm{map}_{\mathcal{C}}(X,Y)$ *differs substantially from the underlying function spectra* $\mathrm{map}_{\mathrm{Sp}}(X,Y)$.

Proposition 5.26. [KN21b, Proposition 10.3] *The forgetful functor*

$$U : \mathrm{CycSp}_p^{\mathrm{Fr}} \to \mathrm{CycSp}_p \qquad (X,F) \mapsto (X, s_X^p F)$$

is left adjoint to the functor TR *of Definition 5.80.*

Nikolaus and Scholze show that the following are examples.

Example 5.27. Some cyclotomic spectra.

(i) For an $\mathbb{E}_1$*-ring spectrum* R, $\mathrm{THH}(R)$ *is cyclotomic. It is defined as the geometric realization of a cyclic spectrum spelled out in* [NS18, Definition III.2.3] *and in Theorem 5.28.*

(ii) Every spectrum X *has a trivial action of* $\mathbb{T}$ *and a trivial cyclotomic structure whose pth component is given by the composite*

$$X \longrightarrow X^{h\mathrm{C}_p} \xrightarrow{s_X^{\mathrm{C}_p}} X^{t\mathrm{C}_p}$$

where the first map is pullback along $B\mathrm{C}_{p+} \to S^0$ *(see (A.12)) and the second one is as in (4.35). The Segal Conjecture implies that this composite is p-adic completion when* X *is finite. We sometimes denote the resulting cyclotomic spectrum by* X^{triv}.

(iii) Every cyclotomic spectrum is a p-cyclotomic spectrum by restriction.

Theorem 5.28. The cyclotomic structure on $\mathrm{THH}(A)$ *as in Definition 5.21 is the composite map from the geometric realization of the top row to that of the bottom row in the following diagram of cyclic spectra.*

$$
(5.29)\qquad
\begin{array}{ccccccc}
 & & \overset{\mathrm{C}_3}{\circlearrowright} & & \overset{\mathrm{C}_2}{\circlearrowright} & & \\
\cdots & \Rrightarrow & A^{\otimes 3} & \Rrightarrow & A^{\otimes 2} & \rightrightarrows & A \\
 & & \downarrow \Delta_p & & \downarrow \Delta_p & & \downarrow \Delta_p \\
\cdots & \Rrightarrow & ((A^{\otimes 3})^{\otimes p})^{t\mathrm{C}_p} & \Rrightarrow & ((A^{\otimes 2})^{\otimes p})^{t\mathrm{C}_p} & \rightrightarrows & (A^{\otimes p})^{t\mathrm{C}_p} \\
 & & \downarrow \otimes & & \downarrow \otimes & & \downarrow \otimes \\
\cdots & \Rrightarrow & (A^{\otimes 3})^{t\mathrm{C}_p} & \Rrightarrow & (A^{\otimes 2})^{t\mathrm{C}_p} & \rightrightarrows & (A)^{t\mathrm{C}_p},
\end{array}
$$

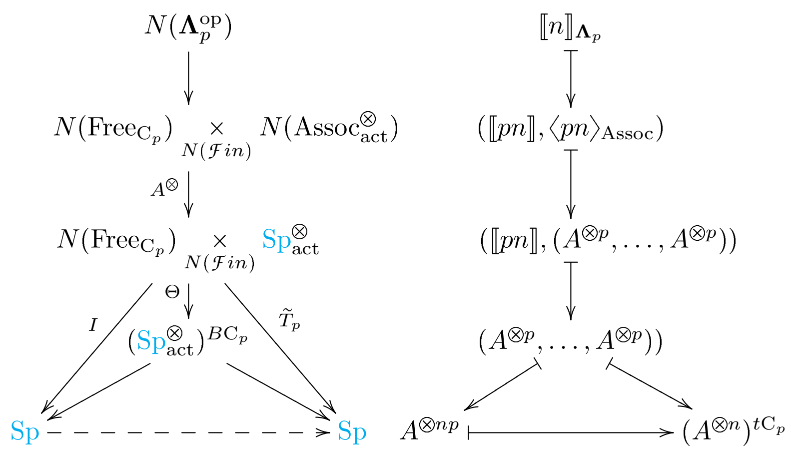
where Δ_p *is the Tate diagonal of Definition 5.18 and the action of* C_p *on the spectra in the bottom row is trivial.*

Proof. The geometric realizations of the first and third rows of (5.29) are $\mathrm{THH}(A)$ and $\mathrm{THH}(A)^{t\mathrm{C}_p}$, and the composite map between them is the desired cyclotomic structure on the former. The second row requires further explanation. The p-fold subdivision endofunctor sd_p of $\mathbf{\Delta}$ (as in Definition 2.55) induces one on $\mathbf{\Lambda}_p$ (in which $\mathbf{\Delta}$ is a wide subcategory), which we follow by the projection onto $\mathbf{\Lambda}$ to give a functor which we abusively denote by

$$
\begin{array}{ccc}
\mathbf{\Lambda}_p^{\mathrm{op}} & \xrightarrow{\quad \mathrm{sd}_p \quad} & \mathbf{\Lambda}^{\mathrm{op}} \\
{[n-1]_{\mathbf{\Lambda}_p} = [\![n]\!]_{\mathbf{\Lambda}_p}} & \longmapsto & [\![pn]\!]_{\mathbf{\Lambda}} = [pn-1]_{\mathbf{\Lambda}}
\end{array}
$$

The object on the lower right is a free C_p-set. We denote the category of such sets by $\mathrm{Free}_{\mathrm{C}_p}$ and a free C_p-set with pn elements by $[\![pn]\!]$. For such a set S, we denote its orbit set by $\overline{S}$. Thus we have a functor $\mathrm{Free}_{\mathrm{C}_p} \to \mathcal{F}in$ sending S to $\overline{S}$, and for each $n > 0$ a composite

(5.30)



Here an object in the third category is a pair $(S, (X_{\overline{s}})_{\overline{s}\in\overline{S}})$ where S is a free C_p-set and each $X_{\overline{s}}$ is a spectrum. This category admits a functor Θ to $(\mathrm{Sp}^{\otimes}_{\mathrm{act}})^{B\mathrm{C}_p}$ by [NS18, Proposition III.3.6], where it is not named. The functors I and $\widetilde{T}_p$ are given by

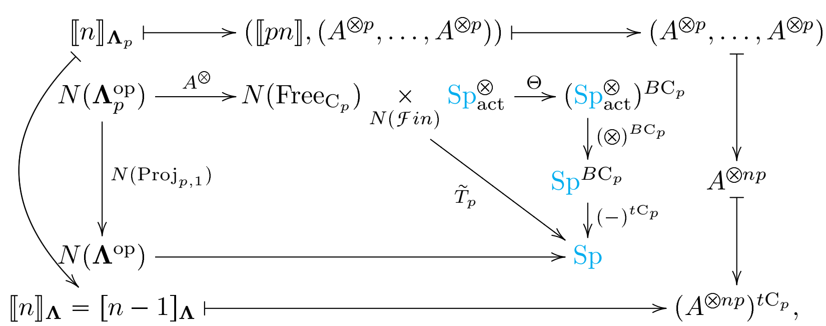
$$(S, (X_{\overline{s}})_{\overline{s}\in\overline{S}}) \mapsto \bigotimes_{\overline{s}\in\overline{S}} X_{\overline{s}} \qquad \text{and} \qquad (S, (X_{\overline{s}})_{\overline{s}\in\overline{S}}) \mapsto \left(\bigotimes_{s\in S} X_s\right)^{t\mathrm{C}_p},$$

where the action of C_p on the smash product on the right is induced by its free action on S. Both functors are lax symmetric monoidal. In [NS18, Lemma III.3.7] they show that I is an initial object in the ∞-category of all such functors satisfying a certain exactness condition. [NS18, Corollary III.3.8] asserts there is a unique natural transformation from I to $\widetilde{T}_p$, which is indicated by the broken arrow in the bottom row of (5.30).

In the image of $[\![n]\!]_{\mathbf{\Lambda}_p}$ in the fourth category of (5.30), there are n spectrum coordinates, and C_p acts on each of them by permuting its factors.

Thus we have a composite

(5.31)



where $\mathrm{Proj}_{p,1}$ is the projection functor of (2.52). The middle row of (5.29) is defined by the bottom functor $N(\mathbf{\Lambda}^{\mathrm{op}}) \to \mathrm{Sp}$ above. □

5.6. **Polygonic spectra.** Polygonic spectra, which are controlled by a truncation set (Definition 5.33) of natural numbers T, are first defined and studied by Krause, McCandless and Nikolaus in [KMN23]. The case where $T = \{1, p\}$ is reviewed in [BHLS23, §2.1.3], and it figures in their definition of the Dehn twist trivialization of [BHLS23, §4.2].

Here is the topological analog of (2.4) and (2.5).

Definition 5.32. *The* **topological Hochschild homology** $\mathrm{THH}(R; M)$ *of an* $\mathbb{E}_1$*-ring spectrum* R **with coefficients** *in an* R*-bimodule* M *is the geometric realization of a simplicial spectrum informally depicted by*

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; M \otimes R \otimes R \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; M \otimes R \;\substack{\longrightarrow\\\longrightarrow}\; M.$$

The simplicial structure here does not extend to a cyclic one, so the spectrum $\mathrm{THH}(R, M)$ does not have a cyclotomic structure. The point of polygonic spectra, which we now define, is to provide a substitute for such a structure, as explained in [KMN23].

For them, part of the data is the following.

Definition 5.33. [KMN23, Definition 2.1 and Example 2.2]. *A* **truncation set** T *is a set of positive integers closed under divisors. Examples include*

$$
\begin{aligned}
\langle n \rangle &:= \{m \in \mathbb{N}_{>0} : m | n\} && \text{for } n \in \mathbb{N}_{>0}\\
\langle \infty \rangle &:= \{m \in \mathbb{N}_{>0}\}\\
\langle p^\infty \rangle &:= \left\{p^k : k \geqslant 0\right\} && \text{for } p \text{ prime}\\
T/n &:= \{t \in T : nt \in T\} && \text{for a truncation set } T \text{ and } n \in \mathbb{N}_{>0}.
\end{aligned}
$$

In [KMN23] and [BHLS23] the first three truncation sets are denoted by $\langle - \rangle$. We use the notation $\langle - \rangle$ instead to avoid conflicting with the notation of Definition D.1.

Definition 5.34. [KMN23, Definition 2.4]. *A* T**-typical polygonic spectrum** $X_{\pentagon}$ *consists of the following data:*

(i) A T*-indexed collection of spectra* $\{X_t : t \in T\}$*, where each* X_t *is an object of* $\mathrm{Sp}^{B\mathrm{C}_t}$.

(ii) For every prime number p *and integer* t *with* $pt \in T$*, a* C_t*-equivariant map of spectra* $\varphi_{p,t} : X_t \to (X_{pt})^{t\mathrm{C}_p}$*, where the Tate construction on* X_{pt} *carries the residual action of* $\mathrm{C}_{pt}/\mathrm{C}_p \cong \mathrm{C}_t$.

For $T = \langle p^k \rangle$, $T = \langle p^\infty \rangle$ *and* $T = \langle \infty \rangle$*, we call these* p^k*-,* p^∞*- and* **integer polygonic spectra.**

Example 5.35. Some polygonic spectra. [KMN23, Example 1.2]

(i) A $\langle 1 \rangle$*-typical polygonic spectrum is an ordinary spectrum.*

(ii) For a cyclotomic spectrum X *with structure maps* $\varphi_p : X \to X^{t\mathrm{C}_p}$ *for each prime* p*, for any* T *we can define a constant* T*-typical polygonic spectrum* $(i_T X)_{\pentagon}$ *by* $(i_T X)_t = X$ *for all* $t \in T$*. Since* X *is a* $\mathbb{T}$*-spectrum it is also a* C_t*-spectrum for each* $t \in T$*, and the required maps* $\varphi_{p,t}$ *are induced by the structure maps* φ_p*. See Theorem 5.46.*

(iii) The topological Hochschild homology of an $\mathbb{E}_1$*-ring spectrum* R *with coefficients in an* R*-bimodule* M *canonically admits the structure of an integer*

polygonic spectrum $\mathrm{THH}(R;M)_{\pentagon}$ *with*

$$\mathrm{THH}(R;M)_t := \mathrm{THH}(R, M^{\otimes_R t})$$

for every $t \geqslant 1$. *This is the motivating example for* [KMN23]. *Their main result is Theorem 5.39.*

(iv) $\mathbb{L} := L_{\langle p^\infty \rangle}\mathbb{S}$ *is the* p^∞*-polygonic spectrum whose* p^j*th component (in the notation of Definition 4.49) is*

$$\mathbb{L}_{p^j} := \mathbb{S}[\mathbb{T}/\mathrm{C}_{p^j}].$$

This spectrum is $L_{\langle p^\infty \rangle} i_{\langle p^\infty \rangle}\mathbb{S}$ *in the notation of Theorem 5.46, as noted in the footnote to* [BHLS23, Example 2.5].

It follows that we have a diagram

$$\begin{array}{ccccccccc} \mathbb{L}_1 & & & & \mathbb{L}_p & & & & \mathbb{L}_{p^2} & & & & \cdots \\ & \searrow^{\varphi_{p,1}} & & \swarrow_{\theta^{\mathrm{C}_p}} & & \searrow^{\varphi_{p,p}} & & \swarrow_{\theta^{\mathrm{C}_p}} & & \searrow^{\varphi_{p,p^2}} & & \swarrow & \\ & & \mathbb{L}_p^{t\mathrm{C}_p} & & & & \mathbb{L}_{p^2}^{t\mathrm{C}_p} & & & & \mathbb{L}_{p^3}^{t\mathrm{C}_p} & & \end{array} \tag{5.36}$$

in which θ^{C_p} *is the map of Definition 4.36. The image of this under the functor* Υ *of Definition 5.37, namely the wedge of all the* $\mathbb{S}[\mathbb{T}/\mathrm{C}_{p^j}]$*, corepresents the functor* TR *of Definition 5.80 in* CycSp_+*, the* ∞*-category of bounded below cyclotomic spectra, by* [BHLS23, Lemma 2.6].

(v) For a p-polygonic spectrum

$$X_{\pentagon} = (X_1, X_p, \varphi_{p,1} : X_1 \to X_p^{t\mathrm{C}_p}),$$

[BHLS23, page 15] *defines*

$$X_{\pentagon}^{\Phi\mathrm{C}_p} := X_1, \quad X_{\pentagon}^{\Phi e} := X_p, \quad X_{\pentagon}^{h\mathrm{C}_p} := X_p^{h\mathrm{C}_p}, \quad X_{\pentagon}^{t\mathrm{C}_p} := X_p^{t\mathrm{C}_p},$$

and $X_{\pentagon}^{\mathrm{C}_p}$ *is the pullback in*

$$\begin{array}{ccc} X_{\pentagon}^{\mathrm{C}_p} & \longrightarrow & X_1 \\ \big\downarrow & \lrcorner & \big\downarrow \varphi_{p,1} \\ X_p^{h\mathrm{C}_p} & \xrightarrow{\ s_{X_p}^{\mathrm{C}_p}\ } & X_p^{t\mathrm{C}_p}, \end{array}$$

where $s_{X_p}^{\mathrm{C}_p}$ *is the map of (4.35).*

Definition 5.37. The cyclotomic spectrum associated with a p^∞**-polygonic spectrum.** [KMN23, Construction 2.20] *For the data*

$$X_{\pentagon} = \left\{ X_{p^j}, \varphi_{p,p^j} : X_{p^j} \to (X_{p^{j+1}})^{t\mathrm{C}_p} : j \geqslant 0 \right\},$$

we define

$$\Upsilon X_{\pentagon} := \bigoplus_{j \geqslant 0} X_{p^j}$$

with structure map $\varphi_p^{\Upsilon X_{\pentagon}} : \Upsilon X_{\pentagon} \to (\Upsilon X_{\pentagon})^{t\mathrm{C}_p}$ *induced by the maps* φ_{p,p^j}.

One can define the cyclotomic spectrum associated with an integer polygonic spectrum in a similar way.

Definition 5.38. ∞-Categories of polygonic spectra. [KMN23, Definition 2.6 and Example 2.9]. *The ∞-category of T-typical polygonic spectra* PgcSp_T *is the lax equalizer as in Definition 5.14 where*

$$\mathcal{C} = \prod_{t \in T} \mathrm{Sp}^{BC_t},$$
$$\mathcal{D} = \prod_{\substack{p \in T \\ p \text{ prime}}} \prod_{s \in T/p} \mathrm{Sp}^{BC_s},$$

and the pth components of the two functors $\mathcal{C} \to \mathcal{D}$ are the identity and the functor

$$F_p : \prod_{t \in T} \mathrm{Sp}^{BC_t} \to \prod_{s \in T/p} \mathrm{Sp}^{BC_s}$$

which for each $s \in T/p$ is induced by the composite

$$\prod_{u \in T} \mathrm{Sp}^{BC_u} \xrightarrow{\mathrm{pr}} \mathrm{Sp}^{BC_{ps}} \xrightarrow{(-)^{tC_{ps}}} \mathrm{Sp}^{BC_s}.$$

For truncation sets $T' \subseteq T$ there is a restriction functor

$$\mathrm{PgcSp}_T \to \mathrm{PgcSp}_{T'}.$$

We abbreviate $\mathrm{PgcSp}_{\langle \infty \rangle}$ *by* PgcSp, *and (as indicated in* [BHLS23, Definition 2.8]*).*

The ∞-category of p-polygonic spectra $\mathrm{PgcSp}_{\langle p \rangle}$ *is the pullback in*

$$\begin{array}{ccc} \mathrm{PgcSp}_{\langle p \rangle} & \longrightarrow & \mathrm{Sp}^{\Delta^1} \\ \downarrow & \lrcorner & \downarrow \mathrm{ev}_1 \\ \mathrm{Sp}^{BC_p} & \xrightarrow{(-)^{tC_p}} & \mathrm{Sp}. \end{array}$$

Theorem 5.39. Polygonic spectra and THH with coefficients. [KMN23, Theorem 6.31] *Let R denote an $\mathbb{E}_1$-ring and let* BMod_R *denote the ∞-category of two sided R-module spectra. The topological Hochschild homology functor with coefficients*

$$\mathrm{THH}(R; -) \colon \mathrm{BMod}_R \to \mathrm{Sp}$$

of Definition 5.32 canonically refines to a functor of ∞-categories

$$\mathrm{THH}(R; -)_{\pentagon} \colon \mathrm{BMod}_R \to \mathrm{PgcSp}$$

with $\mathrm{THH}(R; -)_t$ as in Example 5.35(iii).

We get a similar functor to PgcSp_T *for any T by restriction of truncation sets.*

Definition 5.40. Some functors to and from $\mathrm{PgcSp}_{\langle p \rangle}$. [BHLS23, Definition 2.9]

(i) The functor $\mathrm{res}_{\pentagon} : \mathrm{CycSp} \to \mathrm{PgcSp}_{\langle p \rangle}$ *sends a cyclotomic spectrum X to the p-polygonic spectrum*

$$(X, X, \varphi_p : X \to X^{tC_p}).$$

(ii) We denote by

$$\mathrm{res}_{\varphi} : \mathrm{PgcSp}_{\langle p \rangle} \to \mathrm{Sp}^{\Delta^1}$$

the lax symmetric monoidal functor sending $(X_1, X_p,\ \varphi_{p,1} : X_1 \to X_p^{tC_p})$ *to the morphism* $\varphi_{p,1}$.

Definition 5.41. *Let* EQ *be the* **equalizer category** $\bullet \rightrightarrows \bullet$, *and let* $\mathcal{C}^{\mathrm{EQ}}$ *denote the category of equalizer diagrams in* $\mathcal{C}$.

An object $R \rightrightarrows S$ in $\mathrm{Alg}(\mathrm{Sp})^{\mathrm{EQ}}$ is an R-bimodule structure on the ring spectrum S, so the functor $\mathrm{THH}(R;-)_{\pentagon}$ of Theorem 5.39 leads to

$$\mathrm{THH}_{\pentagon} \colon \mathrm{Alg}(\mathrm{Sp})^{\mathrm{EQ}} \to \mathrm{PgcSp}.$$

Lemma 5.42. [BHLS23, Lemma 2.16] *There is a commuting diagram of lax symmetric monoidal functors*

$$\begin{array}{ccccc}
\mathrm{Alg}(\mathrm{Sp}) & \xrightarrow{\mathrm{THH}} & \mathrm{CycSp} & & \\
\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{\pentagon}} & \searrow & \\
 & & & & \mathrm{Sp}^{\Delta^1} \\
\mathrm{Alg}(\mathrm{Sp})^{\mathrm{EQ}} & \xrightarrow[\mathrm{THH}_{\pentagon}]{} & \mathrm{PgcSp}_{\langle p \rangle}, & \nearrow_{\mathrm{res}_{\varphi}} &
\end{array}$$

where the left vertical map sends an $\mathbb{E}_1$*-algebra* R *to the constant diagram* $R \rightrightarrows R$.

Lemma 5.43. [BHLS23, Lemma 2.17] *Suppose we are given* $(A, B) \in \mathrm{Alg}(\mathrm{Sp})^{E\mathbb{Q}}$ *and* $V \in \mathrm{Alg}(\mathrm{Sp})$ *such that the underlying spectrum of* V *is a dualizable. Then the natural map*

$$\mathrm{THH}_{\pentagon}(\mathbb{S}; V)\ \otimes\ \mathrm{THH}_{\pentagon}(A; B)\ \longrightarrow\ \mathrm{THH}_{\pentagon}\big(A;\, V \otimes B\big),$$

coming from the lax symmetric monoidal structure on $\mathrm{res}_{\varphi}\,\mathrm{THH}(-;-)$, *is an isomorphism.*

Definition 5.44. [McC24, Notation 1.1.1] *The* **spectral affine line** *is the* $\mathbb{E}_\infty$*-ring*

$$\mathbb{S}[x] := \Sigma^\infty_+ \mathbb{N}$$

with multiplication induced by addition in $\mathbb{N}$. $\mathbb{S}[x]/x^n$ *is the following pushout in* $\mathbb{E}_\infty$*-rings*

$$\begin{array}{ccc}
\mathbb{S}[x] & \xrightarrow{x \mapsto x^n} & \mathbb{S}[x] \\
{\scriptstyle x \mapsto 0}\downarrow & & \downarrow \\
\mathbb{S} & \longrightarrow & \mathbb{S}[x]/x^n
\end{array} \tag{5.45}$$

The reduced topological Hochschild homologies of $\mathbb{S}[x]$ *and* $\mathbb{S}[x]/x^n$, $\widetilde{\mathrm{THH}}(\mathbb{S}[x])$ *and* $\widetilde{\mathrm{THH}}(\mathbb{S}[x]/x^n)$, *are the fibers of the maps of cyclotomic spectra*

$$\mathrm{THH}(\mathbb{S}[x]) \to \mathrm{THH}(\mathbb{S}) = \mathbb{S} \qquad \textit{induced by the left map in (5.45)}$$

$$\textit{and} \qquad \mathrm{THH}(\mathbb{S}[x]/x^n) \to \mathbb{S} \qquad \textit{induced by a similar map } \mathbb{S}[x]/x^n \to \mathbb{S}$$

with the trivial cyclotomic structure on $\mathbb{S}$.

For a cyclotomic spectrum X,

$$X \widehat{\otimes} \widetilde{\mathrm{THH}}^{\mathrm{cont}}(\mathbb{S}[\![x]\!]) := \lim_n \big(X \otimes \widetilde{\mathrm{THH}}(\mathbb{S}[x]/x^n)\big)$$

$$\textit{and} \qquad X \widehat{\otimes}\, \Omega \widetilde{\mathrm{THH}}^{\mathrm{cont}}(\mathbb{S}[\![x]\!]) := \lim_n \big(X \otimes \Omega \widetilde{\mathrm{THH}}(\mathbb{S}[x]/x^n)\big).$$

where the limits on the right are induced by the evident maps $\mathbb{S}[x]/x^{n+1} \to \mathbb{S}[x]/x^n$.

For a connective $\mathbb{E}_1$*-ring* R,

$$R[x]/x^n := R \otimes \mathbb{S}[x]/t^n.$$

Note that the $\mathbb{E}_\infty$-ring $\mathbb{S}[x]$ is not the free $\mathbb{E}_\infty$-ring on one generator. However the underlying $\mathbb{E}_1$-ring of $\mathbb{S}[x]$ is the free $\mathbb{E}_1$-ring on one generator since $\mathbb{N}$ is the free $\mathbb{E}_1$-monoid on one generator in the 1-category of spaces.

On [McC24, page 5] McCandless asserts without proof that

$$\widetilde{\mathrm{THH}}(\mathbb{S}[x]) \simeq \bigoplus_{r>0} \mathbb{S}[\mathbb{T}/\mathrm{C}_r].$$

The computation is discussed by Catherine Li in [HYn24, Example 2.11].

Theorem 5.46. The cyclotomic/polygonic adjunction. [KMN23, Theorem 2.24] *For a truncation set* T*, let* i_T : CycSp $:\to$ PgcSp_T *be the functor given in Example 5.35(ii). It has left and right adjoints* L_T *and* R_T*. We omit the subscript when* $T = \langle\infty\rangle$.

For a cyclotomic spectrum X

$$\begin{aligned} &Li(X) \simeq X \otimes \widetilde{\mathrm{THH}}(\mathbb{S}[x]) \\ \textit{and}\quad &Ri(X) \simeq X \hat{\otimes}\, \Omega\widetilde{\mathrm{THH}}^{\mathrm{cont}}(\mathbb{S}[\![x]\!]) := \Omega \lim_n \left(X \otimes \widetilde{\mathrm{THH}}(\mathbb{S}[x]/x^n)\right). \end{aligned}$$

A formula for R_T is given in [KMN23, Definition 2.23].

5.7. **Epicyclic spaces and spectra.** Recall Theorem 2.45, which says that the geometric realization of a cyclic space has an action of the circle group $\mathbb{T}$. In this subsection we will state a similar result for epicyclic spaces in ∞-categorical language, where by Definition 2.68 an epicyclic object in a category $\mathcal{C}$ is a $\mathcal{C}$-valued functor on $\widetilde{\mathbf{\Lambda}}^{\mathrm{op}}$.

Definition 5.47. [McC24, Definition 2.1.14], **An epicyclic object in an ∞-category** $\mathcal{C}$ *is a map of simplicial sets* $N(\widetilde{\mathbf{\Lambda}}^{\mathrm{op}}) \to \mathcal{C}$*, that is a* $\mathcal{C}$*-valued presheaf on* $\widetilde{\mathbf{\Lambda}}$ *as in Definition 5.1(vii).*

For the rotation-power monoid $\mathscr{M}$ of Definition 2.70, we have a one object topological category $\mathcal{B}\mathscr{M}$ with $\mathscr{M}$ as endomorphism space. It contains the self dual $\mathcal{B}\mathbb{T}$ as a subcategory. Recall that a functor $\mathcal{B}\mathbb{T} \to \mathcal{C}$ is an object X in $\mathcal{C}$ equipped with an action of $\mathbb{T}$ and hence with each of its finite subgroups C_r. When $\mathcal{C}$ has finite limits and colimits, the limit and colimit of the functor restricted to $\mathcal{B}\mathrm{C}_r$ are $X^{h\mathrm{C}_r}$ and $X_{h\mathrm{C}_r}$.

Definition 5.48. [McC24, Definition 2.1.2]. *For an* ∞*-category* $\mathcal{C}$*, the* ∞**-category of objects in** $\mathcal{C}$ **with Frobenius lifts** *is*

$$\mathcal{C}^{\mathrm{Fr}} := \mathrm{Fun}(\mathcal{B}\mathscr{M}^{\mathrm{op}}, \mathcal{C}) = \mathcal{P}_{\mathcal{C}}(\mathcal{B}\mathscr{M}),$$

the category of $\mathcal{C}$*-valued presheaves on* $\mathcal{B}\mathscr{M}$*, where* $\mathscr{M}$ *is the rotation-power monoid of Definition 2.70. We omit the subscript when* $\mathcal{C} = \mathcal{S}$*, the* ∞*-category of topological spaces.*

Such a functor defines an object X in $\mathcal{C}$ with a $\mathbb{T}$-action since $\mathcal{B}\mathbb{T} \subseteq \mathcal{B}\mathscr{M}$. If $\mathcal{C}$ has finite limits and colimits, there are $\mathbb{T}$-equivariant maps $\psi_k : X \to \rho_k^* X^{h\mathrm{C}_r}$

for each $k > 0$, for ρ_k^* as in Definition 1.3. These are compatible in that for two integers $k_1, k_2 > 0$, we have a commutative diagram

$$\begin{array}{ccccc} \rho_{k_1}^* X^{h\mathrm{C}_1} & \xleftarrow{\psi_{k_1}} & X & \xrightarrow{\psi_{k_2}} & \rho_{k_2}^* X^{h\mathrm{C}_2} \\ \Big\downarrow{\scriptstyle \rho_{k_1}^*(\psi_{k_2})^{h\mathrm{C}_{k_1}}} & & & & \Big\downarrow{\scriptstyle \rho_{k_2}^*(\psi_{k_1})^{h\mathrm{C}_2}} \\ \rho_{k_1k_2}^*(X^{h\mathrm{C}_{k_1}})^{h\mathrm{C}_{k_2}} & & \xrightarrow{\simeq} & & \rho_{k_2k_1}^*(X^{h\mathrm{C}_{k_2}})^{h\mathrm{C}_{k_1}}. \end{array}$$

Proposition 5.49. [McC24, Proposition 2.2.1]. *The forgetful functor*

$$\mathcal{P}(\mathcal{B}\mathscr{M}) \to \mathcal{P}(\mathcal{B}\mathbb{T})$$

from spaces with Frobenius lefts to $\mathbb{T}$-spaces is conservative and preserves small limits and colimits.

Theorem 5.50. [McC24, Proposition 2.1.15]. **Geometric realizations of cyclic and epicyclic objects.** *Let $\mathcal{C}$ be an ∞-category that admits geometric realizations. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{P}_{\mathcal{C}}(\widetilde{\mathbf{\Lambda}}) & \longrightarrow & \mathcal{P}_{\mathcal{C}}(\mathcal{B}\mathscr{M}) = \mathcal{C}^{\mathrm{Fr}} \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{P}_{\mathcal{C}}(\mathbf{\Lambda}) & \longrightarrow & \mathcal{P}_{\mathcal{C}}(\mathcal{B}\mathbb{T}) = \mathcal{C}^{B\mathbb{T}} \end{array}$$

in which the vertical arrows are induced by the inclusions $\mathbf{\Lambda} \to \widetilde{\mathbf{\Lambda}}$ and $\mathbb{T} \to \mathscr{M}$, and each horizontal arrow is geometric realization of the corresponding simplicial object (induced by the inclusion functors $\mathbf{\Delta} \to \mathbf{\Lambda} \to \widetilde{\mathbf{\Lambda}}$) in $\mathcal{C}$.

Example 5.51. [McC24, Example 2.2.2]. **The free loop space.** *For a space X we define an epicyclic space by*

$$[\![n]\!]_{\widetilde{\mathbf{\Lambda}}} \mapsto \mathrm{Map}_{\mathcal{S}}(|N([\![n]\!]_{\widetilde{\mathbf{\Lambda}}})|, X) \simeq \mathcal{L}(X),$$

so we may regard the free loop space $\mathcal{L}X$ as a space with Frobenius lift by Theorem 5.50. This functor is constant on the objects of $\widetilde{\mathbf{\Lambda}}^{\mathrm{op}}$ but not on its morphisms. The self-map of $\mathcal{L}X$ determined by the cyclic operator τ_n is induced by the self-map of $\mathbb{T}$ given by rotation by $2\pi/(n+1)$ as in §2.5. The self-map of $\mathcal{L}X$ determined by the epicyclic operator π_*^k of (2.65) is induced by the self-map of $\mathbb{T}$ given by the kth power map. This is compatible with the action of $\mathscr{M}$ on $\mathcal{L}X$ of (2.71).

Definition 5.52. *An* $\mathbb{E}_1$**-monoid** M **is a map of simplicial sets** $N(T_{\mathrm{Assoc}}^{\mathrm{op}}) \to \mathcal{S}$, *where T_{Assoc} is the opposite category of Lawvere's algebraic theory of monoids, which is described in Appendix C. Its* **epicyclic bar construction** $B^{\mathrm{epi}}M$ *as in* [McC24, Definition 2.2.4] *is the geometric realization of the epicyclic space given by the composite functor*

$$\begin{array}{ccccc} N(\widetilde{\mathbf{\Lambda}}^{\mathrm{op}}) & \xrightarrow{j^{\mathrm{op}}} & N(T_{\mathrm{Assoc}}^{\mathrm{op}}) & \xrightarrow{M} & \mathcal{S} \\ [\![n]\!] & \longmapsto & \langle n\rangle, & & \end{array}$$

while its **cyclic bar construction** $B^{\mathrm{cyc}}M$ *as in Definition 2.39 is that of*

$$N(\mathbf{\Lambda}^{\mathrm{op}}) \longrightarrow N(\widetilde{\mathbf{\Lambda}}^{\mathrm{op}}) \xrightarrow{j^{\mathrm{op}}} N(T_{\mathrm{Assoc}}^{\mathrm{op}}) \xrightarrow{M} \mathcal{S}.$$

The **epicyclic topological Hochschild homology** $\mathrm{THH}^{\mathrm{epi}}(\mathcal{C})$ *of a small* ∞-*category* $\mathcal{C}$ *is the geometric realization of the epicyclic space given by*

$$[\![n]\!] \mapsto \mathrm{Fun}([\![n]\!], \mathcal{C})^{\simeq},$$

where the superscript on the right denotes the maximal Kan complex within the given simplicial set.

Lemma 5.53. [McC24, Proposition 2.2.7] **The epicyclic bar construction and** THH**.** *For an* $\mathbb{E}_1$*-monoid* M*, let* $\mathcal{B}M$ *denote the* ∞*-category with one object and* M *as endomorphisms. Then there is an equivalence of spaces with Frobenius lifts*

$$B^{\mathrm{epi}}M \to \mathrm{THH}^{\mathrm{epi}}(\mathcal{B}M).$$

5.8. **Topological cyclic homology** TC **and friends.**

Definition 5.54. [NS18, Definition II.1.8] *and* [BHLS23, Definition 2.4]. *Let* $(X, (\varphi_p)_{p\in\mathbb{P}})$ *be a cyclotomic spectrum as in Definition 5.23.*

(i) The **integral topological cyclic homology** $\mathrm{TC}(X)$ **of** X *is the mapping spectrum*

$$\mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, X) \in \mathrm{Sp}$$

for $\mathbb{S}^{\mathrm{triv}}$ *as in Example 5.27(ii).*

(ii) The p**-typical topological cyclic homology** $\mathrm{TC}(X,p)$ *is the mapping spectrum*

$$\mathrm{map}_{\mathrm{CycSp}_p}(\mathbb{S}^{\mathrm{triv}}, X) \in \mathrm{Sp}_p.$$

(iii) Let $R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$ *be an associative ring spectrum. (For a discrete ring* R*, the associative ring spectrum is the Eilenberg-MacLane spectrum* HR*.) Then*

$$\mathrm{TC}(R) := \mathrm{TC}(\mathrm{THH}(R)), \qquad \mathrm{TC}(R,p) := \mathrm{TC}(\mathrm{THH}(R),p),$$
$$\mathrm{TC}^-(R) := \mathrm{TC}^-(\mathrm{THH}(R)), \qquad \mathrm{TC}^-(R,p) := \mathrm{TC}^-(\mathrm{THH}(R),p),$$
$$\mathrm{TP}(R) := \mathrm{TP}(\mathrm{THH}(R)) \quad \text{and} \quad \mathrm{TP}(R,p) := \mathrm{TP}(\mathrm{THH}(R),p)$$

for $\mathrm{TC}^-(-)$, $\mathrm{TC}^-(-,p)$, $\mathrm{TP}(-)$ *and* $\mathrm{TP}(-,p)$ *as in Definition 4.37, namely*

$$\mathrm{TC}^-(X) := X^{h\mathbb{T}}, \qquad \mathrm{TC}^-(X,p) := X^{h\mathrm{C}_{p^\infty}}$$
$$\mathrm{TP}(X) := X^{t\mathbb{T}} \quad \text{and} \quad \mathrm{TP}(X,p) := X^{t\mathrm{C}_{p^\infty}}.$$

Remark 5.55. TC **as a type of fixed point functor.** *A cyclotomic spectrum* X *is a* $\mathbb{T}$*-spectrum with additional structure, as is the sphere spectrum* $\mathbb{S}$*. In the category* $\mathcal{T}^G$ *of pointed* G*-spaces, the fixed point set* X^G *of a pointed* G*-space* X *is the morphism object* $\mathcal{T}^G(S^0, X)$ *by Proposition A.5, where the symmetric monoidal unit* S^0 *has trivial* G*-action. Definition 5.54 has the same flavor. See Remark A.4.*

From Definition 5.54(i) we deduce that

$$\mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, X) = \mathrm{TC}(X) = \mathrm{map}_{\mathrm{Sp}}(\mathbb{S}, \mathrm{TC}(X)),$$

so we have the adjunction of [NS18, Proposition IV.4.14]

$$\mathrm{Sp} \underset{\mathrm{TC}}{\overset{(-)^{\mathrm{triv}}}{\rightleftarrows}} \mathrm{CycSp}. \tag{5.56}$$

If X is an $\mathbb{E}_\infty$-ring spectrum, so is $\mathrm{TC}(X)$, and there is a similar adjunction to (5.56) between the corresponding categories of $\mathbb{E}_\infty$-ring objects.

The following is the best known result of Nikolaus-Scholze and is a major breakthough.

Theorem 5.57. How to compute TC**.** [NS18, Proposition II.1.9]

(i) Let $(X, (\varphi_p)_{p\in\mathbb{P}})$ *be a cyclotomic spectrum. There is a functorial fiber sequence*

$$\mathrm{TC}(X) \longrightarrow X^{h\mathbb{T}} \xrightarrow{(\varphi_p^{h\mathbb{T}}-\mathrm{can})_{p\in\mathbb{P}}} \prod_{p\in\mathbb{P}} (X^{t\mathrm{C}_p})^{h\mathbb{T}},$$

where the maps are given by

$$\varphi_p^{h\mathbb{T}} : X^{h\mathbb{T}} \to (X^{t\mathrm{C}_p})^{h\mathbb{T}}$$

and the pth component of can *is the composite in the diagram*

$$\begin{array}{ccccc} \mathrm{TC}^-(X) & = & X^{h\mathbb{T}} & \xrightarrow{\mathrm{can}_p} & \left(X^{t\mathrm{C}_p}\right)^{h\mathbb{T}} \\ & & \simeq\downarrow & & \uparrow \left(s_X^{\mathrm{C}_p}\right)^{h\mathbb{T}}, \\ & & \left(X^{h\mathrm{C}_p}\right)^{h(\mathbb{T}/\mathrm{C}_p)} & \xrightarrow{\simeq} & \left(X^{h\mathrm{C}_p}\right)^{h\mathbb{T}} \end{array} \tag{5.58}$$

where the left vertical map is the residual action equivalence of Definition A.1, the lower equivalence comes from the p-th root isomorphism $\mathbb{T}/\mathrm{C}_p \cong \mathbb{T}$ *of Definition 1.3, and* $s_X^{\mathrm{C}_p}$ *is as in (4.35).*

(ii) Let (X, φ_p) *be a p-cyclotomic spectrum. There is a functorial fiber sequence*

$$\mathrm{TC}(X,p) \longrightarrow X^{h\mathrm{C}_{p^\infty}} \xrightarrow{(\varphi_p)^{h\mathrm{C}_{p^\infty}}-\mathrm{can}_p} (X^{t\mathrm{C}_p})^{h\mathrm{C}_{p^\infty}}$$

with notation as in (i).

The last statement above identifies $\mathrm{TC}(X,p)$ as an equalizer. A similar identification in terms of TR, the subject of §5.12, is Theorem 5.88.

For a cyclotomic spectrum X, we have $\mathbb{T}$-equivariant maps $\varphi_p : X \to X^{t\mathrm{C}_p}$ from Definition 5.23 and $s_p : X^{h\mathrm{C}_p} \to X^{t\mathrm{C}_p}$ from (4.35). Applying the functor $(-)^{h\mathrm{C}_{p^i}}$ to each gives maps

$$X^{h\mathrm{C}_{p^i}} \xrightarrow{\varphi_{p^{i+1}}} (X^{t\mathrm{C}_p})^{h\mathrm{C}_{p^i}}$$

$$\text{and} \qquad (X^{h\mathrm{C}_p})^{h\mathrm{C}_{p^i}} = X^{h\mathrm{C}_{p^{i+1}}} \xrightarrow{s_{p^{i+1}}} (X^{t\mathrm{C}_p})^{h\mathrm{C}_{p^i}},$$

and $(X^{t\mathrm{C}_p})^{h\mathrm{C}_{p^i}} \simeq X^{t\mathrm{C}_{p^{i+1}}}$ when X is bounded below by [NS18, Lemma II.4.1].

5.9. **The cyclotomic spectra** $\mathrm{THH}(\mathbb{Z}/p)$ **and** $\mathrm{THH}(\mathbb{Z})$**.** Here we will report on the results of [NS18, §IV.4], which the reader should consult for details. They denote our $\mathrm{THH}(\mathbb{Z}/p)$ (as in Definition 5.54(iii)) by $\mathrm{THH}(H\mathbb{F}_p)$. $\mathrm{TC}(\mathbb{Z}/p)$ is first computed in [HM97, Theorem B], and redone by Nikolaus and Scholze using techniques from [BM94].

The latter paper makes use of three "skeleton spectral sequences," one for each of the spectra in the bottom row of (4.38). The Tate version is a whole plane spectral sequence that may or may not converge. The upper and lower half plane portions of it are related to the homotopy orbit and homotopy fixed point spectra. This is explained nicely following the proof of [BM94, Theorem 2.15] and later by Alice Hedenlund and John Rognes in [HR24].

5.9.1. *Toward* $\mathrm{TC}(\mathbb{Z}/p)$. It follows from Theorem 5.57 that in order to find $\mathrm{TC}(\mathbb{Z}/p)$, we must first determine $\mathrm{TC}^-(\mathbb{Z}/p) := \mathrm{THH}(\mathbb{Z}/p)^{h\mathbb{T}}$.

Proposition 5.59. [NS18, Proposition IV.4.6]

$$\pi_i \mathrm{THH}(\mathbb{Z}/p)^{h\mathbb{T}} = \begin{cases} \mathbb{Z}_p & \text{for } i \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

and as a ring,

$$\pi_* \mathrm{THH}(\mathbb{Z}/p)^{h\mathbb{T}} = \mathbb{Z}_p[\tilde{u}, v]/(\tilde{u}v - p)$$

with $\tilde{u} \in \pi_2$ *and* $v \in \pi_{-2}$*, where* $\tilde{u}$ *maps to the class* $u \in \pi_2 \mathrm{THH}(\mathbb{Z}/p)$ *of Theorem 3.18(i).*

The proof makes use of the homotopy fixed point spectral sequence, one of the three skeleton spectral sequences alluded to above, for which

$$E_2^{i,j} = H^i(B\mathbb{T}; \pi_{-j} \mathrm{THH}(\mathbb{Z}/p)) \implies \pi_{-i-j} \mathrm{THH}(\mathbb{Z}/p)^{h\mathbb{T}}.$$

See [HR24, Chapter 5] for more details. They identify the $E_2^{i,j}$ (which they denote by $E^2_{-i,-j}$) in [HR24, Theorem 5.14].

Since $H^* B\mathbb{T} = H^* \mathbb{C}P^\infty$ and $\pi_* \mathrm{THH}(\mathbb{Z}/p)$ are both even dimensional, $E_2^{i,j}$ is nontrivial only when i and j are both even, and the spectral sequence collapses. The equation $\tilde{u}v = p$ is the subject of [NS18, Lemma IV.4.7].

Corollary 5.60. *The action of* $\mathbb{T}$ *on* $\mathrm{THH}(\mathbb{Z}/p)$ *is nontrivial.*

Proof. For any $\mathbb{T}$-action (including the trivial one) on H/p, the homotopy fixed point spectral sequence collapses without any additive extensions for degree reasons, showing that $\pi_*(H/p)^{h\mathbb{T}}$ is an $\mathbb{F}_p$-vector space, namely

$$\pi_i (H/p)^{h\mathbb{T}} \cong H^{-i}(\mathbb{C}P^\infty; \mathbb{Z}/p).$$

If the $\mathbb{T}$-action were trivial on

$$\mathrm{THH}(\mathbb{Z}/p) \simeq \bigoplus_{j \geqslant 0} \Sigma^{2j} H/p, \tag{5.61}$$

then $\pi_* \mathrm{THH}(\mathbb{Z}/p)^{h\mathbb{T}}$ would also be a graded $\mathbb{F}_p$-vector space. By Proposition 5.59, we find that this graded abelian group is torsion free instead. It follows that the splitting of (5.61) is *not* $\mathbb{T}$-equivariant. □

There is a similar spectral sequence for the Tate construction (see [HR24, Chapter 6]), which yields the following.

Corollary 5.62. [NS18, Propositions IV.4.6 and IV.4.9] *As a ring,*

$$\pi_* \mathrm{THH}(\mathbb{Z}/p)^{t\mathbb{T}} = \mathbb{Z}_p[v^{\pm 1}]$$

and for all even integers i*, the map*

$$\pi_i \mathrm{THH}(\mathbb{Z}/p)^{h\mathbb{T}} \cong \mathbb{Z}_p \to \pi_i \mathrm{THH}(\mathbb{Z}/p)^{t\mathbb{T}} \cong \mathbb{Z}_p$$

induced by $\varphi_p^{h\mathbb{T}}$ *is injective. For* $i \leqslant 0$*, it is an isomorphism, while for* $i = 2j \geqslant 0$*, it has image* $p^j \mathbb{Z}_p$*.*

In other words, the ring of Proposition 5.59 is embedded into that of Corollary 5.62 by sending $\tilde{u}$ to pv^{-1}. With this result in hand, an easy application of Theorem 5.57 gives the following.

Corollary 5.63. [NS18, Corollary IV.4.10] *The homotopy type of* $\mathrm{TC}(\mathbb{Z}/p)$ *is*

$$\mathrm{TC}(\mathbb{Z}/p) \simeq H\mathbb{Z}_p \oplus \Sigma^{-1} H\mathbb{Z}_p.$$

This is proved by looking at the long exact sequence of homotopy groups associated with the fiber sequence of Theorem 5.57(ii) for $X = \mathrm{THH}(\mathbb{Z}/p)$. The second and third spectra have even dimensional homotopy groups, so for each integer i we get a 4-term sequence

$$\begin{array}{ccc} 0 \longrightarrow \pi_{2i}\mathrm{TC}(\mathbb{Z}/p) \longrightarrow & \pi_{2i}\mathrm{THH}(\mathbb{Z}/p)^{h\mathbb{T}} = \mathbb{Z}_p & \\ & \Big\downarrow {\scriptstyle \mathrm{can}-\varphi_p^{h\mathbb{T}}} & \\ & \mathbb{Z}_p = \pi_{2i}\mathrm{THH}(\mathbb{Z}/p)^{t\mathbb{T}} & \longrightarrow \pi_{2i-1}\mathrm{TC}(\mathbb{Z}/p) \longrightarrow 0. \end{array}$$

When $i \neq 0$ the vertical homomorphism is the difference between the identity on $\mathbb{Z}_p$ and a map divisible by p, and therefore an isomorphism. For $i = 0$, it is trivial, and the result follows.

Remark 5.64. The simplest instance of chromatic redshift. *Recall that* H/p *has* fp-*type* -1 *(as explained in Example 1.9), and we see that*

$$\mathrm{TC}(\mathbb{Z}/p) := \mathrm{TC}(\mathrm{THH}(\mathbb{Z}/p))$$

has fp-*type* 0*. Note that* $\mathrm{THH}(\mathbb{Z}/p)$ *itself does not have any* fp-*type because its smash product with any nontrivial finite spectrum has infinitely many nontrivial homotopy groups.*

It follows from Corollary 5.63 that there is a unique map (up to unit p-adic scalar)

$$H\mathbb{Z}_p \to \mathrm{TC}(\mathbb{Z}/p) := \mathrm{TC}(\mathrm{THH}(\mathbb{Z}/p)) \tag{5.65}$$

that is surjective in homotopy since $H\mathbb{Z}_p$ is the connective cover $\tau_{\geqslant 0}\mathrm{TC}(\mathbb{Z}/p)$. Under the adjunction of (5.56) this determines a map of cyclotomic spectra $H\mathbb{Z}_p^{\mathrm{triv}} \to \mathrm{THH}(\mathbb{Z}/p)$.

Note that since the underlying spectrum of $\mathrm{THH}(\mathbb{Z}/p)$ is a wedge of evenly suspended copies of $H\mathbb{F}_p$, a map to it from $H\mathbb{Z}_p$ is given by a sequence of even dimensional mod p cohomology classes in the latter with suitable multiplicative properties. We do not know which such sequence our map corresponds to, but we do know this.

Proposition 5.66. A $\mathbb{T}/\mathrm{C}_p$-equivariant equivalence. [NS18, Corollary IV.4.13] *The* $\mathbb{T}$*-equivariant map of* $\mathbb{E}_\infty$*-algebras*

$$H\mathbb{Z}_p^{\mathrm{triv}} \longrightarrow \mathrm{THH}(\mathbb{Z}/p)$$

adjoint to (5.65) induces a $\mathbb{T}/\mathrm{C}_p$*-equivariant equivalence of* $\mathbb{E}_\infty$*-algebras*

$$H\mathbb{Z}_p^{t\mathrm{C}_p} \;\simeq\; \mathrm{THH}(\mathbb{Z}/p)^{t\mathrm{C}_p} =: \mathrm{TP}(\mathbb{Z}/p).$$

In particular,

$$\pi_*\mathrm{THH}(\mathbb{Z}/p)^{t\mathrm{C}_p} \;\cong\; \mathbb{F}_p[v^{\pm 1}].$$

Moreover, the $\mathbb{T} \simeq \mathbb{T}/\mathrm{C}_p$*-equivariant map of* $\mathbb{E}_\infty$*-algebras*

$$\varphi_p\colon \mathrm{THH}(\mathbb{Z}/p) \longrightarrow \mathrm{THH}(\mathbb{Z}/p)^{t\mathrm{C}_p}$$

identifies $\mathrm{THH}(\mathbb{Z}/p)$ *with the connective cover*

$$\tau_{\geqslant 0}\,\mathrm{THH}(\mathbb{Z}/p)^{t\mathrm{C}_p} \;\simeq\; \tau_{\geqslant 0}\,H\mathbb{Z}_p^{t\mathrm{C}_p}.$$

5.9.2. *The Bökstedt-Madsen computation of* $\mathrm{TC}(\mathbb{Z}_p)$. Nikolaus and Scholze do not determine $\mathrm{TC}(H\mathbb{Z}_p)$, but it is treated earlier by Bökstedt and Madsen in [BM94]. In his MathSciNet review Peter May says

> This difficult and important paper gives a pioneering application of equivariant stable homotopy theory to calculations in algebraic K-theory.

The authors determine not just the homotopy groups but the homotopy type of $\mathrm{TC}(\mathbb{Z}_p)$, relating it to BU and $\mathrm{Im}\, J$. They show (without using that terminology since it had not been invented yet) that it has fp-type 1 as in Definition 1.7(ii). See [BM94, (0.7)] for a precise statement. The groups "agree with [the answer] predicted by a generalized version of the Lichtenbaum-Quillen conjecture formulated by Dwyer and Friedlander [DF85]." Bökstedt and Madsen's proof relies on [BM94, Conjecture 4.3], which was proved while their paper was in press by Stavros Tsalidis in [Tsa97].

[BM94] also anticipates the *redshift philosophy* introduced by Christian Ausoni and John Rognes in [AR02], which says that algebraic K-theory and the related functor TC each raise chromatic height (or fp-type as in Definition 1.7) by one. In this case the input is the integer Eilenberg-MacLane spectrum, which has fp-type 0, while the spectra associated with BU and $\mathrm{Im}\, J$ have fp-type 1. The case where the input is the mod p Eilenberg-MacLane spectrum is discussed in §5.9.1.

The first theorem about redshift at all heights is that of Dylan Wilson and Hahn [HW22] proved 28 years later.

There is a complementary notion of *chromatic blueshift* having to do with the Tate construction's lowering chromatic height; see §4.5.2.

5.10. **t-structures and boundedness.** The original definition of a t-structure on a triangulated category is due to Alexander Beilinson, Joseph Bernstein and Pierre Deligne, [BBD82, Definition 1.3.1]. A **t-structure** on a stable ∞-category $\mathcal{C}$ is a system of full sub-∞-categories $\mathcal{C}_{\geqslant n}$ and $\mathcal{C}_{\leqslant n}$ for $n \in \mathbb{Z}$ with certain properties spelled out in [Lur17, Definition 1.2.1.4] and in [AN21, Appendix A]. These properties imply that the subcategories are determined by $\mathcal{C}_{\geqslant 0}$ and $\mathcal{C}_{\leqslant 0}$, the **aisle** and **co-aisle**. $\mathcal{C}_{\geqslant 0}$ is an example of a **prestable ∞-category**, a notion studied in [Lur18b, Appendix C], that abstracts the properties of the ∞-category of connective spectra.

$\mathcal{C}_{\geqslant n}$ and $\mathcal{C}_{\leqslant n}$ are sometimes called the subcategories of **n-connective objects** and **n-coconnective objects**. Objects that are in $\mathcal{C}_{\geqslant n}$ for some n are said to be **bounded below**, and those in $\mathcal{C}_{\leqslant n}$ for some n are said to be **bounded above**. The full subcategories of such objects are denoted by $\mathcal{C}_{+}$ and $\mathcal{C}_{-}$. The subcategory $\mathcal{C}_{\geqslant 0} \cap \mathcal{C}_{\leqslant 0}$ is called **the heart** $\mathcal{C}^{\heartsuit}$. Its homotopy category is known to be abelian. For $m \leqslant n$, let

$$\mathcal{C}_{[m,n]} := \mathcal{C}_{\geqslant m} \cap \mathcal{C}_{\leqslant n}.$$

Let $\mathcal{C}$ be a stable ∞-category $\mathcal{C}$ with t-structure. One has has a **truncation functor** $\tau_{\leqslant n} : \mathcal{C} \to \mathcal{C}_{\leqslant n}$ generalizing the classical nth Postnikov section. The fiber of the map $X \to \tau_{\leqslant n} X$ is denoted by $\tau_{>n} X := \tau_{\geqslant n+1} X$, the generalization of the n-connected cover of X. The functor $\tau_{>n} : \mathcal{C} \to \mathcal{C}_{>n}$ is also called a truncation functor.

We need the following informal definition, which is discussed in much more detail by Weibel in [Wei94, Chapter 10].

Definition 5.67. The derived category $\mathcal{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$. *Let* $\mathrm{Ch}(\mathcal{A})$ *denote the category of chain complexes with objects in* $\mathcal{A}$*. One can define homology and chain homotopy in the usual way. Let* $\mathcal{K}(\mathcal{A})$ *be the category whose objects are chain complexes and whose morphisms are chain homotopy classes of chain maps. A* **quasi-isomorphism** *is a morphism in* $\mathrm{Ch}(\mathcal{A})$ *or* $\mathcal{K}(\mathcal{A})$ *that induces an isomorphism in homology. (This notion is weaker than chain homotopy equivalence.)* $\mathcal{D}(\mathcal{A})$ *is the category obtained from* $\mathcal{K}(\mathcal{A})$ *by formally inverting all quasi-isomorphisms.*

The stable ∞-category of spectra Sp has the **Postnikov t-structure** in which the subcategories are those of spectra with trivial homotopy groups in dimensions outside the range indicated by the subscripts. Its heart is the category of Eilenberg-MacLane spectra with homotopy groups concentrated in dimension 0, which is known to be equivalent to the derived category of abelian groups $\mathcal{D}(\mathcal{A}b)$ of Definition 5.67. Its homotopy category is $\mathcal{A}b$. There is a similar t-structure on Sp^{BG} for any compact Lie group G, in which connectivity and coconnectivity is that of the underlying spectrum.

Definition 5.68. Exactness with respect to t-structures. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be stable ∞-categories each equipped with a t-structure. A functor* $\mathcal{C} \to \mathcal{D}$ *is*

- **left exact** *if sends* $\mathcal{C}_{\geqslant 0}$ *to* $\mathcal{D}_{\geqslant 0}$*,*
- **right exact** *if sends* $\mathcal{C}_{\leqslant 0}$ *to* $\mathcal{D}_{\leqslant 0}$ *and*
- **t-exact** *if both conditions hold, meaning that it sends* $\mathcal{C}^{\heartsuit}$ *to* $\mathcal{D}^{\heartsuit}$*.*

Definition 5.69. Canonical vanishing. *Suppose we are given a* $\mathbb{T}$*-spectrum* X *which is bounded below. We say that:*

(i) [BHLS23, Definition 2.18(1)] *Recall the map* $S_X^G : X^{hG} \to X^{tG}$ *of (4.35) for* $G \subseteq \mathbb{T}$*.* X *satisfies* **weak canonical vanishing** *with parameter* b *(*$\mathrm{WCV}(\leqslant b)$ *for short) if for each* $H = \mathrm{C}_{p^j}$ *with* $1 \leqslant j \leqslant \infty$*, the composite*
$$\tau_{>b} X^{hH} \longrightarrow X^{hH} \xrightarrow{s_X^H} X^{tH}$$
is null.

(ii) [BHLS23, Definition 2.18(2)] *Recall the Euler class of Definition 4.49,* $a_{(1)} : \mathbb{S}^{-(1)} \to \mathbb{S}$*, and let*
$$a_{(1)}^d : X \to \Sigma^{d(1)} X$$
be the smash product of its dth smash power with the twisted suspension $\Sigma^{d(1)} X$*.* X *satisfies* **strong canonical vanishing** *with parameter* b *(*$\mathrm{SCV}(\leqslant b)$ *for short) if there exists some* $d \geqslant 0$ *for which the composition*
$$\tau_{>b} X \longrightarrow X \xrightarrow{a_{(1)}^d} \Sigma^{d(1)} X$$
is $\mathbb{T}$*-equivariantly null.*

They show in [BHLS23, Lemma 2.19] that (ii) implies (i).

5.11. **The surprising Antieau-Nikolaus t-structure on CycSp.** There is a t-structure on CycSp_p due to Nikolaus and Ben Antieau [AN21]. In [BHLS23] it is introduced in §2.1.4, and discussed further in §2.2 and §2.4.

The Antieau-Nikolaus connectivity of a cyclotomic spectrum is the Postnikov connectivity of the underlying spectrum, but cyclotomic coconnectivity is much more interesting, as Theorem 5.71 illustrates. In any case $\mathrm{CycSp}_{p,+}$ (denoted in [AN21] by $\mathbf{CycSp}_p^-$) is the full subcategory of cyclotomic spectra underlain by spectra that are bounded below in the Postnikov sense.

Theorem 5.70. [AN21, Theorem 1] *The ∞-category $\mathrm{CycSp}_{p,+}$ of connective p-typical cyclotomic spectra is the connective part of a unique t-structure on CycSp_p, the ∞-category of p-typical cyclotomic spectra as in Definition 5.24(ii). The heart $\mathrm{CycSp}_p^\heartsuit$ is equivalent to the abelian category of derived V-complete p-typical Cartier modules, to be defined in Definition 5.72(i).*

In their words,

> The existence and uniqueness of such a t-structure is a formal consequence of the fact that $\mathrm{CycSp}_{p,+}$ is presentable and is closed under colimits and extensions in CycSp_p. The difficult part of the theorem is the identification of the heart.
>
> Recall Bökstedt's [Theorem 3.18], which says that
>
> $$\pi_*\mathrm{THH}(\mathbb{F}_p) \cong \mathbb{F}_p[b],$$
>
> a polynomial ring on a degree 2 generator. More generally, using the vanishing of the cotangent complex, one deduces that
>
> $$\pi_*\mathrm{THH}(k) \cong k[b]$$
>
> for any perfect ring k. Our interest in the cyclotomic t-structure was piqued by the discovery of the next result.

Theorem 5.71. [AN21, Theorem 2] *If k is a perfect ring of characteristic p, then* $\mathrm{THH}(k) \in \mathrm{CycSp}_p^\heartsuit$.

Again in the words of [AN21],

> Despite the higher homotopy groups, $\mathrm{THH}(k)$ is discrete [meaning in the heart of the t-structure] as a cyclotomic spectrum. On the Cartier module side of the story, when k is a perfect ring of characteristic p, $\mathrm{THH}(k)$ corresponds to $W(k)$, the ring of p-typical Witt vectors over k, with its Witt vector Verschiebung and Frobenius operations. The fact that $\mathrm{THH}(k)$ is in $\mathrm{CycSp}_p^\heartsuit$ is consistent with the fact, due to Hesselholt–Madsen [HM97, Theorem B], that for perfect fields of characteristic p, that $\pi_i\mathrm{TC}(k) = 0$ for $i > 0$ [see Corollary 5.63]. However, the theorem is much stronger: it says that for any cyclotomic spectrum X with $\pi_i X = 0$ for $i < 0$, one has
>
> $$\mathrm{Map}_{\mathrm{CycSp}_p}\big(\Sigma^k X,\, \mathrm{THH}(k)\big) \simeq \mathrm{pt} \qquad \text{for } k > 0.$$

For any p-typical cyclotomic spectrum X, the homotopy groups with respect to the t-structure of Theorem 5.70 are denoted by $\pi_i^{\mathrm{cyc}} X$. These are objects in the homotopy category of $\mathrm{CycSp}_p^\heartsuit \subseteq \mathrm{CycSp}_p$. Thus, they can be considered either as p-typical cyclotomic spectra, with underlying spectrum with S^1-action and Frobenius

$$\varphi\colon \pi_i^{\mathrm{cyc}} X \longrightarrow \big(\pi_i^{\mathrm{cyc}} X\big)^{tC_p},$$

or as derived V-complete p-typical Cartier modules under the equivalence of Theorem 5.70.

It is known that this t-structure is not compatible with filtered colimits; see [AN21, Example 3.27].

5.11.1. *Cartier modules.* Before describing the Antieau-Nikolaus t-structure more explicitly, we discuss the relevant abelian category, the homotopy category of its heart, and a related category of spectra.

Definition 5.72. Two flavors of Cartier modules. [AN21, Definition 3.1]

(i) *A* p**-typical Cartier module** *is an abelian group* M *equipped with endomorphisms* $\mathbf{V}$ *and* $\mathbf{F}$ *(the* **Verschiebung** *and* **Frobenius** *maps) such that* $\mathbf{FV} = p$. *(We do not require that* $\mathbf{VF} = p$.*)*

Such a module M *is* **derived V-complete** [AN21, Definition 3.24] *if the map*

$$M \to \lim_n M/\mathbf{V}^n$$

is an equivalence in the derived category of abelian groups $\mathcal{D}(\mathcal{A}b)$ *as in Definition 5.67. We will sometimes refer to such an object as a p-typical* **algebraic** *Cartier module to distinguish it from what comes next.*

(ii) *A* p**-typical topological Cartier module** *is a* $\mathbb{T}$*-spectrum* X *with a* $\mathbb{T}$*-equivariant factorization of the* C_p*-norm of (4.34)*

$$\rho_p^* X_{h\mathrm{C}_p} \xrightarrow{\ \mathbf{V}\ } X \xrightarrow{\ \mathbf{F}\ } \rho_p^* X^{h\mathrm{C}_p}, \tag{5.73}$$

where $X_{h\mathrm{C}_p}$ *and* $X^{h\mathrm{C}_p}$ *each have a residual action of* $\overline{\mathbb{T}} := \mathbb{T}/\mathrm{C}_p \cong \mathbb{T}$. *The isomorphism* $\rho_p : \mathbb{T} \to \overline{\mathbb{T}}$ *is the pth root map of Definition 1.3.*

X *is* **V-complete** [AN21, Definition 3.20] *if the limit of the tower*

$$\cdots \xrightarrow{\ \rho_{p^2}^*\mathbf{V}\ } \rho_{p^2}^* X_{h\mathrm{C}_{p^2}} \xrightarrow{\ \rho_p^*\mathbf{V}\ } \rho_p^* X_{h\mathrm{C}_p} \xrightarrow{\ \mathbf{V}\ } X$$

is contractible.

The maps $\mathbf{F}$ and $\mathbf{V}$ here are related to the maps in Definition 2.16 and to the maps $\pi_{k,L}$ and $\pi_{k,R}$ of Definition 5.80, but the former is not the restriction F of Definition A.1. We use the bold font to avoid confusion.

Remark 5.74. [AN21, §4.1] *A p-typical topological Cartier module is a p-cyclotomic spectrum with additional structure. The Frobenius map* $\mathbf{F} : X \to X^{h\mathrm{C}_p}$ *can be composed with the map* $s^p : X^{h\mathrm{C}_p} \to X^{t\mathrm{C}_p}$ *of (4.35) to give a cyclotomic structure on* X. *This is the same as a spectrum with Frobenius lifts as in* [McC24, Definition 2.1.2] *and Definition 5.24(iii).*

Example 5.75. Some Cartier modules.

(i) *For a p-typical algebraic Cartier module* M *as in Definition 5.72(i), consider the Eilenberg-MacLane spectrum* HM *with trivial* $\mathbb{T}$*-action. Then* $\mathbf{V}$ *and* $\mathbf{F}$ *induce endomorphisms* $H\mathbf{V}$ *and* $H\mathbf{F}$ *of* HM, *and we have*

$$\begin{array}{ccccccc}
HM_{h\mathrm{C}_p} & \longrightarrow & (HM_{h\mathrm{C}_p})_{\leqslant 0} & & (HM^{h\mathrm{C}_p})_{\geqslant 0} & \longrightarrow & HM^{h\mathrm{C}_p} \\
\downarrow\simeq & & \downarrow\simeq & & \uparrow\simeq & & \uparrow\simeq \\
HM \wedge B & & HM \xrightarrow{H\mathbf{V}} & HM & \xrightarrow{H\mathbf{F}} HM & & \mathrm{map}_{\mathrm{Sp}}(B, HM),
\end{array}$$

where $B := \Sigma^\infty B\mathrm{C}_{p+}$, $\mathrm{map}_{\mathrm{Sp}}(-,-)$ *is as in (5.10), and the composite* $HM_{h\mathrm{C}_p} \to HM^{h\mathrm{C}_p}$ *is the norm* $\mathrm{Nm}_{\mathrm{C}_p}$ *of (4.28). This makes* HM *a p-typical topological Cartier module.*

(ii) *For a p-typical topological Cartier module X, one has endomorphisms* $\mathbf{V}$ *and* $\mathbf{F}$ *in* $\pi_*^{\mathrm{cyc}} X$ *induced by the composites*

$$X \xrightarrow{i_*} X_{h\mathrm{C}_p} \xrightarrow{\mathbf{V}} X \qquad \text{and} \qquad X \xrightarrow{\mathbf{F}} X^{h\mathrm{C}_p} \xrightarrow{i^*} X.$$

for i_* and i^* as in (A.14) and (A.11). These make $\pi_* X$ a graded p-typical Cartier module.

(iii) *For a p-typical topological Cartier module X, let $X/\mathbf{V}$ denote the cofiber of the $\mathbb{T}$-equivariant map $\mathbf{V} : \rho_p^* X_{h\mathrm{C}_p} \to X$, where ρ_p^* is as in Definition 1.3. It is a p-cyclotomic spectrum but need not be a topological Cartier module. Consider the diagram*

$$\begin{array}{ccccccc} \rho_p^* X_{h\mathrm{C}_p} & \xrightarrow{\mathbf{V}} & X & \xrightarrow{j} & X/\mathbf{V} & \xrightarrow{\varphi_p} & \\ \| & & \downarrow{\scriptstyle \mathbf{F}} & & \downarrow{\scriptstyle \mathbf{F}'} & & (X/\mathbf{V})^{t\mathrm{C}_p} \\ \rho_p^* X_{h\mathrm{C}_p} & \xrightarrow{\mathrm{Nm}^X_{\mathrm{C}_p}} & \rho_p^* X^{h\mathrm{C}_p} & \xrightarrow{s_p^X} & X^{t\mathrm{C}_p} & \xrightarrow{j^{t\mathrm{C}_p}} & \end{array}$$

where the top and bottom rows are cofiber sequences and $\mathbf{F}'$ is induced by $\mathbf{F}$. Then the map $\varphi_p := j^{t\mathrm{C}_p}\mathbf{F}'$ is a cyclotomic structure on $X/\mathbf{V}$. The fiber of $j^{t\mathrm{C}_p}$ is $(X_{h\mathrm{C}_p})^{t(\overline{\mathrm{C}}_p)}$ (where $\overline{\mathrm{C}}_p := \mathrm{C}_{p^2}/\mathrm{C}_p$ acting residually on $X_{h\mathrm{C}_p}$ as in Definition A.1(i)), which is contractible by Lemma 5.15(i) when X is bounded below.

Definition 5.76. [AN21, Definition 3.6] *The* **∞-category of p-typical topological Cartier modules** TCart_p *is the pullback*

$$\begin{array}{ccc} \mathrm{TCart}_p & \longrightarrow & \left(\mathrm{Sp}^{\mathbb{T}}\right)^{\mathbf{\Delta}^2} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_1, d_1)} \\ \mathrm{Sp}^{\mathbb{T}} & \xrightarrow{(id, \mathrm{Nm}_{\mathrm{C}_p})} & \mathrm{Sp}^{\mathbb{T}} \times \left(\mathrm{Sp}^{\mathbb{T}}\right)^{\mathbf{\Delta}^1} \\ X & \longmapsto & (X, \mathrm{Nm}_{\mathrm{C}_p} : X_{h\mathrm{C}_p} \to X^{h\mathrm{C}_p}) \end{array}$$

where the right arrow has the form

$$\begin{array}{ccccc} & X_1 & & & \\ {\scriptstyle v}\nearrow & & \searrow{\scriptstyle f} & \longmapsto & (X_1, n : X_0 \to X_2). \\ X_0 & \xrightarrow{n} & X_2 & & \end{array}$$

We will write an object in this category as $(X, \mathbf{V}_X, \mathbf{F}_X, \sigma_X)$, where X is a $\mathbb{T}$-spectrum,

$$\mathbf{V}_X : X_{h\mathrm{C}_p} \to X, \qquad \mathbf{F}_X : X \to X^{h\mathrm{C}_p},$$

and σ_X is a 2-simplex corresponding to a factorization $\mathrm{Nm}_{\mathrm{C}_p} \simeq \mathbf{F}_X \cdot \mathbf{V}_X$.

The following is a conseqence of the above and a similar statement about topological Cartier modules proved by Mathew as [Mat21, Proposition 6.5].

Lemma 5.77. [BHLS23, Lemma 2.12] *A filtered colimit of cyclotomic spectra bounded in the range $[a, b]$ is itself bounded in the range $[a, b + 3]$.*

Definition 5.78. [BHLS23, Definition 2.25] *For $X \in \mathrm{CycSp}$ and $b \in \mathbb{Z}$, we say that X satisfies the* **Segal condition** *with parameter b (*$\mathrm{Segal}(\leqslant b)$ *for short) if the fiber of the map*

$$X \xrightarrow{\varphi} X^{tC_p}$$

is cyclotomically b-truncated.

This condition is so named because the Segal conjecture implies that the map in question is an equivalence when X is a p-local finite complex with trivial cyclotomic structure.

The conditions of Definitions 5.69 and 5.78 are related as follows.

Proposition 5.79. [BHLS23, Proposition 2.30] *Let $X \in \mathrm{CycSp}_p$ be a bounded below.*

(i) If X satisfies $\mathrm{SCV}(\leqslant b)$*, then X satisfies* $\mathrm{WCV}(\leqslant b)$*.*
(ii) If X satisfies $\mathrm{WCV}(\leqslant b)$ *and* $\mathrm{Segal}(\leqslant b)$*, then $X \in \mathrm{CycSp}_{\leqslant b}$.*
(iii) If $X \in \mathrm{CycSp}_{\leqslant b}$, then X satisfies $\mathrm{Segal}(\leqslant b)$*.*
(iv) If $X \in \mathrm{CycSp}_{[c,b]}$ and p^m acts by zero on X, then X satisfies

$$\mathrm{SCV}\big(\leqslant b + 2(b-c+1)m\big).$$

5.12. **Topological restriction homology** TR. In [BHLS23, §2] this functor is discussed and figures in the proofs of [BHLS23, Theorem 3.22] (our Theorem 6.7) and [BHLS23, Lemma 4.26], which is needed for [BHLS23, Theorem C], our [Rav26, Theorem 5.3].

For a cyclotomic Ω-spectrum X, Blumberg and Mandell [BM15, Definition 6.3] define

$$\mathrm{TR}^k(X,p) := X^{\mathrm{C}_{p^k}} \text{ for } k \geqslant 0 \qquad \text{and} \qquad \mathrm{TR}(X,p) := \operatorname*{holim}_k \mathrm{TR}^k(X),$$

which they call a "mapping microscope." They also define a global version.

When X is bounded below, the above definition is equivalent to the following one by [McC24, Theorem 3.3.12], which is used in [BHLS23], where we leave the prime implicit.

Definition 5.80. Topological restriction homology [AN21, Example 3.4 and Construction 3.18] *and* [BHLS23, Definition 2.4]*. For a cyclotomic spectrum X, the* **spectrum** $\mathrm{TR}(X)$ *(which turns out to be cyclotomic) is the limit of the diagram*

$$\begin{array}{ccccccccc} X & & X^{h\mathrm{C}_p} & & X^{h\mathrm{C}_{p^2}} & & X^{h\mathrm{C}_{p^3}} & & \\ & \searrow^{\varphi_p} \quad \swarrow_{s^p} & & \searrow^{\varphi_{(p^2)}} \quad \swarrow_{s^{(p^2)}} & & \searrow^{\varphi_{(p^3)}} \quad \swarrow_{s^{(p^3)}} & & \searrow^{\varphi_{(p^4)}} & \\ & X^{t\mathrm{C}_p} & & (X^{t\mathrm{C}_p})^{h\mathrm{C}_p} & & (X^{t\mathrm{C}_p})^{h\mathrm{C}_{p^2}} & & & \cdots, \end{array} \tag{5.81}$$

where φ_p is the cyclotomic structure map for X as in Definition 5.23(i) and s^p is as in (4.39). The other maps are defined by

$$\varphi_{(p^{i+1})} := (\varphi_p)^{h\mathrm{C}_{p^i}} \qquad \text{and} \qquad s^{(p^{i+1})} := (s^p)^{h\mathrm{C}_{p^i}} \qquad \text{for } i > 0.$$

For each $k \geqslant 0$, the **spectrum** $\mathrm{TR}^k(X)$ *is the limit of the following subdiagram of (5.81).*

$$\begin{array}{ccccccc} X & & X^{h\mathrm{C}_p} & & \cdots & & X^{h\mathrm{C}_{p^k}} \\ & \searrow^{\varphi_p} \quad \swarrow_{s^p} & & \searrow^{\varphi_{(p^2)}} \quad \swarrow & & \searrow \quad \swarrow_{s^{(p^k)}} & \\ & X^{t\mathrm{C}_p} & & (X^{t\mathrm{C}_p})^{h\mathrm{C}_p} & & (X^{t\mathrm{C}_p})^{h\mathrm{C}_{p^{k-1}}} & \end{array} \tag{5.82}$$

For $k > 0$, let $\pi_{k,L} : \mathrm{TR}^k(X) \to \mathrm{TR}^{k-1}(X)$ be projection away from the last two factors, and let $\pi_{k,R} : \mathrm{TR}^k(X) \to \mathrm{TR}^{k-1}(X)^{h\mathrm{C}_p}$ be projection away from the first two factors, as in (5.85) and (5.86).

The limit of (5.82) consists of sequences $(x_0, x_1, \dots, x_k)$, where $x_i \in X^{h\mathrm{C}_{p^i}}$ with $s_{p^i}(x_i) = \varphi_{p^i}(x_{i-1})$ for $1 \leqslant i \leqslant k$. It is the equalizer of

$$X \vee X^{h\mathrm{C}_p} \vee \cdots \vee X^{h\mathrm{C}_{p^k}} \underset{\varphi}{\overset{s}{\rightrightarrows}} X^{t\mathrm{C}_p} \vee (X^{t\mathrm{C}_p})^{h\mathrm{C}_p} \vee \cdots \vee (X^{t\mathrm{C}_p})^{h\mathrm{C}_{p^{k-1}}}$$

in which $s(x_0)$ and $\varphi(x_k)$ are each understood to be the basepoint. The limit of (5.81) consists of infinite sequences $(x_0, x_1, \dots)$ satisfying similar conditions.

$\mathrm{TR}^k(X)$ as defined above is denoted by $X^{\mathrm{C}_{p^k}}$ in [KN21b, Definition 9.1]. They justify this notation by proving in [KN21b, Proposition 9.2] that when X is bounded below, there is an orthogonal $\mathbb{T}$-spectrum whose genuine C_{p^k} fixed point is $\mathrm{TR}^k(X)$.

There is a $\mathbb{T}$-equivariant map $\pi : \mathrm{TR}(X) \to X$ given by

$$(x_0, x_1, \dots) \mapsto x_0$$

and a $\mathbb{T}$-equivariant equivalence

$$\Phi : \mathrm{TR}(X) \to X \times_{X^{t\mathrm{C}_p}} \mathrm{TR}(X)^{h\mathrm{C}_p}$$

given by the pullback diagram

$$\begin{array}{ccccc} (x_0, x_1, x_2, \dots) & \longmapsto & & & (x_1, x_2, \dots) \\ \Big\downarrow & \mathrm{TR}(X) & \xrightarrow{\pi_{k,R}} & \mathrm{TR}(X)^{h\mathrm{C}_p} & \Big\downarrow \\ & {\scriptstyle \pi}\Big\downarrow & \lrcorner & \Big\downarrow{\scriptstyle s^p \pi^{h\mathrm{C}_p}} & \\ & X & \xrightarrow{\varphi_p} & X^{t\mathrm{C}_p} & s^p(x_1) \\ & & & & \shortparallel \\ x_0 & \longmapsto & & & \varphi_p(x_0). \end{array} \tag{5.83}$$

Remark 5.84. TR **and topological Cartier modules.** *The maps $\pi_{k,L}$ and $\pi_{k,R}$ are* **F** *and* **V** *in Definition 5.72. For a cyclotomic spectrum X, the spectrum $\mathrm{TR}(X)$ of Definition 5.80 has a topological Cartier module structure induced by maps between diagrams similar to that of (5.81) as follows. Recall that $\mathrm{TR}(X)$ is defined to be the limit of that diagram.*

For **V** *we have*

$$\begin{array}{ccccccccc} \mathrm{pt} & & X_{h\mathrm{C}_p} & & (X^{h\mathrm{C}_p})_{h\mathrm{C}_p} & & \cdots \\ & \searrow \quad \mathrm{pt} \quad \swarrow & & \searrow \ (X^{t\mathrm{C}_p})_{h\mathrm{C}_p} \ \swarrow & & \searrow \ ((X^{t\mathrm{C}_p})^{h\mathrm{C}_p})_{h\mathrm{C}_p} \ \swarrow & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ X & & X^{h\mathrm{C}_p} & & X^{h\mathrm{C}_{p^2}} & & \cdots \\ & \searrow \quad X^{t\mathrm{C}_p} \quad \swarrow & & \searrow \ (X^{t\mathrm{C}_p})^{h\mathrm{C}_p} \ \swarrow & & \searrow \ (X^{t\mathrm{C}_p})^{h\mathrm{C}_{p^2}} \ \swarrow & \end{array} \tag{5.85}$$

where each vertical arrow save the first two is $\mathrm{Nm}_{\mathrm{C}_p}$. The bottom two rows are those of (5.81), and the top two rows, ignoring the first entry of each, comprise its image under the functor $(-)_{h\mathrm{C}_p}$.

For $\mathbf{F}$ *we have*

$$
(5.86)\qquad
\begin{array}{ccccccccccccc}
X & & & & X^{hC_p} & & & & X^{hC_{p^2}} & & & & X^{hC_{p^3}} & & \\
\downarrow & \searrow & & \swarrow & \downarrow & \searrow & & \swarrow & \downarrow & \searrow & & \swarrow & \downarrow & \searrow & \\
 & & X^{tC_p} & & & & (X^{tC_p})^{hC_p} & & & & (X^{tC_p})^{hC_{p^2}} & & & & \cdots \\
\mathrm{pt} & & \downarrow & & X^{hC_p} & & \downarrow & & X^{hC_{p^2}} & & \downarrow & & X^{hC_{p^3}} & & \\
 & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & \\
 & & \mathrm{pt} & & & & (X^{tC_p})^{hC_p} & & & & (X^{tC_p})^{hC_{p^2}} & & & & \cdots ,
\end{array}
$$

where each vertical arrow save the first two is the identity map. Here the top two rows are those of (5.81), and the bottom two rows, ignoring the first entry of each, form its image under the functor $(-)^{hC_p}$*.*

Thus we can regard TR *as a functor from* CycSp_p *to* TCart_p *that preserves bounded below objects.*

A global form of a polygonal analog of TR is defined in [KMN23, Definition 1.3]. It is corepresented in PgcSp by $i\mathbb{S}$, the constant $\mathbb{S}$-valued polygonal spectrum.

It follows that

$$\mathrm{TR}(X) = \lim_{\pi_{k,L}} \mathrm{TR}^k(X),$$

and the maps $\pi_{k,R}$ assemble into a map $\pi_R : \mathrm{TR}(X) \to \mathrm{TR}(X)^{hC_p}$. The maps $\pi_{k,L}$ assemble into a map $\pi_L : \mathrm{TR}(X) \to \mathrm{TR}(X)$.

Theorem 5.87. A fully faithful right adjoint. [AN21, Theorem 3.21] *and* [BHLS23, Lemma 2.11]. *The functor* $\mathrm{TR} : \mathrm{CycSp}_{p,+} \to \mathrm{TCart}_{p,+}$ *on the bounded below subcategory is fully faithful and t-exact with left adjoint* $X \mapsto X/\mathbf{V}$*. The essential image is the full subcategory of* $\mathbf{V}$*-complete bounded below p-typical topological Cartier modules.*

This implies the following, which can be viewed as an ∞-categorical reformulation of [BHM93, Definition 5.12(i) and Remark 5.13].

Theorem 5.88. The relation between TC **and** TR**.** $\mathrm{TC}(X,p)$ *is the equalizer in*

$$\mathrm{TC}(X,p) \longrightarrow \mathrm{TR}(X) \underset{1}{\overset{\pi_L}{\rightrightarrows}} \mathrm{TR}(X).$$

6. THH of cochains on the circle: the foot in the door

In this section we will explain how the machinery developed above can be used to approach the telescope conjecture. More details will be given in [Rav26].

6.1. **$\mathbb{S}$-cochains on the circle.** The category (or ∞-category) of spectra is cotensored over that of topological spaces, pointed or not. This means that for a spectrum E and a space X one can define a spectrum E^X in which the nth component is the mapping space $\mathrm{Map}(X, E_n)$ if X is pointed, and $\mathrm{Map}(X_+, E_n)$ otherwise. The same goes for both naive and genuine G-spectra, for a pointed or unpointed G-space X.

The authors of [BHLS23] refer to $\mathbb{S}^X$ for an unpointed space X as the spectrum of **$\mathbb{S}$-cochains on** X. When X is a finite CW-complex, it is more traditionally known as $\mathbb{D}X$, the **Spanier-Whitehead dual of** X as in (5.11).

Let $R \in \mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z}}$ be a p-complete $\mathbb{E}_1$-ring spectrum with an action of the integers $\mathbb{Z}$. Then one has a diagram of cyclotomic spectra

$$(6.1)\qquad \mathrm{THH}(R^{h\mathbb{Z}}) \to \mathrm{THH}(R^{h(p\mathbb{Z})}) \to \mathrm{THH}(R^{h(p^2\mathbb{Z})}) \to \cdots \to \mathrm{THH}(R).$$

It is the image of the evident diagram of subgroups of $\mathbb{Z}$,

$$\mathbb{Z} \hookleftarrow p\mathbb{Z} \hookleftarrow p^2\mathbb{Z} \hookleftarrow \cdots \hookleftarrow e$$

under the contravariant functor $\mathrm{THH}(R^{h(-)})$. There is a similar diagram with $\mathbb{Z}$ replaced by the p-adic integers $\mathbb{Z}_p$. The two are used interchangeably in [BHLS23] since they assume that their spectra (including $\mathbb{S}$) are p-adically complete. Half of that paper (§3, 4 and 5) is devoted to the study of the cyclotomic spectra in (6.1).

When the action of $\mathbb{Z}$ on R is trivial, then by (A.12) we have

$$R^{h(p^i\mathbb{Z})} \simeq R^{B(p^i\mathbb{Z})_+} \simeq R \vee \Sigma^{-1}R.$$

When $R = \mathbb{S}$, this spectrum is that of **cochains on the circle** (the circle being the space $B(p^i\mathbb{Z}) \cong B(\mathbb{Z})$), also known as the **dual circle**. It is the subject of [BHLS23, §3], where they say

> The key idea governing our analysis is that, since $\mathbb{S}^{B\mathbb{Z}}$ is the $\mathbb{S}$-cochains on $B\mathbb{Z}_p$, $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ is controlled by the geometry the free loop space of $B\mathbb{Z}_p$.

That free loop space is described in §2.10. The connection between THH and the free loop space is evident in Proposition 3.11, where the suspension spectrum of $\mathcal{L}X$ is related to a Bökstedt functor. The underlying cyclotomic spectrum of $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ is studied by Malkiewich in [Mal17], and the underlying commutative algebra is studied by Levy and David Jongwon Lee in [LL23, Lemma 4.6].

Malkiewich studies $\mathbb{S}^{B\mathbb{Z}}$, which he calls $\mathbb{D}S^1$, as an $\mathbb{E}_1$-ring spectrum and proves the following.

Theorem 6.2. THH **of the dual circle.** [Mal17, Corollary 1.3] *As genuine* $\mathbb{T}$-*spectra,*

$$\mathrm{THH}(\mathbb{S}^{B(p^k\mathbb{Z})}) \simeq \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) \simeq \mathbb{S} \oplus \bigoplus_{r\geqslant 1} \Sigma^{-1}\mathbb{S}[\mathbb{T}/\mathrm{C}_r] \tag{6.3}$$

for $\mathbb{S}[\mathbb{T}/\mathrm{C}_r]$ *as in Definition 4.49.*

Completing both $\mathbb{Z}$ *and* $\mathbb{S}$ *at* p *gives*

$$\mathbb{S}_p \oplus \bigoplus_{k\geqslant 1}\left(\prod_{s\in\mathbb{Z}_p^\times} \Sigma^{-1}\mathbb{S}[\mathbb{T}/\mathrm{C}_{p^k}]_p\right).$$

In both cases the CW-spectrum in question has a single cell in dimension -1 and infinitely many in dimension 0. Note that by (2.84),

$$\begin{aligned}\Sigma_+^{\infty-1}\mathcal{L}S^1 &\simeq \bigoplus_{r\in\mathbb{Z}} \Sigma^{-1}\mathbb{S}[\mathbb{T}/\mathrm{C}_{|r|}]\\ &\simeq \left(\bigoplus_{r\leqslant -1} \Sigma^{-1}\mathbb{S}[\mathbb{T}/\mathrm{C}_{|r|}]\right) \vee \mathbb{S} \vee \left(\bigoplus_{r\geqslant 1} \Sigma^{-1}\mathbb{S}[\mathbb{T}/\mathrm{C}_{|r|}]\right),\end{aligned}$$

which is similar to the expression of (6.3).

On the other hand we know by Example 3.16 that $\mathrm{THH}(\mathbb{S}) \simeq \mathbb{S}$. This means the coassembly map

$$\epsilon : \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) \to \mathrm{THH}(\mathbb{S})^{B\mathbb{Z}} \simeq \mathbb{S}^{B\mathbb{Z}} \simeq \mathbb{S} \oplus \Sigma^{-1}\mathbb{S}$$

is very far from an equivalence. *This proposition will enable us to show that various* TC *coassembly maps are not isomorphisms.*

For the next theorem we need some notation.

- For a profinite set X, let $C^0(X)$ denote the ring of continuous (meaning locally constant) $\mathbb{F}_p$-valued functions on X.
- For a commutative perfect $\mathbb{F}_p$-algebra A (such as $C^0(X)$ or a finite field), let $\mathbb{W}(A)$ denote the commutative ring spectrum of spherical Witt vectors as in [Lur18a, Example 5.2.7], which is typically a countable coproduct of p-adic sphere spectra. Let Perf_p denote the category of commutative perfect $\mathbb{F}_p$-algebras, making $\mathbb{W}$ a functor from it to CAlgSp, the category of commutative ring spectra. It is known ([BSY22, Proposition 2.2]) to have a right adjoint, the **tilting functor** $(-)^\flat$, which is inverse limit along Frobenius on $\pi_0(-)/p$.
- Let $L_{\langle p^\infty \rangle}\mathbb{S}_p$ be the polygonic spectrum of Example 5.35(iv). Then the cyclotomic spectrum

$$\mathbb{L} := \Upsilon L_{\langle p^\infty \rangle}\mathbb{S}_p = \bigoplus_{j\geqslant 0} \mathbb{S}[\mathbb{T}/\mathrm{C}_{p^j}], \tag{6.4}$$

  for Υ as in Definition 5.37, is the corepresenting object in CycSp for TR. This is proved in [BHLS23, Lemma 2.6]. Note that by (2.84), $\mathbb{L}$ is a summand of $\Sigma^\infty_+ \mathcal{L}S^1$.

Example 6.5. Continuous $\mathbb{F}_p$-valued functions on the p-adic integers. *Each $a \in \mathbb{Z}_p$ can be written uniquely as*

$$a = \sum_{j\geqslant 0} a_j p^j \qquad \text{with } a_j^p = a_j.$$

By abuse of notation we can regard each coefficient a_j as a continuous $\mathbb{F}_p$-valued function (by reducing mod p) that factors though the quotient group $\mathbb{Z}/p^{j+1}$. It follows that

$$\begin{aligned} C^0(\mathbb{Z}/p^j) &\cong \mathbb{F}_p[a_0, a_1, \ldots, a_{j-1}]/(a_k^p - a_k : 0 \leqslant k < j), \\ C^0(\mathbb{Z}_p) &\cong \operatorname*{colim}_j C_0(\mathbb{Z}/p^j) \cong \mathbb{F}_p[a_0, a_1, \ldots]/(a_k^p - a_k : k \geqslant 0), \\ C^0(p^k\mathbb{Z}_p) &\cong C_0(\mathbb{Z}_p)/(a_0, \ldots, a_{k-1}), \\ \text{and} \qquad C^0(\mathbb{Z}_p^\times) &\cong (a_0) \subseteq C^0(\mathbb{Z}_p). \end{aligned}$$

The ring $C^0(\mathbb{Z}/p^j)$ has rank p^j as an $\mathbb{F}_p$-vector space.

Proposition 6.6. The fibers of two coassembly maps.

(i) [BHLS23, Proposition 3.19] *The fiber of the coassembly map (see Definition 5.7)*

$$\epsilon : \mathrm{THH}(\mathbb{S}_p^{B\mathbb{Z}_p}) \to \mathrm{THH}(\mathbb{S}_p)^{B\mathbb{Z}_p}$$

is

$$\Sigma^{-1}\mathbb{L} \otimes \mathbb{W}(C^0(\mathbb{Z}_p^\times))$$

for $\mathbb{L}$ as in (6.4) and $C^0(\mathbb{Z}_p^\times)$ as in Example 6.5.

(ii) [BHLS23, Corollary 3.21] *For a connective ring spectrum R, the thick subcategory generated by the fiber of $\epsilon : \mathrm{TC}(R^{B\mathbb{Z}_p}) \to \mathrm{TC}(R)^{B\mathbb{Z}_p}$ contains*

$$\mathbb{W}(C^0(\mathbb{Z}_p^\times)) \otimes \mathrm{TC}(R).$$

In particular ϵ is not an equivalence when $\mathrm{TC}(R)$ is nontrivial.

An important step toward the disproof of the telescope conjecture is the following, which is a consequence of Proposition 6.6.

Theorem 6.7. The $T(n+1)$-local K-theory coassembly map for the trivial $\mathbb{Z}$-action. [BHLS23, Theorem 3.22 for $X = \mathbb{S}$] *Let R be a $T(n)$-local $\mathbb{E}_1$-ring spectrum for $n \geqslant 1$. If $\mathrm{K}_{T(n+1)}(R)$ (see Definition 1.12) is nontrivial, then the coassembly map (Definition 5.7) for the trivial action of $\mathbb{Z}$ on R,*

$$\epsilon : \mathrm{K}_{T(n+1)}(R^{B\mathbb{Z}}) \to \mathrm{K}_{T(n+1)}(R)^{B\mathbb{Z}},$$

is not *an equivalence.*

6.2. *K***-theory and** TC**.** In [BHLS23, §6.1] they explain why $T(n+1)$-localized algebraic K-theory and $T(n+1)$-localized TC coincide for examples of interest. For connective rings, we have the following.

Theorem 6.8. [LMMT24, Purity Theorem] *and* [CMNN24, Cor. 4.11]. *Let R be a connective $\mathbb{E}_1$-algebra. For $n \geqslant 1$, the $(T(n) \oplus T(n+1))$-localization map and the cyclotomic trace induce equivalences*

$$\mathrm{K}_{T(n+1)}\big(L_{T(n)\oplus T(n+1)}R\big) \xleftarrow[\simeq]{\eta} \mathrm{K}_{T(n+1)}(R) \xrightarrow[\simeq]{\mathrm{Trc}} \mathrm{TC}_{T(n+1)}(R).$$

To disprove the telescope conjecture, we will need to understand the topological cyclic homologies of fixed points of $\mathbb{Z}$-actions on connective $\mathbb{E}_1$-algebras. Such fixed points are (-1)-connective, but often not 0-connective, so the above theorem does not apply.

In order to get around this we will use Theorem 6.10 below, which requires the following.

Definition 6.9. [LT19, Definition 3.1] *A localizing invariant as in Definition 5.9*

$$E \colon \mathrm{Cat}_\infty^{\mathrm{perf}} \longrightarrow \mathrm{Sp}$$

(with source as in Definition 5.1(viii)) is a **truncating invariant** *if for every connective $\mathbb{E}_1$-ring spectrum R, the canonical map*

$$E(R) \longrightarrow E(H\pi_0(R))$$

is an equivalence. Here $E(R)$ is understood to be the value of E on the ∞-category Mod_R of Definition 5.8(vii) and similarly for the Eilenberg-MacLane ring spectrum $H\pi_0(R)$.

In other words, the truncating invariant E does not see the positive dimensional homotopy groups of R.

Theorem 6.10. [Lev22, Theorem B] *Let R_0 and R_1 be connective $\mathbb{E}_1$-algebras with $\mathbb{Z}$-action. Let*

$$f \colon R_0 \longrightarrow R_1$$

be a 1-connective (meaning it induces an isomorphism in π_0), $\mathbb{Z}$-equivariant $\mathbb{E}_1$-algebra map. For any truncating invariant E, the induced map

$$E(R_0^{h\mathbb{Z}}) \longrightarrow E(R_1^{h\mathbb{Z}})$$

is an equivalence.

Corollary 6.11. [BHLS23, Corollary 6.3] *For $n \geqslant 1$, let R be a $T(n+1)$-acyclic, connective $\mathbb{E}_1$-algebra with a $\mathbb{Z}$-action. The coassembly map ϵ, the $T(n)$-localization* *map η, and the cyclotomic trace* Trc *fit into a commuting diagram*

$$\begin{array}{ccccc}
\mathrm{K}_{T(n+1)}(L_{T(n)}R^{h\mathbb{Z}}) & \xleftarrow[\simeq]{\mathrm{K}_{T(n+1)}(\eta)} & \mathrm{K}_{T(n+1)}(R^{h\mathbb{Z}}) & \xrightarrow[\simeq]{\mathrm{Trc}} & \mathrm{TC}_{T(n+1)}(R^{h\mathbb{Z}}) \\
\downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} \\
\mathrm{K}_{T(n+1)}(L_{T(n)}R)^{h\mathbb{Z}} & \xleftarrow[\simeq]{\mathrm{K}_{T(n+1)}(\eta)} & \mathrm{K}_{T(n+1)}(R)^{h\mathbb{Z}} & \xrightarrow[\simeq]{\mathrm{Trc}} & \mathrm{TC}_{T(n+1)}(R)^{h\mathbb{Z}}
\end{array}$$

where each horizontal map is an equivalence.

The coassembly maps in Corollary 6.11 need not be equivalences. Indeed Theorem 6.7 says the middle one is not an equivalence when the group action is trivial and $\mathrm{K}_{T(n+1)}(R)$ is nontrivial.

If we knew that the $K(n+1)$-local analog of the coassembly map of Theorem 6.7 was an equivalence for $n \geqslant 1$, we would know that the telescope conjecture is false. What we do know is two steps removed from this. There is a *particular* R, namely $L_{T(n)}BP\langle n\rangle$, with a *nontrivial* action of $\mathbb{Z}$ for which the coassembly map is a $K(n+1)$-local but not a $T(n+1)$-local equivalence. This will be discussed in [Rav26], where we will see that a crucial ingredient is [BMCSY25, Theorem C].

6.3. **Coming attractions.** In the next paper we will talk about how to relax the triviality hypothesis for the group action of Theorem 6.7.

Definition 6.12. Local unipotence. *For an action of $\mathbb{Z}$ on an abelian group A, let $\Psi : A \to A$ be the automorphism induced by a generator of $\mathbb{Z}$. Then the action is* **unipotent** *if $\Psi - 1$ is a nilpotent endomorphism of A. An action of $\mathbb{Z}$ on a spectrum R is* **locally unipotent** *if the induced action onn each homotopy group is unipotent.*

Such group actions are studied in [BHLS23, §4]. The specific $\mathbb{Z}$-action on $BP\langle n\rangle$ by Adams operations is the subject of [BHLS23, §5].

In [BHLS23, §6] they show that for $n \geqslant 1$ and $k \geqslant 0$, the $T(n+1)$-localized coassembly map

$$\epsilon : \mathrm{K}_{T(n+1)}\left(BP\langle n\rangle^{hp^k\mathbb{Z}}\right) \longrightarrow \mathrm{K}_{T(n+1)}(BP\langle n\rangle)^{hp^k\mathbb{Z}}$$

is not an equivalence, but becomes one after $K(n+1)$-localization. Thus, the functors $\mathrm{K}_{T(n+1)}$ and $\mathrm{K}_{K(n+1)}$ differ, so the telescope conjecture fails.

Here we repeat the words of [BHLS23] that we quoted in §1.1.

> We do this by looking at the coassembly map from two highly divergent perspectives, which are connected via trace theorems:
> (1) From the perspective of locally unipotent $\mathbb{Z}$-actions on ring spectra, the results of [BHLS23, §4] tell us that the coassembly map cannot be an isomorphism.
> (2) From the perspective of cyclotomic redshift of [BMCSY25], the map
> $$L_{T(n)}BP\langle n\rangle^{hp^k\mathbb{Z}} \longrightarrow L_{T(n)}BP\langle n\rangle$$
> splits after base change to the maximal abelian extension of the $K(n)$-local sphere, and therefore the coassembly map is a $K(n+1)$-local isomorphism.

etoolbox

Appendix A. Some equivariant homotopy theory

A.1. Residual action and restriction.

Definition A.1. *Let G be a group acting on a space or spectrum X, and let $H \subseteq G$ be a subgroup.*

(i) When H is normal in G, the action of G on X induces **residual actions** *of G/H on X^H and X_H, and a* **residual action homeomorphisms**

$$\mathrm{R}_H^G : X^G \to (X^H)^{G/H} \qquad \text{and} \qquad \hat{\mathrm{R}}_H^G : (X_H)_{G/H} \to X_G.$$

We also have **residual action homotopy equivalences**

$$h\mathrm{R}_H^G : X^{hG} \to (X^{hH})^{h(G/H)} \qquad \text{and} \qquad h\hat{\mathrm{R}}_H^G : (X_{hH})_{h(G/H)} \to X_{hG}.$$

(ii) For arbitrary $H \subseteq G$, the **restriction maps**

$$\begin{array}{lcl} \mathrm{F}_H^G : X^G \to X^H & \text{and} & \hat{\mathrm{F}}_H^G : X_H \to X_G \\ h\mathrm{F}_H^G : X^{hG} \to X^{hH} & \text{and} & h\hat{\mathrm{F}}_H^G : X_{hH} \to X_{hG} \end{array}$$

are defined by the fact that any point fixed by G is fixed by its subgroup H and any H-orbit is part of a G-orbit.

In both cases the indices may be omitted when they are clear from the context. In this paper the groups will always be finite cyclic p-groups, and we will sometimes write

$$\mathrm{R} = \mathrm{R}_p := \mathrm{R}_{\mathrm{C}_p}^{\mathrm{C}_{p^{n+1}}} \qquad \text{and} \qquad \mathrm{F} = \mathrm{F}_p := \mathrm{F}_{\mathrm{C}_{p^n}}^{\mathrm{C}_{p^{n+1}}}.$$

This notation was introduced by Madsen in [Mad94a] and has been used in most other papers on this subject since then. Since the words 'residual' and 'restriction' both begin with the same letter, the choice is not obvious, and occasionally one sees R used for the restriction map instead. The maps were originally denoted by Φ and D respectively in [BHM93], [BM94] and [Mad94b]. The letters in Definition A.1 stand for **restriction** (in a different sense) and **Frobenius**, as explained by Hesselholt and Madsen in [HM97, page 3]. They are related to similarly named maps in the theory of Witt vectors, which Serre describes in [Ser79, §II.6].

The following is elementary.

Proposition A.2. F **and** R **commute.** *Let $K \lhd H \lhd G$ be groups acting on a space or spectrum X. Then the following diagram commutes.*

$$\begin{array}{ccc} X^G & \xrightarrow[\cong]{\mathrm{R}_K^G} & (X^K)^{G/K} \\ \Big\downarrow{\scriptstyle \mathrm{F}_H^G} & & \Big\downarrow{\scriptstyle \mathrm{F}_{H/K}^{G/K}} \\ X^H & \xrightarrow[\cong]{\mathrm{R}_K^H} & (X^K)^{H/K}. \end{array}$$

For a prime p, integer $n > 0$ and groups $\mathrm{C}_p \lhd \mathrm{C}_{p^n} \lhd \mathrm{C}_{p^{n+1}}$, Proposition A.2 gives

$$\begin{array}{ccc} X^{\mathrm{C}_{p^{n+1}}} & \xrightarrow[\cong]{\mathrm{R}} & (X^{\mathrm{C}_p})^{\mathrm{C}_{p^{n+1}}/\mathrm{C}_p} \\ \Big\downarrow{\scriptstyle \mathrm{F}} & & \Big\downarrow{\scriptstyle \mathrm{F}} \\ X^{\mathrm{C}_{p^n}} & \xrightarrow[\cong]{\mathrm{R}} & (X^{\mathrm{C}_p})^{\mathrm{C}_{p^n}/\mathrm{C}_p}. \end{array} \tag{A.3}$$

Remark A.4. *When the group is $\mathbb{T}$, we have fixed point sets for all of its finite subgroups, and there is a global analog of (A.3) that we do not need here. These maps are also discussed by Blumberg and Mandell in* [BM15, Definition 6.1] *where they are used to define functors* TR, TF *and* TC *on fibrant cyclotomic spectra as homotopy limits obtained by iterating* R, F *or both. They give p-typical versions, which they denote by* $\mathrm{TR}(-;p)$ *and so on, in* [BM15, Definition 6.3]*. Then they show that the right derived functors of* $\mathrm{TR}(-;p)$ *(*[BM15, Theorem 6.5]*) and* $\mathrm{TC}(-;p)$ *(*[BM15, Theorem 6.8]*) are corepresentable by p-cyclotomic spectra described in* [BM15, Constructions 6.4 and 6.6]*. They do this for the global* TR *in* [BM15, Theorem 6.12]*, which* [McC24] *cites as justification for his claim about* THH *of the spectral affine line. Their proof makes use of the definition of* TR *as a homotopy limit (or "mapping microscope") of fixed point sets.*

In [BM15, Construction 5.11] *they define mapping spaces $F^h_{Cyc}(X,Y)$ in the p-cyclotomic category that anticipates Definition 5.24. They show it is the categorical mapping space when X is cofibrant and Y is fibrant. Presumably when we pass the the ∞-categorical world, we need not worry about derived functors or fibrancy and cofibrancy. See Remark 5.55.*

A.2. **Fixed point spaces and orbit spaces.** For a compact Lie group G (such as $\mathbb{T}$), let $\mathcal{T}_G$ denote the category of pointed G-spaces (assumed to be compactly generated and weak Hausdorff for technical reasons) where the basepoint is fixed by G, and continuous (but not necessarily equivariant) pointed maps. This category is enriched over itself: for pointed G-spaces X and Y, the morphism set $\mathcal{T}_G(X,Y)$ equipped with the compact-open topology is itself a pointed G-space, where for an element $\gamma \in G$ and a map $f : X \to Y$, $\gamma(f)$ is defined to be $\gamma f \gamma^{-1}$. The map f is equivariant iff $\gamma(f) = f$ for all $\gamma \in G$, so the fixed point set $\mathcal{T}_G(X,Y)^G$ (which does not have a G-action) is the pointed space of equivariant maps from X to Y. Thus we can define the category $\mathcal{T}^G$ of pointed G-spaces and equivariant maps by $\mathcal{T}^G(X,Y) := \mathcal{T}_G(X,Y)^G$. It is enriched over the category of pointed spaces $\mathcal{T}$ rather than over $\mathcal{T}_G$ or itself.

A pointed G-space can also be regarded as a $\mathcal{T}$-valued functor on $\mathcal{B}G$, the one object topological category with an automorphism for each element of G, with composition given by the binary operation of G. (A category is topological if each of its morphism sets has a topology with suitable continuity conditions.) One then sees that the fixed point space X^G and orbit space X_G are the limit and colimit of this functor.

Proposition A.5. The fixed point set as a mapping space. *For a pointed G-space X, the fixed point set X^G is also $\mathcal{T}^G(S^0,X)$, where the group action on S^0 is trivial. More generally for each subgroup $H \subseteq G$, $X^H = \mathcal{T}^G(G/H_+,X)$.*

The analog of this in the ∞-category Sp^{BG} of Definition 5.1(vi) is

$$\mathrm{map}_{\mathrm{Sp}^{BG}}(\Sigma^\infty G/H_+, Y) \cong Y^{hH}.$$

In particular for a naive $\mathbb{T}$-spectrum Y,

$$\mathrm{map}_{\mathrm{Sp}^{B\mathbb{T}}}(\mathbb{S}[\mathbb{T}/\mathrm{C}_{p^k}], Y) \cong Y^{h\mathrm{C}_{p^k}},$$

where $\mathbb{S}[\mathbb{T}/\mathrm{C}_{p^k}]$ is as in Definition 4.49.

There is a way to construct a contractible G-space with free G-action, commonly denoted by EG; it is unique up to equivariant homotopy equivalence. Its orbit space is the classifying space BG.

Example A.6. The case $G = \mathbb{T}$. *We regard $\mathbb{T}$ as the multiplicative group of complex numbers on the unit circle. For each $k > 0$, it acts freely by scalar multiplication of S^{2k-1}, the space of unit vectors in $\mathbb{C}^k$. The orbit space $S^{2k-1}_{\mathbb{T}}$ is the complex projective space $\mathbb{C}P^{k-1}$.*

We have equivariant maps

$$\cdots \longrightarrow S^{2k-1} \longrightarrow S^{2k+1} \longrightarrow S^{2k+3} \longrightarrow \cdots \tag{A.7}$$

with contractible colimit. This is our contractible free $\mathbb{T}$-space $E\mathbb{T}$, It also serves as a contractible H-space for any subgroup $H \subseteq \mathbb{T}$. For $H = \mathrm{C}_r$, the orbit space $S^{2k-1}_{\mathrm{C}_r}$ is a lense space, and $E\mathbb{T}_{\mathrm{C}_r}$ is the Eilenberg-MacLane space $B\mathrm{C}_r = K(\mathrm{C}_r, 1)$.

The diagram of $\mathbb{T}$-orbit spaces for (A.7) is

$$\cdots \longrightarrow \mathbb{C}\mathrm{P}^{k-1} \longrightarrow \mathbb{C}\mathrm{P}^{k} \longrightarrow \mathbb{C}\mathrm{P}^{k+1} \longrightarrow \cdots,$$

where $\mathbb{C}\mathrm{P}^k$ is k-dimensional complex projective space, the space of complex lines through the origin in $\mathbb{C}^{k+1}$. The colimit is $\mathbb{C}\mathrm{P}^\infty = K(\mathbb{Z}, 2)$.

If we replace $\mathbb{T}$ by its subgroup $\mathrm{C}_2 = \{\pm 1\}$, the orbit space diagram becomes

$$\cdots \longrightarrow \mathbb{R}\mathrm{P}^{2k-1} \longrightarrow \mathbb{R}\mathrm{P}^{2k+1} \longrightarrow \mathbb{R}\mathrm{P}^{2k+3} \longrightarrow \cdots,$$

for which the colimit is

$$\mathbb{R}\mathrm{P}^\infty = \operatorname*{colim}_m \mathbb{R}\mathrm{P}^m = B\mathrm{C}_2, \tag{A.8}$$

where $\mathbb{R}\mathrm{P}^m$ is m-dimensional real projective space, the space of real lines through the origin in $\mathbb{R}^{m+1}$. Note that $\mathbb{C}^{k+1}$ as a real vector space is $\mathbb{R}^{2k+2}$.

Then we define the *homotopy fixed point space* as

$$X^{hG} := \mathcal{T}^G(EG_+, X), \tag{A.9}$$

where EG_+ denotes EG with a disjoint base point. This may or may not be homotopy equivalent to X^G. Its homotopy type is known to be independent of the choice of EG. The map $j : EG_+ \to S^0$ of (4.23) induces a map to X^{hG} from $\mathcal{T}^G(S^0, X) = X^G$, so one might think of X^{hG} as a fattened up version of X^G.

Consider the G-equivariant maps

$$G_+ \xrightarrow{\ i\ } EG_+ \xrightarrow{\ j\ } S^0, \tag{A.10}$$

where $i$sends G to some orbit of EG and j send EG to the nonbase point in S^0 as in (4.23). Applying the functor $\mathcal{T}^G(-, X)$ gives maps

$$X \xleftarrow{\ i^*\ } X^{hG} \xleftarrow{\ j^*\ } X^G. \tag{A.11}$$

When the action of G on X is trivial, an equivariant map $EG \to X$ factors through the orbit space $EG_G = BG$, so

$$X^{hG} \simeq \mathcal{T}(BG_+, X). \tag{A.12}$$

One also has the *homotopy orbit space*

$$X_{hG} := EG_+ \wedge_G X = EG \times_G X, \tag{A.13}$$

which is defined as follows. The smash product $EG_+ \wedge X$ is the quotient of $EG \times X$ by $EG \times \{x_0\}$ for the base point $x_0 \in X$. The diagonal action of G on $EG \times X$

induces one on this quotient, and X_{hG} is its orbit space. The projection map $EG \times X \to EG$ leads to a map

$$X_{hG} \to EG \times_G (*) = (EG)_G = BG$$

with fiber X_G. Here the map $j : EG_+ \to S^0$ induces one from X_{hG} to

$$S^0 \wedge_G X = X_G,$$

so X_{hG} is a space over X_G.

Applying the functor $- \wedge_G X$ to (A.10) gives maps

$$X \xrightarrow{i_*} X_{hG} \xrightarrow{j_*} X_G. \qquad \text{(A.14)}$$

A.3. **Equivariant homotopy groups.**

Definition A.15. Twisted loop spaces, twisted suspensions and equivariant homotopy groups. *For a finite dimensional orthogonal representation V of a compact Lie group G, we denote by S^V the one point compactification of V (its* **representation sphere***), and by $S(V)$ its space of unit vectors. For a pointed G-space X, let*

$$\Omega^V X := \mathcal{T}_G(S^V, X) \qquad \text{and} \qquad \Sigma^V X := S^V \wedge X,$$

the **twisted loop space** *and* **twisted suspension** *of X. When $V^G = 0$, we denote the inclusion of fixed points $S^0 \to S^V$ by a_V.*

For each closed subgroup $H \subseteq G$, let

$$\pi_V^H X := \pi_0 \mathcal{T}^H(S^V, X),$$

the set of homotopy classes of H-equivariant maps $S^V \to X$, where we are regarding S^V and X as pointed H-spaces by restricting the G-action to H. We omit the superscript H when it is the trivial subgroup. If the action of H on V is trivial, we write $\pi_V^H X$ as $\pi_{|V|}^H X$.

The set $\pi_V^H X$ has a natural group structure when the fixed point vector space V^H is nontrivial. This group is abelian when $|V^H| > 1$. Thus we have homotopy groups indexed by such representations for each subgroup. The set $\pi_V^H X$ depends only on the action of H (rather than G) on V and X.

A theorem of Bredon [Bre67] says that a map $f : X \to Y$ of path connected pointed G-CW complexes (see [HHR21, Definition 8.4.4]) is an equivariant homotopy equivalence iff it induces an isomorphism is π_*^H (meaning the integer graded groups) for each $H \subseteq G$.

It is known that for any module M over the finite group $\pi_0 G$, there is an Eilenberg-MacLane G-spectrum $\mathrm{H}M$ with

$$\pi_i^H \mathrm{H}M = \begin{cases} M & \text{for } i = 0 \\ 0 & \text{for nonzero integers } i \end{cases} \qquad \text{(A.16)}$$

for each closed subgroup $H \subseteq G$. A similar statement is true if we replace M by a Mackey functor $\underline{M}$, but we do not need this here. Note that we are requiring only that $\mathrm{H}M$ has just one nontrivial *integer graded* homotopy group. In general there will be other representations V for which $\pi_V^H HM$ is nontrivial. For the case of finite G, see [HHR21, §9.1H].

From the sequence

$$\mathrm{C}_m \subseteq \mathrm{C}_{mn} \subseteq \mathbb{T}$$

we get

$$\mathrm{C}_{mn}/\mathrm{C}_m \to \mathbb{T}/\mathrm{C}_m \to \mathbb{T}/\mathrm{C}_{mn}$$

corepresenting

$$(i^{\mathbb{T}}_{\mathrm{C}_{mn}}X)^{\mathrm{C}_m} \leftarrow X^{\mathrm{C}_m} \leftarrow X^{\mathrm{C}_{mn}}$$

Appendix B. The ∞-category of genuine G-spectra

In [NS18, Definition II.2.5] Nikolaus and Scholze say $G\mathrm{Sp}$ is the ∞-category obtained from the simplicial set $N(\mathrm{Sp}^G)$ (for Sp^G as in Definition 4.12) by inverting stable equivalances. This process is treated in Lurie's Kerodon [`https://kerodon.net`] as follows.

Definition B.1. [`https://kerodon.net/tag/01M5`] *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories and let $\mathcal{W}$ be a collection of morphisms in $\mathcal{C}$. We say that F* **exhibits $\mathcal{D}$ as a strict localization of $\mathcal{C}$ with respect to $\mathcal{W}$** *if, for every category $\mathcal{E}$, precomposition with F induces a bijection*

$$\{\textit{Functors } \mathcal{D} \to \mathcal{E}\}$$
$$\downarrow$$
$$\left\{\textit{Functors } \mathcal{C} \to \mathcal{E} \textit{ carrying each } w \in \mathcal{W} \textit{ to an isomorphism in } \mathcal{E}\right\}.$$

It turns out that such a category $\mathcal{D}$ is uniquely determined by the category $\mathcal{C}$ and the morphism collection $\mathcal{W}$, and we will denote it by $\mathcal{C}[\mathcal{W}^{-1}]$. Explicitly, the category $\mathcal{C}[\mathcal{W}^{-1}]$ can be constructed from $\mathcal{C}$ by adjoining a new morphism $w^{-1} : Y \to X$ for each morphism $w : X \to Y$ of $\mathcal{W}$, and imposing the relations $w^{-1} \cdot w = 1_X$ and $w \cdot w^{-1} = 1_Y$.

Now we generalize the situation above, replacing the categories $\mathcal{C}$ and $\mathcal{D}$ by simplicial sets $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ (which Lurie calls $\mathcal{C}$ and $\mathcal{D}$, like the categories above) not required to be nerves, meaning simplicial sets to which each map of an inner horn $\partial\mathbf{\Delta}^n_i$ with $0 < i < n$ (see Definition 2.21) has a unique extension to the full simplex $\mathbf{\Delta}^n$. The domain simplicial set $\widetilde{\mathcal{C}}$ comes equipped with a collection of edges $\widetilde{\mathcal{W}}$. The target category $\mathcal{E}$ above gets replaced by an ∞-category $\mathcal{E}$.

Proposition B.3 says that for each $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{W}}$ there is an ∞-category $\mathcal{D}$ that does the job that $\mathcal{D}$ does in Definition B.1.

Definition B.2. [`https://kerodon.net/tag/01MP`] *Let $F : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{D}}$ be a morphism of simplicial sets and let $\widetilde{\mathcal{W}}$ be a collection of edges in $\widetilde{\mathcal{C}}$. We say that F* **exhibits $\widetilde{\mathcal{D}}$ as a localization of $\widetilde{\mathcal{C}}$ with respect to $\widetilde{\mathcal{W}}$** *if, for every ∞-category $\mathcal{E}$, the precomposition map $\mathrm{Fun}(\widetilde{\mathcal{D}}, \mathcal{E}) \circ F \to \mathrm{Fun}(\widetilde{\mathcal{C}}, \mathcal{E})$ is fully faithful, and its essential image is the full subcategory of functors that send edges in $\mathcal{W}$ to isomorphisms in $\mathcal{E}$.*

Lurie denotes this essential image by

$$\mathrm{Fun}(\widetilde{\mathcal{C}}[\widetilde{\mathcal{W}}^{-1}], \mathcal{E})$$

without defining $\widetilde{\mathcal{C}}[\widetilde{\mathcal{W}}^{-1}]$. The next result says there is an ∞-category having the properties one might expect of this undefined simplicial set.

Proposition B.3. Existence of localizations. [https://kerodon.net/tag/01N0] *Let $\widetilde{\mathcal{C}}$ be a simplicial set and let $\widetilde{\mathcal{W}}$ be a collection of edges of $\widetilde{\mathcal{C}}$. Then there exists an ∞-category $\mathcal{D}$ and a morphism of simplicial sets $F : \widetilde{\mathcal{C}} \to \mathcal{D}$ which exhibits $\mathcal{D}$ as a localization of $\widetilde{\mathcal{C}}$ with respect to $\widetilde{\mathcal{W}}$.*

Remark B.4. Not in HTT. *As far as we can tell, no similar result is proved in* [Lur09]*. In §5.2.7 Lurie says the following.*

> *Suppose we are given an ∞-category $\mathcal{C}$ and a collection S of morphisms of $\mathcal{C}$ which we would like to invert. In other words, we wish to find an ∞-category $S^{-1}\mathcal{C}$ equipped with a functor $\eta : \mathcal{C} \to S^{-1}\mathcal{C}$ which carries each morphism in S to an equivalence and is in some sense universal with respect to these properties. One can give a general construction of $S^{-1}\mathcal{C}$ using the formalism of §3.1.1. ... However, this construction is generally very difficult to analyze, and the properties of $S^{-1}\mathcal{C}$ are very difficult to control. For example, it might be the case that $\mathcal{C}$ is locally small and $S^{-1}\mathcal{C}$ is not.*
>
> *Under suitable hypotheses on S (see §5.5.4), there is a drastically simpler approach: we can find the desired ∞-category $S^{-1}\mathcal{C}$* inside *$\mathcal{C}$ as the full subcategory of S-local objects of $\mathcal{C}$.*

Finding the localized category inside the original one was the approach taken by Bousfield in his theorem about the localization of a model category. See [Rav23, §10] *and* [HHR21, Chapter 6] *for more discussion.*

Recall that a *relative category* $(\mathcal{C}, \mathcal{W})$ consists of a category $\mathcal{C}$ and a wide subcategory $\mathcal{W}$, which can be identified with its morphism collection, which we also denote by $\mathcal{W}$. This morphism collection, since it is that of a subcategory, contains all identities and is closed under composition.

Corollary B.5. The ∞-category for a relative category. *Let $(\mathcal{C}, \mathcal{W})$ be a relative category, let $\widetilde{\mathcal{C}} = N(\mathcal{C})$, and let $\widetilde{\mathcal{W}}$ be the collection of edges in $\widetilde{\mathcal{C}}$ corresponding to the collection $\mathcal{W}$ of morphisms in $\mathcal{C}$. Then there exists an ∞-category $\mathcal{C}[\mathcal{W}^{-1}]$ and a morphism of simplicial sets $F : N(\mathcal{C}) \to \mathcal{C}[\mathcal{W}^{-1}]$ which exhibits $\mathcal{C}[\mathcal{W}^{-1}]$ as a localization of $N(\mathcal{C})$ with respect to $\widetilde{\mathcal{W}}$ as in Definition B.2.*

Definition B.6. *The* **∞-category of orthogonal G-spectra** Sp^G *is that obtained as in Corollary B.5 where $\mathcal{C}$ the category of of orthogonal G-spectra,* Sp^G *as in Definition 4.12, and $\mathcal{W}$ is the collection of stable equivalences as in Definition 4.17.*

For trivial G the ∞-category Sp^G, that of orthogonal spectra, is not the same as Lurie's Sp (as in [Lur17, Definition 1.4.3.1]), which might be called the ∞-category of Ω-spectra.

Appendix C. Some universal algebra

In [McC24, Construction 2.2.3] McCandless mentions Lawvere's theory of monoids T_{Assoc}, referring to [Law63b]. More details can be found in [EW67]. We need it in Definition 5.52.

Definition C.1. [EW67, §4] *An* **algebraic theory** *is a category T whose objects are finite sets $\langle n \rangle = \{1, 2, \ldots, n\}$ for $n \geqslant 0$ as in Definition D.1, which contains the category of finite sets $\mathcal{F}in$ as a wide subcategory. We will denote the singleton $\langle 1 \rangle$*

by I and the empty set $\langle 0\rangle$ by $\varnothing$. Given morphisms $\phi_i : I \to \langle p\rangle$ in T for $1 \leqslant i \leqslant n$, there is a unique morphism

$$\phi : \langle n\rangle \to \langle p\rangle$$

such that for $1 \leqslant i \leqslant p$, ϕ_i is the composition

$$\begin{array}{ccccc} I & \xrightarrow{i} & \langle n\rangle & \xrightarrow{\phi} & \langle p\rangle \\ 1 & \longmapsto & i. & & \end{array}$$

Here $i : I \to \langle n\rangle$ denotes the map sending the single element of I to $i \in \langle n\rangle$. We will sometimes denote ϕ by $(\phi_1, \ldots, \phi_n)$, and say that the ϕ_i are its **components**.

In other words the object $\langle n\rangle$ the n-fold coproduct of the object I. This implies there is a unique morphism $\langle 0\rangle \to \langle n\rangle$ in T, as in the subcategory $\mathcal{F}in$. However unlike $\mathcal{F}in$, T could have morphisms $\langle n\rangle \to \langle 0\rangle$ for $n > 0$.

Definition C.2. [EW67, §5 and §6] *A* **T-algebra** *A is a presheaf on T, that is a contravariant* Set*-valued functor, that converts coproducts to products. This means $\langle n\rangle \mapsto A_n := A_1^{\times n}$, where $A_1 := A(I)$.*

We also require that for a morphism $\phi : \langle n\rangle \to \langle p\rangle$ in $\mathcal{F}in$, the induced map

$$A(\phi) : A_1^{\times p} \to A_1^{\times n} \qquad \text{is} \qquad (x_1, \ldots, x_p) \mapsto (x_{\phi(1)}, \ldots x_{\phi(n)}).$$

A morphism of T-algebras is a natural transformation of such functors.

A morphism $\phi : I \to \varnothing$ in T defines an element

$$\phi_A := A(\phi)(A_0) \in A_1. \tag{C.3}$$

The category of T-algebras is denoted by $T^\natural$. (It is denoted by $T^\flat$ in [EW67]*.)*

The **free T-algebra on k generators** *(denoted by A_k in* [EW67, §6]*) is the Yoneda functor*

$$よ_k^T := T(-, \langle k\rangle).$$

The Yoneda lemma says that

$$T^\natural(よ_k^T, A) = A_k \cong A_1^{\times k} = \mathrm{Set}(\langle k\rangle, A_1).$$

Thus a k-tuple of elements in A_1 determines a T-algebra morphism $よ_k^T \to A$, and all such morphisms arise in this way. This justifies the term "free T-algebra on k generators."

Lawvere's definition of an algebraic theory in [Law63b], and the one used in [McC24, Construction 2.2.3], was the opposite category of that of Definition C.1, so for him a T-algebra was a covariant product preserving Set-valued functor. The same goes for Borceaux's [Bor94, Definition 3.3.1]. We will use the Eilenberg-Wright definition.

The following is originally due to Lawvere [Law63a] and is proved as [Bor94, Proposition 3.2.9].

Theorem C.4. The categorical structure of an algebraic theory. *An algebraic theory T as in Definition C.1 is equivalent to the category of finitely generated free T-algebras as in Definition C.2. The equivalence sends $\langle k\rangle$ to $よ_k^T$.*

In [EW67, §7] the authors defined a **free theory** $T = \mathcal{F}in[\Omega]$ on a sequence of sets

$$\Omega = \{\Omega_0, \Omega_1, \dots\} \qquad \text{with} \qquad \Omega_n \subseteq T(I, \langle n\rangle). \tag{C.5}$$

Roughly speaking, it is the smallest algebraic theory that contains the additional morphisms (to those of $\mathcal{F}in$) of (C.5).

They assigned a **degree** to each morphism in $\mathcal{F}in[\Omega]$ such that morphisms in $\mathcal{F}in$ have degree zero, the degree of a morphism out of $\langle n\rangle$ is the sum of the degrees of its n components, and precomposition with a morphism in some Ω_n increases degree by 1.

They used induction on degree to prove the following.

Theorem C.6. [EW67, §7] **The free theory on the morphism set Ω.** *There is a unique algebraic theory $\mathcal{F}in[\Omega]$ with the following properties.*

(i) For each morphism $\phi : I \to \langle p\rangle$ of positive degree there is a unique $k \geqslant 0$ with a unique factorization

$$I \xrightarrow{\ \omega\ } \langle k\rangle \xrightarrow{\ \psi\ } \langle p\rangle \qquad \text{with } \omega \in \Omega_k.$$

(ii) For any algebraic theory T' containing the morphisms of (C.5), there is a unique morphism $\mathcal{F}in[\Omega] \to T'$ of theories.

(iii) Given a set A_1 and functions $\overline{\omega} : A_1^{\times n} \to A_1$ for all $\omega \in \Omega_n$ and $n \geqslant 0$, there exists a unique $\mathcal{F}in[\Omega]$-algebra structure on A such that

$$A(\omega)(x_1, ..., x_n) = \overline{\omega}(x_1, ..., x_n).$$

In [EW67, §8] they specified how to construct the quotient of a free algebraic theory $\mathcal{F}in[\Omega]$

Definition C.7. *A* **congruence** *Q in an algebraic theory T is a family of equivalence relations $\sim$ in the sets $T(\langle n\rangle, \langle p\rangle)$ such that*

(i) Given a diagram

$$\langle m\rangle \xrightarrow{\ \psi\ } \langle n\rangle \underset{\phi_2}{\overset{\phi_1}{\rightrightarrows}} \langle p\rangle \xrightarrow{\ \gamma\ } \langle q\rangle$$

in T, $\phi_1 \sim \phi_1$ implies $\gamma\phi_1 \sim \gamma\phi_1$ and $\phi_1\psi \sim \phi_1\psi$. Congruence plays nicely with composition.

(ii) For

$$I \xrightarrow{\ i\ } \langle n\rangle \underset{\phi_2}{\overset{\phi_1}{\rightrightarrows}} \langle p\rangle,$$

if $\phi_1 i \sim \phi_2 i$ for $1 \leqslant i \leqslant n$, then $\phi_1 \sim \phi_2$. Two morphisms are congruent iff each of their components are.

(iii) If $\phi_1, \phi_2 : I \to \langle p\rangle$ are in $\mathcal{F}in$ with $\phi_1 \sim \phi_2$, then $\phi_1 = \phi_2$. Distinct morphisms in the subcategory $\mathcal{F}in$ are never congruent.

These conditions enable one to define an algebraic theory T/Q whose morphism sets are the sets of congruence classes in the morphism sets in T. A T/Q-algebra is a T-algebra A in which

$$A(\phi_1)(x_1, \dots, x_p) = A(\phi_2)(x_1, \dots, x_p) \qquad \text{whenever } \phi_1 \sim \phi_2.$$

This makes $(T/Q)^\natural$ a subcategory of $T^\natural$.

Example C.8. *In a T-algebra A, one has a set $A_1 = A(I)$, and each morphism $I \to \langle n \rangle$ in T determines an* n**-ary operation**, *meaning a map $A_1^{\times n} \to A_1$.*

(i) If $T = \mathcal{F}in$, the least interesting case, one has n such maps. The resulting operations merely define the coordinates of an n-tuple. The corresponding "algebras" are sets.

(ii) We will construct the theory T_{sg} whose algbras are semigroups, meaning sets with an associative binary operation. We need a morphism $\pi : I \to \langle 2 \rangle$ outside of $\mathcal{F}in$ for the binary operation. Thus we define Ω as in Theorem C.6 by

$$\Omega_n = \begin{cases} \{\pi\} & \text{for } n = 2 \\ \varnothing & \text{otherwise} \end{cases}$$

Then a $\mathcal{F}in[\Omega]$-algebra is a set equipped with a binary operation. We need a congruence to make it associative. The following diagram must commute in the quotient theory

$$\begin{array}{ccc} I & \xrightarrow{\pi} & \langle 2 \rangle \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle ((1,2)\pi,3)} \\ \langle 2 \rangle & \xrightarrow[(1,(2,3)\pi)]{} & \langle 3 \rangle. \end{array}$$

Here we are using the notation of Definition C.1 for the two morphisms $\langle 2 \rangle \to \langle 3 \rangle$.

(iii) We will construct the theory T_{Assoc} whose algebras are monoids, meaning semigroups with an identity element. We will derive to from T_{sg} by adjoining a morphism $e : I \to \varnothing$, which defines the identity element e_A as in (C.3). Then we must pass to a suitable quotient to insure that e_A has the desired porperties. Let $e_1 : I \to I$ be the composite

$$I \xrightarrow{e} \varnothing \xrightarrow{\sigma} I.$$

where $\sigma : \varnothing \to I$ is the unique such morphism.

Then the following diagram must commute in T_{Assoc}.

$$\begin{array}{ccc} I & \xrightarrow{\pi} & \langle 2 \rangle \\ {\scriptstyle \pi}\downarrow & \searrow^{1} & \downarrow{\scriptstyle (e_1,1)} \\ \langle 2 \rangle & \xrightarrow[(1,e_1)]{} & I. \end{array}$$

Appendix D. Symmetric monoidal ∞-categories

The structure of a symmetric monoidal ∞-category $\mathcal{C}$ is not as simple as a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with the expected properties. It is disussed at length by Lurie leading up to [Lur17, Definition 2.0.0.7], by Moritz Groth in [Gro10, 3.2], and by Nikolaus-Scholze in [NS18, Appendix A].

We begin with three definitions in *ordinary category theory* that will motivate Lurie's definition of a symmetaric monoidal ∞-category.

Definition D.1. *The* **categories of finite sets** $\mathcal{F}in$ **and pointed finite sets** $\mathcal{F}in_*$ *have as objects the sets*

$$\begin{aligned} & \langle n \rangle := \{1, \dots, n\} \\ \text{and} \quad & \langle n \rangle_* := \{0, 1, \dots, n\} \end{aligned} \tag{D.2}$$

respectively, with 0 as basepoint in $\mathcal{F}in_*$. *A morphism* $f : \langle n \rangle_* \to \langle m \rangle_*$ *is a map of sets sending 0 to 0. For* $1 \leqslant i \leqslant n$, *we define* $\varrho^i : \langle n \rangle_* \to \langle 1 \rangle_*$ *by*

$$\varrho^i(j) := \begin{cases} 1 & \textit{for } j = i \\ 0 & \textit{otherwise.} \end{cases} \tag{D.3}$$

A morphism f *in* $\mathcal{F}in_*$ *as above is* **active** *if it sends each nonzero element in the domain to one in the codomain. It is* **inert** *if the preimage of each nonzero nonzero element in the codomain is a singleton. Such a map defines an injection* $\langle n \rangle \to \langle m \rangle$ *sending each element to its preimage under* f.

Note that

$$\langle n \rangle_* \xrightarrow{f} \langle m \rangle_* \qquad \rightsquigarrow \qquad \langle n \rangle_* \supseteq f^{-1}(\langle m \rangle) =: S \xrightarrow{\alpha} \langle m \rangle, \tag{D.4}$$

i.e., f leads to a partially defined map $\langle n \rangle \to \langle m \rangle$. The subset S above is all of $\langle n \rangle$ iff f is active, so the wide subcategory of active morphisms is isomorphic to $\mathcal{F}in$.

We will give a short introduction to operads, of which the following is a special case, in [Rav26, Appendix B].

Definition D.5. [Lur17, Definition 4.1.1.3] *The* **associative operad** $\mathrm{Assoc}^\otimes$ *is a category whose objects are those of* $\mathcal{F}in_*$. *When* $\langle n \rangle_*$ *as in (D.2) is an object of* $\mathrm{Assoc}^\otimes$, *we write it as* $\langle n \rangle_{\mathrm{Assoc}}$. *A morphism*

$$\tilde{f} : \langle n \rangle_{\mathrm{Assoc}} \to \langle m \rangle_{\mathrm{Assoc}}$$

is a map $f : \langle n \rangle_* \to \langle m \rangle_*$ *in* $\mathcal{F}in_*$ *together with a linear ordering on each inverse image* $f^{-1}(i) \subseteq \langle n \rangle_*$ *for* $1 \leqslant i \leqslant m$. *For a composite*

$$\begin{array}{ccccc} & & j & \longmapsto & i \\ \langle n \rangle_{\mathrm{Assoc}} & \xrightarrow{\tilde{f}} & \langle m \rangle_{\mathrm{Assoc}} & \xrightarrow{\tilde{g}} & \langle \ell \rangle_{\mathrm{Assoc}}, \end{array}$$

in $\mathrm{Assoc}^\otimes$, *we define a linear ordering on* $(gf)^{-1}(i) = f^{-1}(g^{-1}(i))$ *for* $1 \leqslant i \leqslant \ell$ *as follows. For each* $j \in g^{-1}(i)$ *we have a linear ordering of* $f^{-1}(j)$ *associated with* $\tilde{f}$, *and the* j*s themselves have an ordering associated with* $\tilde{g}$. *These lead to a lexicographic ordering on* $(gf)^{-1}(i)$ *and thus determine the composite* $\tilde{g}\tilde{f}$ *in* $\mathrm{Assoc}^\otimes$.

The subcategory $\mathrm{Assoc}^\otimes_{\mathrm{act}} \subseteq \mathrm{Assoc}^\otimes$ *is the wide subcatgory whose morphisms project to active morphisms in* $\mathcal{F}in_*$, *namely*

$$\mathrm{Assoc}^\otimes_{\mathrm{act}} := \mathrm{Assoc}^\otimes \times_{\mathcal{F}in_*} \mathcal{F}in. \tag{D.6}$$

We denote by $V : \mathrm{Assoc}^\otimes \to \mathcal{F}in_*$ *the functor that assigns to each object in* $\mathrm{Assoc}^\otimes$ *its underlying pointed finite set.*

Remark D.7. Associative algebras as functors. *A symmetric monoidal functor* A *from* $\mathrm{Assoc}^\otimes$ *to a symmetric monoidal category* $(\mathcal{C}, \otimes)$ *defines an associative algebra* $A\langle 1 \rangle$ *in* $\mathcal{C}$, *and each such algebra determines such a functor sending* $\langle n \rangle$ *to* $A^{\otimes n}$. *The* ∞*-categorical analog of this statement is a special case of* [Lur17, Proposition 2.2.4.9].

Definition D.8. [Lur17, Construction 2.0.0.1] *Let* $(\mathcal{C}, \otimes)$ *be a symmetric monoidal category. The* **sequence category** $\mathcal{C}^\otimes$ *has as objects finite (possibly empty) sequences of objects* C_i *in* $\mathcal{C}$ *for* $1 \leqslant i \leqslant n$, *which we denote by* $(C_1, \ldots, C_n)$. *A morphism*

$$(C_1, \ldots, C_n) \to (C'_1, \ldots, C'_m)$$

consists of a subset $S \subseteq \langle n\rangle_*$ *as in (D.4), a map of finite sets* $\alpha : S \to \langle m\rangle_*$*, and a collection of morphisms*

$$f_j : \bigotimes_{i\in\alpha^{-1}(j)} C_i \to C'_j \qquad \text{for } 1 \leqslant j \leqslant m, \tag{D.9}$$

where each of the tensor products is defined up to canonical isomorphism by the monoidal structure on $\mathcal{C}$*. In the composite*

$$\begin{array}{c} (C_1,\dots,C_n) \xrightarrow{f} (C'_1,\dots,C'_m) \xrightarrow{g} (C''_1,\dots,C''_\ell) \\ \langle n\rangle_* \supseteq S \xrightarrow{\alpha} \langle m\rangle_* \\ \langle m\rangle \supseteq T \xrightarrow{\beta} \langle \ell\rangle_* \end{array}$$

The subset $U \subseteq \langle n\rangle$ *associated with* gf *is* $\alpha^{-1}(T) = \alpha^{-1}\beta^{-1}(\langle m\rangle)$*, and the map* $\gamma : U \to \langle\ell\rangle$ *is* $\beta\alpha$*.*

We have a forgetful functor

$$\begin{array}{ccc} \mathcal{C}^\otimes & \xrightarrow{\text{pr}} & \mathcal{F}in_* \\ (C_1,\dots,C_n) & \longmapsto & \langle n\rangle_*. \end{array} \tag{D.10}$$

We denote by $\mathcal{C}^\otimes_{\langle n\rangle}$ *the full subcategory of sequences of length* n*. The full subcategory of active morphisms is*

$$\mathcal{C}^\otimes_{\text{act}} := \text{pr}^{-1}(\mathcal{F}in),$$

where $\mathcal{F}in$ *is understood to be the wide subcategory of active morphisms in* $\mathcal{F}in_*$*.*

Example D.11. The associative operad as a sequence category. *When* $\mathcal{C}$ *is the category with a single object and a single morphism, and hence a unique monoidal structure which is symmetric,* $\mathcal{C}^\otimes$ *is the associative operad* $\text{Assoc}^\otimes$*. Each* $\mathcal{C}^\otimes_{\langle n\rangle}$ *has a single object which we can identify with* $\langle n\rangle_{\text{Assoc}}$*. The linear ordering of each* $f^{-1}(i)$ *is that inherited from the notational ordering of the set* $\langle n\rangle_*$*.*

Proposition D.12. Properties of the forgetful functor to $\mathcal{F}in_*$**.**
Let pr *be the forgetful functor of (D.10).*

(i) pr *is a* **Grothendieck op-fibration of categories** *(see* [HHR21, Definition 2.8.1]*), meaning that for every object*

$$C = (C_1,\dots,C_n) \in \mathcal{C}^\otimes_{\langle n\rangle}$$

and every morphism $f : \langle n\rangle_* \to \langle m\rangle_*$ *in* $\mathcal{F}in_*$*, there exists a morphism*

$$\overline{f} : C \to C' = (C'_1,\dots,C'_m)$$

which covers f *(meaning* $\text{pr}(\overline{f}) = f$*), and is universal in the sense that composition with* $\overline{f}$ *induces a bijection*

$$\begin{array}{c} \text{Hom}_{\mathcal{C}^\otimes}(C',C'') \\ \downarrow\cong \\ \text{Hom}_{\mathcal{C}^\otimes}(C,C'') \times_{\text{Hom}_{\mathcal{F}in_*}(\langle n\rangle_*,\langle\ell\rangle_*)} \text{Hom}_{\mathcal{F}in_*}(\langle m\rangle_*,\langle\ell\rangle_*) \end{array} \tag{D.13}$$

for every object $C'' = (C''_1,\dots,C''_\ell)$ *in* $\mathcal{C}^\otimes_{\langle\ell\rangle}$*. (Such a morphism can be constructed using the morphisms* f_j *of (D.9).)*

(ii) $\mathcal{C}^{\otimes}_{\langle 1\rangle}$ *is equivalent to* $\mathcal{C}$*, and functors associated with the morphisms* ϱ^i *of (D.3) lead to an equivalence of* $\mathcal{C}^{\otimes}_{\langle n\rangle}$ *with* $\mathcal{C}^{\times n}$ *for all* $n \geqslant 1$*.*

The punchline. Now we come to a brilliant observation of Lurie. *Suppose we forget how the sequence category* $\mathcal{C}^{\otimes}$ *was constructed, and assume only that it is equipped with a* $\mathcal{F}in_*$*-valued functor* pr *having analogs of the two properties in Proposition D.12.*

Then we have full subcategories $\mathcal{C}^{\otimes}_{\langle n\rangle} := \mathrm{pr}^{-1}\langle n\rangle_*$ for all $n \geqslant 0$, and we can define $\mathcal{C}$ to be $\mathcal{C}^{\otimes}_{\langle 1\rangle}$. In (D.13), the objects C, C' and C'' lie in $\mathcal{C}^{\otimes}_{\langle n\rangle}$, $\mathcal{C}^{\otimes}_{\langle m\rangle}$ and $\mathcal{C}^{\otimes}_{\langle \ell\rangle}$ respectively. For (ii) we can still require that functors associated with the morphisms ϱ^i of (D.3) lead to an equivalence of $\mathcal{C}^{\otimes}_{\langle n\rangle}$ with $\mathcal{C}^{\times n}$ for all $n \geqslant 1$.

For example

- $\mathcal{C}^{\otimes}_{\langle 0\rangle}$ has one object, and the unique map $\langle 0\rangle_* \to \langle 1\rangle_*$ induces a functor $\mathcal{C}^{\otimes}_{\langle 0\rangle} \to \mathcal{C}$ identifying the unit object.
- The unique active map $f : \langle 2\rangle_* \to \langle 1\rangle_*$ leads to the monoidal structure $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.
- Let $t : \langle 2\rangle_* \to \langle 2\rangle_*$ be the automorphism that interchanges 1 and 2. The identity $ft = f$ leads to the symmetry condition.
- The unique active map $\langle 3\rangle_* \to \langle 1\rangle_*$ can be factored as the composite of order preserving active maps $\langle 3\rangle_* \to \langle 2\rangle_* \to \langle 1\rangle_*$ in two different ways, which leads to the associativity condition.

Further relations in $\mathcal{F}in_*$ lead to further structure in $\mathcal{C}$ that would be tedious to spell out explicitly. The functor pr gives us a painfree and obvious way to specify it.

This suggests replacing $\mathcal{C}^{\otimes}$ by an ∞-category $\mathcal{C}^{\otimes}$ and the forgetful functor pr by a map of simplicial sets to $N(\mathcal{F}in_*)$, the nerve of Definition 2.25.

Definition D.14. [Lur17, Definition 2.0.0.7] *and* [NS18, Definition A.1]. *A* **symmetric monoidal ∞-category** *is an ∞-category* $\mathcal{C}^{\otimes}$ *(the* **total space***) with a coCartesian fibration (see* [Lur09, Definition 2.4.2.1]*) of simplicial sets*

$$\mathrm{pr} : \mathcal{C}^{\otimes} \to N(\mathcal{F}in_*),$$

in which we define $\mathcal{C}^{\otimes}{}_{\langle n\rangle} := \mathrm{pr}^{-1}\langle n\rangle_*$*, having the following property For each* $n \geqslant 0$*, the morphisms* ϱ^i *of (D.3) induce functors*

$$\varrho^i_! : \mathcal{C}^{\otimes}{}_{\langle n\rangle} \to \mathcal{C}^{\otimes}{}_{\langle 1\rangle} \qquad \text{for } \leqslant i \leqslant n$$

which determine an equivalence $\mathcal{C}^{\otimes}{}_{\langle n\rangle} \simeq (\mathcal{C}^{\otimes}{}_{\langle 1\rangle})^n$*.*

The **underlying ∞-category of** $\mathcal{C}^{\otimes}$ *is* $\mathcal{C} := \mathcal{C}^{\otimes}{}_{\langle 1\rangle}$*.*

Lurie then goes on to say

> One of our main goals in this book is to show that Definition 2.0.0.7 is reasonable: that is, it provides a robust generalization of the classical theory of symmetric monoidal categories.

As in Remark D.7, the ∞-category of associative algebras in $\mathcal{C}$ is

$$\mathrm{Alg}(\mathcal{C}) := \mathrm{Fun}^{\otimes}(\mathrm{Assoc}^{\otimes}, \mathcal{C}),$$

the ∞-category of symmetric monoidal functors; see [Lur17, Proposition 2.2.4.9].

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