CYCLOTOMIC EXTENSIONS IN STABLE HOMOTOPY THEORY

DOUGLAS C. RAVENEL

Contents

1. Introduction 2
2. The players 7
2.1. The chromatic landscape 7
2.2. The Burklund-Hahn-Levy-Schlank counterexample 8
2.3. Nontrivial coassembly maps 9
2.4. The Lichtenbaum-Quillen property 11
3. Semiadditivity 11
4. Chromatic cyclotomic extensions 17

Date: May 18, 2026.

4.1. Some classical number theory 18
4.1.1. Ramified cyclotomic extensions 18
4.1.2. Unramified extensions 20
4.2. Rognes' Galois theory for commutative ring spectra 20
4.3. Higher cyclotomic extensions and cyclotomic completion 22
4.4. The surprising appearance of Eilenberg-MacLane spaces 24
4.5. Chromatic cyclotomic redshift 26
5. Locally unipotent $\mathbb{Z}$-actions 28
5.1. Trivializing a locally unipotent $\mathbb{Z}$-action 28
5.2. Adams operations on $BP\langle n\rangle$ 31
6. The main counterexample 33
Appendix A. Some ∞-categorical constructions 35
A.1. Miscellaneous definitions 35
A.1.1. Augmented simplicial objects 35
A.1.2. Limits and colimits 36
A.1.3. Truncation and connectivity 37
A.1.4. Accessibile and presentable ∞-categories 38
A.1.5. The coskeleton functor 39
A.1.6. Localization 41
A.2. Groupoids and the Čech nerve 42
A.3. Sheaves and presheaves 44
A.4. ∞-topoi 45
A.5. Grothendieck topologies and ∞-sheaves 48
A.6. All things hyper: hypercompletion, hypercoverings, hyperdescent and hypersheaves 52
A.7. F-descent and F-covers 55
A.8. Finite sets acted on by a profinite group 56
Appendix B. Operads 58
B.1. The early work of May and Stasheff. 58
B.2. Some generalizations 63
B.3. Trees 64
B.4. Free operads and coproduct operads 66
B.5. The Boardman-Vogt tensor product 67
References 70

1. Introduction

This expository paper is a companion to [Rav26], in which we discuss cyclotomic spectra. Both papers are intended to shed light on the recent resolution of the telescope conjecture by Robert Burklund, Jeremy Hahn, Ishan Levy and Tomer Schlank (hereafter referred to as BHLS) in [BHLS23]. Their proof involves both cyclotomic spectra, the subject of [Rav26], and cyclotomic extensions of spectra, the subject of this paper.

Cyclotomic spectra have an action of the circle group S^1, which we denote by $\mathbb{T}$, for torus. The subspectra fixed by finite subgroups of $\mathbb{T}$ are required to have certain properties. Such spectra come up in algebraic K-theory and its close relatives topological Hochschild homology THH and topological cyclic homology TC.

Higher cyclotomic extensions of commutative ring spectra are analogous to Galois extensions of p-adic number fields (or rings of integers thereof) obtained by adjoining roots of unity. When we want to distinguish between these two forms of cyclotomy, we will use the terms **smooth** for the spectra of [Rav26] and **discrete** for the extensions of this paper.

For the reader's convenience we note that

- [BHLS23, Theorem A] is our Theorem 6.3.
- [BHLS23, Theorem B] is our Theorem 5.11.
- [BHLS23, Theorem C] is our Theorem 5.3
- [BHLS23, Theorems D and E] are not covered here.

Remark 1.1. *As in* [Rav26] *we follow the font convention of* [BHLS23] *by denoting the algebraic K-theory of a ring spectrum R by* $\mathrm{K}(R)$ *(Quillen's* K*) to avoid confusion with its nth Morava K-theory $K(n)_*(R)$ (Jack's K).*

Here, as in [Rav26] *and* [Rav23]*, we use cyan colored symbols to denote ∞-categories that are not necessarily ordinary categories.*

In §2 we introduce various functors of interest (§2.1), describe the BHLS counterexample to the telescope conjecture (§2.2), review methods for showing that certain telescopically localized coassembly maps are not equivalences (§2.3), and discuss the Lichtenbaum-Quillen property in §2.4.

§3 is about **semiadditivity**, which concerns the following sitation. Suppose we have a group G acting on a space, spectrum or object in a suitable ∞-category $\mathcal{C}$. This action is equivalent to a functor $F : BG \to \mathcal{C}$, where BG is the one object ∞-category associated with G. Such a functor determines the choice of an object X in $\mathcal{C}$ since BG has a single object. This functor has a colimit X_{hG}, the homotopy orbit object, and a limit X^{hG}, the homotopy fixed point object.

When G is finite one can define a so called norm map $\mathrm{Nm}_X : X_{hG} \to X^{hG}$. It is known to be an equivalence when $\mathcal{C} = \mathrm{Sp}_{K(n)}$, the ∞-category of $K(n)$-local spectra. An ∞-category is said to be **1-semiadditive** (see Definition 3.5) if it has a similar property for all functors to it from BG. In [HL13] Mike Hopkins and Jacob Lurie generalize from BG, a path connected space for which π_1 is finite and all other homotopy groups vanish, to an **m-finite space** as in Definition 3.2, meaning one with finitely many path components, each of which has finite homotopy groups that vanish above dimension m. They show that for a $\mathcal{C}$-valued functor from such a space one can still define a norm map, and they say that $\mathcal{C}$ is **m-semidditive** if the norm is always an equivalence. $\mathcal{C}$ is said to be **∞-semidditive** if it is m-semidditive for all m.

They prove that $\mathrm{Sp}_{K(n)}$ has this property. In [CSY22, Theorem A], Schlank, Shachar Carmeli and Lior Yanovski show the same for the (possibly) larger ∞-category $\mathrm{Sp}_{T(n)}$. In [CSY22, Theorem B] (our Theorem 3.7) they show that $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$ are respectively minimal and maximal among 1-semiadditive ∞-categories. In [CSY22, Theorem C] (our Theorem 3.9) they show that 1-semiadditivity is equivalent to ∞-semiadditivity.

The original telescope conjecture asserted the equality $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$, both of which are ∞-semiadditive. This suggests that further study of such ∞-categories could lead to insight about the conjecture, which has indeed proven to be the case.

In §4, which is about **chromatic cyclotomic extensions**, we learn that between $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$ there is an intermediate ∞-semiadditive ∞-category $\mathrm{Sp}_{\mathrm{Cyclo}(n)}$, that of **height n cyclotomically complete spectra** as in Definition 4.17.

In order to define such extensions, we first review the cyclotomic extensions of classical number theory in §4.1. These are Galois extensions of the field $\mathbb{Q}_p$ of p-adic numbers, and their rings of integers, obtained by adjoining roots of unity.

In §4.2 we review Rognes' Galois theory for commutative ring spectra, the stable homotopy theoretic analog of classical Galois theory. A pivotal example of a faithful (as in Definition 4.6) profinite Galois extension in this sense (see Proposition 4.8) is the map $\mathbb{S}_{K(n)} \to E_n(\overline{\mathbb{F}}_p)$ from the $K(n)$-local sphere spectrum to extended height n Morava E-theory. Its Galois group is the extended Morava stabilizer group $\mathbb{G}_n$. One might say that $E_n(\overline{\mathbb{F}}_p)$ is the algebraic closure of $\mathbb{S}_{K(n)}$.

It would be nice to have a similar algebraic closure of the telescopic analog $\mathbb{S}_{T(n)}$, but no telescopic analog of either $E_n(\overline{\mathbb{F}}_p)$ or $\mathbb{G}_n$ is known or even thought to exist. On the other hand, by [CSY24, Theorems A] (quoted here as Theorem 4.10) we do have a telescopic analog of every *abelian* extension of $\mathbb{S}_{K(n)}$, including the maximal one. These are treated in §4.3.

The abelianization of $\mathbb{G}_n$ is known to be isomorphic to that of the absolute Galois group of $\mathbb{Q}_p$, namely $\mathbb{Z}_p^\times \times \widehat{\mathbb{Z}}$. The second factor, the profinite integers, is generated by a lift of the Frobenius element in the absolute Galois group of $\mathbb{F}_p$.

It has a quotient isomorphic to $(\mathbb{Z}/p^j)^\times$ obtained by sending the Frobenius generator of $\widehat{Z}$ to the identity element. This corresponds to the cyclotomic extension $\mathbb{Q}_p \to \mathbb{Q}_p[\omega_{p^j}]$ obtained by adjoining a primitive p^jth root of unity.

The analogous quotient to $\mathbb{G}_n$ corresponds to faithful Galois extensions

$$\mathbb{S}_{K(n)} \to \mathbb{S}_{K(n)}[\omega_{p^j}^{(n)}] \qquad \text{and} \qquad \mathbb{S}_{T(n)} \to \mathbb{S}_{T(n)}[\omega_{p^j}^{(n)}]. \tag{1.2}$$

These are the chromatic and telescopic analogs of the classical cyclotomic extensions of §4.1. Each is obtained by adjoining a **higher root of unity** as in Definition 4.26, which is as follows. For a commutative ring spectrum R, such as $\mathbb{S}_{K(n)}$ or $\mathbb{S}_{T(n)}$, one has a topological abelian group of units $R^\times$

underlain by the subspace of $\Omega^\infty R$ comprising the path components indexed by the invertible elements in the ring $\pi_0 R$. An ordinary p^jth root of unity in R is the image of a generator under a monomorphism $\mathrm{C}_{p^j} \to R^\times$. A **height n p^jth root of unity** in R is a suitable map $\mathrm{C}_{p^j} \to \Omega^n R^\times$, which is adjoint to a map $B^n\mathrm{C}_{p^j} \to R^\times$. The domain of the latter is the n-finite Eilenberg-MacLane space $K(\mathbb{Z}/p^j, n)$, hence the relevance of the n-semiadditivity of §3. Remarkably it turns out that

$$\mathbb{S}_{T(n)}[\omega_{p^j}^{(n)}] \simeq \mathbb{S}_{T(n)} \otimes K((\mathbb{Z}/p^j)^\times, n)_+$$

and similarly for its $K(n)$-localization.

Taking the colimit as $j \to \infty$ in (1.2) yields profinite Galois extensions as in Definition 4.12,

$$\mathbb{S}_{K(n)} \to \mathbb{S}_{K(n)}[\omega_{p^\infty}^{(n)}] =: R_n \qquad \text{and} \qquad \mathbb{S}_{T(n)} \to \mathbb{S}_{T(n)}[\omega_{p^\infty}^{(n)}] =: R_n^{\mathrm{Fin}}.$$

The former is known to be faithful but the latter is not. In terms of Bousfield classes, as in [Bou79a] and [Rav92, Definition 7.2.1], we have

$$\langle \mathbb{S}_{K(n)} \rangle = \langle R_n \rangle \leqslant \langle R_n^{\mathrm{Fin}} \rangle \leqslant \langle \mathbb{S}_{T(n)} \rangle, \tag{1.3}$$

which should be compared with (4.20). The three corresponding localization functors are denoted by L_n, L_n^{Cyclo} and L_n^{Fin}. In Definition 4.17 we say that a $T(n)$-local spectrum is **cyclotomically complete** if it is R_n^{Fin}-local. We do not know whether the first inequality of (1.3) is strict, but the counterexample of [BHLS23] shows that the second one is. Cyclotomic completion figures prominently in their proof.

The subject of §4.5 is the charmingly named chromatic cyclotomic redshift, "the final key idea in giving a counter-example to the telescope conjecture" [BHLS23, §6.3]. The term "redshift" refers to the increase of chromatic height caused by algebraic K-theory or some related functor. Conversely "blueshift" refers to the reduction of chromatic height by the Tate construction, which is the starting point for the semiadditivity of §3. Theorem 4.33 says that for a $T(n)$-local commutative ring spectrum R,

$$\mathrm{K}_{T(n+1)}(R[\omega_{p^\infty}^{(n)}]) \simeq \mathrm{K}_{T(n+1)}(R)[\omega_{p^\infty}^{(n+1)}].$$

In §5 we discuss locally unipotent $\mathbb{Z}$-actions as in Definition 5.1, which are relevant to the main counterexample of [BHLS23]. The local unipotence condition on a group action can be thought of as "nearly trivial."

Suppose we have a spectrum X acted on by a group G, to which we apply a functor F. There is a **coassembly map** as in [Rav26, Definition 5.6],

$$\epsilon : F(X^{hG}) \to F(X)^{hG}, \tag{1.4}$$

which is an equivalence if F preserve limits. [BHLS23, Theorem 3.22], quoted here as Theorem 5.2, says that if R is a $T(n)$-local $\mathbb{E}_1$-ring spectrum for $n \geqslant 1$ for which $\mathrm{K}_{T(n+1)}(R)$ is nontrivial, then the map

$$\epsilon : \mathrm{K}_{T(n+1)}(R^{B\mathbb{Z}}) \to \mathrm{K}_{T(n+1)}(R)^{B\mathbb{Z}},$$

the coassembly map for the trivial $\mathbb{Z}$-action, is *not* an equivalence.

In the main counterexample of [BHLS23], the ring of interest is

$$R = L_{T(n)}BP\langle n\rangle$$

equipped with a locally unipotent $\mathbb{Z}$-action by certain Adams operations described in §5.2. In order to apply Theorem 5.2 we need to show that this nontrivial group action is functorially related to one that is trivial. We have two tools at our disposal:

- replace the group $\mathbb{Z}$ by a subgroup $p^k\mathbb{Z}$ and
- smash with a suitable finite spectrum.

Theorem 5.3 says that for $k \gg 0$ we can smash with a finite spectrum of type $n+2$ and "trivialize" the $\mathbb{Z}$-action on $\mathrm{THH}(R)$ in the sense that its homotopy fixed point set behaves like that of the trivial action. Corollary 5.7 and Theorem 5.11 are similar statements for $\mathrm{TC}(R)$ and $\mathrm{TC}_{T(n+1)}(BP\langle n\rangle)$.

In §6 we sketch the proof that the map of (1.4) is an equivalence for $F = \mathrm{K}_{K(n+1)}$ but not for $F = \mathrm{K}_{T(n+1)}$, thus showing that the two functors are different.

We need Theorem 5.11 as well as Corollary 5.7 because Theorem 6.1 and Corollary 6.2 pertain only to *connective* ring spectra. The latter says that for a $T(n+1)$-acyclic, connective $\mathbb{E}_1$-algebra R with a $\mathbb{Z}$-action, such as $BP\langle n\rangle$, there is a diagram

$$\begin{array}{ccc}
\mathrm{K}_{T(n+1)}(L_{T(n)}R^{h(p^k\mathbb{Z})}) & \xrightarrow{\ \epsilon\ } & \mathrm{K}_{T(n+1)}(L_{T(n)}R)^{h(p^k\mathbb{Z})} \\
\simeq\uparrow & & \simeq\uparrow \\
\mathrm{K}_{T(n+1)}(R^{h(p^k\mathbb{Z})}) & \xrightarrow{\ \epsilon\ } & \mathrm{K}_{T(n+1)}(R)^{h(p^k\mathbb{Z})} \\
\simeq\downarrow & & \simeq\downarrow \\
\mathrm{TC}_{T(n+1)}(R^{h(p^k\mathbb{Z})}) & \xrightarrow{\ \epsilon\ } & \mathrm{TC}_{T(n+1)}(R)^{h(p^k\mathbb{Z})},
\end{array}$$

This means we can replace the functor $\mathrm{K}_{T(n+1)}L_{T(n)}$ with the more accessible $\mathrm{TC}_{T(n+1)}$.

Theorem 6.6 is the specialization of the above to the case where $R = BP\langle n\rangle$ for $n \geqslant 1$ with its Adams operations. In that case it can be shown the the coassembly map is height $n+1$ cyclotomic completion. Hence the coassembly map on

$$\mathrm{K}_{\mathrm{Cyclo}(n+1)}(BP\langle n\rangle^{h(p^k\mathbb{Z})})$$

is an equivalence, while the middle coassembly map above is not. Thus the functors $L_{T(n_1)}$ and $L_{\mathrm{Cyclo}(n+1)}$, so the telescope conjecture is false.

There are two appendices, each added to clarify a needed but complicated definition. Appendix A covers various ∞-categorical notions including that of a hypersheaf (Definition A.58), which is needed in Proposition 4.29 and Theorem 4.33. Appendix B covers operads in order to describe the multiplicative structure preserved by the Adams operations on $BP\langle n\rangle$ discussed in §5.2.

It is a pleasure to acknowledge helpful conversations with Tomer Schlank, Ishan Levy, Jeremy Hahn, Robert Burklund, Hari Rau-Murthy, Siddharth Gurumurthy, John Rognes, Mike Mandell, Inbar Klang, Liam Keenan, and Mike Hopkins.

2. The players

2.1. The chromatic landscape.

Definition 2.1.

(i) [Rav92, Definition 1.5.3.] *A p-local finite spectrum X* **has type** *n if $K(n)_*X \neq 0$ and $K(m)_*X = 0$ for all $m < n$. We will sometimes denote such a spectrum by $F(n)$.*

(ii) [MR99, §3] *A p-complete bounded below spectrum Y* **has** fp**-type** *n if it has finite type and for each finite spectrum $F(n+1)$ of type $n+1$ as above, $F(n+1) \otimes Y$ is π-finite, meaning that is has only finitely many nontrivial homotopy groups, each of which is finite.*

In both cases we say the spectrum has **(chromatic) height** *n.*

It is known [Rav84, Theorem 2.11] that for finite spectra X, $K(m)_*X = 0$ implies $K(m-1)_*X = 0$, but this is far from true for infinite CW-spectra. For example there is a infinite spectrum $BP\langle n\rangle$ for which $K(m)_*BP\langle n\rangle$ is trivial for $m > n$ but nontrivial for $m \leqslant n$, and this spectrum has fp-type n. An infinite spectrum need not have an fp-type, but a finite one always has a type.

The telescope conjecture is about the relation between Bousfield localizations (originally defined by Pete Bousfield in [Bou79b]) with respect to $K(n)$, the nth Morava K-theory, and "the" telescope $T(n)$. The quotation marks refer to the following consequences of the periodicity theorem of Jeff Smith and Hopkins [HS98]:

- For each prime p and positive integer n (the chromatic height) there are p-local finite spectra of type n as in Definition 2.1(i).
- Each type n finite spectrum $F(n)$ admits a map $v : F(n) \to \Sigma^{-d}F(n)$ for some $d > 0$ inducing an isomorphism in $K(n)$ homology. The cofiber of v has type $n+1$, so it is possible to construct finite spectra of all types by induction on n.

We denote the filtered colimit obtained by iterating v, the **height n telescope**, by $T(n)$. For a given prime p and height n, neither $F(n)$ nor v is unique, although for a given finite $F(n)$, the homotopy type of its telescope $T(n)$ is known to be independent of the choice of v. Better still, the Bousfield localization functor $L_{T(n)}$ associated with $T(n)$ is known to be independent of the choice of $F(n)$ as well. Hence it is customary to omit $F(n)$ and v from the notation.

The original conjecture of [Rav84] was that the functors $L_{T(n)}$ and $L_{K(n)}$ are the same, which was then known to be the case for $n = 0$ and $n = 1$. A few years later (1989) it became apparent that the statement was likely to be false for $n > 1$. The author made several unsuccessful attempts to disprove it, one of which is the subject of [MRS01]. None of them were anything like the work of BHLS.

There is an equality of Bousfield classes [Rav92, Definition 7.2.1 and Theorem 7.3.2 (d)],

$$\langle E_n \rangle = \langle E(n) \rangle = \bigoplus_{0 \leqslant m \leqslant n} \langle K(m) \rangle. \tag{2.2}$$

Definition 2.3. The functors L_n and L_n^{Fin}. *The former is Bousfield localization with respect to $E(n)$, periodic Johnson-Wilson theory, or equivalently with respect to E_n, Morava E-theory. We denote the ∞-category of $E(n)$-local spectra by $L_n\mathrm{Sp}$.*

L_n^{Fin} is Bousfield localization with respect to

$$\bigoplus_{0 \leqslant m \leqslant n} T(m).$$

This functor is independent of the choice of the $T(m)$s. We denote the corresponding category of local spectra by $L_n^{\mathrm{Fin}}\mathrm{Sp}$.

The notation L_n^{Fin} is used because it is known that the fiber of the map $X \to L_n^{\mathrm{Fin}} X$ is the colimit of all *finite* $E(n)$-acyclic spectra mapping to X. See Haynes Miller's [Mil92] and [BMCSY25, §2] for more discussion.

Both L_n and L_n^{Fin} are known to be **smashing** [Rav92, Theorem 7.5.6], meaning that for any spectrum X,

$$L_n X \simeq X \wedge L_n \mathbb{S} \qquad \text{and} \qquad L_n^{\mathrm{Fin}} X \simeq X \wedge L_n^{\mathrm{Fin}} \mathbb{S}.$$

In Definition 4.17 we will define the functors L_n^{Cyclo} (height n cyclotomic completion) and $L_{\mathrm{Cyclo}(n)}$ that interpolate respectively between L_n^{Fin} and L_n, and between $L_{T(n)}$ and $L_{K(n)}$.

As in [Rav26] we will use the following notation.

Definition 2.4. Telescopic, cyclotomically complete and chromatic localizations of K-theory and TC. *For an $\mathbb{E}_1$-ring spectrum R and $n \geqslant 0$,*

$$\mathrm{K}_{T(n)}(R) := L_{T(n)}\mathrm{K}(R), \qquad \mathrm{TC}_{T(n)}(R) := L_{T(n)}\mathrm{TC}(R),$$
$$\mathrm{K}_{\mathrm{Cyclo}(n)}(R) := L_{\mathrm{Cyclo}(n)}\mathrm{K}(R), \qquad \mathrm{TC}_{\mathrm{Cyclo}(n)}(R) := L_{\mathrm{Cyclo}(n)}\mathrm{TC}(R),$$

where $L_{\mathrm{Cyclo}(n)}$ is as in Definition 4.17 below,

$$\mathrm{K}_{K(n)}(R) := L_{K(n)}\mathrm{K}(R), \qquad \text{and} \qquad \mathrm{TC}_{K(n)}(R) := L_{K(n)}\mathrm{TC}(R).$$

The notation $\mathrm{K}_{T(n)}$ is used in [BMCSY25], but not in most other papers in this area.

2.2. **The Burklund-Hahn-Levy-Schlank counterexample.** The authors of [BHLS23] show that for each $n \geqslant 1$ and each prime p, there is a p-local $\mathbb{E}_1$-ring spectrum R of chromatic height n such that $\mathrm{TC}_{K(n+1)}(R)$ and $\mathrm{TC}_{T(n+1)}(R)$ are distinct. More precisely, for each prime p and each integer $n > 0$, they consider a form of the Johnson-Wilson spectrum $BP\langle n \rangle$

(which has fp-type n), originally defined by Dave Johnson and Steve Wilson in [JW73] and revisited half a century later by Dylan Wilson (no relation to Steve) and Hahn in [HW22]. In [BHLS23, §5] the authors define an action of the additive group of integers $\mathbb{Z}$, and hence of its subgroups $p^k\mathbb{Z}$ for $k > 0$, via Adams operations, which we will discuss in §5.2. They prove that the $T(n+1)$-local K-theoretic coassembly map of [Rav26, Definition 5.6]

$$\epsilon : \mathrm{K}_{T(n+1)}(BP\langle n\rangle^{h(p^k\mathbb{Z})}) \to \mathrm{K}_{T(n+1)}(BP\langle n\rangle)^{h(p^k\mathbb{Z})} \tag{2.5}$$

is not an equivalence, but becomes one after $K(n+1)$-localization. In other words, the algebraic K-theory functor on ring spectra with $\mathbb{Z}$-action does does not commute with passage to homotopy fixed points, even $T(n+1)$-locally, but its does so commute $K(n+1)$-locally. Thus, the telescope conjecture fails at heights greater than 1. In their words,

> We do this by looking at the coassembly map from two highly divergent perspectives, which are connected via trace theorems:
>
> (1) From the perspective of locally unipotent $\mathbb{Z}$-actions on ring spectra, the results of [BHLS23, §4] tell us that the coassembly map cannot be an isomorphism. [In §2.3 we will say that it is **nontrivial** when it is not an isomorphism.]
>
> (2) From the perspective of cyclotomic redshift of [BMCSY25], the map
>
> $$L_{T(n)}BP\langle n\rangle^{h(p^k\mathbb{Z})} \longrightarrow L_{T(n)}BP\langle n\rangle$$
>
> splits after base change to the maximal abelian extension of the $K(n)$-local sphere [see Definition 4.12], and therefore the coassembly map is a $K(n+1)$-local isomorphism.

Remark 2.6. Why $BP\langle n\rangle$? *While $BP\langle n\rangle$ has less structure than its relatives E_n and $E_n(\overline{\mathbb{F}}_p)$, it has the advantage of being connective. (The homotopy groups of these three spectra are given in (4.1).) We need connectivity to apply* [BHLS23, Corollary 3.21] *(quoted as* [Rav26, Proposition 6.6]*) to show that a certain $T(n+1)$-local coassembly map is not an equivalence.*

In that statement the $\mathbb{Z}$-action on the connective ring spectrum is assumed to be trivial, but the action here is nontrivial. We only know that it is locally unipotent as in Definition 5.1. Such actions are sketched in §5. In Theorem 5.11, we will see that our action on $BP\langle n\rangle$ becomes trivial after smashing with a suitable finite complex.

2.3. **Nontrivial coassembly maps.** The relevant results of [BHLS23, §4] are reported in [Rav26, §6]. They have to do with a connective ring spectrum R on which $\mathbb{Z}$ acts trivially, leading to a similar actions on $\mathrm{THH}(R)$ and

$\mathrm{TC}(R)$. It turns out that the coassembly maps ϵ in these cases are far from being isomorphisms.

[BHLS23, Proposition 3.19], quoted as [Rav26, Proposition 6.6], identifies the fiber of coassembly in the THH case when $R = \mathbb{S}_p$, the p-local sphere spectrum. It also guarantees the nontriviality of the fiber in the TC case for general R provided that $\mathrm{TC}(R)$ is nontrivial.

The example of interest is a *nontrivial* action of $\mathbb{Z}$ (via Adams operations) on the connective ring spectrum $R = BP\langle n\rangle$. This action is locally unipotent (Definition 5.1) after p-adic completion, and such actions are discussed here in §5. Theorem 5.3 and Corollary 5.7 say that if such a spectrum has certain properties (that $BP\langle n\rangle$ is known to have), then smashing it with a type $n+2$ finite complex renders the Z-action trivial on $\mathrm{THH}(R)$ and on $\mathrm{TC}(R)$ respectively. Hence we know that the $T(n+1)$-local coassembly maps are nontrivial.

BHLS take their discrete cyclotomic extensions and apply functors such as algebraic K-theory, TC and THH, which are defined in terms of smooth cyclotomy. Their cyclotomic completion (Definition 4.17) has to do with discrete cyclotomy, while cyclotomic boundedness (in relation to the Antieau-Nikolaus t-structure of [AN21]) has to do with smooth cyclotomy. Most of their proof takes place in the ∞-category CycSp of smoothly cyclotomic spectra as in [Rav26, Definition 5.24].

[Rav26] ends by quoting [BHLS23, Theorem 3.22 for $X = \mathbb{S}$], as [Rav26, Theorem 6.7], which says that a map similar to (2.5) is not an equivalence. In it $BP\langle n\rangle$ is replaced by a $T(n)$-local ring spectrum R for which $\mathrm{K}_{T(n+1)}(R)$ (see Definition 2.4) is nontrivial, equipped with a *trivial* action of $\mathbb{Z}$.

The object of this paper is twofold:

(i) To outline the proof that the chromatic analog of (2.5), the coassembly map

$$\mathrm{K}_{K(n+1)}(BP\langle n\rangle^{h(p^k\mathbb{Z})}) \to \mathrm{K}_{K(n+1)}(BP\langle n\rangle)^{h(p^k\mathbb{Z})},$$

is an equivalence for a certain nontrivial action of $\mathbb{Z}$ by Adams operations. We will see in (4.20) that there is an intermediate functor between $K(n)$ and $T(n)$ localizations, height n cyclotomic completion $L_{\mathrm{Cyclo}(n)}$ as in Definition 4.17. It turns out that the map of (2.5) with $T(n+1)$-localization replaced by height $n+1$ cyclotomic completion is also an equivalence. See Theorem 6.6.

(ii) To show that said group action (which is locally unipotent as in Definition 5.1), when restricted to the subgroup $p^k\mathbb{Z}$ for $k \gg 0$, is close enough to trivial to show that (2.5) *is not an equivalence.* A TC version of this is given in Theorem 5.11, and Theorem 6.6 relates it to algebraic K-theory.

Then we can use the fact that when $\mathbb{Z}$ acts trivially on a ring spectrum R for which $\mathrm{K}_{T(n+1)}(R)$ (see Definition 2.4) is nontrivial, the coassembly map

$$\epsilon : \mathrm{K}_{T(n+1)}(R^{B\mathbb{Z}}) \to \mathrm{K}_{T(n+1)}(R)^{B\mathbb{Z}},$$

is not an equivalence by [BHLS23, Theorem 3.22 for $X = \mathbb{S}$], which is quoted as [Rav26, Theorem 6.7].

2.4. The Lichtenbaum-Quillen property.

Definition 2.7. [BHLS23, Definition 4.2] *Let $n \geqslant -1$. We say that an $\mathbb{E}_1$-ring R has the* **height n Lichtenbaum-Quillen property** *if* $\mathrm{THH}(R)$ *is bounded below and, for each finite spectrum F of type $n+2$, the spectrum $F \otimes \mathrm{THH}(R)$ is bounded above in the Postnikov sense.*

The bounded above condition means that $F \otimes \mathrm{THH}(R)$ is a colimit of spectra with finite Postnikov systems (its m-connected covers), so all of its Morava K-theories vanish. Since $K(m)_*F = 0$ iff $m \leqslant n+1$, we must have $K(m)_*\mathrm{THH}(R) = 0$ for $m \geqslant n+2$.

This condition on $F \otimes \mathrm{THH}(R)$ is also known to be equivalent to $F \otimes \mathrm{THH}(R)$ being bounded above in the Antieau-Nikolaus t-structure of [AN21], which is described in [Rav26, §5.11]. Recall that the cyclotomic spectrum $\mathrm{THH}(H\mathbb{F}_p)$, the subject of [Rav26, §5.9], is bounded above in the Antieau-Nikolaus t-structure, but its underlying spectrum is not bounded above in the (classical) Postnikov t-structure.

Theorem 2.8. The Lichtenbaum-Quillen property and localized TC. [HW22, Theorem 3.4.1] *For R as in Definition 2.7 with finite type,* $\mathrm{TC}(R)$ *has* fp*-type at most $n+1$. In particular, the map*

$$\mathrm{TC}(R) \longrightarrow \mathrm{TC}_{T(n+1)}(R)$$

is truncated, meaning that the homotopy groups of its fiber are bounded above.

3. Semiadditivity

Semiadditivity (along with ambidexterity, which we do not need here) is first introduced in the unpublished masterpiece [HL13] of Hopkins and Lurie. Much of the subsequent work in this area is due to Carmeli, Schlank and Yanovski. Their three papers, [CSY22], [CSY21] and [CSY24], appeared as preprints in the stated order, even though [CSY21] was published before [CSY22]. They also wrote [BCSY24] with Toby Barthel and [BMCSY25] with Shay Ben-Moshe.

In [CSY21] they say

> The localizations $L_{K(n)}$ and $L_{T(n)}$ are known to possess several rather special and remarkable properties. Among them are the vanishing of the Tate construction for finite group actions [CM17, HS96, GS96, Kuh04]; see Theorem 3.1. In [HL13], Hopkins and Lurie reinterpret this Tate vanishing property as 1-semiadditivity (see Definition 3.5), and vastly

> generalize it by showing that the ∞-categories $\mathrm{Sp}_{K(n)}$ are ∞-semiadditive; see Theorem 3.3(i). In turn, this is exploited to obtain new structural results for $\mathrm{Sp}_{K(n)}$.
>
> In [CSY22, Theorem B, our Theorem 3.7] the authors extended the results of [HL13] by classifying all the higher semiadditive localizations of Sp with respect to homotopy rings. First, for all such localizations, 1-semiadditivity was shown to be equivalent to ∞-semiadditivity. Second, the telescopic localizations $L_{T(n)}$, for various primes p and heights n, were shown to be precisely the maximal examples of such localizations (while the $L_{K(n)}$ are the minimal).
>
> Concisely put, among localizations of spectra with respect to homotopy rings, the higher semiadditive property singles out precisely the monochromatic localizations, which are parameterized by the chromatic height.

I must say that I had no idea that $K(n)$-localization had such properties when I wrote [Rav84]!

The starting point of this theory is the following result of Mark Hovey and Hal Sadofsky, which is related to [MR87, Conjecture 12]. See Remark 3.12. It is stated in terms of the Tate construction $(-)^{tG}$ for a finite group G, which was originally defined by John Greenlees and Peter May in [GM95, Introduction] (quoted as [Rav26, (4.26)]) to be the cofiber of a map from the homopty orbit spectrm to the homotopy fixed point spectrum.

Theorem 3.1. Algebraic blueshift. [HS96, Theorem 1.1] *For a finite spectrum X with trivial action by a finite group G, the Bousfield class of $(L_nX)^{tG}$ is that of $L_{n-1}X$.*

We now know that the finiteness of X and the triviality of the G-action are unnecessary hypotheses.

The term **blueshift** refers to the lowering of chromatic height, and hence of "wavelength" or periodicity. The term **redshift** refers to the raising of chromatic height, which we will see in §4.5.

If a finite spectrum X has type n, meaning that $L_{n-1}X$ is contractible but L_nX is not, Theorem 3.1 says that the Tate spectrum of $L_nX = L_{K(n)}X$ is contractible, making the norm map $\mathrm{Nm}_G : X_{hG} \to X^{hG}$ of [Rav26, (4.28)] (of which the cofiber is the Tate spectrum) an equivalence. To put it another way, for a type n finite spectrum X with trivial G-action, the norm map Nm_G, which appears in Greenlees-May's "Tate diagram" of [Rav26, (4.35)], induces an isomorphism in $K(n)_*(-)$. A related result of Greenlees and Sadofsky [GS96, Theorem 1.1], from the same year as the Hovey-Sadofsky result, says that the Tate spectrum for $K(n)$ itself (with trivial G-action) is contractible.

If we assume instead that X is $K(n)$-local, then so is the homotopy fixed point spectrum X^{hG}, although the homotopy orbit spectrum X_{hG} need not be. [HL13, Theorem 0.0.1] says that for *any action* of G on X, the norm map exhibits X^{hG} as a $K(n)$-localization of X_{hG}. Their goal is to generalize this result in a way that involves the following.

Definition 3.2. π-finiteness, truncation and connectivity of spaces. *A space (or Kan complex) X is* **π-finite** *if it has finitely many path connected components, each of its homotopy groups $\pi_i(X, x)$ for any base point (or vertex) x is finite, and for each x they vanish for $i \gg 0$. It is* **m-finite** *if these groups vanish for $i > m$.*

It is **m-truncated** *for $m \geqslant 0$ if its homotopy groups vanish above dimension m. They need not be finite in low dimensions. It is* **(-1)-truncated** *if it is empty or contractible, and* **(-2)-truncated** *if it is contractible.*

It is **m-connective** *for $m \geqslant 0$ if its homotopy groups vanish below dimension m.*

For a prime p, a **p-space** *is one whose homotopy groups are all finite p-groups.*

An **m-finite (respectively m-connective or π-finite) map** *$q : A \to B$ is one for which $q^{-1}(b)$ is m-finite m-connective or(π-finite) for all $b \in B$.*

An **m-finite limit or colimit** *is one that is indexed by an m-finite space.*

For an ∞-categorical generalization of the above see Definition A.4.

The word **truncation** here refers to the process of killing homotopy groups above dimension m by attaching k-cells for $k \geqslant m + 2$. This process is also known as **passing to the mth Postnikov section**.

The definition of m-finiteness above is such that the space of paths between any pair of points in an m-finite space is $(m - 1)$-finite, even when $m \leqslant 0$.

A map $q : A \to B$ is (-2)-finite if it is an equivalence, it is (-1)-finite if the preimage of each component of B is either empty or is mapped equivalently to it, it is 0-finite if the preimage of each component of B is equivalent to a finite covering of it, and so on.

As before let $\mathrm{Sp}_{K(n)}$ be the ∞-category of $K(n)$-local spectra. Then a $K(n)$-local spectrum X with G action is a functor $\rho : BG \to \mathrm{Sp}_{K(n)}$ for BG the one object ∞-category associated with G as in [Rav26, Definition 5.1]. Its limit and colimit are X^{hG} and $L_{K(n)}X_{hG}$. The norm map of [Rav26, (4.28)] extends uniquely to $L_{K(n)}X_{hG}$ (since the target is $K(n)$-local), and we can ask if this extension is an equivalance.

The space BG for a finite group G has

$$\pi_i BG = \begin{cases} G & \text{for } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The key insight of [HL13] *is that similar statements hold not just for functors $BG \to \mathrm{Sp}_{K(n)}$, but for functors $X \to \mathrm{Sp}_{K(n)}$ where X is π-finite as in Definition 3.2.* The case $X = B^m \mathrm{C}_{p^j}$ for $m, j > 0$ is of particular interest.

Theorem 3.3. The Hopkins-Lurie norm. *Let X be a π-finite space or Kan complex.*

(i) [HL13, Theorem 0.0.2] *For any functor* $\rho : X \to \mathrm{Sp}_{K(n)}$, *there is a canonical equivalence*

$$\mathrm{Nm}_X : \operatorname*{colim}_X \rho \xrightarrow{\simeq} \lim_X \rho.$$

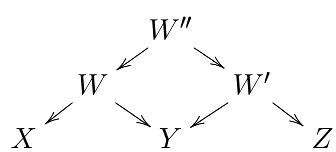
(ii) [CSY22, Theorem A] *The same holds for any functor* $X \to \mathrm{Sp}_{T(n)}$.

This norm map is studied more generally by Lurie in [Lur17, 6.1.6]. For a suitable ∞-category $\mathcal{C}$ and a map of Kan complexes $f : X \to Y$, one has a functor $f^* : \mathcal{C}^Y \to \mathcal{C}^X$ (sometimes called a pullback functor) with right and left adjoints $f_*, f_! : \mathcal{C}^X \to \mathcal{C}^Y$ (as in Definition 4.23), sometimes called the pushforward and extraordinary pushforward. When $Y = \mathbf{\Delta}^0$, $f^* : \mathcal{C} \to \mathcal{C}^X$ is the diagonal functor, and its adjoints are the homotopy colimit and homotopy limit functors. When in addition $X = BG$ for a finite group G, they are the homotopy orbit and homotopy fixed point functors.

The definition of this map requires some "intricate categorical constructions" given in [HL13, §4]. An easier way to do it is given by Yonatan Harpaz in [Har20]. He defines an ∞-category Span_m (Definition 3.4) of spans of m-finite spaces and shows that it has a certain universal property with respect to m-semiadditive ∞-categories, as in Definition 3.2.

Here is Harpaz' definition. In [Har20, Theorem 4.1] he shows that this category has a universal property that is convenient for the study of m-semiadditive ∞-categories as in Definition 3.5.

Definition 3.4. The mth Harpaz ∞-category. [Har20, Definition 2.12] *The symmetric monoidal ∞-category* $(\mathrm{Span}_m, \times, \mathrm{pt})$ *of spans of m-finite spaces has m-finite spaces as in Definition 3.2 as objects. A morphism from X to Y is a diagram (called a* **span***) $X \leftarrow W \to Y$ (as is a morphism from Y to X) in which W is also m-finite. Its composite with $Y \leftarrow W' \to Z$ is $X \leftarrow W'' \to Z$, which is derived from the diagram*



in which the square is a pullback diagram.

Definition 3.5. *An ∞-category $\mathcal{C}$ is m-***semiadditive** *if for every functor $\rho : X \to \mathcal{C}$ from an m-finite space X as in Definition 3.2, there is an equivalence Nm_X as in Theorem 3.3. It is ∞-***semiadditive** *if it is m-semiadditive for all $m > 0$.*

Infinite semiadditivity for $\mathcal{C}$ does not imply that there is an equivalence Nm_X for a functor $\rho : X \to \mathcal{C}$ for *any* space X, but only for spaces X with finite Postnikov towers and finite homotopy groups.

Example 3.6. The meaning of m-semiadditivity for small m.

- (-2)*-semiadditivity means that the limit and colimit of a functor from the trivial category (the one with a single object and a single morphism) are equivalent. Such a functor is simply the choice of an object in* $\mathcal{C}$*, with both the limit and colimit being the object itself. Hence* every ∞-category is (-2)-semiadditive.
- (-1)*-semiadditivity means that in addition the limit and colimit are equivalent for any functor from the empty category. There is only one such functor, and its limit and colimit are terminal and initial objects in* $\mathcal{C}$*. Hence* $\mathcal{C}$ is (-1)-semiadditive iff it is pointed.
- 0*-semiadditivity means in addition that the limit and colimit are equivalent for any functor from a finite discrete category. This means that finite products and coproducts are the same,* which is the case in any stable ∞-category.
- 1*-semiadditivity means that* for any finite group G, the Greenlees-May norm map $\mathrm{Nm}_G^X : X_{hG} \to X^{hG}$ of [Rav26, (4.28)] is an equivalence for any X-valued functor from BG*, which defines a* G*-action on* X*. Here* BG *denotes the nerve of the one object category* $\mathcal{B}G$ *associated with* G*. This Kan complex has a single vertex, so a functor from the corresponding* BG *(meaning a map of simplicial sets) identifies a single object* X *in* $\mathcal{C}$ *and defines a* G*-action on it. Hence the Tate object* X^{tG} *(the cofiber of* Nm_G^X*) is contractible.*
- *Higher semiadditivity means that one can replace* BG *for a finite group* G *as above by an* m*-finite space and still have contractible Tate objects.*

Thus Theorem 3.3 says that $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$ are ∞-semiadditive. It turns out that certain other localizations of Sp also have this property.

Theorem 3.7. [CSY22, Theorem B] *Let* R *be a non-zero* p*-local homotopy ring spectrum. The* ∞*-category* Sp_R *is* 1*-semiadditive if and only if there exists a (necessarily unique) integer* $n \geqslant 0$ *such that*

$$\mathrm{Sp}_{K(n)} \subseteq \mathrm{Sp}_R \subseteq \mathrm{Sp}_{T(n)}\,. \tag{3.8}$$

As noted in [CSY22, paragraph after Theorem B and proof of Theorem 5.4.7], the Nilpotence Theorem of [DHS88, Theorem 1] (also treated in [Rav92]) implies that Sp_R is 1-semiadditive if and only if $R \otimes H\mathbb{F}_p = 0$ and there is exactly one integer $n \geqslant 0$ for which $R \otimes K(n) \neq 0$. Namely, Sp_R is 1-semiadditive if and only if R is supported at a unique (finite) chromatic height. In (4.16) we will see an example of such an R, namely a certain infinite Galois extension of $\mathbb{S}_{T(n)}$.

This means that $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$ are respectively minimal and maximal among 1-semiadditive ∞-categories of R-local spectra. *Now that we know these two are distinct for* $n \geqslant 2$*, we would like to know the structure of the poset of such* ∞*-categories* Sp_R*.*

Theorem 3.9. [CSY22, Theorem C] *Let $R \in \mathrm{Sp}$ be a homotopy ring spectrum. The ∞-category Sp_R of R-local spectra is 1-semiadditive if and only if it is ∞-semiadditive.*

There is no requirement above that R be local at any prime. When it is, we can say much more.

Theorem 3.10. [CSY22, Theorem D]
Let R be a non-zero p-local homotopy ring spectrum. The following are equivalent:

- *(i)* $R \otimes H\mathbb{F}_p = \mathrm{pt}$ *and there is exactly one integer* $n \geqslant 0$ *for which* $R \otimes K(n) \neq \mathrm{pt}$.
- *(ii) There exists a (necessarily unique) integer $n \geqslant 0$ such that (3.8) holds.*
- *(iii) Either $\mathrm{Sp}_R = \mathrm{Sp}_{H\mathbb{Q}}$, or the functor $\Omega^\infty \colon \mathrm{Sp}_R \to \mathcal{S}_*$ admits a retract, meaning a Bousfield-Kuhn functor as in* [Bou87] *and* [Kuh89].
- *(iv) Sp_R is 1-semiadditive.*
- *(v) Sp_R is ∞-semiadditive.*

Theorem 3.11. [CSY22, Theorem E] *Let R be a non-zero p-local homotopy ring spectrum, and let $n \geqslant 0$. Then the following are equivalent:*

- *(i)* $R \otimes K(m) \simeq \mathrm{pt}$ *for all* $m > n$.
- *(ii)* $R \otimes F(n+1) \simeq \mathrm{pt}$ *for a finite p-local spectrum $F(n+1)$ of type $n+1$ as in Definition 2.1(i).*
- *(iii)* $R \otimes \Sigma^\infty A \simeq \mathrm{pt}$ *for every n-connected π-finite space A.*

Not all nonzero p-local homotopy ring spectra have the properties of Theorem 3.11, $H\mathbb{F}_p$ itself being an obvious exception. Such a spectrum with vanishing mod p and rational homology cannot be connective.

Returning to π-finite spaces, each one has a finite Postnikov tower. This means we need to understand (among others) the cases $X = BG$ for a finite group G, and

$$X = B^m \mathrm{C}_p = K(\mathbb{Z}/p, m).$$

We know that $K(n)_*BG$ has finite rank [Rav82], and a formula for its Euler characteristic is given in [HKR00]. It is concentrated in even dimensions when G is abelian, but there are counterexamples for nonabelian G due to Igor Kriz and Kevin Lee [KL00].

The Morava K-theory of each Eilenberg-MacLane space with finite abelian homotopy group is computed in [RW80]; see §4.4. In each case it is concentrated in even dimensions for $n > 0$. We also know by [HRW98] that the Morava K-theory of a finite Postnikov tower is that of the corresponding product of Eilenberg-MacLane spaces. In other words, the k-invariants of such a tower are invisible to Morava K-theory.

In any case, Hopkins and Lurie have formal methods to reduce showing their norm map is an equivalence in the case where $X = B^m\mathrm{C}_p$ and the functor ρ of Definition 3.5 is constant with value $K(n)$. This amounts to showing that the map induces an isomorphism $K(n)_iX \to K(n)^{-i}X$ for each i.

Remark 3.12. Three notions of height. *A finite p-local spectrum X of type n has $K(m)_*X = 0$ iff $m < n$ since it is known that the nontriviality of $K(m)_*X$ implies that of $K(m+1)_*X$ for any finite X. The situation for the ring spectra of Theorem 3.11 is in a sense the opposite: they are $K(m)$-acyclic for all $m > n$ instead of for $m < n$. The authors of* [CSY22] *observe*

> *We obtain an equivalence of three different notions of* **height** *$\leqslant n$ for a homotopy ring:*
>
> *(i) the* **algebraic** *one using Morava K-theories,*
> *(ii) the* **geometric** *one using finite complexes, and*
> *(iii) the* **categorical** *one using π-finite spaces.*
>
> *The categorical height of a spectrum (i.e. the minimal d for which condition Theorem 3.11(iii) holds) was considered, using different terminology, by Bousfield in* [Bou82]*. The most prominent example of such R is $K(n)$, which by* [RW80]*, has categorical height n. Bousfield's work also implies that for all $n \geqslant 0$, the spectrum $T(n)$ has some finite categorical height, but determining its precise value had been an open question. This can be now settled using Theorem 3.11; as the algebraic and geometric heights of $T(n)$ are known to be equal to n, the categorical height must be n as well.*

The blueshift of Theorem 3.1 is algebraic in the sense of (i) above. One could also speak of geometric and categorical blueshift. The former is the subject of [MR87, Conjecture 12].

4. Chromatic cyclotomic extensions

A theorem of Ethan Devinatz and Hopkins [DH04, Theorem 1] says that $\mathbb{S}_{K(n)} := L_{K(n)}\mathbb{S}$, the localization of the sphere spectrum at height n Morava K-theory, is the homotopy fixed point set $E_n(\overline{\mathbb{F}}_p)^{h\mathbb{G}_n}$. Here $E_n(\overline{\mathbb{F}}_p)$ is the Lubin-Tate spectrum constructed by Paul Goerss, Hopkins and Haynes Miller, (see [GH04] and [Rez98]) and $\mathbb{G}_n$ is the extended height n Morava stabilizer group of [Mor85].

For future reference recall that

$$
\begin{aligned}
\pi_* BP &= \mathbb{Z}_{(p)}[v_1, v_2, \dots] \qquad \text{with } |v_i| = 2(p^i - 1)\\
\pi_* BP\langle n\rangle &= \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n]\\
\pi_* E(n) &= v_n^{-1}\pi_* BP\langle n\rangle\\
\pi_* E_n &= W(\mathbb{F}_{p^n})[\![u_1, \dots, u_{n-1}]\!][u^{\pm 1}] \tag{4.1}\\
&\qquad \text{with } |u| = 2,\ |u_i| = 0,\ v_n = u^{p^n-1} \text{ and } v_i = u^{p^i-1}u_i;\\
&\qquad \text{see Remark 4.27.}\\
\pi_* E_n(\overline{\mathbb{F}}_p) &= W(\overline{\mathbb{F}}_p)[\![u_1, \dots, u_{n-1}]\!][u^{\pm 1}]
\end{aligned}
$$

Using ideas of John Rognes [Rog08], and Andy Baker and Birgit Richter [BR08], the Morava spectrum $E_n(\overline{\mathbb{F}}_p)$ can be viewed as a Galois extension of $\mathbb{S}_{K(n)}$ with Galois group $\mathbb{G}_n$, the subject of Proposition 4.8. This means that for each normal subgroup $N \subseteq \mathbb{G}_n$ there is a Galois extension of $L_{K(n)}\mathbb{S}$ with Galois group $G = \mathbb{G}_n/N$. We are interested in the cases where G is abelian.

We will begin with a review of abelian extensions of the p-adic numbers in §4.1. In §4.2 we will review Rognes' Galois theory for commutative ring spectra, the subject of [Rog08]. In §4.3 we will discuss *higher* cyclotomic extensions. §4.4 and §4.5 will cover the role of Eilenberg-MacLane spaces and cyclotomic redshift respectively.

4.1. **Some classical number theory.** Recall the classical description of the abelian extensions of the p-adic numbers $\mathbb{Q}_p$. A concise summary of the relevant material on local fields (meaning finite extensions of the p-adic numbers $\mathbb{Q}_p$) can be found in Xavier Caruso's [Car19, §1.1]. The definitive introduction is Jean-Pierre Serre's [Ser79].

4.1.1. *Ramified cyclotomic extensions.* We know that every finite abelian extension is contained in some cyclotomic extension, that is the extension obtained by adjoining a primitive mth root of unity for some integer $m > 2$. The most interesting case for us is $m = p^j$ for some $j > 0$, and we denote such a primitive root of unity by ω_{p^j}. This extension has degree $(p-1)p^{j-1}$ and Galois group $(\mathbb{Z}/p^j)^\times$. Consider the diagram

$$
\begin{array}{ccccccccccc}
 & & & & \omega_{p^j} & \longmapsto & \omega_{p^{j+1}}^p & & & & \\
\mathbb{Q}_p & \to & \mathbb{Q}_p[\omega_p] & \to \cdots \to & \mathbb{Q}_p[\omega_{p^j}] & \to & \mathbb{Q}_p[\omega_{p^{j+1}}] & \to \cdots \to & \mathbb{Q}_p[\omega_{p^\infty}]\\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow\\
\mathbb{Z}_p & \to & \mathbb{Z}_p[\omega_p] & \to \cdots \to & \mathbb{Z}_p[\omega_{p^j}] & \to & \mathbb{Z}_p[\omega_{p^{j+1}}] & \to \cdots \to & Z_p[\omega_{p^\infty}]\\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow\\
\mathbb{Z}_p & \to & \mathbb{Z}_p[\mathrm{C}_p] & \to \cdots \to & \mathbb{Z}_p[\mathrm{C}_{p^j}] & \to & \mathbb{Z}_p[\mathrm{C}_{p^{j+1}}] & \to \cdots \to & Z_p[\mathrm{C}_{p^\infty}]
\end{array} \tag{4.2}
$$

in which the top row is the **ramified cyclotomic tower** of field extensions, the second row shows their rings of integers, and the third is the evident diagram of p-adic group rings with a generator of the cyclic group C_{p^j} of order p^j mapping to ω_{p^j}. We denote the colimits of the three rows by $\mathbb{Q}_p[\omega_{p^\infty}]$, $\mathbb{Z}_p[\omega_{p^\infty}]$ and $\mathbb{Z}_p[C_{p^\infty}]$, where $C_{p^\infty} \subseteq \mathbb{T}$ is the Prüffer group of [Rav26, Definition 5.23 (ii)]. The Galois group of $\mathbb{Q}_p[\omega_{p^\infty}]$ over $\mathbb{Q}_p$ is $\mathbb{Z}_p^\times$, the multiplicative group of p-adic units.

There is a split short exact sequence

$$1 \longrightarrow \mathbb{Z}_p \xrightarrow{e} \mathbb{Z}_p^\times \longrightarrow T_p \longrightarrow 1, \tag{4.3}$$

where $T_p \subset \mathbb{Z}_p^\times$ is the torsion subgroup, which consists of roots of unity, namely $\mu_2 \cong \{\pm 1\}$ for $p = 2$, and $\mu_{p-1} \cong C_{p-1}$, the group of $(p-1)$th roots of unity, for p odd. The map e is the p-adically convergent exponential function given by

$$e(x) = \begin{cases} \exp(4x) = \sum_{k \geqslant 0} \dfrac{(4x)^k}{k!} = 1 + 4x + 8x^2 + \cdots & \text{for } p = 2 \\ \exp(px) = \sum_{k \geqslant 0} \dfrac{(px)^k}{k!} = 1 + px + \dfrac{p^2x^2}{2} + \cdots & \text{for } p > 2. \end{cases}$$

Note that the power series $\exp(2x)$ does not converge 2-adically.

For future reference we note that $\mathbb{Z}_p^\times$ has a dense finitely generated subgroup

$$\begin{aligned} \mathbb{Z}_p^{\times,\text{fg}} &:= \begin{cases} \langle e(1), -1 \rangle & \text{for } p = 2 \\ \langle e(1), \zeta_{p-1} \rangle & \text{for } p > 2 \end{cases} \\ &\qquad \text{for a primitive } (p-1)\text{th root of unity } \zeta_{p-1} \\ &\cong \mathbb{Z}_p \times T_p. \end{aligned} \tag{4.4}$$

We will see this group in Proposition 4.18.

Let

$$\varpi_j := \omega_{p^j} - 1 \in \mathbb{Z}_p[\omega_{p^j}].$$

Then we find that $\varpi_j^{(p-1)p^{j-1}}$ is a unit multiple of p, so

$$(p) = (\varpi_j)^{(p-1)p^{j-1}}.$$

The maximal ideal $(p) \subseteq \mathbb{Z}_p$ becomes a power of the maximal ideal (ϖ_j) when we pass to $\mathbb{Z}_p[\omega_{p^j}]$. This is called **ramification**. The p-adic integers $\mathbb{Z}_p$ and the p-adic numbers $\mathbb{Q}_p$ have discrete valuations $||-||$ in which $||p^i|| = i$. They extend to valuations on $\mathbb{Z}_p[\omega_{p^j}]$ and $\mathbb{Q}_p[\omega_{p^j}]$ in which $||\varpi_j|| = 1/(p-1)p^{j-1}$.

The lower vertical maps in (4.2) are induced by certain group monomorphisms $C_{p^j} \to \mathbb{Z}_p[\omega_{p^j}]^\times$. Such a map, along with a choice of generator of C_{p^j}, determines a p^jth root of unity in $\mathbb{Z}_p[\omega_{p^j}]$. These maps are known to split after inverting p, as explained in [CSY24, page 3514]. For any ring R in which p is invertible, $R[C_{p^j}]$ is isomorphic as a ring to $R[C_{p^{j-1}}] \times R[\omega_{p^j}]$.

4.1.2. *Unramified extensions.* For each $d > 1$, the p-adic numbers also has an **unramified** (meaning that the ideal (p) remains maximal) abelian extension whose ring of integers is the Witt ring $\mathbb{W}(\mathbb{F}_{p^d})$. Its Galois group is C_d. It is also the cyclotomic extension $\mathbb{Q}_p[\omega_{p^d-1}]$ obtained by adjoining (p^d-1)th roots of unity. The degree of the corresponding extension of $\mathbb{Q}$ is the Euler totient $\varphi(p^d-1)$, which larger than d. The cyclotomic polynomial $\Phi_{p^d-1}(x)$ is irreducible over $\mathbb{Q}$ but has an irreducible factor of degree d over $\mathbb{Q}_p$. For example when $(p,d) = (3,2)$, we have

$$\Phi_8(x) = x^4 + 1 = \left(x^2 - \sqrt{-2}x - 1\right)\left(x^2 + \sqrt{-2}x - 1\right),$$

and the reader can verify that $\sqrt{-2}$ can be regarded a 3-adic integer, as can the square root of any integer congruent to 1 mod 3.

The algebraic closure $\overline{\mathbb{F}}_p$, which is the colimit of the finite fields $\mathbb{F}_{p^d}$, has Galois group $\widehat{\mathbb{Z}}$, the profinite integers. We know that the Galois group of the maximal abelian extension of $\mathbb{Q}_p$ is $\mathbb{Z}_p^\times \times \widehat{\mathbb{Z}}$.

4.2. **Rognes' Galois theory for commutative ring spectra.** Rognes' definitions are stated in terms of spectra localized at a homology theory E such as Morava K-theory. His spectra are $\mathbb{S}$-algebras in the sense of [EKMM97] but we are treating them in terms of orthogonal spectra. We are not aware of a published definition in the language of ∞-categories, but one is hinted at by Akhil Mathew, Niko Naumann and Justin Noel in [MNN17].

Definition 4.5. Galois extensions of E-local commutative ring spectra. [Rog08, Definition 4.1.3]

- *Let $A \to B$ be a map of E-local commutative ring spectra, making B a commutative A-algebra, and let G be an E-locally stably dualizable group (for example a finite group) acting continuously on B from the left through commutative A-algebra maps.*
- *Let $i\colon A \to B^{hG}$ be the map to the homotopy fixed point spectrum $B^{hG} = F(EG_+, B)^G$ that is right adjoint to the composite map*

$$A \wedge EG_+ \to A \to B,$$

that collapses the contractible free G-space EG to a point.
- *Let $h\colon B \wedge_A B \to F(G_+, B)$ be the canonical map to the product (cotensor) $F(G_+, B)$ that is right adjoint to the composite map*

$$B \wedge_A B \wedge G_+ \to B \wedge_A B \to B,$$

induced by the action $B \wedge G_+ = G_+ \wedge B \to B$ of G on B, followed by the A-algebra multiplication $B \wedge_A B \to B$ in B.

A map of E-local commutative ring spectra $A \to B$ is an **E-local G-Galois extension** *if i and h are both E-local equivalences. Such an extension is* **finite**, **abelian** *or* **finite abelian** *if G is.*

Definition 4.6. Properties of Galois extensions.

(i) [Rog08, Definition 4.3.1] *A map of commutative ring spectra $A \to B$ is* **faithful** *if for every A-module N, the condition $N \wedge_A B \simeq *$ implies that $N \simeq *$.*

(ii) [Rog08, Definition 9.1.1] *The map $A \to B$ is* **separable** *if the multiplication map $\mu\colon B \wedge_A B \to B$ admits a bimodule section up to homotopy.*

(iii) [Rog08, Definition 10.2.1] *A commutative ring spectrum B is* **connected**, *in the sense of algebraic geometry, if its space of idempotents $\mathcal{E}(B)$ is weakly equivalent to the two-point space $\{0,1\}$.*

Then his main theorem is

Theorem 4.7. The stable homotopy theoretic Galois correspondence. [Rog08, Theorem 1.2]

Let $A \to B$ be a faithful E-local G-Galois extension of commutative ring spectra.

(i) For each subgroup $H \subset G$, the map $C = B^{hH} \to B$ is a faithful E-local H-Galois extension, and the map $A \to C$ is separable.

(ii) For each normal subgroup $H \subset G$, the map $A \to C = B^{hH}$ is a faithful E-local G/H-Galois extension.

If B is connected, then

(iii) The Galois group G is weakly equivalent to the mapping space $\mathrm{CAlg}_A(B,B)$ of commutative A-algebra self-maps of B.

(ov) For each factorization $A \to C \to B$ of the G-Galois extension, with $A \to C$ separable and $C \to B$ faithful, there exists a subgroup $H \subset G$ such that $C \simeq B^{hH}$ as an A-algebra over B.

In other words, for a faithful E-local G-Galois extension $A \to B$ with B connected, there is a bijective contravariant Galois correspondence $H \longleftrightarrow C \simeq B^{hH}$ between the subgroups $H \subset G$ and the weak equivalence classes of separable A-algebras C mapping faithfully to B.

The inverse correspondence takes C to $H = \pi_0\mathrm{CAlg}_C(B,B)$, the group of connected components of the mapping space of commutative C-algebra self-maps of B.

Proposition 4.8. A faithful profinite Galois extension. [Rog08, Proposition 5.4.9] *The map $\mathbb{S}_{K(n)} \to E_n(\overline{\mathbb{F}}_p)$ is a $K(n)$-local pro-Galois extension with Galois group $\mathbb{G}_n$, and both the extension and the associated $\mathbb{G}_n$–action are faithful. We have a short exact sequence*

$$1 \longrightarrow \mathbb{S}_n \longrightarrow \mathbb{G}_n \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \longrightarrow 1, \tag{4.9}$$

where $\mathbb{S}_n$ has a pro-p open subgroup of index $p^n - 1$.

Recall that $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ and $\mathbb{S}_n$ are isomorphic respectively to the profinite integers $\widehat{\mathbb{Z}}$ and the group of units $\mathcal{O}_D^\times$ in the ring of integers $\mathcal{O}_D$ of the division algebra D over $\mathbb{Q}_p$ with Hasse invariant $1/n$.

4.3. **Higher cyclotomic extensions and cyclotomic completion.** We do not have an analog of Proposition 4.8 for $\mathbb{S}_{T(n)}$. However by a theorem of Carmeli, Schlank and Yanovski we do know that for each *abelian* Galois extension of $\mathbb{S}_{K(n)}$, there is an analogous extension of $\mathbb{S}_{T(n)}$.

We also know that the abelianization of the absolute Galois group of $\mathbb{Q}_p$, which controls the abelian extensions of that field reviewed in §4.1, is isomorphic to $\mathbb{Z}_p^\times \times \widehat{\mathbb{Z}}$. The same is true of the abelianization $\mathbb{G}_n^{\mathrm{ab}}$ of the extended Morava stabilizer group $\mathbb{G}_n$, while the abelianization of $\mathbb{S}_n$ is $\mathbb{Z}_p^\times$.

In the former case the projections onto $\mathbb{Z}_p^\times$ and $\widehat{\mathbb{Z}}$ correspond respectively to the maximal totally ramified and unramified abelian extensions of $\mathbb{Q}_p$.

In the latter case this means that the abelian extensions of $\mathbb{S}_{K(n)}$ can also be thought of as "chromatic cyclotomic extensions," hence the title of [CSY24]. We will be interested in the chromatic analogs of the ramified extensions of (4.2) rather than the unramified ones of §4.1.2.

Theorem 4.10. Lifting abelian extensions of $\mathbb{S}_{K(n)}$ to $\mathbb{S}_{T(n)}$. [CSY24, Theorems A and 5.31] *For every $K(n)$-local finite abelian extension $\mathbb{S}_{K(n)} \to R$ in $\mathrm{Sp}_{K(n)}$, there exists a $T(n)$-local finite abelian extension $\mathbb{S}_{T(n)} \to R^{\mathrm{Fin}}$ in $\mathrm{Sp}_{T(n)}$ (having the same Galois group), such that*

$$L_{K(n)} R^{\mathrm{Fin}} \simeq R.$$

When $G = (\mathbb{Z}/p^j)^\times$ for $j > 0$, we denote this extension by $\mathbb{S}_{T(n)}[\omega_{p^j}^{(n)}]$. For $X \in \mathrm{Sp}_{T(n)}$ we define

$$X[\omega_{p^j}^{(n)}] := X \otimes \mathbb{S}_{T(n)}[\omega_{p^j}^{(n)}]. \tag{4.11}$$

Definition 4.12. Two infinite abelian extensions of $\mathbb{S}_{K(n)}$ and $\mathbb{S}_{T(n)}$. *Let*

$$\begin{aligned} R_n = \mathbb{S}_{K(n)}[\omega_{p^\infty}^{(n)}] &:= \operatorname*{colim}_j \mathbb{S}_{K(n)}[\omega_{p^j}^{(n)}], \\ \mathbb{S}_{K(n)}^{\mathrm{ab}} &:= \mathbb{SW}(\overline{\mathbb{F}}_p) \otimes \mathbb{S}_{J(n)}[\omega_{p^\infty}^{(n)}], \\ R_n^{\mathrm{Fin}} = \mathbb{S}_{T(n)}[\omega_{p^\infty}^{(n)}] &:= \operatorname*{colim}_j \mathbb{S}_{T(n)}[\omega_{p^j}^{(n)}] \\ \textit{and} \qquad \mathbb{S}_{T(n)}^{\mathrm{ab}} &:= \mathbb{SW}(\overline{\mathbb{F}}_p) \otimes \mathbb{S}_{T(n)}[\omega_{p^\infty}^{(n)}], \end{aligned}$$

where $\mathbb{SW}(\overline{\mathbb{F}}_p)$ denotes the evident Moore spectrum. $\mathbb{S}_{K(n)}^{\mathrm{ab}}$ and $\mathbb{S}_{T(n)}^{\mathrm{ab}}$ are the **maximal abelian extensions** *of $\mathbb{S}_{K(n)}$ and $\mathbb{S}_{T(n)}$.*

Remark 4.13. *$\mathbb{S}_{K(n)}[\omega_{p^\infty}^{(n)}]$ and $\mathbb{S}_{K(n)}^{\mathrm{ab}}$ are known to be a faithful Galois extensions of $\mathbb{S}_{K(n)}$.*

Example 4.14. Westerland's spectrum. *The spectrum $\mathbb{S}_{K(n)}[\omega_{p^\infty}^{(n)}]$ for $n > 0$ is the ring R_n studied by Craig Westerland in* [Wes17, Theorem 1.2].

Namely, it is the (continuous) homotopy fixed points of E_n for the action the kernel $S\mathbb{G}_n^{\pm}$ of

$$\det_{\pm} : \mathbb{G}_n \longrightarrow \mathbb{Z}_p^{\times},$$

whence it is a $\mathbb{Z}_p^{\times}$–Galois extension of $\mathbb{S}_{K(n)}$. This map is the extension of the determinant homomorphism on the subgroup $\mathbb{S}_n$ (as in (4.9)) to $\mathbb{G}_n$ given by sending the Frobenius generator F of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ to $(-1)^{n-1}$. This means the homomorphisn is trivial on the subgroup $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^n})$ topologically generated by F^n.

He describes R_n as a close relative of $L_{K(n)}\Sigma^{\infty}K(\mathbb{Z}_p, n+1)_+$. Note that the fiber sequence

$$K(\mathbb{Q}_p, n) \to K(\mathbb{Q}_p/Z_p, n) \to K(\mathbb{Z}_p, n+1)$$

leads to an equivalence

$$L_{K(n)}\Sigma^{\infty}K(\mathrm{C}_{p^\infty}, n)_+ \to L_{K(n)}\Sigma^{\infty}K(\mathbb{Z}_p, n+1)_+. \tag{4.15}$$

since $K(\mathbb{Q}_p, n)$ is $K(n)$-acyclic for $n > 0$.

We will say more about Eilenberg-MacLane spaces in §4.4.

For each finite j, the map

$$\mathbb{S}_{T(n)} \to \mathbb{S}_{T(n)}[\omega_{p^j}^{(n)}]^{h(\mathbb{Z}/p^j)^{\times}}$$

is an equivalence, meaning that the corresponding Galois extension is faithful. This *does not* imply that the Galois extension

$$\mathbb{S}_{T(n)} \to \mathbb{S}_{T(n)}[\omega_{p^\infty}^{(n)}] =: R_n^{\mathrm{Fin}} \tag{4.16}$$

is faithful.

Definition 4.17. Height n cyclotomic completion. *We denote localization with respect to R_n^{Fin} by L_n^{Cyclo} and the corresponding category by $L_n^{\mathrm{Cyclo}}\mathrm{Sp}$. A $T(n)$-local spectrum is* **cyclotomically complete** *if it is R_n^{Fin}-local (see (4.16)).*

The analog of $L_{K(n)}$ is

$$L_{\mathrm{Cyclo}(n)} := L_{T(n)}L_n^{\mathrm{Cyclo}},$$

and we denote the corresponding category by $\mathrm{Sp}_{\mathrm{Cyclo}(n)}$.

$\mathrm{Sp}_{\mathrm{Cyclo}(n)}$ is denoted by $\widehat{\mathrm{Sp}}_{T(n)}$ in [BCSY24] and by $(\mathrm{Sp}_{T(n)})^{\wedge}_{\mathrm{cyc}}$ in [BHLS23, page 4].

Proposition 4.18. The height n cyclotomically complete sphere. [BCSY24, Proposition 6.19] *The restriction of the functor L_n^{Cyclo} to $\mathrm{Sp}_{T(n)}$ is smashing with the unit object*

$$(R_n^{\mathrm{Fin}})^{h\mathbb{Z}_p^{\times,\mathrm{fg}}} \in \mathrm{CAlg}(\mathrm{Sp}_{\mathrm{Cyclo}(n)}) \qquad \text{for } \mathbb{Z}_p^{\times,\mathrm{fg}} \text{ as in (4.4).}$$

We do not know if the functor L_n^{Cyclo} is smashing in the larger category Sp like L_n and L_n^{Fin} are known to be.

Remark 4.19. Why the finitely generated subgroup? *Note that the subgroup $\mathbb{Z}_p^{\times,\mathrm{fg}} \subseteq \mathbb{Z}_p^{\times}$ is not closed. Devinatz-Hopkins theory* [DH04, Theorem 1] *concerns closed subgroups of $\mathbb{G}_n$ and the category of $K(n)$-local spectra, but we are now in the category of $T(n)$-local spectra. In that world we need to replace $\mathbb{Z}_p^{\times}$ by a dense finitely generated subgroup in the description of the unit object.*

It is stated without explicit proof in [BCSY24, page 94] that $K(n)$-local spectra are height n cyclotomically complete. By Theorem 3.7 (originally [CSY22, Theorem B]) this is the case because $K(m)_* R_n^{\mathrm{Fin}} = 0$ iff $m > n$ and $R_n^{\mathrm{Fin}} \otimes H\mathbb{F}_p = 0$.

Thus, using the notation of Definitions 2.3 and 4.17, we have categorical inclusions and natural transformations of restricted functors (see (1.3))

$$\begin{array}{ccccc} L_n & \Longleftarrow & L_n^{\mathrm{Cyclo}} & \Longleftarrow & L_n^{\mathrm{Fin}} \\ L_n\mathrm{Sp} & \hookrightarrow & L_n^{\mathrm{Cyclo}}\mathrm{Sp} & \hookrightarrow & L_n^{\mathrm{Fin}}\mathrm{Sp} \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Sp}_{K(n)} & \hookrightarrow & \mathrm{Sp}_{\mathrm{Cyclo}(n)} & \hookrightarrow & \mathrm{Sp}_{T(n)} \\ L_{K(n)} & \Longleftarrow & L_{\mathrm{Cyclo}(n)} & \Longleftarrow & L_{T(n)}. \end{array} \tag{4.20}$$

For $n = 1$, the three categories in the second row are the same (as are the ones in the third row) since the telescope conjecture is known to be true for $n = 0$ and $n = 1$. For $n \geqslant 2$, $T(n)$-local spectra are now known not to be cyclotomically complete by [BHLS23, Theorem A], quoted here as Theorem 6.3.

Theorem 4.21 ([BMCSY25, Proposition 4.17] and [BHLS23, Theorem 6.18])**.** *For $n \geqslant 0$, let $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$. The map*

$$\mathrm{K}_{T(n+1)}(R) \longrightarrow \mathrm{K}_{T(n+1)}\Big(\mathbb{S}_{T(n)}[\omega_{p^\infty}^{(n)}]^{hT_p} \otimes R\Big)^{h\mathbb{Z}},$$

for $T_p, \mathbb{Z} \subseteq \mathbb{Z}_p^{\times}$ as in (4.4), exhibits the target as the height $n+1$ cyclotomic completion (as in Definition 4.17) of the source, where the tensor products are taken in $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$.

Corollary 4.22. [BHLS23, Corollary 6.19] *For $n \geqslant 0$, the map*

$$\mathbb{S}_{T(n)} \longrightarrow \mathbb{S}_{T(n)}[\omega_{p^\infty}^{(n)}]$$

is a $\mathrm{K}_{\mathrm{Cyclo}(n+1)}$-cover (for $\mathrm{K}_{\mathrm{Cyclo}(n+1)}$ as in Definition 2.4) in the sense of Definition A.67.

4.4. **The surprising appearance of Eilenberg-MacLane spaces.**

Definition 4.23. Space indexed products and coproducts. [CSY21, §1.4(3)] *Given an ∞-category $\mathcal{C}$ and a space A, the map $A \to \mathrm{pt}$ induces a functor $A^* : \mathcal{C} \to \mathcal{C}^A := \mathrm{Fun}(A, \mathcal{C})$ sending an object X in $\mathcal{C}$ to the* *constant X-valued functor on $\mathcal{A}$, the ∞-category associated with A as in* [Rav26, Definition 5.1(ii)]. *For suitable $\mathcal{C}$, A^* has left and right adjoints $A_!$ and A_*, the colimit and limit of the functor $\mathcal{A} \to \mathcal{C}$. We define*

$$X[A] := A_! A^* X \qquad \text{and} \qquad X^A := A_* A^* X, \tag{4.24}$$

which are respectively covariant and contravariant in A. They come equipped with a counit or fold map $\nabla : X[A] \to X$ and a unit or diagonal map $\Delta : X \to X^A$.

Example 4.25. *When A is a finite set with k elements, then $X[A]$ and X^A are the k-fold coproduct and product of X. When $A = BG$ for a topological group G, then $X[A]$ and X^A are the homotopy orbit space X_{hG} and the homotopy fixed point set X^{hG}.*

It turns out that in the $K(n)$-local and $T(n)$-local worlds for $n > 0$, the height n analogs of $\mathbb{Z}_p[\mathrm{C}_{p^j}]$ in (4.2) are the spectra

$$\begin{aligned} & \mathbb{S}_{K(n)}[B^n \mathrm{C}_{p^j}] = L_{K(n)} \Sigma^\infty B^n \mathrm{C}_{p^j} \\ \text{and} \qquad & \mathbb{S}_{T(n)}[B^n \mathrm{C}_{p^j}] = L_{T(n)} \Sigma^\infty B^n \mathrm{C}_{p^j}. \end{aligned}$$

The analogs of $\mathbb{Z}_p[\omega_{p^j}]$ are summands of these.

Here the space $B^n \mathrm{C}_{p^j}$ is the nth iterated classifying space of the cyclic group of order p^j, more traditionally known as the Eilenberg-MacLane space $K(\mathbb{Z}/p^j, n)$, first introduced in [EM45]. We saw the colimit as $j \to \infty$ in Example 4.14.

To see why these spaces arise, suppose we replace the ring $\mathbb{Z}_p[\omega_{p^\infty}]$ in (4.2) by a commutative ring R in a symmetric monoidal ∞-category $\mathcal{C}$, for example an $\mathbb{E}_\infty$-ring spectrum. Its multiplicative group of units $R^\times$ can have nontrivial iterated loop spaces, unlike the totally disconnected space $\mathbb{Z}_p[\omega_{p^\infty}]^\times$.

Definition 4.26. [CSY24, Definition 4.2] *A* **p^jth root of unity of height n**, *$\omega_{p^j}^{(n)} \in R$, is the image of a generator of C_{p^j} under a group homomorphism $\mathrm{C}_{p^j} \to \Omega^n R^\times$.*

Such a map is adjoint to a map $B^n \mathrm{C}_{p^j} \to R^\times$. *This notion of height differs from that in the theory of formal group laws.* See Remark 3.12.

Higher roots of unity are discussed in [CSY24, §4.1]. They are also used in [BCSY24] to construct higher height analogues of the discrete Fourier transform and Kummer theory. In [BMCSY25, Theorems 4.26 and 4.28] respectively the authors show that these constructions play nicely with the functor $\mathrm{K}_{T(n+1)}$.

The Morava K-theories of the spaces $B^n \mathrm{C}_{p^j}$, for all heights (in the formal group law sense) and all values of n and j, were computed long ago by Steve Wilson and the author in [RW80] for odd primes, and by Johnson and Wilson in [JW85, Appendix] for $p = 2$. We also did it for the $K(n)$-equivalent spaces $B^{n+1}\mathbb{Z}$ and $B^n \mathrm{C}_{p^\infty}$; see (4.15). The alternative computation of Hopkins and Jacob Lurie in [HL13, §2], which works for all primes, is explained by Sanath Devalapurkar in [Dev18]. An interesting arithmetic interpretation of our result is given by Victor Buchstaber and Andrey Lazarev in [BL07].

In particular we know that $K(n)_*B^n\mathrm{C}_{p^j}$ has rank p^j (as a module over $K(n)_*$) and is concentrated in dimensions divisible by

$$|v_n|/(p-1) = 2(1+p+\cdots+p^{n-1}).$$

This means that in the $K(n)$-local world, $\Sigma^\infty_+ B^n\mathrm{C}_{p^j}$ "looks like" a CW-complex with p^j cells evenly distributed in such dimensions, with an extra one in dimension 0. This generalizes that fact (the case $n=0$) that $\Sigma^\infty_+\mathrm{C}_{p^j}$ (and hence its rationalization) is a wedge of p^j copies of the (rationalized) sphere spectrum.

When we wrote [RW80], we had no idea that these spaces would play such a role in chromatic homotopy theory!

Remark 4.27. Periodicity in Morava K-theory. *In the above paragraph we are using the original definition of $K(n)$, for which*

$$\pi_*K(n) = \mathbb{F}_p[v_n^{\pm 1}] \qquad \text{with } |v_n| = 2(p^n-1).$$

This spectrum is $|v_n|$-periodic. One can formally adjoin a (p^n-1)th root u of v_n (with $|u|=2$) to obtain a 2-periodic spectrum $\widehat{K}(n)$. It follows that

$$\widehat{K}(n)_0(X) = \bigoplus_{0\leqslant i<p^n-1} u^{-i}K(n)_{2i}X$$
$$\text{and} \qquad \widehat{K}(n)_1(X) = \bigoplus_{0\leqslant i<p^n-1} u^{-i}K(n)_{2i+1}X.$$

The functors $L_{K(n)}$ and $L_{\widehat{K}(n)}$ are the same, as are the ∞-categories $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{\widehat{K}(n)}$.

In the rest of the paper $K(n)$ will abusively denote the 2-periodic spectrum $\widehat{K}(n)$.

The equality of the ranks of $T(n)_*B^m\mathrm{C}_{p^j}$ and $K(n)_*B^m\mathrm{C}_{p^j}$ is the subject of [CSY22, Lemma 5.1.7].

4.5. **Chromatic cyclotomic redshift.**

Here we recall some results of the insightful work of Ben-Moshe, Carmeli, Schlank and Yanovski, [BMCSY25].

We first look at ∞-sheaves (Definition A.47) on the categories $\mathcal{F}in_{\mathbb{Z}_p}$ and $\mathcal{F}in_{\mathbb{Z}_p^\times}$ (see Definition A.72) of finite sets acted on by the groups of p-adic integers and p-adic units. The general theory of such sheaves for profinite groups G is reviewed in Appendix A.8.

Definition 4.28. [BMCSY25, Definition 5.10] *For $X \in \mathrm{Sp}_{T(n)}$, the* **height n cyclotomic sheaf** *is the contravariant functor on $\mathcal{F}in_{\mathbb{Z}_p^\times}$ given by*

$$(\mathbb{Z}/p^j)^\times \mapsto X[\omega^{(n)}_{p^j}]$$

as in (4.11). Its stalk (see (A.62)) is $X[\omega^{(n)}_{p^\infty}]$.

Proposition 4.29. [BMCSY25, Proposition 5.11] *The cyclotomic sheaf* $\mathbb{S}_T^{(n)}[\omega_{p^{(-)}}^{(n)}]$ *is a continuous* $\mathbb{Z}_p^\times$*–Galois extension, and there is a symmetric monoidal equivalence*

$$\mathrm{Sp}_{T(n)} \xrightarrow{\sim} \mathrm{Mod}_{\mathbb{S}_T^{(n)}[\omega_{p^{(-)}}^{(n)}]}\big(\mathrm{Shv}(\mathcal{F}in_{\mathbb{Z}_p}; \mathrm{Sp}_{T(n)})\big), \qquad X \longmapsto X[\omega_{p^{(-)}}^{(n)}].$$

Moreover, $X \in \mathrm{Sp}_{T(n)}$ *is cyclotomically complete if and only if the height* n *cyclotomic sheaf* $X[\omega_{p^{(-)}}^{(n)}]$ *is a hypersheaf as in Definition A.58.*

Definition 4.30. [BMCSY25, Definition 2.10] *An* L_n^{Fin}**-local** ∞**-category** *is an idempotent complete stable* ∞*-category in which all mapping spectra are* L_n^{Fin}*-local as in Definition 2.3.*

The first theorem involves such an ∞-category acted on by a π-finite p-group G as in Definition 3.2; an ordinary finite p-group is a special case. This means there is a functor $BG \to \mathrm{Cat}_\infty$ (the latter ∞-category is the subject of [Lur09, Chapter 3]) sending the single object of the domain to an ∞-category $\mathcal{C}$, for which the limit is $\mathcal{C}^{hG}$ and the colimit is $\mathcal{C}_{hG}$.

Theorem 4.31. Higher descent. [BMCSY25, Theorem A] *Let* $\mathcal{C}$ *be a* L_n^{Fin}*-local* ∞*-category (and hence a simplicial set) acted on by a* π*-finite* p*-group* G *as in Definition 3.2, and let* $\mathrm{K}_{T(n+1)}$ *be the functor of Definition 2.4. Then the coassembly map*

$$\epsilon : \mathrm{K}_{T(n+1)}(\mathcal{C}^{hG}) \to \mathrm{K}_{T(n+1)}(\mathcal{C})^{hG}$$

and the assembly map

$$\eta : \mathrm{K}_{T(n+1)}(\mathcal{C})_{hG} \to \mathrm{K}_{T(n+1)}(\mathcal{C}_{hG})$$

of [Rav26, Definition 5.6] *are isomorphisms.*

Corollary 4.32. [BMCSY25, Corollary 1.4] *Let* $R \to S$ *be a* $T(n)$*-local* G*-Galois extension where* G *is an* n*-finite* p*-group. Then*

$$\mathrm{K}_{T(n+1)}(R) \longrightarrow \mathrm{K}_{T(n+1)}(S)$$

is a $T(n+1)$*-local* G*-Galois extension.*

A similar theorem and corollary for a discrete finite p-group G are proved by Mathew, Naumann, Noel and Dustin Clausen as [CMNN24, Theorem C and Corollary 4.1].

The following says that when X is a $T(n)$-local ring, the functor $\mathrm{K}_{T(n+1)}$ converts the height n cyclotomic sheaf of Definition 4.28 on R to the height $n+1$ one on $\mathrm{K}_{T(n+1)}R$. For $n = 0$ this is proved by Clausen, Mathew and Bhargav Bhatt in [BCM20].

Theorem 4.33. Cyclotomic redshift and hyperdescent. [BMCSY25, Theorems B and C] *Let* R *be a* $T(n)$*-local ring spectrum. Then there is a* $\mathbb{Z}_p^\times$*-equivariant isomorphism:*

$$\mathrm{K}_{T(n+1)}(R[\omega_{p^\infty}^{(n)}]) \simeq \mathrm{K}_{T(n+1)}(R)[\omega_{p^\infty}^{(n+1)}],$$

and the sheaf on $\mathcal{F}in_{\mathbb{Z}_p^\times}$ given by

$$(\mathbb{Z}/p^j)^\times \mapsto \mathrm{K}_{T(n+1)}(R)[\omega_{p^j}^{(n+1)}]$$

is a hypersheaf as in Definition A.58.

5. Locally unipotent $\mathbb{Z}$-actions

Definition 5.1. Local unipotence. *For an action of $\mathbb{Z}$ on a topological abelian group A, let $\Psi : A \to A$ be the automorphism induced by a generator of $\mathbb{Z}$. Then the action is* **unipotent** *if $\Psi - 1$ is a topologically nilpotent endomorphism of A.*

An action of $\mathbb{Z}$ on a spectrum R is **locally unipotent** *if the induced action on each homotopy group is unipotent. We denote by $\mathrm{Sp}^{B\mathbb{Z},u}$ the ∞-category of spectra X equipped with an action of $\mathbb{Z}$ that is locally unipotent on π_*X.*

If A is p-adically complete, topological nilpotence means that $\Psi - 1$ is nilpotent on A/p^i for each i, which is implied by nilpotence on A/p.

The word "local" above refers to the fact that there need not be a power of the endomorphism that kills all homotopy groups, but each such group is killed by some power of it.

As explained in the introduction, the counterexample to the telescope conjecture involves a nontrivial but locally unipotent action of $\mathbb{Z}$ on $R = L_{T(n)}BP\langle n\rangle$ that is induced by one on $BP\langle n\rangle$ itself.

5.1. Trivializing a locally unipotent $\mathbb{Z}$-action.

Theorem 5.2. The $T(n+1)$-local K-theory coassembly map for the trivial $\mathbb{Z}$-action. [BHLS23, Theorem 3.22] *Let R be a $T(n)$-local $\mathbb{E}_1$-ring spectrum for $n \geqslant 1$ and let X be a spectrum. If $L_{T(n+1)}(X \otimes \mathrm{K}(R))$ (see Definition 2.4) is nontrivial, then the coassembly map of (*[Rav26, Definition 5.6]*) for the trivial action of $\mathbb{Z}$ on R,*

$$\epsilon : L_{T(n+1)}(X \otimes \mathrm{K}(R^{B\mathbb{Z}})) \to L_{T(n+1)}(X \otimes \mathrm{K}(R))^{B\mathbb{Z}},$$

is not *an equivalence. In particular the map*

$$\epsilon : \mathrm{K}_{T(n+1)}(R^{B\mathbb{Z}}) \to \mathrm{K}_{T(n+1)}(R)^{B\mathbb{Z}},$$

is not an equivalence when $\mathrm{K}_{T(n+1)}(R)$ is nontrivial.

Theorem 5.3. Asymptotic constancy for THH. [BHLS23, Theorem C] *Let $\mathbb{EA}_2$ be the operad of Example B.38, and suppose that R is an $\mathbb{EA}_2$-ring spectrum (such as $BP\langle n\rangle$) that is connective, p-complete and of* fp*-type n as in Definition 2.1. Suppose also that it has a locally unipotent action of $\mathbb{Z}$.*

If R has the height n Lichtenbaum-Quillen property of Definition 2.7, then for $k \gg 0$ so does $R^{h(p^k\mathbb{Z})}$, and for a finite spectrum F of type $n+2$, there *is a commutative diagram of cyclotomic spectra*

$$\begin{array}{ccc} F \otimes \mathrm{THH}(R^{h(p^k\mathbb{Z})}) & \xrightarrow{\epsilon} & F \otimes \mathrm{THH}(R)^{h(p^k\mathbb{Z})} \\ \cong \downarrow & & \downarrow \cong \\ F \otimes \mathrm{THH}(R^{B(p^k\mathbb{Z})}) & \xrightarrow{\epsilon} & F \otimes \mathrm{THH}(R)^{B(p^k\mathbb{Z})} \end{array}$$

where the horizontal maps are the coassembly maps of [Rav26, Definition 5.6]*.*

Hence tensoring with F "trivializes" the $\mathbb{Z}$-action on $\mathrm{THH}(R)$ in the sense its homotopy fixed point set behaves like that of the trivial action. This implies a similar statement for $\mathrm{TC}(R)$.

Following [BHLS23, Notations and Conventions], we denote the ∞-category of dualizable spectra by $\mathrm{Sp}^{\diamond}$ (which includes all finite complexes) and that of associative algebras in a symmetric monoidal ∞-category $\mathcal{C}$ by $\mathrm{Alg}(\mathcal{C})$.

Definition 5.4. [BHLS23, Definition 4.10] *We let* $\mathrm{UAlg}(\mathrm{Sp})$ *be the presentably symmetric monoidal ∞-category defined by the pullback square*

$$\begin{array}{ccccc} \mathrm{UAlg}(\mathrm{Sp}) & \longrightarrow & \mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z},u} \times \mathrm{Alg}(\mathrm{Sp}^{\diamond}) & \qquad & (R,V) \\ \downarrow & \lrcorner & \downarrow & & \downarrow \\ \mathrm{Alg}(\mathrm{Sp}) & \xrightarrow[\mathrm{triv}]{} & \mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z},u}. & & R \otimes V \end{array}$$

Thus an object in this ∞-category is a pair (R,V), where R is a ring spectrum with a unipotent $\mathbb{Z}$-action and V is a dualizable ring spectrum on which $\mathbb{Z}$ acts trivially, such that the diagonal action on $R \otimes V$ is trivial.

For the next result we need some notation. Let $C^0(\mathbb{Z}_p)$ denote the ring of continuous (meaning locally constant) $\mathbb{F}_p$-valued functions on the p-adic integers. Any $a \in \mathbb{Z}_p$ can be written uniquely as

$$a = \sum_{k \geqslant 0} a_k p^k \qquad \text{with } a_k^p = a_k.$$

Hence each coefficient a_k is either zero or a $(p-1)$th root of unity. Thus its mod p reduction, which we alos denote (abusively) by a_k, is a continuous $\mathbb{F}_p$-valued function. It turns out that the ring of all such functions is

$$C^0(\mathbb{Z}_p) = \mathbb{F}_p[a_k : k \geqslant 0]/(a_k^p - a_k).$$

See [Rav26, Example 6.5] for more discussion.

Let $\mathbb{W}(C^0(\mathbb{Z}_p))$ denote the commutative ring spectrum of spherical Witt vectors as in [Lur18a, Example 5.2.7]. It is a countable coproduct of p-adic sphere spectra.

The proof of the following occupies seven pages of [BHLS23, §4.2] and makes use of the *Dehn twist* of [BHLS23, §4.2.3].

Theorem 5.5. [BHLS23, Theorem 4.11] *There is a natural transformation* $\theta : M_1 \implies M_2$ *(denoted by* η *in* [BHLS23, Theorem 4.11]*) of lax symmetric monoidal functors*

$$M_1, M_2 : \mathrm{UAlg}(\mathrm{Sp}) \to \mathrm{Sp}^{\Delta^1}$$

given by

$$\begin{aligned} & M_1(R,V) := \mathbb{W}(C^0(\mathbb{Z}_p)) \otimes V \otimes \mathrm{res}_\varphi(\mathrm{THH}(R)^{h\mathbb{Z}}) \\ \textit{and} \qquad & M_2(R,V) := V \otimes \mathrm{res}_\varphi \mathrm{THH}(R^{h\mathbb{Z}}), \end{aligned}$$

such that:

(1) *θ becomes an isomorphism after composing with pullback along $i_0 \colon \Delta^0 \to \Delta^1$.*
(2) *θ becomes an isomorphism upon restricting to the full subcategory of those (R,V) for which the Tate coassembly map*

$$\begin{array}{c} \mathbb{W}(C^0(\mathbb{Z}_p)) \otimes \big((V \otimes \mathrm{THH}(R))^{tC_p}\big) \\ \downarrow \\ \big(\mathbb{W}(C^0(\mathbb{Z}_p)) \otimes V \otimes \mathrm{THH}(R)\big)^{tC_p} \end{array}$$

is an isomorphism.

Here the abusively denoted composite functor $\mathrm{res}_\varphi := \mathrm{res}_\varphi \mathrm{res}_{\pentagon}$ as in [Rav26, Definition 5.40]*, sends a p-cyclotomic spectrum X to its Frobenius map $\varphi_p : X \to X^{tC_p}$, which is an object in the morphism category Sp^{Δ^1}, which is tensored over Sp.*

The first property means that the two morphisms $M_1(R,V)$ and $M_2(R,V)$ have the same domain.

The proof of the next result occupies two pages of [BHLS23, §4.2].

Theorem 5.6. Asymptotic boundedness for THH. [BHLS23, Theorem 4.30] *Let*

$$R \in \mathrm{Alg}_{\mathbb{E}\mathbb{A}_2}(\mathrm{Sp}^{B\mathbb{Z},u})$$

be connective of fp-type $n \geqslant -1$, and let F be a finite spectrum of type $n+2$. Suppose that $F \otimes \mathrm{THH}(R)$ is bounded in the range $[c,b]$ in the sense of the Antieau-Nikolaus t-structure of [AN21]*, which is described in* [Rav26, §5.11].

Then for $k \gg 0$, the spectrum $F \otimes \mathrm{THH}(R^{h(p^k\mathbb{Z})})$ is bounded in the range $[c-1,\, b+3]$, and there is an isomorphism of W_k-modules in cyclotomic spectra

$$F \otimes \mathrm{THH}(R^{h(p^k\mathbb{Z})}) \;\simeq\; F \otimes W_k \otimes \mathrm{THH}(R),$$

where $W_k := \mathrm{THH}(\mathbb{S}^{B(p^k\mathbb{Z})})$.

The spectrum W_k above is described by Cary Malkiewich in [Mal17, Corollary 1.3] (quoted as [Rav26, Theorem 6.2]) as a cyclotomic spectrum with a single underlying cell in dimension -1 and countably many in dimension 0.

Corollary 5.7. Asymptotic constancy for TC. [BHLS23, Corollary 4.33] *Let*

$$R \in \mathrm{Alg}_{\mathbb{E}\mathbb{A}_2}(\mathrm{Sp}^{B\mathbb{Z},u})$$

be connective, of fp-type $n \geqslant 0$, and satisfying the height n Lichtenbaum-Quillen property of Definition 2.7. Fix a finite spectrum F of type $n+1$ with a v_{n+1}-self map v.

Then for $k \gg 0$, there is a commutative diagram of $\mathbb{Z}[v]$-modules as below, where the horizontal maps are the coassembly maps:

$$\begin{array}{ccc} \pi_*(F \otimes TC(R^{h(p^k\mathbb{Z})})) & \xrightarrow{\epsilon} & \pi_*(F \otimes TC(R)^{h(p^k\mathbb{Z})}) \\ \cong\downarrow & & \downarrow\cong \\ \pi_*(F \otimes TC(R^{B(p^k\mathbb{Z})})) & \xrightarrow{\epsilon} & \pi_*(F \otimes TC(R)^{B(p^k\mathbb{Z})}). \end{array}$$

In the above the vertical isomorphisms are not induced by isomorphisms of spectra.

The following is needed in the proof of Theorem 6.6.

Lemma 5.8. The coassembly map and the Čech nerve. [BHLS23, Lemma 6.15] *Let $R \in \mathrm{CAlg}\big(\mathrm{Sp}_{T(n)}^{B\mathbb{Z},u}\big)$ for $n \geqslant 1$. There is a commuting triangle, natural in R,*

$$\begin{array}{ccc} \mathrm{K}_{T(n+1)}(R^{h\mathbb{Z}}) & \xrightarrow{\qquad\epsilon\qquad} & \mathrm{K}_{T(n+1)}(R)^{h\mathbb{Z}} \\ & \searrow \qquad\qquad \nearrow_{\simeq} & \\ & \lim\limits_{\Delta} \mathrm{K}_{T(n+1)}\big(R \otimes_{R^{h\mathbb{Z}}} R^{h\mathbb{Z}^{\bullet+1}}\big) & \end{array}$$

identifying the coassembly map ϵ with the Čech nerve (Definition A.29) of the map $i_R : R^{h\mathbb{Z}} \to R$. In particular, the coassembly map for R is an isomorphism if and only if i_R satisfies $\mathrm{K}_{T(n+1)}$-descent as in Definition A.67.

5.2. **Adams operations on $BP\langle n\rangle$.** The main counterexample of [BHLS23] (see Theorem 6.3) involves an action of the integers on the Johnson-Wilson spectrum $BP\langle n\rangle$ via Adams operations. The construction of such operations is the subject of their §5.

In [BHLS23, Theorem C] (our Theorem 5.3) the hypothesis on R is that it is an algebra over the operad $\mathbb{E}\mathbb{A}_2 := \mathbb{E}_1 \otimes_{\mathrm{BV}} \mathbb{A}_2$ of Example B.38. Here $(- \otimes_{\mathrm{BV}} -)$ denotes the tensor product of operads defined by Michael Boardman and Rainer Vogt in [BV73] and discussed in Appendix B.5. The operads $\mathbb{A}_2$ and $\mathbb{E}_1$ are defined in Definitions B.15 and B.17 respectively. Such a structure is intermediate between $\mathbb{E}_1$ and $\mathbb{E}_2$.

They choose $\mathbb{E}\mathbb{A}_2$ because it is the strongest structure they can establish for the Adams operations Ψ^ℓ on $BP\langle n\rangle$; see [BHLS23, Theorem 5.4 and Remark 5.5]. Even though $BP\langle n\rangle$ itself is known to be an $\mathbb{E}_3$-algebra over the $\mathbb{E}_\infty$-ring $MU_{(p)}$, they can only show that their Adams operation on it, and hence its homotopy fixed point set, has the weaker structure. Recall that Vogt, Morten Brun and Zbigniew Fiedorowicz[BFV07] show that for an $\mathbb{E}_m$-ring spectrum R with $m \geqslant 2$, $\mathrm{THH}(R)$ is an $\mathbb{E}_{m-1}$-ring spectrum. Apparently if R is an $\mathbb{E}\mathbb{A}_2$-algebra, then $\mathrm{THH}(R)$ is an $\mathbb{A}_2$-algebra, but we know of no published proof of this.

The Adams operation we want is denoted by Ψ^ℓ, where $\ell = 3$ when the implicit prime p is 2, and $\ell = 2$ when p is odd. For any p-local unit k one has an infinite loop map $\Psi^k : BU_{(p)} \to BU_{(p)}$ which induces multiplication by k^i in π_{2i} for each $i > 0$. This map can be Thomified to an $\mathbb{E}_\infty$-map $\Psi^k : MU_{(p)} \to MU_{(p)}$ with similar properties. There is a similarly named $\mathbb{E}_\infty$-endomorphism of the Morava spectrum E_n that is related to the formal group law endomorphism $[k](x)$. More details can be found in [BHLS23, §5].

We lose the $\mathbb{E}_\infty$-structure when we pass to $BP\langle n\rangle$, which is only known to be an $\mathbb{E}_3$-$MU_{(p)}$-algebra. There the induced Adams operation is only known to to preserve the still weaker structure of an $\mathbb{E}\mathbb{A}_2$-$MU_{(p)}$-algebra.

The proof of the following runs for twelve pages and makes use of factorization homology in [BHLS23, Proposition 5.11].

Theorem 5.9. The curative effect of $T(n)$-localization on $BP\langle n\rangle$. [BHLS23, Theorem 5.4] *The $\mathbb{E}\mathbb{A}_2$–$MU_{(p)}$-algebra underlying the $\mathbb{E}_3$–$MU_{(p)}$-algebra $BP\langle n\rangle$ admits a lift to an object*

$$BP\langle n\rangle^\Psi \in \mathrm{Alg}_{\mathbb{E}\mathbb{A}_2}\big(\mathrm{Mod}(\mathrm{Sp}^{B\mathbb{Z}}; MU_{(p)}^\Psi)\big)$$

such that:

(i) There is a map

$$\iota\colon BP\langle n\rangle^\Psi \longrightarrow E_n^\Psi$$

in $\mathrm{Alg}_{\mathbb{E}_1}\big(\mathrm{Mod}(\mathrm{Sp}^{B\mathbb{Z}}; MU_{(p)}^\Psi)\big)$.

(ii) There is an identification

$$\begin{array}{ccc} L_{T(n)}BP\langle n\rangle^\Psi & \xrightarrow{\;L_{T(n)}(\iota)\;} & E_n^\Psi \\ {\scriptstyle\cong}\big\downarrow & & \big\| \\ (E_n^\Psi)^{h(\mu_{p^n-1}\rtimes\widehat{\mathbb{Z}})} & \longrightarrow & E_n^\Psi \end{array}$$

in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}^{B\mathbb{Z}})$. *The two spectra other than $L_{T(n)}BP\langle n\rangle^\Psi$ are $\mathbb{E}_\infty$. The subgroup $\mu_{p^n-1} \rtimes \widehat{\mathbb{Z}} \subseteq \mathbb{G}_n$ fits into a diagram with exact rows*

$$\begin{array}{ccccc} \mu_{p^n-1} & \longrightarrow & \mu_{p^n-1}\times n\widehat{\mathbb{Z}} & \longrightarrow & n\widehat{\mathbb{Z}} \\ \big\| & & \big\downarrow & & \big\downarrow \\ \mu_{p^n-1} & \longrightarrow & \mu_{p^n-1}\rtimes\widehat{\mathbb{Z}} & \longrightarrow & \widehat{\mathbb{Z}} \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\cong} \\ \mathcal{O}_D^\times & \longrightarrow & \mathbb{G}_n & \longrightarrow & \mathrm{Gal}(\mathbb{F}_p), \end{array} \tag{5.10}$$

where the bottom row is the short exact sequence of (4.9). In the middle row the Frobenius generator of $\widehat{\mathbb{Z}}$ acts on μ_{p^n-1} by raising each root of unity to its pth power.

(iii) The underlying $\mathbb{Z}$-action on $BP\langle n\rangle$ is locally unipotent in p-complete spectra after p-completion.

Local unipotence is implied by the fact that $\ell^{p-1}-1$ is divisible by p. The generator of $\widehat{\mathbb{Z}}$ acts on $\pi_* E_n$ via the lifting of the Frobenius automorphism of $\overline{\mathbb{F}}_p$ to its Witt ring. It follows that

$$E_n \simeq E_n(\overline{\mathbb{F}}_p)^{h(n\widehat{\mathbb{Z}})};$$

the action of $n\widehat{\mathbb{Z}}$ on the scalars in $\mathbb{W}(\overline{\mathbb{F}}_p)$ fixes $\mathbb{W}(\mathbb{F}_{p^n})$. Meanwhile the action of μ_{p^n-1} on $\pi_* E_n$ is such that

$$\begin{aligned} \pi_*\left(E_n^{h\mu_{p^n-1}}\right) &= W(\mathbb{F}_{p^n})[\![u^{p-1}u_1,\dots,u^{p^{n-1}-1}u_{n-1}]\!][u^{\pm(p^n-1)}] \\ &= W(\mathbb{F}_{p^n})[\![v_1,\dots,v_{n-1}]\!][v_n^{\pm1}] \\ \text{and}\quad \pi_*\left(E_n(\overline{\mathbb{F}}_p)^{h(\mu_{p^n-1}\rtimes\widehat{\mathbb{Z}})}\right) &= \mathbb{Z}_p[\![v_1,\dots,v_{n-1}]\!][v_n^{\pm1}] \\ &= \pi_* L_{T(n)} BP\langle n\rangle. \end{aligned}$$

This means the $\mathbb{EA}_2$-structure of the group action on $BP\langle n\rangle$ becomes an $\mathbb{E}_\infty$-structure upon passage to $L_{T(n)}BP\langle n\rangle$.

The proof of the following occupies most of [BHLS23, §4].

Theorem 5.11. Telescopic TC asymptotic constancy for $BP\langle n\rangle$. [BHLS23, Theorem B] *Fix a telescope $T(n+1)$ of a type $n+1$ p-local finite spectrum $F(n+1)$. Then for all $k \gg 0$, there is a commuting square*

$$\begin{array}{ccc} T(n+1)_*\mathrm{TC}(BP\langle n\rangle^{h(p^k\mathbb{Z})}) & \xrightarrow{\epsilon} & T(n+1)_*\mathrm{TC}(BP\langle n\rangle)^{h(p^k\mathbb{Z})} \\ \cong\downarrow & & \downarrow\cong \\ T(n+1)_*\mathrm{TC}(BP\langle n\rangle^{B(p^k\mathbb{Z})}) & \xrightarrow{\epsilon} & T(n+1)_*\mathrm{TC}(BP\langle n\rangle)^{B(p^k\mathbb{Z})}), \end{array}$$

where the horizontal maps are TC *coassembly maps.*

6. The main counterexample

As in [BHLS23, §6.1], we start with

Theorem 6.1. Telescopic K-theory for connective ring spectra. [LMMT24, Purity Theorem and Corollary 4.30] *and* [CMNN20, Corollary 4.11]. *Let R be a connective $\mathbb{E}_1$-algebra. For $n \geqslant 1$, the $(T(n)\oplus T(n+1))$-localization map and the cyclotomic trace induce isomorphisms*

$$\mathrm{K}_{T(n+1)}\big(L_{T(n)\oplus T(n+1)}R\big) \xleftarrow{\cong} \mathrm{K}_{T(n+1)}(R) \xrightarrow{\cong} \mathrm{TC}_{T(n+1)}(R).$$

With additional hypotheses we have

Corollary 6.2. Connective rings with $\mathbb{Z}$-action. [BHLS23, Corollary 6.3] *For $n \geqslant 1$, let R be a $T(n+1)$-acyclic, connective $\mathbb{E}_1$-algebra with a $\mathbb{Z}$-action. The coassembly map, the $T(n)$-localization map, and the cyclotomic* *trace fit into a commuting diagram*

$$\begin{array}{ccc}
\mathrm{K}_{T(n+1)}(L_{T(n)}R^{h\mathbb{Z}}) & \xrightarrow{\epsilon} & \mathrm{K}_{T(n+1)}(L_{T(n)}R)^{h\mathbb{Z}} \\
\simeq\uparrow & & \simeq\uparrow \\
\mathrm{K}_{T(n+1)}(R^{h\mathbb{Z}}) & \xrightarrow{\epsilon} & \mathrm{K}_{T(n+1)}(R)^{h\mathbb{Z}} \\
\simeq\downarrow & & \simeq\downarrow \\
\mathrm{TC}_{T(n+1)}(R^{h\mathbb{Z}}) & \xrightarrow{\epsilon} & \mathrm{TC}_{T(n+1)}(R)^{h\mathbb{Z}},
\end{array}$$

The ring of interest is $BP\langle n\rangle$ with the $\mathbb{Z}$-action given by the Adams operations of §5.2. While its known multiplicative structure ($\mathbb{EA}_2$) is relatively weak, Theorem 5.9 says that $T(n)$-localization elevates it to $\mathbb{E}_\infty$.

Theorem 6.3. The main counterexample. [BHLS23, Theorem A] *Let p be any prime and $n \geqslant 1$. Then, for all $k \geqslant 0$,*

$$\mathrm{K}_{T(n+1)}\left(BP\langle n\rangle^{h(p^k\mathbb{Z})}\right)$$

is not $K(n+1)$-local. In particular,

$$\mathrm{Sp}_{K(n+1)} \neq \mathrm{Sp}_{T(n+1)}.$$

On [BHLS23, page 5] they say (using our notation)

> Using cyclotomic redshift [see §4.5], and the fact that the $(p^k\mathbb{Z}_p)$-pro-Galois extension
>
> $$L_{T(n)}BP\langle n\rangle^{h(p^k\mathbb{Z})} \longrightarrow L_{T(n)}BP\langle n\rangle$$
>
> is closely related to a [higher] cyclotomic extension [see §4.3], we deduce that there is an equivalence
>
> $$\mathrm{K}_{T(n+1)}(BP\langle n\rangle)^{h(p^k\mathbb{Z})} \simeq \mathrm{K}_{\mathrm{Cyclo}(n+1)}\left(BP\langle n\rangle^{h(p^k\mathbb{Z})}\right)$$
>
> for each $k \geqslant 0$. Thus, in order to prove Theorem 6.3, it suffices to show that the coassembly map
>
> $$\mathrm{K}_{T(n+1)}\left(BP\langle n\rangle^{h(p^k\mathbb{Z})}\right) \longrightarrow \mathrm{K}_{T(n+1)}(BP\langle n\rangle)^{h(p^k\mathbb{Z})}$$
>
> is not an equivalence.

Lemma 6.4. A cyclotomic completeness criterion. [BHLS23, Lemma 6.22] *For $n \geqslant 1$, if R is a $T(n)$-local commutative ring spectrum and there is a map in* $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$

$$S[\omega_{p^\infty}^{(n)}]^{hT_p} \longrightarrow L_{T(n)}\big(W(\overline{\mathbb{F}}_p)\otimes R\big) \qquad \textit{for } T_p \textit{ as in (4.4)},$$

then $\mathrm{K}_{T(n+1)}(R)$ is height $n+1$ cyclotomically complete.

The proof of the following requires a full page and several lemmas in [BHLS23]. It uses the machinery sketched in §4.3. Recall that $\mathrm{K}_{\mathrm{Cyclo}(n+1)}$ (height $n+1$ cyclotomically complete algebraic K-theory) is defined in Definition 2.4.

Proposition 6.5. $\mathrm{K}_{\mathrm{Cyclo}(n+1)}$-descent and cyclotomic completeness. [BHLS23, Prop. 6.24] *For $n \geqslant 1$, let $R = L_{T(n)}BP\langle n\rangle$ with the $\mathbb{E}_\infty$-$\mathbb{Z}$-action of Theorem 5.9. Then for each $k \geqslant 0$, the map*

$$f_k\colon R^{h(p^k\mathbb{Z})} \longrightarrow R$$

is a $\mathrm{K}_{\mathrm{Cyclo}(n+1)}$-cover as in Definition A.67 (also see Example A.71), and $\mathrm{K}_{T(n+1)}(R)$ is height $n+1$ cyclotomically complete as in Definition 4.17.

Theorem 6.3 for a given value of k implies the same for smaller values of k, so it suffices to prove it for $k \gg 0$. It is implied by the following, which is related to Corollary 6.2 and relies on Lemma 5.8.

Theorem 6.6. Coassembly as cyclotomic completion. [BHLS23, Theorem 6.25] *Let $BP\langle n\rangle$ be as in Theorem 5.4. For every prime p, height $n \geqslant 1$, and $k \geqslant 0$, there is a diagram*

$$\begin{array}{ccc}
\mathrm{K}_{T(n+1)}(L_{T(n)}BP\langle n\rangle^{h(p^k\mathbb{Z})}) & \xrightarrow{\epsilon} & \mathrm{K}_{T(n+1)}(L_{T(n)}BP\langle n\rangle)^{h(p^k\mathbb{Z})} \\
\simeq\uparrow & & \simeq\uparrow \\
\mathrm{K}_{T(n+1)}(BP\langle n\rangle^{h(p^k\mathbb{Z})}) & \xrightarrow{\epsilon} & \mathrm{K}_{T(n+1)}(BP\langle n\rangle)^{h(p^k\mathbb{Z})} \\
\simeq\downarrow & & \simeq\downarrow \\
\mathrm{TC}_{T(n+1)}(BP\langle n\rangle^{h(p^k\mathbb{Z})}) & \xrightarrow{\epsilon} & \mathrm{TC}_{T(n+1)}(BP\langle n\rangle)^{h(p^k\mathbb{Z})},
\end{array}$$

where the horizontal maps are coassembly maps as in [Rav26, Definition 5.6]. *These maps are not isomorphisms, but rather exhibit the target as the cyclotomic completion of the source. Hence the coassembly map on*

$$\mathrm{K}_{\mathrm{Cyclo}(n+1)}(BP\langle n\rangle^{h(p^k\mathbb{Z})})$$

is an equivalence, while the middle coassembly map above is not. In particular, this gives a counterexample to the height $n+1$ telescope conjecture.

Appendix A. Some ∞-categorical constructions

A.1. **Miscellaneous definitions.** We need the following definitions for future reference. We suggest skipping this subsection initially and referring to specific items in it as needed.

A.1.1. *Augmented simplicial objects.* Recall that the **simplicial category** $\mathbf{\Delta}$ is that of finite ordered sets and order preserving maps. It has an object $[n] = \{0, 1, \ldots, n\}$ for each integer $n \geqslant 0$. More details can be found in [Rav26, §2.4] and in [GJ99, Chapter I]. Here we consider the **augmented simplicial category** $\mathbf{\Delta}_+$, which is obtained from $\mathbf{\Delta}$ by adding an additional object, the empty set, denoted by $[-1]$. It is an initial object, while $[0]$ is a terminal object. There is a unique morphism $[-1] \to [n]$ for each $n \geqslant 0$, but no morphisms going the other way. Hence the category $\mathbf{\Delta}_+$ is not pointed.

Definition A.1. [Lur09, Definition 6.1.2.2] *A* **simplicial object of an ∞-category** $\mathcal{C}$ *is a functor*

$$V_\bullet\colon \mathrm{N}(\boldsymbol{\Delta})^{\mathrm{op}} \longrightarrow \mathcal{C}.$$

An **augmented simplicial object of** $\mathcal{C}$ *is an extension of* V *to a map*

$$V_\bullet^+\colon \mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}} \longrightarrow \mathcal{C}.$$

We let

$$\mathcal{C}_{\boldsymbol{\Delta}} := \mathrm{Fun}\big(\mathrm{N}(\boldsymbol{\Delta})^{\mathrm{op}}, \mathcal{C}\big)$$

denote the ∞-category of simplicial objects of $\mathcal{C}$. Similarly, we refer to

$$\mathcal{C}_{\boldsymbol{\Delta}_+} := \mathrm{Fun}\big(\mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}}, \mathcal{C}\big)$$

as the ∞-category of augmented simplicial objects of $\mathcal{C}$.

If $V_\bullet$ (or $V_\bullet^+$) is an (augmented) simplicial object of $\mathcal{C}$ and $n \geqslant 0$ (respectively $n \geqslant -1$), we write

$$V_n := V([n]) \in \mathcal{C}.$$

We denote by d_0 the map $V_0 \to V_{-1}$ induced by the map $[-1] \to [0]$.

The simplicial sets Δ^n (the standard n-simplex) and Λ_i^n (its ith horn, meaning its boundary with the interior of ith $(n-1)$-face removed) mentioned below are defined in [Rav26, Definition 2.21] and on [GJ99, page 6].

Definition A.2. Various fibrations. [Lur09, Definition 2.0.0.3] *A morphism $f : X \to S$ of simplicial sets (e.g., a functor of ∞-categories) is*

- *a* **Kan fibration** *if f has the right lifting property with respect to all horn inclusions $\Lambda_i^n \hookrightarrow \Delta^n$ for $0 \leqslant i \leqslant n$;*
- *a* **left fibration** *if the same holds for $0 \leqslant i < n$;*
- *a* **right fibration** *if it holds for $0 < i \leqslant n$;*
- *an* **inner fibration** *(or* **Joyal fibration***) if it holds for $0 < i < n$.*

We will use left fibrations to define left exact functors in Definition A.14.

A.1.2. *Limits and colimits.* Recall that limits and colimits in ∞-categories of spaces or spectra coincide with homotopy limits and colimits, as defined by Bousfield and Dan Kan in [BK72], in the corresponding ordinary categories with homotopy structure.

Limits and colimits in an ∞-category are defined by Joyal in [Joy02, Definition 4.5], which is quoted by Lurie as [Lur09, Definition 1.2.13.4] and again as [Rav26, Definition 5.5].

Definition A.3. [Lur09, Dual of Remark 1.2.13.5] *Let $\mathcal{C}$ be an ∞-category and let $p\colon K \to \mathcal{C}$ be a diagram indexed by a simplicial set K. An extension*

$$\bar{p}\colon K^{\triangleleft} \longrightarrow \mathcal{C}$$

is called a **limit diagram** *if the cone point $\bar{p}(-\infty)$ is a limit of p.*

Here $K^{\triangleleft}$ is the **left cone of** *K, the simplicial set obtained from K by adding a vertex, denoted by $-\infty$, the cone point, with an edge pointing to each* *vertex in K, a 2-simplex for each edge in K and so on. See* [Lur09, §1.2.8 and Notation 1.2.8.4] *for more details.*

Equivalently, $\bar{p}$ is a limit diagram if for every object $X \in \mathcal{C}$, the induced map of mapping spaces

$$\mathrm{Map}_{\mathcal{C}}\big(X, \bar{p}(-\infty)\big) \longrightarrow \lim_{k\in K} \mathrm{Map}_{\mathcal{C}}\big(X, p(k)\big)$$

is an equivalence.

There is a similarly defined right cone $K^{\rhd}$ that is relevant to colimits.

We will see the right cone in **??** and Theorem A.35.

A.1.3. *Truncation and connectivity.*

Definition A.4. Truncated and connective objects and morphisms in an ∞-category. *An object D in an ∞-category $\mathcal{C}$ is* **m-truncated** **(m-connective)** *of for each object C in $\mathcal{C}$ the space of maps $\mathrm{Map}_{\mathcal{C}}(C, D)$ is m-truncated (m-connective). In particular it is* **discrete** *if it is 0-truncated,* **connected** *if it is 1-connective, and a final object if it is (-2)-truncated. It is (-1)-truncated if each such mapping space is either empty or contractible.*

A morphism $f : C \to D$ in $\mathcal{C}$ is **m-truncated** **(m-connective)** *if, for every object $E \in \mathcal{C}$, composition with f induces an m-truncated (m-connective) map of spaces*

$$f_* : \mathrm{Map}_{\mathcal{C}}(E, C) \longrightarrow \mathrm{Map}_{\mathcal{C}}(E, D)$$

which in the latter case is an effective epimorphism as in Definition A.39. It is **∞-connective** *if each such f_* is a weak equivalence.*

The full sub-categories $\tau_{\leqslant m}\mathcal{C}$ and $\tau_{\geqslant m}\mathcal{C}$ of $\mathcal{C}$ are those of m-truncated and m-connective objects. Hence we can regard $\tau_{\leqslant m}$ and $\tau_{\geqslant m}$ as endofunctors in the ∞-category of ∞-categories.

Definition A.5. [Lur09, Def. 5.5.6.8] *Let $f : X \to Y$ be a map of Kan complexes. It is* **m-truncated** *if every homotopy fiber of f (taken over any base point of Y) is m-truncated.*

More generally, let $\mathcal{C}$ be an ∞-category and let $f : C \to D$ be a morphism in $\mathcal{C}$. It is **m-truncated** *if for every object $E \in \mathcal{C}$, the induced map*

$$\mathrm{Map}_{\mathcal{C}}(E, C) \longrightarrow \mathrm{Map}_{\mathcal{C}}(E, D)$$

is a m-truncated map of spaces.

The following is immediate.

Proposition A.6. *A morphism $f : C \to D$ in an ∞-category $\mathcal{C}$ is m-truncated iff the same is true of f as an object in $\mathcal{C}_{/D}$.*

Remark A.7. *An* **m-category***, defined formally in* [Lur09, Definition 2.3.4.1]*, is an ∞-category in which each object is m-truncated. It can also be defined by induction on m as a category enriched over $(m-1)$-categories.*

Thus a (-1)-category has mapping spaces that are contractible, making all objects equivalent and the structure uninteresting. A 0-category has mapping *spaces that are contractible or empty, making it equivalent to a set of equivalence classes of objects, so a 0-category is essentially a set. A 1-category has mapping spaces that are equivalent to sets, making it equivalent to an ordinary category.*

A.1.4. *Accessibile and presentable ∞-categories.*

Definition A.8. [Lur09, Definition 5.3.5.1] *Let $\mathcal{C}$ be a small ∞-category and let κ be a regular cardinal. We let $\mathrm{Ind}_\kappa(\mathcal{C})$, the* **$\kappa$-filtered colimit category of $\mathcal{C}$**, *denote the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by those functors $f : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ which classify right fibrations $\widetilde{\mathcal{C}} \to \mathcal{C}$, where the ∞-category $\widetilde{\mathcal{C}}$ has κ-filtered colimits. In the case where $\kappa = \omega$, we simply write $\mathrm{Ind}(\mathcal{C})$ for $\mathrm{Ind}_\kappa(\mathcal{C})$.*

Accessibility is the subject of [Lur09, §5.4.2].

Definition A.9. [Lur09, Definition 5.4.2.1] *Let κ be a regular cardinal. An ∞-category $\mathcal{C}$ is* **κ-accessible** *if there exists a small ∞-category $\mathcal{C}^0$ and an equivalence*

$$\mathrm{Ind}_\kappa(\mathcal{C}^0) \longrightarrow \mathcal{C}.$$

We say that $\mathcal{C}$ is **accessible** *if it is κ-accessible for some regular cardinal κ.*

The presheaf category $\mathcal{P}(\mathcal{C})$ of Definition A.32 is accessible because $\mathcal{S}$ is.

Proposition A.10. Properties of accessible ∞-categories. [Lur09, Proposition 5.4.2.2] *Let $\mathcal{C}$ be an ∞-category and κ a regular cardinal. The following conditions are equivalent:*

- *(i) The ∞-category $\mathcal{C}$ is κ-accessible.*
- *(ii) The ∞-category $\mathcal{C}$ is locally small and admits κ-filtered colimits, the full subcategory $\mathcal{C}^\kappa \subseteq \mathcal{C}$ of κ-compact objects is essentially small, and $\mathcal{C}^\kappa$ generates $\mathcal{C}$ under small κ-filtered colimits.*
- *(iii) The ∞-category $\mathcal{C}$ admits small κ-filtered colimits and contains an essentially small full subcategory $\mathcal{C}'' \subseteq \mathcal{C}$ consisting of κ-compact objects which generates $\mathcal{C}$ under small κ-filtered colimits.*

Presentable ∞-categories are the subject of [Lur09, §5.5].

Definition A.11. [Lur09, Definition 5.5.0.1] *An ∞-category $\mathcal{C}$ is* **presentable** *if it is accessible and admits small colimits. We denote by Pr^{L} (see* [Lur09, Definition 5.5.3.1]*) the ∞-category of presentable ∞-categories and colimit preserving functors. The ∞-category of such stable ∞-categories is denoted by $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.*

Definition A.12. Continuous functors, compact objects and and compact fibrations. [Lur09, Definition 5.3.4.5] *Let $\mathcal{C}$ be an ∞-category which admits small κ-filtered colimits. A functor $f : \mathcal{C} \to \mathcal{D}$ is* **κ-continuous** *if it preserves κ-filtered colimits.*

For an object C in $\mathcal{C}$, let $j_C : \mathcal{C} \to \mathcal{S}$ denote the functor corepresented by C. If $\mathcal{C}$ admits κ-filtered colimits, then C is **κ-compact** *if j_C is κ-continuous. We will say that C is* **compact** *if it is ω-compact (and $\mathcal{C}$ admits filtered colimits).*

A left fibration (Definition A.2) $\mathcal{C}' \to \mathcal{C}$ is **κ-compact** *if it is classified by a κ-continuous functor $\mathcal{C} \to \mathcal{S}$.*

Definition A.13. Accessible functors. [Lur09, Definition 5.4.2.5] *If $\mathcal{C}$ is an accessible ∞-category, then a functor $F : \mathcal{C} \to \mathcal{C}'$ is* **accessible** *if it is κ-continuous for some regular cardinal κ (and therefore for all regular cardinals $\tau \geqslant \kappa$).*

Definition A.14. Left exact functors. [Lur09, Definition 5.3.2.1] *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories and let κ be a regular cardinal. We say that F is* **κ-left exact** *if, for any left fibration $\mathcal{D}' \to \mathcal{D}$ where $\mathcal{D}'$ is κ-filtered, the ∞-category*

$$\mathcal{C}' = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$$

is also κ-filtered. We say that F is **left exact** *if it is ω-left exact.*

A.1.5. *The coskeleton functor.* Next we define the coskeleton functor, which figures in the definitions of hypercoverings (Definition A.55) and hypersheaves (Definition A.58), which are needed for **??**. Before doing so, here is an observation.

For CW-complexes X and Y, let X^n denote the n-skeleton of X and P^nY the nth Postnikov section of Y. Then we know that

$$\operatorname{Map}(X^n, Y) \simeq \operatorname{Map}(X, P^nY), \tag{A.15}$$

so we have a pair of adjoint endofunctors in the ∞-category of Kan complexes $\mathcal{S}$.

Definition A.16. [Lur09, Notation 6.5.3.1] *For each $n \geqslant 0$, let $\mathbf{\Delta}^{\leqslant n}$ denote the full subcategory of $\mathbf{\Delta}$ spanned by the set of objects $\{[0], \dots, [n]\}$, and we denote the inclusion functor by i_n.*

Similarly let $\mathbf{\Delta}_+^{\leqslant n} \subseteq \mathbf{\Delta}_+$ for $n \geqslant -1$ denote the full subcategory of $\mathbf{\Delta}$ spanned by the set of objects $\{[-1], \dots, [n]\}$.

For a presentable ∞-category $\mathcal{C}$ as in Definition A.11, let

$$\mathcal{C}_{\mathbf{\Delta}} := \operatorname{Fun}(N(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{C}), \qquad \mathcal{C}_{\mathbf{\Delta}_+} := \operatorname{Fun}(N(\mathbf{\Delta}_+)^{\mathrm{op}}, \mathcal{C}),$$
$$\mathcal{C}_{\mathbf{\Delta}^{\leqslant n}} := \operatorname{Fun}\big(N(\mathbf{\Delta}^{\leqslant n})^{\mathrm{op}}, \mathcal{C}\big), \quad \text{and} \quad \mathcal{C}_{\mathbf{\Delta}_+^{\leqslant n}} := \operatorname{Fun}\big(N(\mathbf{\Delta}_+^{\leqslant n})^{\mathrm{op}}, \mathcal{C}\big).$$

Let $\mathbf{\Delta}^{\leqslant -1}$ be the empty category, making $\mathcal{C}_{\mathbf{\Delta}^{\leqslant -1}}$ the trivial ∞-category whose single object is the empty diagram. We define $\mathbf{\Delta}_+^{\leqslant -1}$ to be the trivial category with object the empty set, making $\mathcal{C}_{\mathbf{\Delta}_+^{\leqslant -1}}$ the discrete ∞-category associated with $\mathcal{C}$.

The restriction functor

$$\mathrm{sk}_n := (N(i_n)^{\mathrm{op}})^* : \mathcal{C}_{\mathbf{\Delta}} \longrightarrow \mathcal{C}_{\mathbf{\Delta}^{\leqslant n}}$$

has left and right adjoints i_n^* and $i_n^!$ given by left and right Kan extensions along the inclusion

$$N(i_n)^{\mathrm{op}} : N(\mathbf{\Delta}^{\leqslant n})^{\mathrm{op}} \hookrightarrow N(\mathbf{\Delta})^{\mathrm{op}},$$

and similarly for the augmented case.

(Goerss and Jardine [GJ99, IV.3.2] denote our $\mathbf{\Delta}^{\leqslant n}$ by $\mathbf{\Delta}_n$ and call a contravariant Set*-valued functor on functor on it an n-**truncated simplicial set**. They call our* sk_n *the n-**truncation functor**. These notions are also discussed 30 years earlier by Mike Artin and Barry Mazur in [AM69, §1].)*

For an object X in $\mathcal{C}_{\mathbf{\Delta}}$,

$$(i_n^*X)_m = \operatorname*{colim}_{[m]\to[k]} X_k \qquad \text{and} \qquad (i_n^!X)_m = \lim_{[k]\to[m]} X_k,$$

where the indicated morphisms in both cases are in $\mathbf{\Delta}$ with $k \leqslant n$. For $m \leqslant n$, each object is X_n. For $m > n$, each map $[k] \to [m]$ factors though $[n]$, so $(i_n^!X)_m$ is a certain subobject of the $\binom{m+1}{n+1}$-fold product of X_n. There are maps

$$(i_n^!X)_m \to X_n^{\binom{m+1}{n+1}} \qquad \text{and} \qquad \coprod_{\binom{m+1}{n+1}} X_n \to (i_n^*X)_m,$$

which are respectively a monomorphism (meaning a (-1)-truncated morphism) and an effective epimorphism as in Definition A.39.

*The n-**coskeleton functor** $\mathrm{cosk}_n : \mathcal{C}_{\mathbf{\Delta}} \to \mathcal{C}_{\mathbf{\Delta}}$ is the composition*

$$\mathcal{C}_{\mathbf{\Delta}} \xrightarrow{\ \mathrm{sk}_n\ } \mathcal{C}_{\mathbf{\Delta}^{\leqslant n}} \xrightarrow{\ i_n^!\ } \mathcal{C}_{\mathbf{\Delta}}.$$

In particular we have

$$\begin{aligned}(\mathrm{cosk}_{n-1}(X))_n &= \big\{(x_0,\dots,x_n) \in (X_{n-1})^{n+1} \,\big|\\ &\qquad\qquad d_i(x_j) = d_{j-1}(x_i) \text{ for all } 0 \leqslant i < j \leqslant n\big\}\\ &=: M_n(X) \qquad \text{the } n\textbf{th matching object} \text{ of } X,\end{aligned} \tag{A.17}$$

where $d_i : X_{n-1} \to X_{n-2}$ is the ith face map.

Example A.18. *In the augmented case for small n, we have*

$$\begin{aligned} M_0(X) &= (\mathrm{cosk}_{-1}(X))_0 = X_{-1}\\ M_1(X) &= (\mathrm{cosk}_0(X))_1 = \big\{(x_0,x_1) \in (X_0)^2 \,\big|\, d_0(x_1) = d_0(x_0)\big\}\\ &= X_0 \times_{X_{-1}} X_0\\ M_2(X) &= (\mathrm{cosk}_1(X))_2\\ &= \big\{(x_0,x_1,x_2) \in (X_1)^3 \,\big|\\ &\qquad d_0(x_1) = d_0(x_0),\ d_0(x_2) = d_1(x_0),\ d_1(x_2) = d_1(x_1)\big\}. \end{aligned}$$

We have an adjunction

$$\mathcal{C}_{\mathbf{\Delta}} \underset{i_n^!}{\overset{\mathrm{sk}_n}{\rightleftarrows}} \mathcal{C}_{\mathbf{\Delta}^{\leqslant n}} \quad (\bot) \tag{A.19}$$

with a unit

$$\eta_{X,n} : X_\bullet \to i_n^! \operatorname{sk}_n X_\bullet =: \operatorname{cosk}_n X_\bullet \tag{A.20}$$

for each object $X_\bullet$ in $\mathcal{C}_{\mathbf{\Delta}}$. The target of sk_{-1} is the trivial category $\mathcal{C}_{\mathbf{\Delta}^{\leqslant -1}}$. The image of its one object under $i_{-1}^!$ is a constant functor of $\mathbf{\Delta}$ whose value is a terminal object in $\mathcal{C}$, which exists because $\mathcal{C}$ is presentable as in Definition A.11.

The unit map $(\eta_{X,n-1})_n : X_n \to M_n(X)$ of (A.20) sends an n-simplex K in X_n to its $(n+1)$-tuple of $(n-1)$-faces. It is required to be onto in Definition A.55.

Example A.21. The case $\mathcal{C} = \mathrm{Set}$. *For a simplicial set X with geometric realization $|X|$, the adjunctions of (A.15) and (A.19) imply that*

$$|\operatorname{cosk}_n X| \simeq P^n|X|.$$

For $n = m - 1$, we have

$$(\operatorname{cosk}_{n-1}(X))_n = \operatorname{Map}_{\mathcal{C}_{\mathbf{\Delta}}}(\partial\mathbf{\Delta}^n, X_\bullet) \simeq \Omega^{n-1}X. \tag{A.22}$$

A.1.6. *Localization.* Localization functors are discussed in [Lur09, §5.2.7].

Definition A.23. [Lur09, Definition 5.2.7.2] *A functor $f : \mathcal{C} \to \mathcal{D}$ between ∞-categories is a* **localization** *if f has a fully faithful right adjoint.*

The following definition specializes to one of Bousfield in the ordinary category of spaces.

Definition A.24. [Lur09, Definition 5.5.4.1] *Let $\mathcal{C}$ be an ∞-category and let S be a collection of morphisms of $\mathcal{C}$. We say that an object $Z \in \mathcal{C}$ is* **S-local** *if, for every morphism $s\colon X \to Y$ belonging to S, composition with s induces a homotopy equivalence.*

$$\operatorname{Map}_{\mathcal{C}}(Y, Z) \longrightarrow \operatorname{Map}_{\mathcal{C}}(X, Z).$$

A morphism $f\colon X \to Y$ of $\mathcal{C}$ is an **S-equivalence** *if, for every S-local object Z, composition with f induces a homotopy equivalence*

$$\operatorname{Map}_{\mathcal{C}}(Y, Z) \longrightarrow \operatorname{Map}_{\mathcal{C}}(X, Z).$$

Definition A.25. Topological localizations. [Lur09, Definition 6.2.1.4] *Let $\mathcal{C}$ be a presentable ∞-category and let S be a strongly saturated class of morphisms of $\mathcal{C}$; see* [Lur09, Definition 5.5.4.5]. *We will say that S is* **topological** *if the following conditions are satisfied:*

(i) There exists a subclass $S_0 \subseteq S$ consisting of monomorphisms such that S_0 generates S as a strongly saturated class of morphisms.
(ii) Given a pullback diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

in $\mathcal{C}$ for which f belongs to S, the morphism f' also belongs to S.

We will say that a localization $L\colon \mathcal{C} \to \mathcal{D}$ is **topological** *if the collection S of all morphisms $f\colon X \to Y$ in $\mathcal{C}$ such that Lf is an equivalence is topological.*

A.2. **Groupoids and the Čech nerve.** The material in this subsection is taken from [Lur09, §6.1.2] unless otherwise indicated.

Definition A.26. [Lur09, §6.1.2]. *Let $\mathcal{C}$ be an ordinary category which admits finite limits. A* **groupoid object** *of $\mathcal{C}$ is a functor*

$$F\colon \mathcal{C} \longrightarrow \mathbf{Gpd}$$

from $\mathcal{C}$ to the category **Gpd** *of small groupoids, which has the following properties:*

(i) There exists an object $X_0 \in \mathcal{C}$ and a natural (in C) identification of $\mathrm{Hom}_{\mathcal{C}}(C, X_0)$ with the set of objects of the groupoid $F(C)$ for each $C \in \mathcal{C}$.
(ii) There exists an object $X_1 \in \mathcal{C}$ and a natural identification of $\mathrm{Hom}_{\mathcal{C}}(C, X_1)$ with the set of morphisms in the groupoid $F(C)$ for each $C \in \mathcal{C}$.

For example, a groupoid object in Set is simply a groupoid.

In Lurie's words,

> Giving a groupoid object of a category $\mathcal{C}$ is equivalent to giving a pair of objects $X_0 \in \mathcal{C}$ (the "object classifier") and $X_1 \in \mathcal{C}$ (the "morphism classifier"), together with a collection of maps which relate X_0 to X_1 and satisfy appropriate identities, imitating the usual axiomatics of category theory.
>
> These identities can be very efficiently encoded using the formalism of simplicial objects. For every $n \geqslant 0$, let $[n]$ denote the category associated to the linearly ordered set $\{0, \ldots, n\}$, and consider the functor
>
> $$F_n\colon \mathcal{C} \longrightarrow \mathbf{Set}$$
>
> defined by
>
> $$F_n(C) = \mathrm{Hom}_{\mathbf{Cat}}([n], F(C)).$$
>
> By assumption, F_0 and F_1 are representable by objects $X_0, X_1 \in \mathcal{C}$. Since $\mathcal{C}$ is stable under finite limits, it follows that
>
> $$F_n = F_1 \times_{F_0} \cdots \times_{F_0} F_1 \qquad \text{with } n \text{ factors}$$
>
> is representable by an object
>
> $$X_n = X_1 \times_{X_0} \cdots \times_{X_0} X_1.$$
>
> The objects X_n assemble into a simplicial object $X_\bullet$ of $\mathcal{C}$. We can think of this construction as a generalization of the process which associates to every groupoid D its nerve $N(D)$

(a simplicial set). Moreover, as in the classical case, the association $F \mapsto X_\bullet$ is fully faithful. In other words, we can identify groupoid objects of $\mathcal{C}$ with the corresponding simplicial objects.

Of course, not every simplicial object $X_\bullet$ of $\mathcal{C}$ arises via this construction. This is true if and only if certain additional conditions are met; for instance, the diagram

$$\begin{array}{ccc} X_2 & \xrightarrow{d_2} & X_1 \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ X_1 & \xrightarrow[d_1]{} & X_0 \end{array}$$

must be Cartesian.

We now describe a similar construction in ∞-categories.

Definition A.27. [Lur09, Definition 6.1.2.7] *A* **groupoid object in an ∞-category** $\mathcal{C}$ *is a simplicial object* $V_\bullet : \mathrm{N}(\boldsymbol{\Delta})^{\mathrm{op}} \to \mathcal{C}$ *satisfying any of eight equivalent conditions listed in* [Lur09, Proposition 6.1.2.6].

Of Lurie's conditions, the last and easiest to verify is the following. For every $n \geqslant 0$ and every partition $[n] = S \cup S'$ such that $S \cap S'$ consists of a single element s, the diagram

$$\begin{array}{ccc} V_n = V([n]) & \longrightarrow & V(S) = V_{|S|-1} \\ \downarrow & & \downarrow \\ V_{|S'|-1} = V(S') & \longrightarrow & V(\{s\}) = V_0 \end{array}$$

is a pullback square in $\mathcal{C}$, where the morphisms are induced by the evident inclusions of subsets of $[n]$.

Let $\boldsymbol{\Delta}^{\leqslant n} \subseteq \boldsymbol{\Delta}$ and $\boldsymbol{\Delta}_+^{\leqslant n} \subseteq \boldsymbol{\Delta}_+$ denote the full subcategories of Definition A.16. Hence $\boldsymbol{\Delta}_+^{\leqslant 0} \subseteq \boldsymbol{\Delta}_+$ and $\boldsymbol{\Delta}_+^{\leqslant 1} \subseteq \boldsymbol{\Delta}_+$ are the categories

$$[-1] \longrightarrow [0] \qquad \text{and} \qquad [-1] \longrightarrow [0] \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; [1].$$

$\mathrm{N}(\boldsymbol{\Delta}_+^{\leqslant 0})^{\mathrm{op}}$ is the simplicial 1-simplex $\boldsymbol{\Delta}^1$ of [Rav26, Definition 2.21], which is self-dual. This means a $\mathcal{C}$-valued functor V on it is equivalent to a morphism $u : V_0 \to V_{-1}$ in $\mathcal{C}$.

Proposition A.28. [Lur09, Proposition 6.1.2.11] *Let* $\mathcal{C}$ *be an ∞-category and let*

$$V_\bullet : \mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}} \to \mathcal{C}$$

be an augmented simplicial object of $\mathcal{C}$ *as in Definition A.1.*

The following conditions are equivalent:

(i) The augmented simplicial object $V_\bullet$ *is a right Kan extension of its restriction to* $\mathrm{N}(\boldsymbol{\Delta}_+^{\leqslant 0})^{\mathrm{op}}$*. The latter is determined by the morphism* $u : V_0 \to V_{-1}$.

(ii) The underlying simplicial object $V_\bullet$ is a groupoid object of $\mathcal{C}$ as in Definition A.27, and the diagram

$$V_\bullet|\mathrm{N}(\mathbf{\Delta}_+^{\leqslant 1})^{\mathrm{op}}$$

has a subdiagram

$$\begin{array}{ccc} V_1 & \xrightarrow{d_1} & V_0 \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle u} \\ V_0 & \xrightarrow{u} & V_{-1} \end{array}$$

which is a pullback square in $\mathcal{C}$, where the d_is are the face maps of [Rav26, Definition 2.19].

This means that in the homotopy category $h\mathcal{C}$ we have a groupoid object as in Definition A.26 in which $X_n = V_n$ for each $n \geqslant 0$.

Definition A.29. *An augmented simplicial object*

$$V_\bullet \in \mathcal{C}_{\mathbf{\Delta}_+}$$

for an ∞-category $\mathcal{C}$ is a **Čech nerve** *if it satisfies the equivalent conditions of Proposition A.28. In this case, $V_\bullet$ is determined up to equivalence by the map*

$$u\colon V_0 \to V_{-1},$$

and we will also say that $V_\bullet$ is the **Čech nerve of** *u.*

Example A.30. The classical Čech nerve. [Ale27] *Let*

$$\mathscr{U} = \{U_\alpha \subseteq X\},$$

be an open covering of a space X. Let $V_\bullet$ be the simplicial space with

$$\begin{aligned} V_n &:= \coprod_{\alpha_0,\dots,\alpha_n} U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \\ &= V_0 \times_X V_0 \times_X \cdots \times_X V_0 \qquad \text{with } n+1 \text{ factors.} \end{aligned}$$

The nondegenerate subspace $V_n' \subseteq V_n$ for $n > 0$ is the disjoint union of $(n+1)$-fold intersections in which no two adjacent indices α_i are the same.

The relevant map u for Definition A.29 is

$$V_0 := \coprod_\alpha U_\alpha \to X =: V_{-1}.$$

A.3. **Sheaves and presheaves.** Recall that a presheaf $\mathcal{F}$ on an ordinary category $\mathcal{C}$ is a contravariant functor with values in the category of sets or some variant of it. It is a sheaf if it converts certain colimits in the domain category to limits (or homotopy limits) in the codomain.

Example A.31. Sheaves on the poset category of open subsets. *Suppose the domain of the presheaf $\mathcal{F}$ is the poset category $\mathcal{U}(X)$ of open subsets of a topological space X. Then the colimit (i.e., pushout) of the diagram*

$$U_i \leftarrow U_i \cap U_j \to U_j$$

is $U_i \cup U_j$*, and the sheaf condition is that* $\mathcal{F}(U_i \cup U_j)$ *is the limit (pullback) of*

$$\mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_j) \leftarrow \mathcal{F}(U_j).$$

Equivalently $\mathcal{F}(X)$ *is the equalizer of*

$$\mathcal{F}(X) \longrightarrow \prod_{\alpha} \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

whenever $\{U_\alpha\}$ *is an open cover of* X*.* $\mathcal{F}(X)$ *is also the limit of the cosimplicial diagram* $\mathcal{F}(V_\bullet)$ *for* $V_\bullet$ *as in Example A.30.*

We will look at this again in Example A.48.

The following is the subject of [Lur09, §5.1].

Definition A.32. *For a small* ∞*-category* $\mathcal{C}$ *the* **∞-category of presheaves** $\mathcal{P}(\mathcal{C})$ *is that of space valued contravariant functors* $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$*.*

Definition A.33. *For a small* ∞*-category* $\mathcal{C}$*, the* **Yoneda embedding** *is the functor*

$$よ : \mathcal{C} \to \mathcal{P}(\mathcal{C})$$

that sends an object C *to the functor* $\mathrm{Map}_{\mathcal{C}}(-, C)$*. We call* $よ(C)$ *the* **Yoneda presheaf** *of the object* C*.*

The symbol よ above is the Japanese hiragana character "yo," the first syllable of Yoneda's name.

The ∞-categorical definition of a sheaf is more complicated than the one in ordinary category theory. Before giving it in Definition A.47, we must introduce ∞-topoi in Appendix A.4 and Grothendieck topologies in Appendix A.5.

A.4. **∞-topoi.** In classical category theory a topos is a category that "looks like" the category of sets. A good reference is the book [MLM94] by Moerdijk and Saunders Mac Lane. (You may notice that we are spelling Mac Lane's name in two different ways. This is because he did so himself in [EM45] and [MLM94].) See also the disussion in [Lur09, §6.1].

For any category $\mathcal{C}$ one has the presheaf category $\mathcal{P}(\mathcal{C})$ of contravariant Set-valued functors on $\mathcal{C}$. It is a short step away from being a topos, and every topos is derived from a presheaf category by a certain construction. Such constructions are related to Grothendieck topologies; see Definition A.41.

In [Lur09, page xii] Lurie says

> Roughly speaking, an ∞-topos is an ∞-category which "looks like" the ∞-category of [spaces $\mathcal{S}$]. We will show that this intuition is justified in the sense that it is possible to reconstruct a large portion of classical homotopy theory inside an arbitrary ∞-topos. In other words, an ∞-topos is a world in which one can "do" homotopy theory (much as an ordinary topos can be regarded as a world in which one can "do" other types of mathematics).

Definition A.34. [Lur09, Definition 6.1.0.4] *An ∞-category $\mathcal{X}$ is an* **∞-topos** *if there exists a small ∞-category $\mathcal{C}$ with $\mathcal{S}$-valued presheaf category $\mathcal{P}(\mathcal{C})$ as in Definition A.32, and an accessible left exact localization functor (see Definitions A.13, A.14 and A.23) $\mathcal{P}(\mathcal{C}) \to \mathcal{X}$.*

It is **quasi-compact** [Lur18b, Definition A.2.0.12] *if every covering of $\mathcal{X}$ has a finite subcovering: that is, for every effective epimorphism $\coprod_{i\in I} U_i \to 1$ in $\mathcal{X}$ (where 1 is the final object of $\mathcal{X}$), there exists a finite subset $I_0 \subseteq I$ such that the map*

$$\coprod_{i\in I_0} U_i \longrightarrow 1$$

is also an effective epimorphism. We say that an object $X \in \mathcal{X}$ is quasi-compact if the ∞-topos $\mathcal{X}_{/X}$ is quasi-compact.

In Lurie's words,

> the class of ∞-topoi is defined to be the smallest collection of ∞-categories which contains $\mathcal{S}$ and is stable under certain constructions (left exact localizations and the formation of functor categories).

The terminology in the following is explained in [Lur09, §6.1].

Theorem A.35. [Lur09, Theorem 6.1.0.6] *Let $\mathcal{X}$ be an ∞-category. The following conditions are equivalent:*

(i) The ∞-category $\mathcal{X}$ is an ∞-topos.

(ii) The ∞-category $\mathcal{X}$ is presentable as in Definition A.11, and for every small simplicial set K and every natural transformation $\alpha : p \to q$ of diagrams $p, q : K^{\triangleright} \to \mathcal{X}$, the following condition is satisfied: If q is a colimit diagram and $\alpha|_K$ is a Cartesian transformation, then p is a colimit diagram if and only if α is a Cartesian transformation.

(iii) The ∞-category $\mathcal{X}$ satisfies the following ∞-categorical analogues of Giraud's axioms:
   - *(a) $\mathcal{X}$ is presentable.*
   - *(b) Colimits in $\mathcal{X}$ are universal.*
   - *(c) Coproducts in $\mathcal{X}$ are disjoint.*
   - *(d) Every groupoid object of $\mathcal{X}$ is effective.*

Our next job is to define effective epimorphisms in an ∞-topos, which we will do in Definition A.39. The following discussion from [Lur09, §6.1.1] about effective epimorphisms in an ordinary category is helpful.

> Recall that if X is an object in an (ordinary) category $\mathcal{C}$, then an **equivalence relation** on X is an object R of $\mathcal{C}$ equipped with a map
>
> $$p \colon R \longrightarrow X \times X$$
>
> such that for any object S, the induced map
>
> $$\mathrm{Hom}_{\mathcal{C}}(S, R) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(S, X) \times \mathrm{Hom}_{\mathcal{C}}(S, X)$$

exhibits $\mathrm{Hom}_{\mathcal{C}}(S,R)$ as an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(S,X)$.

If $\mathcal{C}$ admits finite limits, then it is easy to construct equivalence relations in $\mathcal{C}$: given any map $f\colon X \to Y$ in $\mathcal{C}$, the induced map

$$X \times_Y X \longrightarrow X \times X$$

is an equivalence relation on X.

If the category $\mathcal{C}$ admits finite colimits, then one can attempt to invert this process: given an equivalence relation R on X, one can form the coequalizer of the two projections $R \rightrightarrows X$ to obtain an object which we denote by X/R. In the category of sets, one can recover R as the fiber product $X \times_{X/R} X$. In general, this need not occur: one always has

$$R \subseteq X \times_{X/R} X,$$

but the inclusion may be strict (as subobjects of $X \times X$). If equality holds, then R is said to be an **effective equivalence relation**, and the map $X \to X/R$ is said to be an **effective epimorphism**.

Definition A.36. *A* **simplicial resolution in an ∞-topos** $\mathcal{X}$ *is an augmented simplicial object (as in Definition A.1)*

$$U_\bullet^+ : N(\mathbf{\Delta}_+)^{op} \longrightarrow \mathcal{X}$$

in which U_{-1}^+ is a colimit of its underlying simplicial object

$$U_\bullet = U_\bullet^+\big|_{N(\mathbf{\Delta})^{op}}.$$

Example A.37. *For a surjection of sets $u : V_0 \to V_{-1}$, in the Čech nerve $V_\bullet$ (see Example A.30) we have*

$$\begin{aligned} V_n &= V_0 \times_{V_{-1}} \times \cdots \times_{V_{-1}} V_0 \qquad \textit{with } n+1 \textit{ factors} \\ &= \left\{(v_0,\dots,v_n) \in V_0^{n+1} : u(v_0) = \cdots = u(v_n)\right\} \\ &= \coprod_{x \in V_{-1}} (u^{-1}(x))^{n+1}. \end{aligned}$$

This is a simplicial resolution of V_{-1}.

This should not be confused with the simplicial object of Definition A.27, which is a groupoid object and is controlled by two different maps $V_1 \to V_0$.

Proposition A.38. [Lur09, Corollary 6.2.3.5] *Let $f : U \to X$ be a morphism in an ∞-topos $\mathcal{X}$. The following conditions are equivalent:*

(i) Viewing f as an object of the ∞-category $\mathcal{X}_{/X}$, the truncation $\tau_{\leqslant -1}(f)$ is a final object of $\mathcal{X}_{/X}$.

(ii) The Čech nerve $\check{C}(f)$ of Definition A.29 is a simplicial resolution of X.

Definition A.39. *An* **effective epimorphism** *is a morphism in an ∞-topos satisfying the conditions of Proposition A.38. An* **effective monomorphism** *is one that is (-1)-truncated as in Definition A.5.*

Note that any morphism to a final object satisfies these conditions and is therefore an effective epimorphism.

In the category Set, which is the simplest example of an ∞-topos, every morphism $f : U \to X$ is 0-truncated as in Definition A.5. Its (-1)-truncation, in which the preimage of each element of X is replaced by a singleton if it is nonempty, is the inclusion of its image into X. The map f is onto iff this is the identity map and a final object in $\mathrm{Set}_{/X}$, hence condition (i) above. The identification of Čech nerve of such an f as a simplicial resolution is the subject of Example A.37, hence condition (ii).

A.5. Grothendieck topologies and ∞-sheaves.

Definition A.40. Sieves. [Lur09, Definition 6.2.2.1] *A* **sieve on an ∞-category** $\mathcal{C}$ *is a full subcategory $\mathcal{C}^{(0)} \subseteq \mathcal{C}$ having the property that if $f : C \to D$ is a morphism in $\mathcal{C}$ and D belongs to $\mathcal{C}^{(0)}$, then C also belongs to $\mathcal{C}^{(0)}$. In other words the sub-∞-category $\mathcal{C}^{(0)}$ is closed under precomposition with morphisms in $\mathcal{C}$.*

A **sieve on an object** $C \in \mathcal{C}$ *is a sieve on the ∞-category $\mathcal{C}_{/C}$ of morphisms to C. Given a morphism $f : D \to C$ and a sieve $\mathcal{C}^{(0)}_{/C}$ on C, we let $f^*\mathcal{C}^{(0)}_{/C}$ denote the unique sieve on D such that $f^*\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/D}$ and $\mathcal{C}^{(0)}_{/C}$ determine the same sieve on $\mathcal{C}_{/f}$.*

Hence a sieve $\mathcal{C}^{(0)}$ is the full subcategory spanned by some collection of objects along with all objects mapping to them. A sieve $\mathcal{C}^{(0)}_{/C}$ on an object C consists of a collection of morphisms to C (which need not include the identity) along with all morphisms to their domains.

Observe that if $F : \mathcal{C} \to \mathcal{D}$ is a functor between ∞-categories and $\mathcal{D}^{(0)} \subseteq \mathcal{D}$ is a sieve on $\mathcal{D}$, then

$$F^{-1}\mathcal{D}^{(0)} = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}^{(0)}$$

is a sieve on $\mathcal{C}$. Moreover, if F is an equivalence, then F^{-1} induces a bijection between sieves on $\mathcal{D}$ and sieves on $\mathcal{C}$.

Definition A.41. [Lur09, more of Definition 6.2.2.1] *A* **Grothendieck topology** τ *on an ∞-category $\mathcal{C}$ consists of a specification, for each object C of $\mathcal{C}$, of a collection of sieves on C which we refer to as* **covering sieves**. *These collections are required to satisfy the following properties:*

(i) For every object C of $\mathcal{C}$, the sieve $\mathcal{C}_{/C}$ is a covering sieve.

(ii) If $f : C \to D$ is a morphism in $\mathcal{C}$ and $\mathcal{C}^{(0)}_{/D}$ is a covering sieve on D, then $f^*\mathcal{C}^{(0)}_{/D}$ is a covering sieve on C.

(iii) Let C be an object of $\mathcal{C}$, let $\mathcal{C}^{(0)}_{/C}$ be a covering sieve on C, and let $\mathcal{C}^{(1)}_{/C}$ be an arbitrary sieve on C. Suppose that for each morphism $f : D \to C$ belonging to $\mathcal{C}^{(0)}_{/C}$, the pullback $f^*\mathcal{C}^{(1)}_{/C}$ is a covering sieve on D. Then $\mathcal{C}^{(1)}_{/C}$ is a covering sieve on C.

Such a topology is **finitary** [Lur18b, Definition A.3.1.1] *if for every object $C \in \mathcal{C}$ and every covering sieve $\mathcal{C}^{(0)}_{/C}$ on C, there exists a finite collection of morphisms*

$$\{ C_i \to C \mid 1 \leqslant i \leqslant n \} \subseteq \mathcal{C}^{(0)}_{/C}$$

which generate a covering sieve of C (in other words, the smallest sieve $\mathcal{C}^{(1)}_{/C}$ containing each $C_i \to C$ is also a covering sieve on C).

An ∞**-site** $(\mathcal{C}, \tau)$ *is an ∞-category equipped with a Grothendieck topology.*

Remark A.42. *One can define a Grothendieck topology τ on an ordinary category $\mathcal{C}$ in the same way. Such a pair $(\mathcal{C}, \tau)$ is a called a* **site**.

Example A.43. *Any ∞-category $\mathcal{C}$ may be equipped with the* **trivial Grothendieck topology** *in which a sieve $\mathcal{C}^{(0)}_{/C}$ on an object C of $\mathcal{C}$ is covering if and only if*

$$\mathcal{C}^{(0)}_{/C} = \mathcal{C}_{/C}.$$

In other words, the covering sieve for each object C consists of all *morphisms to it.*

A presheaf $\mathcal{F} \in \mathcal{P}(\mathcal{C})$ defines a collection of objects on which it is nonempty, and we denote the corresponding sieve by $\mathcal{C}^{(0)}(\mathcal{F})$. Conversely, given a sieve $\mathcal{C}^{(0)} \subseteq \mathcal{C}$, there is a unique map of simplicial sets $f : \mathcal{C} \to \Delta^1$ such that $\mathcal{C}^{(0)}$ is the preimage of $\{0\}$. This construction determines a bijection between sieves on $\mathcal{C}$ and functors $f : \mathcal{C} \to \Delta^1$, and we may identify Δ^1 with the full subcategory of $\mathcal{S}^{op}$ spanned by the objects $\varnothing$ and Δ^0 in the category of Kan complexes. Since every (-1)-truncated Kan complex is equivalent to either $\varnothing$ or Δ^0, we conclude:

Lemma A.44. [Lur09, Lemma 6.2.2.4] *For every small ∞-category $\mathcal{C}$, the construction*

$$\mathcal{F} \longmapsto \mathcal{C}^{(0)}(\mathcal{F})$$

determines a bijection between the set of equivalence classes of (-1)-truncated objects (Definition A.4) of $\mathcal{P}(\mathcal{C})$ and the set of all sieves on $\mathcal{C}$.

Remark A.45. *The (-1)-truncation of a presheaf $\mathcal{F} \in \mathcal{P}(\mathcal{C})$ is defined by*

$$(\tau_{\leqslant -1}\mathcal{F})(C) = \begin{cases} \varnothing & \text{for } \mathcal{F}(C) := \varnothing \\ \Delta^0 & \text{otherwise.} \end{cases}$$

Following [Lur09, §6.2.2], we now introduce a relative version of the above construction.

Let $C \in \mathcal{C}$ be an object and let $i : \mathcal{F}_C \to \text{よ}(C)$ be a monomorphism in $\mathcal{P}(\mathcal{C})$, meaning a natural transformation of presheaves inducing a monomorphism (i.e., (-1)-truncated map) of Kan complexes $\mathcal{F}_C(D) \to \mathrm{Map}_{\mathcal{C}}(D, C)$ for each object $D \in \mathcal{C}$. We will replace the the set of (-1)-truncated presheaves in Lemma A.44 by the set of such monomorphisms. This means we need a replacement for the set of sieves on the right.

Let $\mathcal{C}_{/C}(\mathcal{F}_C)$ denote the full subcategory of $\mathcal{C}_{/C}$ spanned by those objects $f : D \to C$ of $\mathcal{C}_{/C}$ such that there exists a commutative triangle

$$\begin{array}{ccc} & \mathcal{F}_C & \\ \nearrow & & \searrow^{i} \\ \text{よ}(D) & \xrightarrow[\text{よ}(f)]{} & \text{よ}(C) \end{array}$$

in $\mathcal{P}(\mathcal{C})$. Applying the three functors in the diagram above to an object X in $\mathcal{C}$ gives a diagram of spaces

$$\begin{array}{ccc} & \mathcal{F}_C(X) & \\ \nearrow & & \searrow^{i_X} \\ \mathrm{Map}_{\mathcal{C}}(X, D) & \xrightarrow[f_*]{} & \mathrm{Map}_{\mathcal{C}}(X, C). \end{array}$$

The morphism f must be chosen to that the indicated lifting exists and is natural in X. As usual, both diagrams need commute only up to homotopy.

It is easy to see that $\mathcal{C}_{/C}(\mathcal{F}_C)$ is a sieve on C and that equivalent subobjects of $\text{よ}(C)$ lead to the same sieve.

Proposition A.46. [Lur09, Proposition 6.2.2.5] *Let $\mathcal{C}$ be a small ∞-category containing an object C. The construction described above yields a bijection*

$$(i : \mathcal{F}_C \to \text{よ}(C)) \longmapsto \mathcal{C}_{/C}(\mathcal{F}_C)$$

from the set of monomorphisms to $\text{よ}(C)$ to the set of sieves on C.

Definition A.47. ∞-sheaves. [Lur09, Definition 6.2.2.6] *Let $(\mathcal{C}, \tau)$ be an ∞-site and let S be the collection of all monomorphisms $i : \mathcal{F}_C \to \text{よ}(C)$ which correspond (as in Proposition A.46) to covering sieves*

$$\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$$

for all objects C in $\mathcal{C}$. An object $\mathcal{F} \in \mathcal{P}(\mathcal{C})$ (Definition A.32) is an **∞-sheaf** *or* **sheaf** *if it is S-local. We let $\mathrm{Shv}(\mathcal{C}, \tau)$ denote the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by the S-local objects. We will often omit τ from the notation.*

For a presentable category $\mathcal{D}$ (Definition A.11) we let

$$\mathrm{Shv}(\mathcal{C}; \mathcal{D}) := \mathrm{Shv}(\mathcal{C}) \otimes \mathcal{D} \in \mathrm{Pr}^{\mathrm{L}}$$

denote the category of $\mathcal{D}$-valued sheaves on $\mathcal{C}$. This can alternatively be defined as the full subcategory of presheaves that satisfy descent, namely the sheaf condition (see [Lur18b, Remark 1.3.1.6]*).*

Lurie omits the τ from his notation for this subcategory.

$\mathcal{F}$ is S-local if for each monomorphism $i : \mathcal{F}_C \to \text{よ}(C)$ corresponding to a covering sieve, the map

$$i^* : \mathrm{Map}_{\mathcal{P}(\mathcal{C})}(\text{よ}(C), \mathcal{F}) \to \mathrm{Map}_{\mathcal{P}(\mathcal{C})}(\mathcal{F}_C, \mathcal{F})$$

is an equivalence.

Example A.48. Sheaves on the poset category of a topological space revisited. *We will show how this definition plays out in Example A.31. Let $\mathcal{C}$ be the ordinary category $\mathcal{U}(X)$. We give it the Grothendieck topology in which the covering sieves on U are those sieves $\{U_\alpha \subseteq U\}$ for which $U = \bigcup_\alpha U_\alpha$.*

The Yoneda presheaf $\text{よ}(U)$ of Definition A.33 assigns to each open subset U' its set of embeddings into U. This set is a singleton if $U' \subseteq U$ and the empty set otherwise, so $\text{よ}(U)$ is (-1)-truncated.

There is a presheaf monomorphism $i : \mathcal{F}_U \to \text{よ}(U)$ if $\mathcal{F}_U$ is supported by, and is a singleton on, a collection of open subsets $U_\alpha \subseteq U$ that is closed under inclusion. Hence each such presheaf $\mathcal{F}_U$ is also (-1)-truncated. $\mathcal{F}_U$ corresponds as in Proposition A.46 to a covering sieve if the union (colimit) of the U_α is U itself. This means that

$$\mathcal{F}_U = \operatorname*{colim}_\alpha \text{よ}(U_\alpha).$$

Let S be the set of all such inclusions $\mathcal{F}_U \to \text{よ}(U)$ for all open subsets U. What does it mean for a presheaf $\mathcal{F}$ to be S-local? *It means that for each such i, the map*

$$i^* : \mathrm{Map}_{\mathcal{P}(\mathcal{U}(X))}(\text{よ}(U), \mathcal{F}) \to \mathrm{Map}_{\mathcal{P}(\mathcal{U}(X))}(\mathcal{F}_U, \mathcal{F})$$

is an equivalence, meaning a bijection of sets. The Yoneda lemma identifies the domain with $\mathcal{F}(U)$, since a morphism in the presheaf category is a natural transformation of Set*-valued functors on $\mathcal{U}(X)$. Hence the set on the right must be the same. Thus we must have*

$$\begin{aligned}
\mathcal{F}(\operatorname*{colim}_\alpha U_\alpha) &= \mathcal{F}(U) = \mathrm{Map}_{\mathcal{P}(\mathcal{U}(X))}(\mathcal{F}_U, \mathcal{F}) \\
&= \mathrm{Map}_{\mathcal{P}(\mathcal{U}(X))}(\operatorname*{colim}_\alpha \text{よ}(U_\alpha), \mathcal{F}) \\
&= \lim_\alpha \mathrm{Map}_{\mathcal{P}(\mathcal{U}(X))}(\text{よ}(U_\alpha), \mathcal{F}) \\
&= \lim_\alpha \mathcal{F}(U_\alpha) \qquad \textit{also by the Yoneda lemma.}
\end{aligned}$$

Thus being S-local means the presheaf $\mathcal{F}$ converts colimits of open subsets (such as pushouts) to limits. This is the classical sheaf condition.

Proposition A.49. Topological localizations and Grothendieck topologies. [Lur09, Proposition 6.2.2.17] *Let $\mathcal{C}$ be a small ∞-category. Then Grothendieck topologies on $\mathcal{C}$ are in bijective correspondence with equivalence classes of topological localizations (Definition A.25) of the presheaf ∞-category $\mathcal{P}(\mathcal{C})$.*

The following takes Lurie four pages to prove.

Proposition A.50. A source of ∞-topoi. [Lur09, Proposition 6.2.2.7] *For an ∞-site $(\mathcal{C}, \tau)$, $\mathrm{Shv}(\mathcal{C}, \tau)$ is a topological localization (Definition A.25) of $\mathcal{P}(\mathcal{C})$. In particular, $\mathrm{Shv}(\mathcal{C}, \tau)$ is an ∞-topos.*

Definition A.51. [Lur09, Definition 6.3.1.1] *Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi. A* **geometric morphism** *from $\mathcal{X}$ to $\mathcal{Y}$ is a functor*

$$F_* : \mathcal{X} \to \mathcal{Y}$$

which admits a left adjoint F^* that is left exact as in Definition A.14.

Such functors are studied in [Lur09, §6.3.1], where they are typically denoted by f_* and f^*.

Proposition A.52. [Lur09, Proposition 6.3.1.9]. *Let*

$$F_* : \mathcal{X} \to \mathcal{Y}$$

be a geometric morphism between ∞-topoi having a left adjoint

$$F^* : \mathcal{Y} \to \mathcal{X}.$$

Then F^* and F_* carry m-truncated objects (Definition A.4) to m-truncated objects and m-truncated morphisms to m-truncated morphisms, for any integer $m \geqslant -2$. Moreover, there is a canonical equivalence of functors

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{F^*} & \mathcal{X} \\ {\scriptstyle \tau_{\leqslant m}}\downarrow & & \downarrow{\scriptstyle \tau_{\leqslant m}} \\ \tau_{\leqslant m}\mathcal{Y} & \xrightarrow{F^*} & \tau_{\leqslant m}\mathcal{X} \end{array}$$

where $\tau_{\leqslant m}$ is the truncation functor of Definition A.4.

A.6. All things hyper: hypercompletion, hypercoverings, hyperdescent and hypersheaves.

Definition A.53. [Lur09, §6.5] *An ∞-topos $\mathcal{X}$ (Definition A.34) is* **hypercomplete** *if every ∞-connective morphism (Definition A.4) in it is an equivalence.*

Recall that a morphism $f : X \to Y$ in $\mathcal{X}$ is ∞-connective iff for each object W, the induced map

$$f_* : \mathrm{Map}_{\mathcal{X}}(W, X) \to \mathrm{Map}_{\mathcal{X}}(W, Y)$$

is a weak equivalence of Kan complexes. Thus each such map has an inverse, but in general it need not be induced by a map from Y to X. Hence the condition above is nontrivial.

Definition A.54. [Lur09, §6.5] *Let S be the set of ∞-connective morphisms in an ∞-topos $\mathcal{X}$. An object Z is* **hypercomplete** *if it is S-***local** *as in Definition A.24. The* **hypercompletion** *$\widehat{\mathcal{X}} \subseteq \mathcal{X}$ (which Lurie denotes by $\mathcal{X}^\wedge$) is the full subcategory of such objects. A morphism $f : X \to Y$ is* **hypercomplete** *if it is hypercomplete as in object in the ∞-topos $\mathcal{X}_{/Y}$.*

$\widehat{\mathcal{X}}$ is known to be hypercomplete in the sense of Definition A.53 by [Lur09, Lemma 6.5.2.12]. It is known to contain the full subcategory $\tau_{\leqslant m}\mathcal{X}$ of m-truncated objects (as in Definition A.4) by [Lur09, Lemma 6.5.2.9]. The ∞-category of functors to it from a hypercomplete ∞-topos $\mathcal{Y}$ is known to be isomorphic to that of functors from $\mathcal{Y}$ to $\mathcal{X}$ itself by [Lur09, Proposition 6.5.2.13].

Definition A.55. [Lur09, Definition 6.5.3.2] *and* [AM69, Definition (8.4)]. *Let $\mathcal{X}$ be an ∞-topos. A simplicial object $V_\bullet \in \mathcal{X}_{\mathbf{\Delta}}$ is a* **hypercovering** *(sometimes called a* **hypercover***) of $\mathcal{X}$ if for each $n > 0$, the unit map*

$$(\eta_{V_\bullet,n-1})_n : V_n \longrightarrow M_n(V) = \big(\operatorname{cosk}_{n-1} V_\bullet\big)_n \qquad \text{as in (A.20)}$$

is an effective epimorphism as in Definition A.39. Here $\operatorname{cosk}_{n-1}$ is the functor of Definition A.16, and the codomain is the nth matching object.

We say that $V_\bullet$ is an **effective hypercovering** *of $\mathcal{X}$ if the colimit of $V_\bullet$ is a final object of $\mathcal{X}$.*

An augmented simplicial object $V_\bullet \in \mathcal{X}_{\mathbf{\Delta}_+}$ with $V_{-1} = X$ is an **augmented hypercovering***, or a* **hypercover of** X *if the associated simplicial object is a hypercovering in $\mathcal{X}_{/X}$.*

For an object X in $\mathcal{X}$, the ∞-category $\mathcal{X}_{/X}$ is again an ∞-topos, by [Lur09, Proposition 6.3.5.1], in which the identity map on X is a final object. An effective hypercovering in $\mathcal{X}_{/X}$ is equivalent to a hypercovering in $\mathcal{X}$ with colimit X.

For each $n > 0$, the map $V_n \to V_0^{n+1}$ that sends an n-face to its $(n+1)$-tuple of 0-faces is an effective epimorphism as in Definition A.39.

The Čech nerve $V_\bullet$ of Example A.30 is known to be a hypercovering in which each map $\eta_{V,n} : V_n \to M_n(V)$ is an isomorphism; see [AM69, Enlightenment (8.5)(b)]. A finite skeleton of $V_\bullet$ is usually not a hypercovering.

Definition A.56. *A presheaf $\mathcal{F}$ on a site $(\mathcal{C}, \tau)$ satisfies* **hyperdescent** *if for every augmented hypercovering*

$$U_\bullet \longrightarrow X,$$

the canonical map

$$\lim_{\mathbf{\Delta}} \mathcal{F}(U_\bullet) \longrightarrow \mathcal{F}(X)$$

is an equivalence.

Theorem A.57. [Lur09, Theorem 6.5.3.12] *Let $\mathcal{X}$ be an ∞-topos. The following conditions are equivalent:*

(i) For every $X \in \mathcal{X}$, every hypercovering $U_\bullet$ of $\mathcal{X}_{/X}$ is effective.
(ii) The ∞-topos $\mathcal{X}$ is hypercomplete.

Definition A.58. *A* **hypersheaf** *on an ∞-site $(\mathcal{C}, \tau)$ with values in $\mathcal{S}$ is a sheaf as in Definition A.47 which is hypercomplete (as in Definition A.54) as an object in the presheaf category $\mathcal{P}(\mathcal{C})$. We denote the category of such* *by* $\mathrm{Shv}^{\mathrm{hyp}}(\mathcal{C})$*. The* **category of hypersheaves on $\mathcal{C}$ with values in a presentable ∞-category $\mathcal{D}$** *is*

$$\mathrm{Shv}^{\mathrm{hyp}}(\mathcal{C};\mathcal{D}) := \mathrm{Shv}^{\mathrm{hyp}}(\mathcal{C}) \otimes \mathcal{D} \in \mathrm{Pr}^{\mathrm{L}},$$

where the tensor product of presentable ∞-categories above is as in [Lur17, §4.8.1].

This can alternatively be defined as the full subcategory of presheaves that satisfy hyperdescent as in Definition A.56, the analogue of the sheaf condition for hypercoverings. This follows from [Lur09, Corollary 6.5.3.13] together with the formula for the Lurie tensor product [Lur17, Proposition 4.8.1.17].

Definition A.59. [BMCSY25, page 567] **Hypersheafification** $(-)^{\mathrm{hyp}}$ *is the left adjoint of the inclusion*

$$\mathrm{Shv}^{\mathrm{hyp}}(\mathcal{C};\mathcal{D}) \to \mathrm{Shv}(\mathcal{C};\mathcal{D})$$

for $\mathcal{C}$ and $\mathcal{D}$ as above.

Proposition A.60. Deligne completeness for hypersheaves. [BMCSY25, Proposition 5.1], [Lur18b, Theorem A.4.0.5], *and* [AGV72, Proposition VI.9.0]. *Let $\mathcal{X}$ be an ∞-topos which is locally coherent and hypercomplete. Then $\mathcal{X}$ has enough points. In other words, given a morphism $\alpha : X \to Y \in \mathcal{X}$ which is not an equivalence, there exists a geometric morphism $f^* : \mathcal{X} \to \mathcal{S}$ such that $f^*(\alpha)$ is not an equivalence.*

The following is proved in the discussion in [BMCSY25, pages 567–568].

Proposition A.61. Sheaves on continuous G-sets. *Let G be a profinite group and $\mathcal{C}$ a compactly generated presentable ∞-category. Then*

$$\mathrm{Shv}(\mathcal{F}in_G;\mathcal{C}) \simeq \operatorname*{colim}_{U \trianglelefteq G} \mathcal{C}^{B(G/U)} \in \mathrm{Pr}^{\mathrm{L}},$$

where U ranges over the open normal subgroups of G.

When $\mathcal{C} = \mathcal{S}$ we also have

$$\mathrm{Shv}(\mathcal{F}in_G;\mathcal{S}) \simeq \lim_{U} \mathcal{S}^{B(G/U)} \simeq \lim_{U} B(G/U).$$

This filtered limit is the ∞-category associated with a path connected space BG, which has a canonical basepoint leading to a **stalk** *given by*

$$\mathcal{F}_e := \operatorname*{colim}_{U \trianglelefteq G} \mathcal{F}(G/U). \tag{A.62}$$

Moreover each such space valued sheaf is a hypersheaf.

Definition A.63. [BMCSY25, Definition 5.2] *Let G be a profinite group, let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be ∞-semiadditive as in Definition 3.5, and let $\mathcal{R} \in \mathrm{CAlg}\big(\mathrm{Shv}(\mathcal{F}in_G;\mathcal{C})\big)$. We say that $\mathcal{R}$ is a* **continuous G-Galois extension of $\mathcal{R}(G/G)$** *if for every open normal subgroup $U \trianglelefteq G$, the object*

$$\mathcal{R}(G/U) \in \mathrm{CAlg}(\mathcal{C})^{B(G/U)}$$

is a faithful G-Galois extension of $\mathcal{R}(G/G)$.

Proposition A.64. [BMCSY25, Proposition 5.3] *Let $\mathcal{F}\colon \mathcal{F}in_G^{\mathrm{op}} \to \mathrm{CAlg}(\mathcal{C})$ be a finite product preserving functor such that $\mathcal{F}(G/U)$ is a faithful G/U-Galois extension of $\mathcal{F}(G/G)$ for every open normal subgroup $U \trianglelefteq G$. Then $\mathcal{F}$ satisfies the sheaf condition, and hence is a continuous G-Galois extension.*

Proposition A.65. [BMCSY25, Proposition 5.4] *Let G be a profinite group, let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be semiadditive, and let $\mathcal{R}$ be a continuous G–Galois extension. Then there is a symmetric monoidal equivalence*

$$\mathcal{C} \underset{(-)(G/G)}{\overset{-\otimes\mathcal{R}}{\underset{\perp}{\rightleftarrows}}} \mathrm{Mod}_{\mathcal{R}}(\mathrm{Shv}(\mathcal{F}in_G;\mathcal{C})).$$

Proposition A.66. [BMCSY25, Proposition 5.6] *Let G be a profinite group of finite virtual p-cohomological dimension, and let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be p-local. Let $\mathcal{R}$ be a continuous G-Galois extension with stalk $R := \mathcal{R}_e$ as in (A.62), and let L_R denote Bousfield localization with respect to R in $\mathcal{C}$. For every $M \in \mathrm{Mod}_{\mathcal{R}}(\mathrm{Shv}(\mathcal{F}in_G;\mathcal{C}))$, the presheaf L_RM is a hypersheaf, and the map*

$$M \longrightarrow L_RM$$

exhibits the target as the hypersheafification of the source.

Following [BMCSY25, Notation 5.7], for $X \in \mathrm{Shv}(\mathcal{F}in_G;\mathcal{C})$, we shall abuse notation and also denote the stalk e^*X by X, and for every open $U \leqslant G$, denote by X^{hU} the value of X at $G/U \in \mathcal{F}in_G$.

A.7. F-descent and F-covers.

Definition A.67. F-descent and F-covers. [LZ17, Definition 3.1.1] *and* [BHLS23, Definition 6.6]. *Let $\mathcal{C}$ be an ∞-category admitting pullbacks, $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ a functor, and $u\colon V_0 \to V_{-1}$ a morphism in $\mathcal{C}$. We say that u* **is of F-descent** *if*

$$F \circ (V_\bullet)^{\mathrm{op}}\colon \mathrm{N}(\boldsymbol{\Delta}_+) \longrightarrow \mathcal{D}$$

is a limit diagram in $\mathcal{D}$, where

$$V_\bullet\colon \mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}} \longrightarrow \mathcal{C}$$

is the Čech nerve of u as in Definition A.29. We say that u **is an F-cover** *if every pullback of u in $\mathcal{C}$ is of F-descent.*

We illustrate with three examples.

Example A.68. Ordinary sheaves on a topological space. *Let $\mathcal{C}$ be the poset category $\mathcal{U}(X)$ of open subsets of a topological space X as in Example A.31, in which pullbacks are intersections. Let u be the map of Example A.30 associated with an open covering $\{U_\alpha\}$ of X, and let F be an ordinary sheaf on $\mathcal{U}(X)$. Then u is of F-descent.*

The pullback of u along an open inclusion $A \to X$ is the corresponding map for the open cover of A by its intersections with the U_αs. This is also of F-descent, so u is of universal F-descent.

Example A.69. *The Čech nerve for a map $f : X \to Y$ of spaces or spectra is the simplicial object $V_\bullet$ given by*

$$
\begin{array}{ccccccc}
V_0 & & V_1 & & V_2 & & \\
\| & & \| & & \| & & \\
X & \leftleftarrows\!\!\rightarrow & X \times_Y X & \overset{\leftarrow}{\underset{\leftarrow}{\rightleftarrows}} & X \times_Y X \times_Y X & \cdots & \cdots \\
x_0 & \longmapsto & (x_0, x_0) & & & & \\
x_0 & \longleftarrow & & \longrightarrow & (x_0, x_0, x_1) & & \\
 & & (x_0, x_1) & & & & \\
x_1 & \longleftarrow & & \longrightarrow & (x_0, x_1, x_1) & & \\
 & & (x_1, x_2) & \longleftarrow & & & \\
 & & (x_0, x_2) & \longleftarrow & (x_0, x_1, x_2), & & \\
 & & (x_0, x_1) & \longleftarrow & & &
\end{array}
$$

where the coordinates $x_i \in X$ all have the same image under f.

Applying a contravariant functor F leads to maps

$$F(Y) \to \lim_{\boldsymbol{\Delta}} F(V_\bullet) \to F(X), \tag{A.70}$$

whose composite is $F(f)$, where the limit is the totalization of the cosimplicial object $F(V_\bullet)$. The first map is an equivalence if f is of F-descent. If the same is true for any pullback of f, then f is an F-cover.

Example A.71. Cyclotomically completed algebraic K-theory. *If the functor of interest (such as algebraic K-theory) is covariant, then we have to start in the opposite category of its domain. Let*

$$\mathcal{C} = \mathrm{CAlg}\Big(\mathrm{Sp}_{T(n)}\Big)^{\mathrm{op}}.$$

Pushouts in $\mathrm{CAlg}\Big(\mathrm{Sp}_{T(n)}\Big)$ are smash products over $\mathbb{S}_{T(n)}$, so the opposite category has pullbacks as required in Definition A.67.

Let F be the functor $\mathrm{K}_{\mathrm{Cyclo}(n+1)}$ of Definition 2.4 (which has values in $\mathrm{Sp}_{T(n+1)}$), and let f be opposite of the map

$$\mathbb{S}_{T(n)} \longrightarrow \mathbb{S}_{T(n)}[\omega^{(n)}_{p^\infty}]$$

Then (A.70) becomes

$$\mathrm{K}_{\mathrm{Cyclo}(n+1)}(\mathbb{S}_{T(n)}) \to \lim_{\boldsymbol{\Delta}} \mathrm{K}_{\mathrm{Cyclo}(n+1)}(V_\bullet) \to \mathrm{K}_{\mathrm{Cyclo}(n+1)}(\mathbb{S}_{T(n)}[\omega^{(n)}_{p^\infty}])$$

where $V_\bullet$ is a cosimplicial spectrum built from $\mathbb{S}_{T(n)}[\omega^{(n)}_{p^\infty}]$. The first map is an equivalence by Corollary 4.22. Proposition 6.5 is a similar statement about the map $f_k \colon R^{h(p^k\mathbb{Z})} \longrightarrow R$, where $R = L_{T(n)}BP\langle n\rangle$ equipped with the $\mathbb{Z}$-action of Theorem 5.9.

A.8. **Finite sets acted on by a profinite group.**

Definition A.72. The Grothendieck site of continuous finite G-sets [MLM94, §III.9] *and* [CM21, Definition 4.1]. *For a profinite group G, $\mathcal{F}in_G$ (denoted by $\mathcal{T}_G$ in* [CM21] *and by $\mathscr{T}_G$ in* [BMCSY25]*) is the Grothendieck site defined as follows:*

(1) The underlying category of $\mathcal{F}in_G$ is the category of continuous finite G-sets. Continuity means that each point is fixed by an open (meaning of finite index) subgroup of G.

(2) A family of maps $\{S_i \to S\}_{i\in I}$ forms a covering sieve if it is jointly surjective.

Given an ∞-category $\mathcal{D}$ with all limits, we let $\mathrm{Sh}(\mathcal{F}in_G, \mathcal{D})$ denote the ∞-category of $\mathcal{D}$-valued sheaves on $\mathcal{F}in_G$, as usual. We also write $\mathcal{P}_{\amalg}(\mathcal{F}in_G, \mathcal{D})$ for the ∞-category of presheaves on $\mathcal{F}in_G$ with values in $\mathcal{D}$ which carry finite coproducts in $\mathcal{F}in_G$ to finite products; equivalently, these are $\mathcal{D}$-valued presheaves on the orbit category of G.

If $\mathcal{D}$ is also presentable as in Definition A.11, we define the **stalk** *of a sheaf F by*

$$F_e := \ \mathrm{colim}_{H\subseteq G} F(G/H),$$

where the colimit is over all open subgroups H.

This Grothendieck topology is finitary as in Definition A.41. The category of sheaves of sets on $\mathcal{F}in_G$ is the category of continuous (discrete) G-sets.

Proposition A.73. The sheaf condition for presehaves on $\mathcal{F}in_G$. [Lur18b, Prop. A.3.3.1]. *Let G be a profinite group, and let $\mathcal{F}in_G$ be the site as above. A presheaf F on $\mathcal{F}in_G$ with values in an ∞-category $\mathcal{D}$ with limits is a sheaf if and only if:*

(1) For $X, Y \in \mathcal{F}in_G$, the natural map induces an equivalence

$$F(X \sqcup Y) \simeq F(X) \times F(Y).$$

That is, $F \in \mathcal{P}_{\amalg}(\mathcal{F}in_G, \mathcal{D})$ is coproduct-preserving.

(2) For every surjective map of G-sets $u : T \twoheadrightarrow S$, $F(X)$ is the limit of the Čech nerve of u as in Definition A.29.

Details of the following can be found in [CM21, §4.1]. For a finite group G, $\mathcal{F}in_G$ is the category of finite G-sets. A $\mathcal{D}$-valued sheaf (for $\mathcal{D}$ as above) on it is equivalent to $\mathcal{D}$-valued functor on BG, which amounts to an object in $\mathcal{D}$ equipped with a G-action.

A profinite group G is the limit of its finite quotient groups G/N for open normal subgroups N. Its $\mathcal{D}$-valued sheaf category is

$$\mathrm{Sh}(\mathcal{F}in_G, \mathcal{D}) \simeq \lim_{N\subseteq G} \mathrm{Fun}(B(G/N), \mathcal{D}).$$

where for $N' \leqslant N$, the functor $\mathrm{Fun}\big(B(G/N'), \mathcal{D}\big) \longrightarrow \mathrm{Fun}\big(B(G/N), \mathcal{D}\big)$ is given by $(\cdot)^{h(N/N')}$.

Proposition A.74. Sheafification. [CM21, Proposition 4.12] *Let $\mathcal{D}$ be a presentable ∞-category (Definition A.11) and let G be a profinite group. Suppose that for every open normal subgroup $N \leqslant G$ and each subgroup $K \leqslant G/N$, the limit functor*

$$(\,\cdot\,)^{hK} : \mathrm{Fun}(BK, \mathcal{D}) \longrightarrow \mathcal{D}$$

commutes with filtered colimits of K-objects in $\mathcal{D}$. Let $F \in \mathcal{P}_{\amalg}(\mathcal{F}in_G, \mathcal{D})$ be a product-preserving presheaf on $\mathcal{F}in_G$. Then the sheafification F^{sh} of F is given by the formula

$$F^{\mathrm{sh}}(G/H) = \operatorname*{colim}_{H' \leqslant H} F(G/H')^{h(H/H')},$$

as $H' \leqslant H$ ranges over all open normal subgroups.

Appendix B. Operads

B.1. **The early work of May and Stasheff.** Operads are originally defined by May in [May72]. Light introductions to the topic are given by Eva Belmont [Bel17] and by Jim Stasheff [Sta04]. More comprehensive treatments are given by Martin Markl, Steven Shnider and Stasheff in [MSS02], by Murray Bremner and Vladimir Dotsenko in [BD20], and by Jean-Louis Loday and Bruno Vallette in [LV12].

Definition B.1. [May72, Definitions 1.1 and 3.12]

(i) *A* **non-Σ (or nonsymmetric) operad** $(\mathcal{O}, \gamma)$ *is a collection of spaces $\{\mathcal{O}(j) : j \geqslant 0\}$ where $\mathcal{O}(0)$ is a single point, $\mathcal{O}(1)$ has a special point $1_{\mathcal{O}}$ related to the identity map, and for each $j \geqslant 0$ and each j-tuple $(k_1, \ldots, k_j)$ of nonnegative integers, there are structure maps*

$$\gamma_{(k_1,\ldots,k_j)} : \mathcal{O}(j) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_j) \to \mathcal{O}(k_1 + \cdots + k_j) \tag{B.2}$$

for which the associativity diagram (B.4) below commutes. We will sometimes omit the subscript K on the structure map and the structure map γ in the operad. The integer j is the **arity**, *and a point in $\mathcal{O}(j)$ is an* **operation of arity** j. *The collection $\{\mathcal{O}(j) : j \geqslant 0\}$ is called an* **operadic sequence**.

(ii) *$\mathcal{O}$ is simply an* **operad** *(sometimes called a* **symmetric operad***) if in addition each space $\mathcal{O}(j)$ comes equipped with an action of the symmetric group on j letters, Σ_j satisfying the equivariance condition of (B.5) below. The collection $\{\mathcal{O}(j) : j \geqslant 0\}$ is called a* **symmetric sequence**. *In* [MSS02] *and* [LV12] *it is called a Σ-module and an $\mathbb{S}$-module respectively.*

(iii) *An operad $\mathcal{O}$ is* **Σ-free** *if each $\mathcal{O}(j)$ for $j > 0$ is acted on freely by Σ_j.*

(iv) *It is* **discrete** *if each $\mathcal{O}(j)$ is discrete. For any operad $\mathcal{O}$, $\pi_0\mathcal{O}$ is a discrete operad, and the evident map $\epsilon : \mathcal{O} \to \pi_0\mathcal{O}$ is the* **augmentation** *of $\mathcal{O}$.*

(v) *The* m**th truncation** $\mathcal{O}_{\leqslant m}$ **of** $\mathcal{O}$ *for $m > 0$ is given by*

$$\mathcal{O}_{\leqslant m}(j) = \begin{cases} \mathcal{O}(j) & \text{for } 0 \leqslant j \leqslant m \\ \varnothing & \text{for } j > m. \end{cases}$$

We now spell out May's associativity condition. Given a sequence of positive integers $K = (k_1, \ldots, k_j)$, let

$$|K| := j,\ ||K|| := k_1 + \cdots + k_j \text{ and } \mathcal{O}(K) := \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_j), \tag{B.3}$$

so the structure map of (B.2) is $\gamma_K : \mathcal{O}(|K|) \times \mathcal{O}(K) \to \mathcal{O}(||K||)$.

Now suppose we replace each k_i by a sequence of positive integers

$$M_i = (m_{i,1}, \dots m_{i,k_i}), \qquad \text{with } \widetilde{M} = (M_1, \dots, M_j),$$

and we define

$$|\widetilde{M}| := |K|, \qquad ||\widetilde{M}|| := \sum_{1 \leqslant i \leqslant j} ||M_i|| \qquad \text{and} \qquad \mathcal{O}(\widetilde{M}) := \prod_{1 \leqslant i \leqslant j} \mathcal{O}(M_i).$$

It follows that

$$\begin{aligned} \mathcal{O}(|\widetilde{M}|) &= \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_j) \\ \mathcal{O}(K) \times \mathcal{O}(\widetilde{M}) &= (\mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_j)) \times (\mathcal{O}(M_1) \times \cdots \times \mathcal{O}(M_j)) \\ &= (\mathcal{O}(k_1) \times \mathcal{O}(M_1)) \times \cdots \times (\mathcal{O}(k_j) \times \mathcal{O}(M_j)), \end{aligned}$$

from which we have the map $\gamma'_{\widetilde{M}} := \gamma_{M_1} \times \cdots \times \gamma_{M_j}$ to

$$\mathcal{O}(||M_1||) \times \cdots \times \mathcal{O}(||M_j||).$$

We also have a map

$$\gamma''_{\widetilde{M}} := \gamma_{(||M_1||, \dots, ||M_j||)} : \mathcal{O}(j) \times \mathcal{O}(||M_1||) \times \cdots \times \mathcal{O}(||M_j||) \to \mathcal{O}(||\widetilde{M}||).$$

Then we have a diagram

$$\text{(B.4)} \qquad \begin{array}{ccc} \mathcal{O}(|K|) \times \mathcal{O}(K) \times \mathcal{O}(\widetilde{M}) & \xrightarrow{\gamma_K \times \mathcal{O}(\widetilde{M})} & \mathcal{O}(||K||) \times \mathcal{O}(\widetilde{M}) \\ \Big\downarrow {\scriptstyle \mathcal{O}(|K|) \times \gamma'_{\widetilde{M}}} & & \Big\downarrow {\scriptstyle \gamma_{\widetilde{M}}} \\ \mathcal{O}(|K|) \times \mathcal{O}(||M_1||) \times \cdots \times \mathcal{O}(||M_j||) & \xrightarrow{\gamma''_{\widetilde{M}}} & \mathcal{O}(||\widetilde{M}||), \end{array}$$

which is required to commute.

May's equivariance condition on the structure map γ in the symmetric case is as follows. There is a right action of the symmetric group Σ_j on $\mathcal{O}(j)$ such that the following formulas hold for all $c \in \mathcal{O}(k)$, $d_s \in \mathcal{O}(j_s)$, $\sigma \in \Sigma_k$, and $\tau_s \in \Sigma_{j_s}$:

$$\text{(B.5)} \qquad \begin{aligned} \gamma(c\sigma;\, d_1, \dots, d_k) &= \gamma\big(c;\, d_{\sigma^{-1}(1)}, \dots, d_{\sigma^{-1}(k)}\big)\, \sigma(j_1, \dots, j_k) \\ \text{and} \quad \gamma(c;\, d_1\tau_1, \dots, d_k\tau_k) &= \gamma(c;\, d_1, \dots, d_k)\, (\tau_1 \oplus \cdots \oplus \tau_k), \end{aligned}$$

where $\sigma(j_1, \dots, j_k)$ denotes the permutation of $j = j_1 + \cdots + j_k$ letters that permutes the k blocks of sizes $j_1, \dots, j_k$ according to σ, and where $\tau_1 \oplus \cdots \oplus \tau_k$ is the block sum permutation in Σ_j. More precisely, $\sigma(j_1, \dots, j_k) \in \Sigma_{j_1 + \cdots + j_k}$ is the block permutation sending the sth block of size j_s to the $\sigma(s)$th block, and

$$\tau_1 \oplus \cdots \oplus \tau_k \;\in\; \Sigma_{j_1 + \cdots + j_k}$$

is the block sum permutation acting as τ_s on the sth block.

The following example motivates the associativity condition of (B.4).

Definition B.6. [May72, Definition 1.2] *For an object X in a topologically enriched symmetric monoidal category $\mathcal{C}$ (such as that of pointed spaces), the* **endomorphism operad** *$\mathcal{E}nd_X$ has*

$$\mathcal{E}nd_X(j) := \mathcal{C}(X^j, X),$$

the space of maps from X^j to X. Here the action of Σ_j is induced by its action on X^j by permuting coordinates. Then given maps $g_i : X^{k_i} \to X$ for $1 \leqslant i \leqslant j$, and $f : X^j \to X$, we define $\gamma_K(f, g_1, \ldots, g_j)$ to be the composite

$$\begin{array}{c} X^{k_1} \times X^{k_2} \times \cdots \times X^{k_j} = X^{k_1+\cdots+k_j} \\ \big\downarrow {\scriptstyle (g_1, g_2, \ldots, g_j)} \\ X \times X \times \cdots \times X = X^j \\ \big\downarrow {\scriptstyle f} \\ X. \end{array}$$

Let

$$\circ_i \colon \mathcal{C}(X^n, X) \times \mathcal{C}(X^m, X) \longrightarrow \mathcal{C}\big(X^{n+m-1}, X\big)$$

be given, for $1 \leqslant i \leqslant n$, by

$$\begin{aligned} &(f \circ_i g)(x_1, \ldots, x_{m+n-1}) \\ &\qquad = f\big(x_1, \ldots, x_{i-1}, g(x_i, \ldots, x_{i+m-1}), x_{i+m}, \ldots, x_{n+m-1}\big). \end{aligned}$$

The reader can verify that the diagram corresponding to (B.4) commutes for this operad.

Definition B.7. [May72, §1] *A* **morphism of operads**

$$\phi : (\mathcal{O}, \gamma) \to (\mathcal{P}, \delta)$$

is a sequence of Σ_j-equivariant maps

$$\phi_j : \mathcal{O}(j) \to \mathcal{P}(j)$$

such that $\phi_1(1_{\mathcal{O}}) = 1_{\mathcal{P}}$ and the following diagram commutes, with notation as in (B.3):

$$\begin{array}{ccc} \mathcal{O}(j) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_j) & \xrightarrow{\gamma_K} & \mathcal{O}(||K||) \\ {\scriptstyle \phi_j \times \phi_{k_1} \times \cdots \times \phi_{k_j}} \big\downarrow & & \big\downarrow {\scriptstyle \phi_j} \\ \mathcal{P}(j) \times \mathcal{P}(k_1) \times \cdots \times \mathcal{P}(k_j) & \xrightarrow{\delta_K} & \mathcal{P}(||K||). \end{array}$$

Definition B.8. *Given an operad $\mathcal{O}$, an object X in $\mathcal{C}$ as in Definition B.6 is an* **$\mathcal{O}$-algebra**, *or $\mathcal{O}$* **acts on** *X, if there is a operad morphism (Definition B.7) $\theta : \mathcal{O} \to \mathcal{E}nd_X$. Such a map assigns to each point in $\mathcal{O}(j)$ a map $X^j \to X$. The point $1_{\mathcal{O}} \in \mathcal{O}(1)$ gets sent to the identity map on X. The Σ_j-equivariant map $\mathcal{O}(j) \to \mathrm{Map}_*(X^j, X)$ is adjoint to structure maps*

$$\mathcal{O}(j) \times_{\Sigma_j} X^j \xrightarrow{\theta_j^{\mathcal{O},X}} X, \quad \text{and} \quad X^j \times_{\Sigma_j} \mathcal{O}(j) \xrightarrow{\theta_j^{X,\mathcal{O}}} X. \tag{B.9}$$

For $p \in \mathcal{O}(j)$ and $x_i \in X$ for $1 \leqslant i \leqslant j$, we will sometimes write

$$p(x_1, \ldots, x_j) := \theta_j^{\mathcal{O},X}(p, x_1, \ldots x_j). \tag{B.10}$$

These maps define the structure of X as an $\mathcal{O}$-algebra. We will denote the category of such algebras by $\mathcal{O}[\mathcal{C}]$.

We can make a similar definition in the nonsymmetric case by omitting the symmetric group actions in (B.9).

Definition B.11. $\mathcal{O}$-operations. *Let X be an $\mathcal{O}$-algebra as in Definition B.8. An* **$\mathcal{O}$-operation on** *X is a map $a : X^n \to X^m$ for $n > m$ obtained as follows. Let*

$$n = n_1 + n_2 + \cdots + n_m \qquad \text{with } n_i > 0 \text{ for } 1 \leqslant i \leqslant m,$$

and let $a_i : X^{n_i} \to X$ be a map in $\theta(\mathcal{O}(n_i)) \subseteq \mathcal{C}(X^{n_i}, X)$. Then

$$a := \prod_{1 \leqslant i \leqslant m} (a_i : X^{n_i} \to X).$$

Definition B.12. *Suppose X is both a $\mathcal{P}$-algebra and a $\mathcal{Q}$-algebra for operads $\mathcal{P}$ and $\mathcal{Q}$. Then the two structures* **interchange** *if for each $k, \ell > 0$, the following diagram commutes.*

$$\begin{array}{ccc} (\mathcal{P}(k) \times_{\Sigma_k} X^{k\ell}) \times_{\Sigma_\ell} \mathcal{Q}(\ell) & = & \mathcal{P}(k) \times_{\Sigma_k} (X^{k\ell} \times_{\Sigma_\ell} \mathcal{Q}(\ell)) \\ \downarrow{\scriptstyle \theta_k^{\mathcal{P},X^\ell} \times_{\Sigma_\ell} \mathcal{Q}(\ell)} & & \downarrow{\scriptstyle \mathcal{P}(k) \times_{\Sigma_k} \theta_\ell^{X^k,\mathcal{Q}}} \\ X^\ell \times_{\Sigma_\ell} \mathcal{Q}(\ell) \xrightarrow[\theta_\ell^{X,\mathcal{Q}}]{} & X & \xleftarrow[\theta_k^{\mathcal{P},X}]{} \mathcal{P}(k) \times_{\Sigma_k} X^k \end{array} \tag{B.13}$$

The structure maps θ are those of (B.9). The left and right actions of Σ_k and Σ_ℓ respectively on $X^{kl} = X^{\langle k \rangle \times \langle \ell \rangle}$ are induced by their actions on the indexing set

$$\langle k \rangle \times \langle \ell \rangle := \{1, 2, \ldots k\} \times \{1, 2, \ldots \ell\}.$$

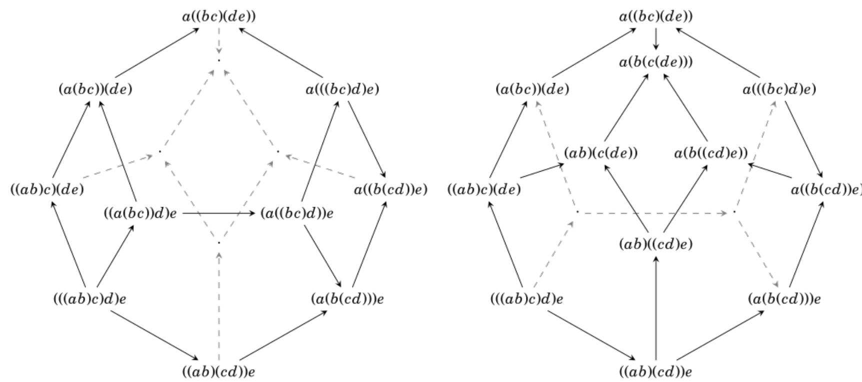
We can make a similar definition for nonsymmetric operads by dropping the action of either or both symmetric groups.

Definition B.14. [May72, Definition 3.1] **Operads for topological monoids and commutative topological monoids.**

(i) Let $\mathcal{M}$, also known as Assoc*, be the discrete operad with $\mathcal{M}(j) = \Sigma_j$ with the evident structure maps. Hence an $\mathcal{M}$-space is a topological monoid.*

(ii) Let $\mathcal{N}$, also known as Comm*, be the discrete operad with $\mathcal{N}(j) = \mathrm{pt}$. with the evident structure maps. Hence an $\mathcal{N}$-space is a commutative topological monoid. $\mathcal{N}$ is a terminal object in the category of operads, i.e., any operad $\mathcal{O}$ admits a unique operad morphism to it. It follows that commutative topological monoid is also an $\mathcal{O}$-space for any operad $\mathcal{O}$.*

Definition B.15. [Sta63] *The* **Stasheff non-Σ operad** $\mathcal{K}$*, also known as* $\mathbb{A}_\infty$*, has* $\mathcal{K}(1) = \mathrm{pt}$ *and* $\mathcal{K}(k)$ *for* $k \geqslant 2$ *is a certain contractible* $(k-2)$*-dimensional polytope called an* **associahedron**. $\mathcal{K}(2)$ *is a point,* $\mathcal{K}(3)$ *is a line segment,* $\mathcal{K}(4)$ *is a pentagon, and* $\mathcal{K}(5)$ *is an* enneahedron *(image from Wikipedia) with nine faces (three disjoint quadrilaterals and six pentagons) and fourteen vertices. Each pentagonal face is a copy of* $\mathcal{K}(4)$.



A space is a $\mathcal{K}$*-algebra if it has a multiplication which is associative up to all higher homotopies. Each edge in* $\mathcal{K}(5)$ *corresponds to a homotopy between the two indicated ways to multiply five points.*

The m**th Stasheff non-Σ operad** $\mathbb{A}_m$ *is the truncation* $\mathcal{K}_{\leqslant m}$ *as in Definition B.1(v).*

Hence every pointed space is an $\mathbb{A}_1$*-algebra. An* $\mathbb{A}_2$*-algebra is a pointed space equipped with a unital multiplication with no associativity condition, and an* $\mathbb{A}_3$*-algebra is a homotopy associative* H*-space with no higher homotopy associativity.*

More information about associahedra can be found in [CSZ15] and [Lod04].

Example B.16. The interchange condition for $\mathcal{Q} = \mathbb{A}_2$. *Since* $\mathbb{A}_2(\ell)$ *is a point for* $\ell \leqslant 2$ *and empty for* $\ell > 2$*, We need consider the diagram of (B.13) only for* $\ell = 1$ *and 2, and we see that it commutes trivially for* $\ell = 1$*. For* $\ell = 2$*, it reads*

$$\begin{array}{ccc}
(\mathcal{P}(k) \times_{\Sigma_k} X^{2k}) \times \mathbb{A}_2(2) & = & \mathcal{P}(k) \times_{\Sigma_k} (X^{2k} \times \mathbb{A}_2(2)) \\
{\scriptstyle \theta_k^{\mathcal{P},X^2} \times \mathbb{A}_2(2)} \downarrow & & \downarrow {\scriptstyle \mathcal{P}(k) \times_{\Sigma_k} \theta_2^{X^k,\mathbb{A}_2}} \\
X^2 \times \mathbb{A}_2(2) \xrightarrow[\theta_2^{X,\mathbb{A}_2}]{} X & \xleftarrow[\theta_k^{\mathcal{P},X}]{} & \mathcal{P}(k) \times_{\Sigma_k} X^k
\end{array}$$

Since $\mathbb{A}_2$ *is not symmetric, there is no* Σ_2*-action as in (B.13). The diagram means that for* $p \in \mathcal{P}(k)$ *and* $x_i \in X$ *for* $1 \leqslant i \leqslant 2k$*, we have*

$$p(x_1, \ldots, x_k) \cdot p(x_{k+1}, \ldots, x_{2k}) = p(x_1 \cdot x_{k+1}, \ldots, x_k \cdot x_{2k}),$$

with notation as in (B.10), where $(- \cdot -)$ *is the monoidal stucture associated with* $\mathbb{A}_2$.

Definition B.17. [May72, Definition 4.1] *The* **little n-cubes operad** $\mathbb{E}_n$, *also known as* $\mathcal{C}_n$, *has* $\mathbb{E}_n(j)$ *being the space of* j*-tuples of rectilinear embeddings of* I^n *into* I^n, *the* n*-cube* $[0,1]^n$, *such that the* j *images of the interior are disjoint. More precisely, a point in* $\mathbb{E}_n(j)$ *is a collection of maps* $e_i : I^n \to I^n$ *for* $1 \leqslant i \leqslant j$ *of the form*

$$e_i(x_1, \dots .x_n) := (a_{i,1} + b_{i,1}x_1, \dots, a_{i,n} + b_{i,n}x_n),$$

where the coefficients $a_{i,k}$ *and* $b_{i,k}$ *satisfy*

$$a_{i,k} \geqslant 0, \qquad b_{i,k} > 0, \qquad a_{i,k} + b_{i,k} \leqslant 1,$$

and the images of the interiors of the I^n*s under the maps* e_k *for* $1 \leqslant k \leqslant j$ *are disjoint.*

This operad is of interest because it acts on $\Omega^n X$ for any pointed space X. The space $\mathbb{E}_n(j)$ has a free action of Σ_j. $\mathbb{E}_1(j)$ is homotopy equivalent to Σ_j, and $\mathbb{E}_n(j)$ is $(n-2)$-connected for $n \geqslant 2$.

It is known that an $\mathbb{E}_1$-structure is equivalent to an $\mathbb{A}_\infty$-structure; see [Lur17, Example 5.1.0.7].

Definition B.18. *Every* $\mathbb{E}_{n+1}$*-algebra is also an* $\mathbb{E}_n$*-algebra.*

Proof. For each map

$$e = e_1 \amalg \cdots \amalg e_k : \coprod_{1 \leqslant i \leqslant k} I^n \to I^n$$

as in Definition B.17, we can form a diagram

$$\begin{array}{ccc} \coprod_{1\leqslant i\leqslant j} I^n & \xrightarrow{e} & I^n \\ \downarrow & & \downarrow \\ \coprod_{1\leqslant i\leqslant j} I^{n+1} & \xrightarrow{\widetilde{e}} & I^{n+1}, \end{array}$$

where the vertical maps send each n-cube to the face of the corresponding $(n+1)$-cube with $x_{n+1} = 0$, and the last coordinate of each map $\widetilde{e}_i$ is x_{n+1}. Thus we get a map $\mathbb{E}_n(j) \to \mathbb{E}_{n+1}(j)$ and the result follows. $\square$

Definition B.19. *The* $\mathbb{E}_\infty$**-operad** *has*

$$\mathbb{E}_\infty(j) := \operatorname*{colim}_n \mathbb{E}_n(j),$$

(which is a contractible free Σ_j*-space) where the maps* $\mathbb{E}_n(j) \to \mathbb{E}_{n+1}(j)$ *are those constructed above.*

B.2. **Some generalizations.** Returning to Definition B.1, the space $\mathcal{O}(1)$ can be regarded as the space of morphisms in a topological category with a single object. The special point in it is the identity morphism and (B.4) says that composition of morphisms is associative. Thus an operad as defined by May is a one object topological category with additional structure.

This suggests three ways of generalizing Definition B.1 so as to accommodate more categories as special cases:

(i) The spaces $\mathcal{O}(j)$ of Definition B.1 could be replaced by objects in a bicomplete closed symmetric monoidal category. This was studied by Max Kelly in [Kel05], but he did not treat the Boardman-Vogt tensor product there. It is also treated in [MSS02, Part II, Chapter 1].

(ii) The point $\mathcal{O}(0)$ could be replaced by a set C of **colors**. We require $\mathcal{O}(0)$ to be a set no matter which symmetric monoidal category the $\mathcal{O}(j)$ are allowed to belong to. The latter are replaced by spaces (or whatever)

$$\mathcal{O}(x_1, \ldots, x_j; y) \qquad \text{for } x_i, y \in C,$$

with the associativity diagram (B.4) suitably modified. The resulting object is a **colored operad**, also known as a **symmetric multicategory**. The term "multicategory" refers to the fact that morphisms are generalized to functions of several variables. See Donald Yau's [Yau16].

(iii) Associated to a colored operad $\mathcal{O}$ is a small category $\mathcal{C}_{\mathcal{O}}$ with object set $\mathcal{O}(0)$ and morphism sets $\mathcal{O}(x; y)$ for $x, y \in \mathcal{O}(0)$. The definition can be modified so that $\mathcal{C}_{\mathcal{O}}$ gets replaced by an ∞-category. Lurie calls such an object an **∞-operad** in [Lur17, Definition 2.1.1.10].

We will only need the first of these.

B.3. **Trees.**

Definition B.20. [MSS02, §1.5], *A* **tree** *is a finite connected contractible graph. We will modify the standard convention that all edges in a graph have two adjacent vertices and delete the vertices with only one adjacent edge. This means that some edges will have only one adjacent vertex and we call these edges* **external edges**. *The edges which are adjacent to two vertices will be called* **internal edges**. *Occasionally it will be convenient to use the standard convention with two vertices adjacent to every edge, in which case we call a vertex adjacent to just one edge an* **external vertex**. *The remaining vertices will be called* **internal vertices**.

All trees are assumed to have at least one edge; the tree with just one edge (and no vertices) is called the **trivial tree**. *A rooted tree is a tree with a distinguished external edge, called the root. The remaining external edges are called leaves. An external vertex adjacent to a leaf will be called a* **leaf vertex**, *and the external vertex adjacent to the root, the* **root vertex**. *A rooted tree has a natural orientation with each edge oriented in the direction of the vertex closest to the root. The root edge is oriented toward the root vertex, but in the case of the trivial tree, this is ambiguous so we have to choose an orientation. In any rooted tree, every vertex is adjacent to a single outgoing edge.*

The **valence** *of an internal vertex is the number of incoming edges. A tree with no vertices of valence one is called* **reduced***. In such a tree the* **distance from the root** *of an internal vertex is one more than the number of internal vertices between it and the root.*

A **corolla** *is a tree with no internal edges and a single internal vertex of valence at least 2. We denote the n-leafed corolla by T_n. A* **binary tree** *is one in which each internal vertex has valence 2. A* **planar tree** *is a tree equipped with an embedding in the plane. A tree that is not so equipped is said to be* **nonplanar***, even though it can be embedded in the plane, unlike a nonplanar graph.*

Unless otherwise indicated, we will assume all trees are reduced, rooted and nonplanar.

Definition B.21. The tree operad. [MSS02, §1.5] *Let $Tree(n)$ for $n > 0$ be the set of isomorphism classes of reduced nonplanar trees with one root and n leaves, with distinct labels, usually the integers 1 through n. The symmetric group acts by permuting the labels. The sequence*

$$Tree = \{Tree(n) : n \geqslant 1\}$$

forms an operad in the category of sets. Given trees $S \in Tree(k)$ and $T \in Tree(j)$, for each $1 \leqslant i \leqslant k$, let

$$S \circ_i T \tag{B.22}$$

be the tree obtained by grafting the root of T to the leaf of S labeled i.

There is a structure map γ_K as in (B.2),

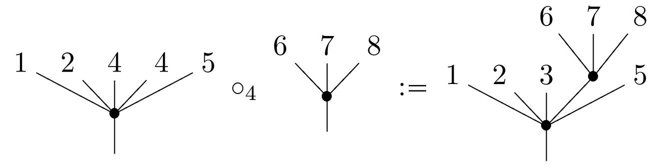
$$\gamma_K : Tree(j) \times Tree(k_1) \times \cdots \times Tree(k_j) \to Tree(k_1 + \cdots + k_j) \tag{B.23}$$

in which, for $1 \leqslant i \leqslant j$, the root of the tree with k_i leaves is grafted onto the ith leaf of the first tree.

The reader should consult [BV73, §1.4 and §2.2] for more information, including the definitions of the terms **twig**, **stump**, **cherry**, **cherry tree**, **fully grown cherry tree**, **planted cherry tree** (with no reference to George Washington), and **copse**.

Remark B.24. How to draw trees. *Linguists draw trees with the root at the top, while botanists draw them with the root at the bottom. We will follow the botanical convention.*

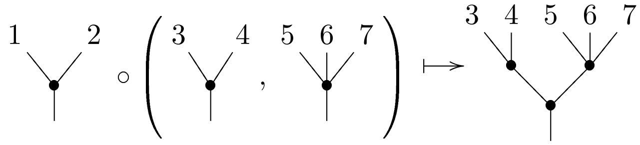
The grafting process of (B.22) is illustrated in (B.25). In each tree the leaves (top vertices) are numerically labeled, and the root is the bottom edge. The two trees on the left each have a single internal vertex. Their **valences**, the number of edges coming in from above, are 5 and 3. The tree on the right has two, with the same valences as the corresponding internal vertices on the left.

(B.25) $T_5 \circ_4 T_3 =$ 

An instance of the structure map of (B.23),

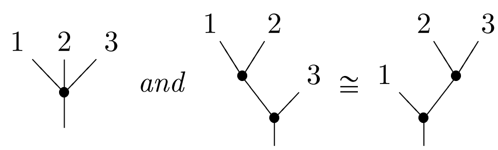
$$\gamma_{(2,3)} : Tree(2) \times (Tree(2) \times Tree(3)) \to Tree(5)$$

is shown here. A similar picture is on [BV73, page 14].

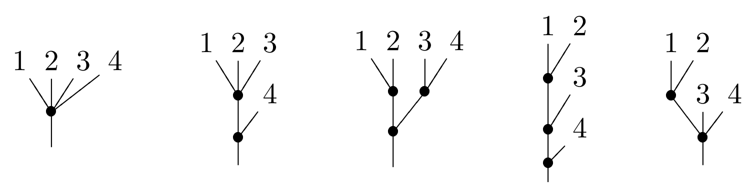


$= (T_2 \circ_2 T_2) \circ_3 T_3$

Example B.26. Some isomorphism classes of trees with few leaves.
There is one 2-leafed tree, T_2.
There are two 3-leafed trees, T_3 and $T_2 \circ_1 T_2 \cong T_2 \circ_2 T_2$ shown here.

(B.27) 

There are five 4-leafed trees, T_4, $T_2 \circ_1 T_3$, $(T_2 \circ_1 T_2) \circ_3 T_2$, $(T_2 \circ_1 T_2) \circ_1 T_2$ and $T_3 \circ_1 T_2$.

(B.28) 

B.4. **Free operads and coproduct operads.**

Definition B.29. The free operad on a symmetric sequence. [MSS02, §II.1.9] *and* [LV12, Theorem 5.5.1]. *For any symmetric sequence $\mathcal{S}$ as in Definition B.1(ii), the free operad $\mathbb{F}(\mathcal{S})$ has*

$$\mathbb{F}(\mathcal{S})(n) \;\cong\; \coprod_{[T]\in Tree(n)} \left(\prod_{v\in V(T)} \mathcal{S}(\mathrm{val}(v)) \right) \Big/ \mathrm{Aut}(T),$$

where:

- *$Tree(n)$ is as in Definition B.21,*
- *$V(T)$ is the set of internal vertices of T,*

- *the valence* $\mathrm{val}(v)$ *is the number of incoming edges at the internal vertex* v*,*
- $\mathrm{Aut}(T)$ *is the group of automorphisms of* T *preserving the root and leaf labels.*

For $\mathcal{S} = \mathcal{P} \amalg \mathcal{Q}$, we have

$$\mathbb{F}(\mathcal{P} \amalg \mathcal{Q})(n) \;\cong\; \coprod_{[T]} \left(\prod_{v \in V(T)} \big(\mathcal{P}(\mathrm{val}(v)) \amalg \mathcal{Q}(\mathrm{val}(v))\big) \right) \Big/ \mathrm{Aut}(T). \tag{B.30}$$

where the coproduct is over all n-leafed trees T. Thus each internal vertex of valence k in T is labelled by a point in $\mathcal{P}$ or $\mathcal{Q}$ of arity k.

Definition B.31. The coproduct of two operads $\mathcal{P}$ *and* $\mathcal{Q}$ *(either of which may or may not be symmetric) is given by*

$$(\mathcal{P} \amalg \mathcal{Q})(n) \;\cong\; \left(\coprod_{[T]} \prod_{v \in V(T)} \big(\mathcal{P}(\mathrm{val}(v)) \amalg \mathcal{Q}(\mathrm{val}(v))\big) \right) \Big/ \sim,$$

where $\sim$ *is the equivalence relation generated by replacing any* $\mathcal{P}$*-only subtree by its composite in* $\mathcal{P}$ *(meaning the corolla with the the same number of leaves), any* $\mathcal{Q}$*-only subtree by its composite in* $\mathcal{Q}$*, and imposing the standard operad relations.*

Example B.32. The coproduct with $\mathbb{A}_1$**.** *For* $\mathcal{Q} = \mathbb{A}_1$ *as in Definition B.15, recall that* $\mathbb{A}_1(1)$ *is a single point and* $\mathbb{A}_1(n)$ *is empty for* $n > 1$*. Since the valence of each internal vertex exceeds 1, the space* $\mathcal{Q}(\mathrm{val}(v))$ *in (B.30) is always empty. It follows that* $\mathcal{P} \amalg \mathbb{A}_1 = \mathcal{P}$*.*

B.5. **The Boardman-Vogt tensor product.** Now suppose we have two operads $(\mathcal{P}, \gamma)$ and $(\mathcal{Q}, \delta)$ as in Definition B.1. Then we have the category $\mathcal{P}[\mathcal{Q}[\mathcal{T}]]$ whose objects are $\mathcal{P}$-algebras in the category of $\mathcal{Q}$-spaces. It is known that there is an operad $\mathcal{R}$ such that this category is equivalent to $\mathcal{R}[\mathcal{T}]$, which is also equivalent to $\mathcal{Q}[\mathcal{P}[\mathcal{T}]]$, the category of $\mathcal{Q}$-algebras in the category of $\mathcal{P}$-spaces. An object in $\mathcal{P}[\mathcal{Q}[\mathcal{T}]] \simeq \mathcal{Q}[\mathcal{P}[\mathcal{T}]]$ is both a $\mathcal{P}$-algebra and a $\mathcal{Q}$-algebra in which the two structure interchange as in Definition B.12.

This leads to a symmetric monoidal structure, the **Boardman-Vogt tensor product** (BV product for short) of [BV73, Definition 2.14, page 41], *on the category of operads*, which we will write as

$$\mathcal{R} = \mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}.$$

Roughly speaking, $\mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}$ is the quotient of the coproduct $\mathcal{P} \amalg \mathcal{Q}$ of Definition B.31 imposed by the interchange requirement.

Example B.33. The BV product does not preserve homotopy equivalence. *We know that the operads* Assoc *of Definition B.14,* $\mathbb{A}_\infty$ *of Definition B.15 and* $\mathbb{E}_1$ *of Definition B.17 are homotopy equivalent. We also know that* $\mathrm{Assoc} \otimes_{BV} \mathrm{Assoc} \cong \mathrm{Comm}$ *by an argument originally due to Beno Eckmann and Peter Hilton* [EH61]*. They showed that if a set is equipped with* *two associative multiplication maps that interchange in the sense of Definition B.12, then they must coincide and be commutative. On the other hand, Gerald Dunn* [Dun88] *shows that* $\mathbb{E}_1 \otimes_{BV} \mathbb{E}_1 \simeq \mathbb{E}_2$*, but* $\mathbb{E}_2$ *is not equivalent to* Comm. *In* [Moe23] *Ieke Moerdijk suggests a solution involving dendroidal sets, which are introduced by him and Ittay Weiss in* [MW07]*, but that is a story for another day.*

We learned the following description of the BV product from the paper [DH14, Definition 1.2] by Bill Dwyer and Kathryn Hess. Like every published definition that we know of, theirs is written with symmetric operads in mind, but it can be modified to include the nonsymmetric case.

Definition B.34. The composite of two symmetric sequences. [DH14, Notation 1.1] *For any two symmetric sequences (see Definition B.1(ii))*

$$X = \{X(n)\}_{n\geqslant 0}, \quad Y = \{Y(n)\}_{n\geqslant 0}$$

of simplicial sets, a representative of a typical element of arity j *in the* **composition product** $X \circ Y$ *of the two sequences is denoted by*

$$(x; y_1, \dots, y_k; \tau),$$

where

$$x \in X(k) \qquad y_s \in Y(j_s) \text{ with } \sum_{s=1}^{k} j_s = j \qquad \text{and } \tau \in \Sigma_j.$$

The right action of $\nu \in \Sigma_j$ *on such an element is given by*

$$(x; y_1, \dots, y_k; \tau) \cdot \nu = (x; y_1, \dots, y_k; \tau\nu).$$

The equivalence relation on representatives of elements of $X \circ Y$ *satisfies*

$$(x; y_1, \dots, y_k; \tau_1 \oplus \cdots \oplus \tau_k) \sim (x; y_1 \cdot \tau_1^{-1}, \dots, y_k \cdot \tau_k^{-1}; \mathrm{Id}),$$

where $\tau_1 \oplus \cdots \oplus \tau_k$ *is the block sum of (B.5), and*

$$(x \cdot \sigma^{-1}; y_1, \dots, y_j; \mathrm{Id}) \sim (x; y_{\sigma(1)}, \dots, y_{\sigma(j)}; \mathrm{Id}),$$

for all $\sigma \in \Sigma_j$ *and* $\tau_s \in \Sigma_{j_s}$ *as above, where* x *and* y_s *are as above.*

If $\mathcal{P}$ *is an operad, it has an equivariant multiplication map*

$$\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$$

as in (B.2). For $p \in \mathcal{P}(k)$ *and* $p_s \in \mathcal{P}(j_s)$ *for* j_s *as above, we write*

$$p(p_1, \dots, p_k) := \gamma(p; p_1, \dots, p_k; \mathrm{Id}) \in \mathcal{P}(j). \tag{B.35}$$

Note that since γ *is equivariant as in (B.5), it is specified by its values on elements of* $\mathcal{P} \circ \mathcal{P}$ *with representatives of the form*

$$(p; p_1, \dots, p_k; \mathrm{Id}).$$

For **operadic sequences** *as in Definition B.1(i) a typical element of arity* n *in the composition product is denoted by simply*

$$(x; y_1, \dots, y_k),$$

with no symmetric group element.

Definition B.36. *The* **Boardman-Vogt tensor product** *of operads $\mathcal{P}$ and $\mathcal{Q}$ is the operad*

$$\mathcal{P} \otimes_{BV} \mathcal{Q}$$

that is the quotient of the coproduct $\mathcal{P} \amalg \mathcal{Q}$ of operads (see Definition B.31) by the equivalence relation generated by

$$\gamma(p; \underbrace{q, \dots, q}_{k}; \mathrm{Id}) \sim \gamma(q; \underbrace{p, \dots, p}_{\ell}; \xi_{k,\ell})$$

for all $p \in P(k)$ and $q \in Q(\ell)$, where $\xi_{k,\ell} \in \Sigma_{k\ell}$ is the transpose permutation that "exchanges rows and columns." That is, for each m with $1 \leqslant m \leqslant k\ell$, there are unique integers i and j with $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant \ell$ for which $m = (i-1)\ell + j \leqslant k\ell$, and we have

$$\xi_{k,\ell}(m) = (j-1)k + i.$$

In the nonsymmetric case the equivalence relation is generated by

$$\gamma(p; \underbrace{q, \dots, q}_{k}) \sim \gamma(q; \underbrace{p, \dots, p}_{\ell}).$$

The only BV product we will need is $\mathbb{EA}_2 := \mathbb{E}_1 \otimes_{\mathrm{BV}} \mathbb{A}_2$, which appears in [BHLS23, Theorem C] and is described in Example B.38.

Example B.37. The Boardman-Vogt unit. *Let $\mathcal{Q}$ be the Stasheff operad $\mathbb{A}_1$ of Definition B.15. In Example B.32 we saw that $\mathcal{P} \amalg \mathbb{A}_1 = \mathcal{P}$ for any operad $\mathcal{P}$. This implies that $\mathcal{P} \otimes_{BV} \mathbb{A}_1 = \mathcal{P}$. A similar argument can be made for $\mathbb{A}_1 \otimes_{BV} \mathcal{Q}$ for any $\mathcal{Q}$. Thus $\mathbb{A}_1$ is the Boardman-Vogt unit.*

Example B.38. *The operad $\mathbb{EA}_2 := \mathbb{E}_1 \otimes_{\mathrm{BV}} \mathbb{A}_2$ (where $\mathbb{A}_2$ and $\mathbb{E}_1$ are as in Definitions B.15 and B.17) is of interest because it appears in* [BHLS23, Theorem C]. *In Section 5 of that paper, the authors construct Adams operations on $BP\langle n\rangle$ (which is known to have an $\mathbb{E}_3$-structure) as $\mathbb{EA}_2$-algebra automorphisms, that being the strongest structure for which their proof works. It is not known to be the strongest structure preserved by their operations.*

An $\mathbb{A}_2$-structure on a spectrum X is a map

$$X \vee (X \wedge X) \to X$$

which is the identity on the first summand and a unital multiplication $\theta_2^{X,\mathbb{A}_2}$ as in (B.9) on the second one.

For an $\mathbb{E}_1$-ring spectrum Y we have the map

$$\theta_j^{\mathbb{E}_1,Y} : \mathbb{E}_1(j)_+ \wedge_{\Sigma_j} Y^{\otimes j} \to Y$$

of (B.9). The space $\mathbb{E}_1(j)$ is homotopy equivalent to Σ_j.

Then for an $\mathbb{EA}_2$-spectrum Z, the following is required to commute as in Definition B.12.

$$\begin{array}{ccc}
(\mathbb{E}_1(j)_+ \times_{\Sigma_j} \Sigma_{2j}/\Sigma_j) \wedge_{\Sigma_{2j}} Z^{\otimes 2j} & & \\
\| & & \\
\mathbb{E}_1(j)_+ \wedge_{\Sigma_j} (Z \wedge Z)^{\otimes j} & \xrightarrow{\theta_j^{\mathbb{E}_1, Z\wedge Z}} & Z \wedge Z \\
{\scriptstyle \mathbb{E}_1(j)_+ \wedge_{\Sigma_j} (\theta_2^{Z,\mathbb{A}_2})^{\otimes j}} \Big\downarrow & & \Big\downarrow {\scriptstyle \theta_2^{Z,\mathbb{A}_2}} \\
\mathbb{E}_1(j)_+ \wedge_{\Sigma_j} Z^{\otimes j} & \xrightarrow{\theta_j^{\mathbb{E}_1, Z}} & Z
\end{array}$$

Hence an $\mathbb{EA}_2$-structure is slightly stronger than an $\mathbb{E}_1$-structure but weaker than an $\mathbb{E}_2$-structure. It is known that for an $\mathbb{EA}_2$-ring R, THH(R) *has an $\mathbb{A}_2$-structure, meaning a unital multiplication with no associativity or commutativity condition.*

References

[AGV72] M. Artin, A. Grothendieck, and J.-L. Verdier, *Théorie des topos et cohomologie étale des schémas. tome 2*, Lecture Notes in Mathematics, vol. 270, Springer-Verlag, Berlin, 1972.

[Ale27] P. Alexandroff, *Über kombinatorische Eigenschaften allgemeiner Kurven*, Mathematische Annalen **96** (1927), 512–554.

[AM69] M. Artin and B. Mazur, *Etale homotopy*, Lecture Notes in Mathematics, vol. No. 100, Springer-Verlag, Berlin-New York, 1969. MR245577

[AN21] B. Antieau and T. Nikolaus, *Cartier modules and cyclotomic spectra*, J. Amer. Math. Soc. **34** (2021), no. 1, 1–78. MR4188814

[BCM20] B. Bhatt, D. Clausen, and A. Mathew, *Remarks on $K(1)$-local K-theory*, Selecta Mathematica (N.S.) **26** (2020), no. 3, Paper No. 39, 16.

[BCSY24] T. Barthel, S. Carmeli, T. M. Schlank, and L. Yanovski, *The chromatic Fourier transform*, Forum Math. Pi **12** (2024), Paper No. e8, 96. MR4728491

[BD20] M. Bremner and V. Dotsenko, *Boardman-Vogt tensor products of absolutely free operads*, Proc. Roy. Soc. Edinburgh Sect. A **150** (2020), no. 1, 367–385. MR4065086

[Bel17] E. Belmont, *A quick introduction to operads*, 2017. Preprint available on author's home page.

[BFV07] M. Brun, Z. Fiedorowicz, and R. M. Vogt, *On the multiplicative structure of topological Hochschild homology*, Algebr. Geom. Topol. **7** (2007), 1633–1650. MR2366174

[BHLS23] R. Burklund, J. Hahn, I. Levy, and T. M. Schlank, *K-theoretic counterexamples to Ravenel's telescope conjecture*, 2023. https://arxiv.org/abs/2310.17459.

[BK72] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin-New York, 1972. MR0365573 (51 #1825)

[BL07] V. Buchstaber and A. Lazarev, *Dieudonné modules and p-divisible groups associated with Morava K-theory of Eilenberg-Mac Lane spaces*, Algebr. Geom. Topol. **7** (2007), 529–564. MR2308956

[BMCSY25] S. Ben-Moshe, S. Carmeli, T. Schlank, and L. Yanovski, *Descent and cyclotomic redshift for chromatically localized algebraic K-theory*, J. Amer. Math. Soc. **38** (2025), no. 2, 521–583. MR4868950

[Bou79a] A. K. Bousfield, *The Boolean algebra of spectra*, Comment. Math. Helv. **54** (1979), no. 3, 368–377. MR543337

[Bou79b] ———, *The localization of spectra with respect to homology*, Topology **18** (1979), no. 4, 257–281. MR551009 (80m:55006)

[Bou82] ———, *On homology equivalences and homological localizations of spaces*, American Journal of Mathematics **104** (1982), 1025–1042.

[Bou87] ———, *Uniqueness of infinite deloopings for K-theoretic spaces*, Pacific J. Math. **129** (1987), no. 1, 1–31. MR901254

[BR08] A. Baker and B. Richter, *Galois extensions of Lubin-Tate spectra*, Homology Homotopy Appl. **10** (2008), no. 3, 27–43. MR2475616

[BV73] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin-New York, 1973. MR0420609

[Car19] X. Caruso, *An introduction to p-adic period rings*, An excursion into p-adic Hodge theory: from foundations to recent trends, 2019, pp. 19–92. MR4363441

[CM17] D. Clausen and A. Mathew, *A short proof of telescopic Tate vanishing*, Proc. Amer. Math. Soc. **145** (2017), no. 12, 5413–5417. MR3717967

[CM21] ———, *Hyperdescent and étale K-theory*, Invent. Math. **225** (2021), no. 3, 981–1076. MR4296353

[CMNN20] D. Clausen, A. Mathew, N. Naumann, and J. Noel, *Descent in algebraic K-theory and a conjecture of Ausoni-Rognes*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 4, 1149–1200. MR4071324

[CMNN24] ———, *Descent and vanishing in chromatic algebraic K-theory via group actions*, Ann. Sci. Éc. Norm. Supér. (4) **57** (2024), no. 4, 1135–1190. MR4773302

[CSY21] S. Carmeli, T. M. Schlank, and L. Yanovski, *Ambidexterity and height*, Adv. Math. **385** (2021), Paper No. 107763, 90. MR4246977

[CSY22] ———, *Ambidexterity in chromatic homotopy theory*, Invent. Math. **228** (2022), no. 3, 1145–1254. MR4419631

[CSY24] ———, *Chromatic cyclotomic extensions*, Geom. Topol. **28** (2024), no. 8, 3511–3564. MR4843744

[CSZ15] C. Ceballos, F. Santos, and G. M. Ziegler, *Many non-equivalent realizations of the associahedron*, Combinatorica **35** (2015), no. 5, 513–551. MR3437894

[Dev18] S. Devalapurkar, *The Morava K-theory of Eilenberg-MacLane spaces*, 2018. https://sanathdevalapurkar.github.io/files/ambidexterity-lecture.pdf.

[DH04] E. S. Devinatz and M. J. Hopkins, *Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups*, Topology **43** (2004), no. 1, 1–47. MR2030586 (2004i:55012)

[DH14] W. Dwyer and K. Hess, *The Boardman-Vogt tensor product of operadic bimodules*, Algebraic topology: applications and new directions, 2014, pp. 71–98. MR3290087

[DHS88] E. S. Devinatz, M. J. Hopkins, and J. H. Smith, *Nilpotence and stable homotopy theory. I*, Ann. of Math. (2) **128** (1988), no. 2, 207–241. MR960945 (89m:55009)

[Dun88] G. Dunn, *Tensor product of operads and iterated loop spaces*, J. Pure Appl. Algebra **50** (1988), no. 3, 237–258. MR938617

[EH61] B. Eckmann and P. J. Hilton, *Group-like structures in general categories. I. Multiplications and comultiplications*, Math. Ann. **145** (1961/62), 227–255. MR136642

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole. MR97h:55006

[EM45] S. Eilenberg and S. MacLane, *Relations between homology and homotopy groups*, Annals of Mathematics. Second Series **46** (1945), 480–509. MR0013131

[GH04] P. G. Goerss and M. J. Hopkins, *Moduli spaces of commutative ring spectra*, Structured ring spectra, 2004, pp. 151–200. MR2125040 (2006b:55010)

[GJ99] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999. MR1711612 (2001d:55012)

[GM95] J. P. C. Greenlees and J. P. May, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. **113** (1995), no. 543, viii+178. MR1230773 (96e:55006)

[GS96] J. P. C. Greenlees and H. Sadofsky, *The Tate spectrum of v_n-periodic complex oriented theories*, Math. Z. **222** (1996), no. 3, 391–405. MR1400199

[Har20] Y. Harpaz, *Ambidexterity and the universality of finite spans*, Proc. Lond. Math. Soc. (3) **121** (2020), no. 5, 1121–1170. MR4133704

[HKR00] M. J. Hopkins, N. J. Kuhn, and D. C. Ravenel, *Generalized group characters and complex oriented cohomology theories*, J. Amer. Math. Soc. **13** (2000), no. 3, 553–594 (electronic). MR2001k:55015

[HL13] M. J. Hopkins and J. Lurie, *Ambidexterity in $K(n)$-local stable homotopy theory*, 2013. https://www.math.ias.edu/~lurie/papers/Ambidexterity.pdf.

[HRW98] M. J. Hopkins, D. C. Ravenel, and W. S. Wilson, *Morava K-theory, Hopf algebras, and spaces K(n)-equivalent to finite Postnikov systems*, Stable and unstable homotopy, 1998, pp. 137–163.

[HS96] M. Hovey and H. Sadofsky, *Tate cohomology lowers chromatic Bousfield classes*, Proc. Amer. Math. Soc. **124** (1996), no. 11, 3579–3585. MR1343699

[HS98] M. J. Hopkins and J. H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) **148** (1998), no. 1, 1–49. MR1652975 (99h:55009)

[HW22] J. Hahn and D. Wilson, *Redshift and multiplication for truncated Brown-Peterson spectra*, Ann. of Math. (2) **196** (2022), no. 3, 1277–1351. MR4503327

[Joy02] A. Joyal, *Quasi-categories and Kan complexes*, J. Pure Appl. Algebra **175** (2002), no. 1-3, 207–222. Special volume celebrating the 70th birthday of Professor Max Kelly. MR1935979

[JW73] D. C. Johnson and W. S. Wilson, *Projective dimension and Brown-Peterson homology*, Topology **12** (1973), 327–353. MR334257

[JW85] ———, *The Brown-Peterson homology of elementary p-groups*, Amer. J. Math. **107** (1985), no. 2, 427–453. MR784291

[Kel05] G. M. Kelly, *On the operads of J. P. May*, Repr. Theory Appl. Categ. **13** (2005), 1–13. MR2177746

[KL00] I. Kriz and K. P. Lee, *Odd-degree elements in the Morava $K(n)$ cohomology of finite groups*, Topology Appl. **103** (2000), no. 3, 229–241. MR2001f:55008

[Kuh04] N. J. Kuhn, *Tate cohomology and periodic localization of polynomial functors*, Invent. Math. **157** (2004), no. 2, 345–370. MR2076926

[Kuh89] ———, *Morava K-theories and infinite loop spaces*, Algebraic topology (Arcata, CA, 1986), 1989, pp. 243–257. MR1000381

[LMMT24] M. Land, A. Mathew, L. Meier, and G. Tamme, *Purity in chromatically localized algebraic K-theory*, J. Amer. Math. Soc. **37** (2024), no. 4, 1011–1040. MR4777639

[Lod04] J.-L. Loday, *Realization of the Stasheff polytope*, Arch. Math. (Basel) **83** (2004), no. 3, 267–278. MR2108555

[Lur09] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR2522659 (2010j:18001)

[Lur17] ———, *Higher algebra*, 2017. https://www.math.ias.edu/~lurie/papers/HA.pdf.

[Lur18a] ———, *Elliptic cohomology II: Orientations*, 2018. https://www.math.ias.edu/~lurie/papers/Elliptic-II.pdf.

[Lur18b] ———, *Spectral algebraic geometry*, 2018. https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf.

[LV12] J.-L. Loday and B. Vallette, *Algebraic operads*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 346, Springer, Heidelberg, 2012. MR2954392

[LZ17] Y. Liu and W. Zheng, *Enhanced six operations and base change theorem for artin stacks*, 2017. https://arxiv.org/abs/1211.5948.

[Mal17] C. Malkiewich, *The topological cyclic homology of the dual circle*, J. Pure Appl. Algebra **221** (2017), no. 6, 1407–1422. MR3599438

[May72] J. P. May, *The geometry of iterated loop spaces*, Springer-Verlag, Berlin-New York, 1972. Lectures Notes in Mathematics, Vol. 271. MR0420610 (54 #8623b)

[Mil92] H. Miller, *Finite localizations*, 1992, pp. 383–389. Papers in honor of José Adem (Spanish). MR1317588

[MLM94] S. Mac Lane and I. Moerdijk, *Sheaves in geometry and logic*, Universitext, Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition. MR1300636 (96c:03119)

[MNN17] A. Mathew, N. Naumann, and J. Noel, *Nilpotence and descent in equivariant stable homotopy theory*, Adv. Math. **305** (2017), 994–1084. MR3570153

[Moe23] I. Moerdijk, *A mysterious tensor product in topology*, Mathematics going forward—collected mathematical brushstrokes, 2023, pp. 209–214. MR4627960

[Mor85] J. Morava, *Noetherian localisations of categories of cobordism comodules*, Ann. of Math. (2) **121** (1985), no. 1, 1–39. MR782555 (86g:55004)

[MR87] M. E. Mahowald and D. C. Ravenel, *Toward a global understanding of the homotopy groups of spheres*, The Lefschetz centennial conference, Part II (Mexico City, 1984), 1987, pp. 57–74. MR893848

[MR99] M. E. Mahowald and C. Rezk, *Brown-Comenetz duality and the Adams spectral sequence*, Amer. J. Math. **121** (1999), no. 6, 1153–1177. MR1719751 (2001a:55008)

[MRS01] M. E. Mahowald, D. C. Ravenel, and P. L. Shick, *The triple loop space approach to the telescope conjecture*, Homotopy methods in algebraic topology (Boulder, CO, 1999), 2001, pp. 217–284. MR1831355

[MSS02] M. Markl, S. Shnider, and J. Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, vol. 96, American Mathematical Society, Providence, RI, 2002. MR1898414

[MW07] I. Moerdijk and I. Weiss, *Dendroidal sets*, Algebr. Geom. Topol. **7** (2007), 1441–1470. MR2366165

[Rav23] D. C. Ravenel, *What is an ∞-category?*, Grad. J. Math. **8** (2023), no. 2, 1–14. MR4694539

[Rav26] ———, *What are cyclotomic spectra and why do we need them?*, 2026. To appear.

[Rav82] ———, *Morava K-theories and finite groups*, Symposium on Algebraic Topology in honor of José Adem (Oaxtepec, 1981), 1982, pp. 289–292. MR676336

[Rav84] ———, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), no. 2, 351–414. MR737778 (85k:55009)

[Rav92] ———, *Nilpotence and periodicity in stable homotopy theory*, Annals of Mathematics Studies, vol. 128, Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith. MR1192553 (94b:55015)

[Rez98] C. Rezk, *Notes on the Hopkins-Miller theorem*, Homotopy theory via algebraic geometry and group representations (evanston, il, 1997), 1998, pp. 313–366. MR2000i:55023

[Rog08] J. Rognes, *Galois extensions of structured ring spectra. Stably dualizable groups*, Mem. Amer. Math. Soc. **192** (2008), no. 898, viii+137. MR2387923

[RW80] D. C. Ravenel and W. S. Wilson, *The Morava K-theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture*, Amer. J. Math. **102** (1980), no. 4, 691–748. MR584466 (81i:55005)

[Ser79] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg. MR554237

[Sta04] J. Stasheff, *What is ... an operad?*, Notices Amer. Math. Soc. **51** (2004), no. 6, 630–631. MR2064150

[Sta63] J. D. Stasheff, *Homotopy associativity of H-spaces. I, II*, Trans. Amer. Math. Soc. **108** (1963), 293–312. **108** (1963), 275-292; ibid. MR158400

[Wes17] C. Westerland, *A higher chromatic analogue of the image of J*, Geom. Topol. **21** (2017), no. 2, 1033–1093. MR3626597

[Yau16] D. Yau, *Colored operads*, Graduate Studies in Mathematics, vol. 170, American Mathematical Society, Providence, RI, 2016. MR3444662

Department of Mathematics, University of Rochester, Rochester, NY 14627
Email address: `dcravenel@gmail.com`