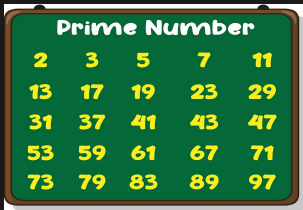


Why are there so many prime numbers?

Doug Ravenel



Prime Number				
2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

SUMS
University of Rochester

September 22, 2026

Why are there so many prime numbers?

Outline

Three big theorems about prime numbers

Euclid's theorem

Dirichlet's theorem

The prime number theorem

Two proofs of Theorem 1

God's proof

Euclid's proof

Primes of the form $4m - 1$

Primes of the form $4m + 1$

Other cases of Dirichlet's theorem

Euler's proof of Theorem 1

The Riemann hypothesis

An AI breakthrough

Some theorems about primes that every mathematician should know



Theorem 1 (Euclid 300 BC)

There are infinitely many prime numbers.

Why are there so many prime numbers?

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Some theorems about primes that every mathematician should know



Theorem 1 (Euclid 300 BC)

There are infinitely many prime numbers.

Euclid's proof is very elementary, and we will give it shortly.

Why are there so many prime numbers?

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Some theorems about primes that every mathematician should know



Theorem 1 (Euclid 300 BC)

There are infinitely many prime numbers.

Euclid's proof is very elementary, and we will give it shortly.

In 1737 Euler found a completely different proof that requires calculus. His method is harder to use but more powerful.

Why are there so many prime numbers?

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Why are there so many prime numbers?



Let a and b be relatively prime positive integers. Then there are infinitely primes of the form $am + b$.

Three big theorems about prime numbers

The prime number theorem

Primes of the form $4m + 1$

The Riemann hypothesis

An AI breakthrough

Dirichlet's theorem

Why are there so many prime numbers?



Theorem 2 (Dirichlet, 1837)

Let a and b be relatively prime positive integers. Then there are infinitely primes of the form $am + b$.

Example. For $a = 10$, b could be 1, 3, 7 or 9. The theorem says there are infinitely many primes of the form $10m + 1$, $10m + 3$, $10m + 7$ and $10m + 9$. For other values of b not prime to 10, there is at most one such prime.

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Dirichlet's proof uses functions of a complex variable.

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Dirichlet's theorem



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Dirichlet's proof uses functions of a complex variable.

We will see how some cases of it can be proved with more elementary methods.

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The prime number theorem

Why are there so many prime numbers?



Gauss
1777–1855



Legendre
1752–1833



Hadamard
1865–1963



Vallée-Poussin
1866–1962

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The following was conjectured by Gauss and Legendre.

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Gauss
1777–1855



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Vallée-Poussin
1866–1962

The following was conjectured by Gauss and Legendre.

Theorem 3 (Hadamard and Vallée-Poussin, 1896, Asymptotic distribution of primes)

Let $\pi(x)$ denote the number of primes less than x . Then

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{x / \ln x} = 1.$$

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The prime number theorem (continued)

Why are there so many prime numbers?

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The prime number theorem (continued)

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Let $\pi(x)$ denote the number of primes less than x . Then

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{x / \ln x} = 1.$$

In other words, the number of primes less than x is roughly $x / \ln x$.

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The prime number theorem (continued)

Why are there so many prime numbers?

Theorem 3 (Hadamard and Vallée-Poussin, 1896, Asymptotic distribution of primes)

Let $\pi(x)$ denote the number of primes less than x . Then

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{x / \ln x} = 1.$$

In other words, the number of primes less than x is roughly $x / \ln x$.

A better approximation is to $\pi(x)$ is the logarithmic integral

$$\text{li}(x) = \int_2^x \frac{dt}{\ln(t)}.$$

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The omniscient being proof of Theorem 1

Here is God's proof that there are infinitely many primes:

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The omniscient being proof of Theorem 1

Why are there so many prime numbers?

Here is God's proof that there are infinitely many primes:

- Look at the positive integers

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...

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Here is God's proof that there are infinitely many primes:

- Look at the positive integers

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...

- See which of them are primes

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...

The omniscient being proof of Theorem 1

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1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...

- See which of them are primes

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...

- Notice that there are infinitely many of them.

The omniscient being proof of Theorem 1

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QED

Euclid's proof

Without God's omniscience, we have to work harder.

*Why are there so
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theorem*

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Euclid's proof

Why are there so many prime numbers?

Without God's omniscience, we have to work harder.

Euclid's proof relies on the *Fundamental Theorem of Arithmetic* (FTA for short), which says that every positive integer can be written as a product of primes in a unique way.

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Euclid's proof

Why are there so many prime numbers?

Without God's omniscience, we have to work harder.

Euclid's proof relies on the *Fundamental Theorem of Arithmetic* (FTA for short), which says that every positive integer can be written as a product of primes in a unique way.

For example,

$$2026 = 2 \cdot 1013 \quad (1013 \text{ is a prime})$$

$$2027 = \text{prime}$$

$$2028 = 2^2 \cdot 3 \cdot 13^2$$

$$2029 = \text{prime}$$

$$2030 = 2 \cdot 5 \cdot 7 \cdot 29$$

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Euclid's proof that there are infinitely many primes

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Here is Euclid's wonderfully elegant argument:

Euclid's proof that there are infinitely many primes

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Here is Euclid's wonderfully elegant argument:

- Let $S = \{p_1, p_2, \dots, p_n\}$ be a finite set of primes.

Euclid's proof that there are infinitely many primes

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Here is Euclid's wonderfully elegant argument:

- Let $S = \{p_1, p_2, \dots, p_n\}$ be a finite set of primes.
- Let $N = p_1 p_2 \dots p_n$, the product of all the primes in S .

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- Let $S = \{p_1, p_2, \dots, p_n\}$ be a finite set of primes.
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- The number $N + 1$ is *not* divisible by any prime in S .
- By the FTA, $N + 1$ is a product of one or more primes not in the set S .

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- Therefore **S is not the set of all the prime numbers.**

Euclid's proof that there are infinitely many primes

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- The number $N + 1$ is *not* divisible by any prime in S .
- By the FTA, $N + 1$ is a product of one or more primes not in the set S .
- Therefore **S is not the set of all the prime numbers.**

This means there are infinitely many primes.

Primes of the form $4m - 1$

We can use Euclid's method to show there are infinitely many prime of the form $4m - 1$.

Why are there so many prime numbers?

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Primes of the form $4m - 1$

Why are there so many prime numbers?

We can use Euclid's method to show there are infinitely many prime of the form $4m - 1$.

- Let $S = \{p_1, \dots, p_n\}$ be a set of such primes, and let N be the product of all of them.

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We can use Euclid's method to show there are infinitely many prime of the form $4m - 1$.

- Let $S = \{p_1, \dots, p_n\}$ be a set of such primes, and let N be the product of all of them.
- The number $4N - 1$ is not divisible by any of the primes in S .

Primes of the form $4m - 1$

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We can use Euclid's method to show there are infinitely many prime of the form $4m - 1$.

- Let $S = \{p_1, \dots, p_n\}$ be a set of such primes, and let N be the product of all of them.
- The number $4N - 1$ is not divisible by any of the primes in S .
- Therefore $4N - 1$ is the product of some primes not in S , all of which are odd and not all of which have the form $4m + 1$.

Primes of the form $4m - 1$

Why are there so many prime numbers?

We can use Euclid's method to show there are infinitely many prime of the form $4m - 1$.

- Let $S = \{p_1, \dots, p_n\}$ be a set of such primes, and let N be the product of all of them.
- The number $4N - 1$ is not divisible by any of the primes in S .
- Therefore $4N - 1$ is the product of some primes not in S , all of which are odd and not all of which have the form $4m + 1$.
- Therefore **S is not the set of all primes of the form $4m - 1$.**

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Primes of the form $4m + 1$

We can try a similar approach to primes of the form $4m + 1$.

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We can try a similar approach to primes of the form $4m + 1$.

- Let $S = \{p_1, \dots, p_n\}$ be a set of such primes, and let N be the product of all of them.

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We can try a similar approach to primes of the form $4m + 1$.

- Let $S = \{p_1, \dots, p_n\}$ be a set of such primes, and let N be the product of all of them.
- The number $4N + 1$ is not divisible by any of the primes in S .

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We can try a similar approach to primes of the form $4m + 1$.

- Let $S = \{p_1, \dots, p_n\}$ be a set of such primes, and let N be the product of all of them.
- The number $4N + 1$ is not divisible by any of the primes in S .
- Therefore $4N + 1$ is the product of some primes not in S , all of which are odd.

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We can try a similar approach to primes of the form $4m + 1$.

- Let $S = \{p_1, \dots, p_n\}$ be a set of such primes, and let N be the product of all of them.
- The number $4N + 1$ is not divisible by any of the primes in S .
- Therefore $4N + 1$ is the product of some primes not in S , all of which are odd.
- However it could be the product of an even number of primes of the form $4m - 1$, eg $21 = 3 \cdot 7$.

Primes of the form $4m + 1$

Why are there so many prime numbers?

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Three big theorems about prime numbers

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An AI breakthrough

We can try a similar approach to primes of the form $4m + 1$.

- Let $S = \{p_1, \dots, p_n\}$ be a set of such primes, and let N be the product of all of them.
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- However it could be the product of an even number of primes of the form $4m - 1$, eg $21 = 3 \cdot 7$. **OOPS.**

How to fix this problem

It turns out that the number $4N^2 + 1$ (instead of $4N + 1$) has to be the product of primes of the form $4m + 1$.

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How to fix this problem

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It turns out that the number $4N^2 + 1$ (instead of $4N + 1$) has to be the product of primes of the form $4m + 1$.

Here are some examples.

N	$4N^2 + 1$	N	$4N^2 + 1$
1	5	9	$325 = 5^2 \cdot 13$
2	17	10	401
3	37	11	$485 = 5 \cdot 97$
4	$65 = 5 \cdot 13$	12	577
5	101	13	677
6	$145 = 5 \cdot 29$	14	$785 = 5 \cdot 157$
7	197	15	$901 = 17 \cdot 53$
8	257	16	$1025 = 5^2 \cdot 41$

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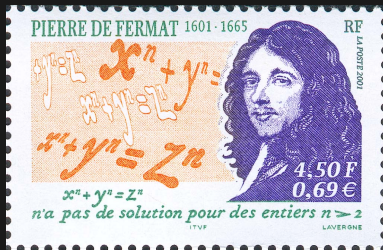
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How to fix this problem (continued)

Why are there so many prime numbers?



To prove this we need some help from Fermat, who is best known for his “Last Theorem.”

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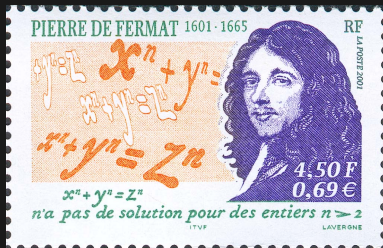
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How to fix this problem (continued)

Why are there so many prime numbers?



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Theorem (Fermat’s Little Theorem, 1640)

If p is a prime, then $x^p - x$ is divisible by p for any integer x .

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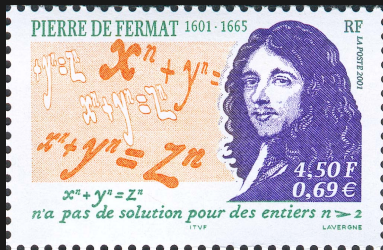
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Since $x^p - x = x(x^{p-1} - 1)$, if x is not divisible by p , then $x^{p-1} - 1$ is divisible by p .

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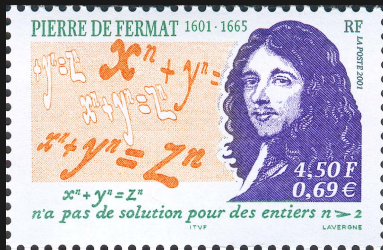
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Since $x^p - x = x(x^{p-1} - 1)$, if x is not divisible by p , then $x^{p-1} - 1$ is divisible by p .

In other words, $x^{p-1} \equiv 1$ modulo p .

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Lemma

Any number of the form $4N^2 + 1$ is a product of primes of the form $4m + 1$.

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How to fix this problem (continued)

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Lemma

Any number of the form $4N^2 + 1$ is a product of primes of the form $4m + 1$.

Proof: Let $x = 2N$, so our number is $x^2 + 1$.

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Proof: Let $x = 2N$, so our number is $x^2 + 1$. Suppose it is divisible by a prime of the form $p = 4m + 3$. This means $x^2 \equiv -1$ modulo p .

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Then $x^{4m} = (x^2)^{2m} \equiv (-1)^{2m} = 1$.

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Fermat's Little Theorem tell us that $x^{p-1} = x^{4m+2} \equiv 1$,

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Fermat's Little Theorem tell us that $x^{p-1} = x^{4m+2} \equiv 1$, but $x^{4m+2} = x^{4m} \cdot x^2 \equiv 1 \cdot -1 = -1$, so we have a contradiction.

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Hence $4N^2 + 1$ is not divisible by any prime of the form $4m + 3$.
QED

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Primes of the form $4m + 1$ again

Here is our second attempt to use Euclid's method, this time with some help from Fermat.

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- Therefore $4N^2 + 1$ is the product of some primes not in S , all of which must have the form $4m + 1$.
- Therefore **S is not the set of all primes of the form $4m + 1$.**

Some other cases of Dirichlet's theorem

Similar methods (involving algebra but no analysis) can be used to prove some but not all cases of Dirichlet's theorem. For example,

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- We can show there are infinitely many primes of the forms $5m + 1$ and $5m - 1$.

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- We can show there are infinitely many primes of the forms $5m + 1$ and $5m - 1$.
- We can show there are infinitely many primes of the forms $5m + 2$ or $5m + 3$, *but not that there are infinitely many of either type alone.*

Euler's proof that there are infinitely many primes

Why are there so many prime numbers?



Leonhard Euler 1707–1783

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Euler considered the infinite series

$$\sum_{n \geq 1} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

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From calculus we know that it converges for $s > 1$ (by the integral test)

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Euler's proof (continued)

Why are there so many prime numbers?

Using FTA, Euler rewrote the series as a product

$$\sum_{n \geq 1} \frac{1}{n^s} = \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \dots\right) \left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \dots\right) \dots$$

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Euler's proof (continued)

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with one factor for each prime. Each factor is a geometric series. The p th factor converges to $1/(1 - p^{-s})$, whenever $s > 0$. Hence

$$\sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

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$$\sum_{n \geq 1} \frac{1}{n^s} = \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \dots\right) \left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \dots\right) \dots$$

with one factor for each prime. Each factor is a geometric series. The p th factor converges to $1/(1 - p^{-s})$, whenever $s > 0$. Hence

$$\sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

If there were only finitely many primes, this would give a finite answer for $s = 1$, contradicting the divergence of the harmonic series.

Why are there so many prime numbers?

Outline

Three big theorems about prime numbers

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Euler's proof (continued)

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Dirichlet used some clever variations of this method to prove his theorem 100 years later.

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Epilogue: The Riemann zeta function.



In his famous 1859 paper *On the Number of Primes Less Than a Given Magnitude*, Georg Bernhard Riemann (1826–1866) studied Euler's series

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as a function of a complex variable s .

Epilogue: The Riemann zeta function (continued)

He showed that the series converges whenever s has real part greater than 1, and that it can be extended as a complex analytic function to all values of s other than 1, where the function has a pole.

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He showed that the series converges whenever s has real part greater than 1, and that it can be extended as a complex analytic function to all values of s other than 1, where the function has a pole.

He showed that the behavior of this function is intimately connected with the distribution of prime numbers.

Epilogue: The Riemann zeta function (continued)

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To learn more about this connection, ask Steve Gonek to give a talk.

When does $\zeta(s)$ vanish?

Riemann showed that $\zeta(s) \neq 0$ for $s = -2, s = -4, s = -6$ and so on.

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Atle Selberg
1917–2007



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Brian Conrey

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An AI breakthrough

An AI breakthrough (continued)

- On August 11 Claude raised the portion to 67.1%.

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CLAUDE
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THANK YOU!