

The 3-primary Arf-Kervaire invariant problem

Mike Hill University of Virginia

Mike Hopkins Harvard University

Doug Ravenel University of Rochester

The 3-primary
Arf-Kervaire invariant
problem

Mike Hill
Mike Hopkins
Doug Ravenel



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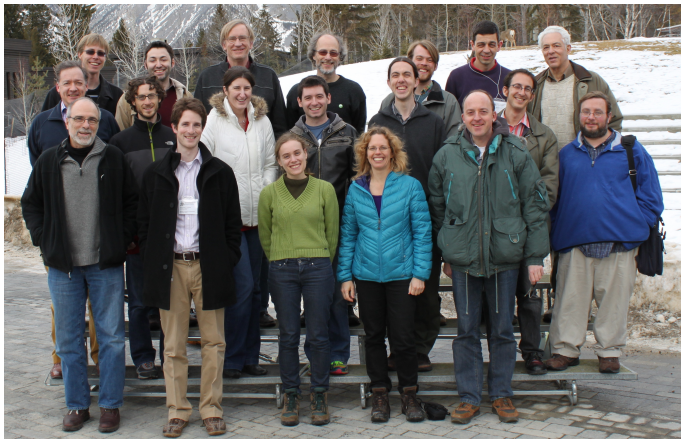
A possible Gap Theorem

The 3-primary Arf-Kervaire invariant problem

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Banff Workshop on Algebraic K-Theory and
Equivariant Homotopy Theory
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The 3-primary Arf-Kervaire invariant problem is still open.



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We have a program for solving it similar to what we did for $p = 2$.

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We have a program for solving it similar to what we did for $p = 2$.



We are missing a crucial ingredient.

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We are missing a crucial ingredient.



Maybe you can find it!

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Defining the problem

The Arf-Kervaire invariant problem for a prime p is to determine the fate of the elements

$$\theta_j = \left\{ \begin{array}{ll} h_j^2 & \text{for } p = 2 \\ b_{j-1} & \text{for } p > 2 \end{array} \right\} \in \text{Ext}_A^{2, 2p^j(p-1)}(\mathbf{Z}/p, \mathbf{Z}/p) \quad (1)$$

where A denotes the mod p Steenrod algebra.

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where A denotes the mod p Steenrod algebra. This Ext group is the E_2 -term for the classical Adams spectral sequence converging to the p -component of the stable homotopy groups of spheres.

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where A denotes the mod p Steenrod algebra. This Ext group is the E_2 -term for the classical Adams spectral sequence converging to the p -component of the stable homotopy groups of spheres.



Frank Adams
1930–1989

In these bidegrees the groups are known to be isomorphic to $\mathbf{Z}/p$ in each case, generated by these elements.

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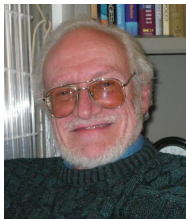
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Introduction (continued)



Bill Browder

Browder's Theorem of 1969 states that for $p = 2$, h_j^2 is a permanent cycle in the Adams spectral sequence if and only if there is a framed manifold with nontrivial Kervaire invariant manifold in dimension $2^{j+1} - 2$.

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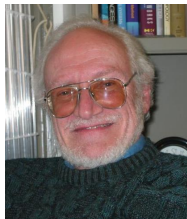
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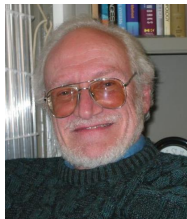
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We recently showed that for $p = 2$, θ_j does not exist for $j \geq 7$.

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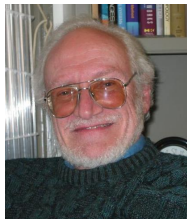
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We recently showed that for $p = 2$, θ_j does not exist for $j \geq 7$.
The case $j = 6$ remains open.

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$$\text{Ext}_{BP_*(BP)}^{2, 2p^j(p-1)} \xrightarrow{\Phi} \text{Ext}_A^{2, 2p^j(p-1)}$$

is onto in all but one case, with

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$$\hat{\theta}_j = \beta_{p^{j-1}/p^{j-1}} \mapsto \begin{cases} 0 & \text{for } j = 1 \text{ and } p = 2 \\ \theta_j & \text{otherwise} \end{cases}$$

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A reformulation of the problem is the following:

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A reformulation of the problem is the following:

Is any element of $\text{Ext}_{BP_*(BP)}^{2,2p^j(p-1)}$ mapping to θ_j a permanent cycle?

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There is no known interpretation of the problem at odd primes in terms of manifolds.

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There is no known interpretation of the problem at odd primes in terms of manifolds. In the late 70s the third author showed for that for $p \geq 5$, the element θ_j for $j > 1$ is not a permanent cycle,

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There is no known interpretation of the problem at odd primes in terms of manifolds. In the late 70s the third author showed for that for $p \geq 5$, the element θ_j for $j > 1$ is not a permanent cycle, while θ_1 is a permanent cycle representing

$$\hat{\theta}_1 = \beta_1 \in \pi_{2p^2-2p-2} S^0.$$

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$$\widehat{\theta}_1 = \beta_1 \in \pi_{2p^2-2p-2} S^0.$$

Modulo some indeterminacy, there are differentials

$$d_{2p-1}(\widehat{\theta}_j) = h_0 \widehat{\theta}_{j-1}^p \quad (2)$$

where $h_0 \in \text{Ext}_A^{1,2p-1}$ represents $\alpha_1 \in \pi_{2p-3} S^0$.

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In order to describe the difficulties at $p = 3$, we need to recall the methods of [HHR] for $p = 2$ and myself for $p \geq 5$.

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Hiroshi Toda
in 2009

In the Adams-Novikov spectral sequence for an odd prime p there is a nontrivial differential

$$d_{2p-1}(\hat{\theta}_2) = \alpha_1 \hat{\theta}_1^p. \quad (3)$$

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We also show that there are relations

$$\hat{\theta}_j \hat{\theta}_2^{p^{j-1}} = \hat{\theta}_{j+1} \hat{\theta}_1^{p^{j-1}}. \quad (4)$$

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Using (3-4) one can deduce that

$$d_{2p-1}(\widehat{\theta}_j) = \alpha_1 \widehat{\theta}_{j-1}^p \quad \text{for all } j \geq 2.$$

The hard part is to use chromatic methods to show that these targets are all nontrivial.

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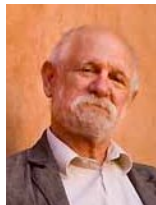
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Jack Morava

We now know (but only suspected in the late 70s) that the extended Morava stabilizer group $\mathbf{G}_n$

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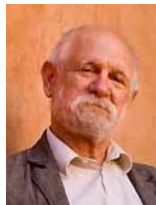
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Jack Morava

We now know (but only suspected in the late 70s) that the extended Morava stabilizer group $\mathbf{G}_n$ acts on the Morava spectrum E_n in such a way that the homotopy fixed point set $E_n^{h\mathbf{G}_n}$ is $L_{K(n)}S^0$,

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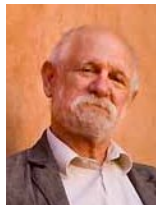
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This is a corollary of the Hopkins-Miller theorem.



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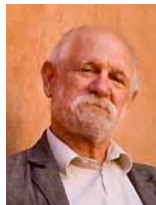
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This is a corollary of the Hopkins-Miller theorem. For any closed subgroup $H \subset \mathbf{G}_n$ there is a homotopy fixed point spectral sequence

$$H^*(H; \pi_* E_n) \implies \pi_* E_n^{hH}$$



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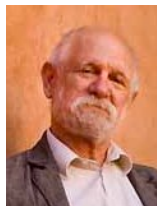
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The group $\mathbf{G}_n$ is known to have a subgroup of order p (unique up to conjugacy) when $p - 1$ divides n .

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The role of the Morava stabilizer group (continued)

The group $\mathbf{G}_n$ is known to have a subgroup of order p (unique up to conjugacy) when $p - 1$ divides n . This leads to a composite homomorphism, [the detection map](#)

$$\mathrm{Ext}_{BP_*(BP)} \longrightarrow H^*(C_p; \pi_* E_{p-1}) \longrightarrow H^*(C_p; \mathbf{F}_{p^{p-1}}[u, u^{-1}])$$

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where the second homomorphism is reduction modulo the maximal ideal in $\pi_* E_{p-1}$ and $|u| = 2$.

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where the second homomorphism is reduction modulo the maximal ideal in $\pi_* E_{p-1}$ and $|u| = 2$. The action of C_p here is trivial, so the target is a bigraded form of the usual mod p cohomology of C_p . For p odd this cohomology is

$$E(\alpha) \otimes P(\beta) \otimes \mathbf{F}_{p^{p-1}}[u, u^{-1}]$$

where $\alpha \in H^1$ and $\beta \in H^2$ each have topological degree 0.

The role of the Morava stabilizer group (continued)

Again we have the detection map

$$\mathrm{Ext}_{BP_*}(BP) \longrightarrow H^*(C_p; \pi_* E_{p-1}) \longrightarrow H^*(C_p; \mathbf{F}_{p^{p-1}}[u, u^{-1}]) \quad (5)$$

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$$\mathrm{Ext}_{BP_*}(BP) \longrightarrow H^*(C_p; \pi_* E_{p-1}) \longrightarrow H^*(C_p; \mathbf{F}_{p^{p-1}}[u, u^{-1}]) \quad (5)$$

We showed that under this map we have

$$\begin{array}{ll} \alpha_1 & \mapsto u^{p-1} \alpha \\ \widehat{\theta}_j \beta_{p^{j-1}/p^{j-1}} & \mapsto u^{p^j(p-1)} \beta \end{array} \quad (6)$$

up to unit scalar.

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up to unit scalar. Hence all monomials in the $\widehat{\theta}_j$ and their products with α_1 have nontrivial images.

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up to unit scalar. Hence all monomials in the $\widehat{\theta}_j$ and their products with α_1 have nontrivial images. This implies that the differentials

$$d_{2p-1}(\widehat{\theta}_j) = \alpha_1 \widehat{\theta}_{j-1}^p$$

are nontrivial as desired.

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To summarize:

- The existence of an element of order p in S_{p-1} leads to the detection map of (5),

$$\begin{array}{ccc} \mathrm{Ext}_{BP_*}(BP) & \longrightarrow & H^*(C_p; \mathbf{F}_{p^{p-1}}[u, u^{-1}]) \\ & & \parallel \\ & & E(\alpha) \otimes P(\beta) \otimes \mathbf{F}_{p^{p-1}}[u, u^{-1}] \end{array}$$

$$\alpha_1 \longmapsto u^{p-1} \alpha$$

$$\hat{\theta}_j \longmapsto u^{p^j(p-1)} \beta$$

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$$\alpha_1 \longmapsto u^{p-1} \alpha$$

$$\hat{\theta}_j \longmapsto u^{p^j(p-1)} \beta$$

- The multiplicative relations among the $\hat{\theta}_j$ and the Toda differential on $\hat{\theta}_2$ lead to differentials on all higher $\hat{\theta}_j$.

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To summarize:

- The existence of an element of order p in S_{p-1} leads to the detection map of (5),

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- The multiplicative relations among the $\hat{\theta}_j$ and the Toda differential on $\hat{\theta}_2$ lead to differentials on all higher $\hat{\theta}_j$. They are nontrivial by the detection data above.

Difficulties at $p = 3$

Why does this approach fail for $p < 5$?

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Why does this approach fail for $p < 5$?

- For $p = 2$, the target of the Toda “differential”,

$$d_3(\hat{\theta}_2) = \alpha_1 \hat{\theta}_1^2 = 0,$$

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so this method does not show that any $\hat{\theta}_j$ fails to be a permanent cycle.

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- The group $\text{Ext}_{BP_*(BP)}^{2,2p^j(p-1)}$ is known to have $[(j-1)/2]$ other generators besides $\hat{\theta}_j$.

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- The group $\text{Ext}_{BP_*(BP)}^{2,2p^j(p-1)}$ is known to have $[(j-1)/2]$ other generators besides $\hat{\theta}_j$. For $p = 3$ these other generators, such as β_7 in the bidegree of $\hat{\theta}_3$, can have nontrivial images under the detection map.

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Difficulties at $p = 3$ (continued)

In order to describe the way out of these difficulties we need to say more about finite subgroups of $\mathbf{G}_n$.

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Difficulties at $p = 3$ (continued)

In order to describe the way out of these difficulties we need to say more about finite subgroups of $\mathbf{G}_n$. It is by definition an extension of the Morava stabilizer group $\mathbf{S}_n$ by $\text{Gal}(\mathbf{F}_{p^n} : \mathbf{F}_p)$.

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In order to describe the way out of these difficulties we need to say more about finite subgroups of $\mathbf{G}_n$. It is by definition an extension of the Morava stabilizer group $\mathbf{S}_n$ by $\text{Gal}(\mathbf{F}_{p^n} : \mathbf{F}_p)$. The Galois group (which is cyclic of order n) is there for technical reasons but plays no role on our calculations. $\mathbf{S}_n$ is the group of units in the maximal order of a certain division algebra over the p -adic numbers $\mathbf{Q}_p$. Its finite subgroups have been classified by Hewett.

$\mathbf{S}_n$ has an element of order p iff $p - 1$ divides n , a condition that is trivial when $p = 2$.

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$\mathbf{S}_n$ has an element of order p iff $p - 1$ divides n , a condition that is trivial when $p = 2$. More generally $\mathbf{S}_n$ has an element of order p^{k+1} iff $p^k(p - 1)$ divides n .

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$$\text{Ext}_{BP_*(BP)} \longrightarrow H^*(C_{p^{k+1}}; \pi_* E_n) \longrightarrow H^*(C_{p^{k+1}}; ?),$$

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$$\text{Ext}_{BP_*(BP)} \longrightarrow H^*(C_{p^{k+1}}; \pi_* E_n) \longrightarrow H^*(C_{p^{k+1}}; ?),$$

for some coefficient ring in the target.

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$$\text{Ext}_{BP_*(BP)} \longrightarrow H^*(C_{p^{k+1}}; \pi_* E_n) \longrightarrow H^*(C_{p^{k+1}}; ?),$$

for some coefficient ring in the target. The naive choice of $\mathbf{F}_{p^n}[u, u^{-1}]$ for this ring turns out not to detect $\hat{\theta}_j$ for $n > p - 1$.

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Again, for n divisible by $p^k(p - 1)$ we have a detection map

$$\mathrm{Ext}_{BP_*(BP)} \longrightarrow H^*(C_{p^{k+1}}; \pi_* E_n) \longrightarrow H^*(C_{p^{k+1}}; ?).$$

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$$\mathrm{Ext}_{BP_*(BP)} \longrightarrow H^*(C_{p^{k+1}}; \pi_* E_n) \longrightarrow H^*(C_{p^{k+1}}; ?).$$

Experience has shown two things:

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$$\mathrm{Ext}_{BP_*(BP)} \longrightarrow H^*(C_{p^{k+1}}; \pi_* E_n) \longrightarrow H^*(C_{p^{k+1}}; ?).$$

Experience has shown two things:

- (i) In order to flush out the spurious elements (which are v_2 -periodic) having the same bidegree as $\widehat{\theta}_j$, we need to have $n > 2$.

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Experience has shown two things:

- (i) In order to flush out the spurious elements (which are v_2 -periodic) having the same bidegree as $\widehat{\theta}_j$, we need to have $n > 2$.
- (ii) In order to detect the $\widehat{\theta}_j$ itself, we need to have n be equal to $p^k(p - 1)$ for some $k \geq 0$, not just be divisible by it.

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- (ii) In order to detect the $\widehat{\theta}_j$ itself, we need to have n be equal to $p^k(p-1)$ for some $k \geq 0$, not just be divisible by it. Then $\widehat{\theta}_j$ will map to an element of order p in a cohomology group isomorphic to $\mathbf{Z}/p^{k+1}$.

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For $p = 2$ these considerations suggest using the group C_8 and $n = 4$, which is the approach used in [HHR].

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For $p = 2$ these considerations suggest using the group C_8 and $n = 4$, which is the approach used in [HHR].

For $p = 3$ we need to use the group C_9 with $n = 6$.

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Difficulties at $p = 3$ (continued)

For the prime 2, our strategy in [HHR] was to construct a ring spectrum Ω with a unit map $S^0 \rightarrow \Omega$ satisfying three properties:

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Difficulties at $p = 3$ (continued)

For the prime 2, our strategy in [HHR] was to construct a ring spectrum Ω with a unit map $S^0 \rightarrow \Omega$ satisfying three properties:

- (i) DETECTION THEOREM. If θ_j exists, its image in $\pi_*\Omega$ is nontrivial.

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For the prime 2, our strategy in [HHR] was to construct a ring spectrum Ω with a unit map $S^0 \rightarrow \Omega$ satisfying three properties:

- (i) DETECTION THEOREM. If θ_j exists, its image in $\pi_*\Omega$ is nontrivial.
- (ii) PERIODICITY THEOREM. $\pi_k\Omega$ depends only on the congruence class of k modulo 256.

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- (ii) PERIODICITY THEOREM. $\pi_k\Omega$ depends only on the congruence class of k modulo 256.
- (iii) GAP THEOREM. $\pi_{-2}\Omega = 0$.

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The nonexistence of θ_j for $j \geq 7$ follows from the fact that its dimension is congruent to -2 modulo 256.

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Ever since the discovery of the Hopkins-Miller theorem, it has been possible to prove that $E_4^{hC_8}$ satisfies the first two of these properties

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The nonexistence of θ_j for $j \geq 7$ follows from the fact that its dimension is congruent to -2 modulo 256.

Ever since the discovery of the Hopkins-Miller theorem, it has been possible to prove that $E_4^{hC_8}$ satisfies the first two of these properties **without the use of equivariant stable homotopy theory**.

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For $p = 3$, the same goes for $E_6^{hC_9}$ with the periodicity dimension being 972 (2 more than the dimension of θ_5) instead of 256.

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Difficulties at $p = 3$ (continued)

For $p = 3$, the same goes for $E_6^{hC_9}$ with the periodicity dimension being 972 (2 more than the dimension of θ_5) instead of 256. If all goes well, we would get a theorem saying θ_j does not exist for $j \geq 5$, leaving the status of θ_4 (in the 322-stem) open.

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For $p \geq 5$, the same holds for $E_{p-1}^{hC_p}$ with periodicity $2p^2(p-1)$, which is 2 more than the dimension of θ_2 .

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For $p \geq 5$, the same holds for $E_{p-1}^{hC_p}$ with periodicity $2p^2(p-1)$, which is 2 more than the dimension of θ_2 . In this case the spectrum also detects the product of α_1 with any monomial in the θ_j s.

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For $p \geq 5$, the same holds for $E_{p-1}^{hC_p}$ with periodicity $2p^2(p-1)$, which is 2 more than the dimension of θ_2 . In this case the spectrum also detects the product of α_1 with any monomial in the θ_j s. As explained above, this enables us to use Toda's differential to show that none of the θ_j for $j > 1$ exists.

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For $p \geq 5$, the same holds for $E_{p-1}^{hC_p}$ with periodicity $2p^2(p-1)$, which is 2 more than the dimension of θ_2 . In this case the spectrum also detects the product of α_1 with any monomial in the θ_j s. As explained above, this enables us to use Toda's differential to show that none of the θ_j for $j > 1$ exists.

We cannot use Toda's differential for $p < 5$ because

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For $p = 3$, the same goes for $E_6^{hC_9}$ with the periodicity dimension being 972 (2 more than the dimension of θ_5) instead of 256. If all goes well, we would get a theorem saying θ_j does not exist for $j \geq 5$, leaving the status of θ_4 (in the 322-stem) open. We already know that θ_1 (in the 10-stem) and θ_3 (in the 106-stem) exist while θ_2 (in the 34-stem) does not.

For $p \geq 5$, the same holds for $E_{p-1}^{hC_p}$ with periodicity $2p^2(p-1)$, which is 2 more than the dimension of θ_2 . In this case the spectrum also detects the product of α_1 with any monomial in the θ_j s. As explained above, this enables us to use Toda's differential to show that none of the θ_j for $j > 1$ exists.

We cannot use Toda's differential for $p < 5$ because

(a) for $p = 2$ its target is trivial, and

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For $p = 3$, the same goes for $E_6^{hC_9}$ with the periodicity dimension being 972 (2 more than the dimension of θ_5) instead of 256. If all goes well, we would get a theorem saying θ_j does not exist for $j \geq 5$, leaving the status of θ_4 (in the 322-stem) open. We already know that θ_1 (in the 10-stem) and θ_3 (in the 106-stem) exist while θ_2 (in the 34-stem) does not.

For $p \geq 5$, the same holds for $E_{p-1}^{hC_p}$ with periodicity $2p^2(p-1)$, which is 2 more than the dimension of θ_2 . In this case the spectrum also detects the product of α_1 with any monomial in the θ_j s. As explained above, this enables us to use Toda's differential to show that none of the θ_j for $j > 1$ exists.

We cannot use Toda's differential for $p < 5$ because

- (a) for $p = 2$ its target is trivial, and
- (b) since we cannot detect products of the θ_j s, we cannot make an inductive argument.

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The proof of the Gap Theorem requires the use of equivariant stable homotopy theory and the slice filtration.

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The proof of the Gap Theorem requires the use of equivariant stable homotopy theory and the slice filtration. The slice filtration is an equivariant analogue of the classical Postnikov filtration.

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The proof of the Gap Theorem requires the use of equivariant stable homotopy theory and the slice filtration. The slice filtration is an equivariant analogue of the classical Postnikov filtration. Analyzing it for a general equivariant spectrum is difficult.

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The proof of the Gap Theorem requires the use of equivariant stable homotopy theory and the slice filtration. The slice filtration is an equivariant analogue of the classical Postnikov filtration. Analyzing it for a general equivariant spectrum is difficult. We do **not** know how to do it directly for the case of interest, the Morava E_6 at $p = 3$ as a C_9 -spectrum,

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The proof of the Gap Theorem requires the use of equivariant stable homotopy theory and the slice filtration. The slice filtration is an equivariant analogue of the classical Postnikov filtration. Analyzing it for a general equivariant spectrum is difficult. We do **not** know how to do it directly for the case of interest, the Morava E_6 at $p = 3$ as a C_9 -spectrum, or for E_4 at $p = 2$ for the group C_8 .

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The proof of the Gap Theorem requires the use of equivariant stable homotopy theory and the slice filtration. The slice filtration is an equivariant analogue of the classical Postnikov filtration. Analyzing it for a general equivariant spectrum is difficult. We do **not** know how to do it directly for the case of interest, the Morava E_6 at $p = 3$ as a C_9 -spectrum, or for E_4 at $p = 2$ for the group C_8 .

We **do** know how to do it for MU_R , which is MU as a C_2 -spectrum via complex conjugation,

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The proof of the Gap Theorem requires the use of equivariant stable homotopy theory and the slice filtration. The slice filtration is an equivariant analogue of the classical Postnikov filtration. Analyzing it for a general equivariant spectrum is difficult. We do **not** know how to do it directly for the case of interest, the Morava E_6 at $p = 3$ as a C_9 -spectrum, or for E_4 at $p = 2$ for the group C_8 .

We **do** know how to do it for $MU_{\mathbf{R}}$, which is MU as a C_2 -spectrum via complex conjugation, and for $N_2^{2^{n+1}} MU_{\mathbf{R}}$, which is underlain by $MU^{(2^n)}$ with a $C_{2^{n+1}}$ -action.

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The proof of the Gap Theorem requires the use of equivariant stable homotopy theory and the slice filtration. The slice filtration is an equivariant analogue of the classical Postnikov filtration. Analyzing it for a general equivariant spectrum is difficult. We do **not** know how to do it directly for the case of interest, the Morava E_6 at $p = 3$ as a C_9 -spectrum, or for E_4 at $p = 2$ for the group C_8 .

We **do** know how to do it for $MU_{\mathbf{R}}$, which is MU as a C_2 -spectrum via complex conjugation, and for $N_2^{2^{n+1}} MU_{\mathbf{R}}$, which is underlain by $MU^{(2^n)}$ with a $C_{2^{n+1}}$ -action. A crucial step here is the **Reduction Theorem**, which says roughly that if we kill all of the underlying homotopy groups in positive dimensions in a certain equivariant way, we get the equivariant Eilenberg-Mac Lane spectrum $H\mathbb{Z}$.

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In order to do a similar thing at an $p = 3$ we need an analog $MU_{\equiv}$ of the C_2 -spectrum MU_R .

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In order to do a similar thing at an $p = 3$ we need an analog MU_{Ξ} of the C_2 -spectrum MU_R . It should be a C_3 -spectrum underlain by $MU^{(2)}$ with two properties:

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In order to do a similar thing at an $p = 3$ we need an analog MU_{Ξ} of the C_2 -spectrum MU_R . It should be a C_3 -spectrum underlain by $MU^{(2)}$ with two properties:

- (i) It has a tractable slice filtration with a certain description.

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In order to do a similar thing at an $p = 3$ we need an analog $MU_{\equiv}$ of the C_2 -spectrum MU_R . It should be a C_3 -spectrum underlain by $MU^{(2)}$ with two properties:

- (i) It has a tractable slice filtration with a certain description.
- (ii) Its geometric fixed point spectrum $MU_{\equiv}^{gC_3}$ is a wedge of suspensions of $H/3$, the mod 3 Eilenberg-Mac Lane spectrum.

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- (i) It has a tractable slice filtration with a certain description.
- (ii) Its geometric fixed point spectrum $MU_{\equiv}^{gC_3}$ is a wedge of suspensions of $H/3$, the mod 3 Eilenberg-Mac Lane spectrum. For $p = 2$ we have $MU_R^{gC_2} = MO$, the unoriented cobordism spectrum, which fits this description.

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We do not know how to construct this spectrum!



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We do not know how to construct this spectrum!

It is our missing piece.



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We do not know how to construct this spectrum!

It is our missing piece. Maybe you can find it.



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Let's suppose the hypothetical $MU_{\equiv}$ exists as described above.

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Let's suppose the hypothetical MU_{Ξ} exists as described above.
For convenience we will work with its BP analog, BP_{Ξ} .

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Let's suppose the hypothetical MU_{Ξ} exists as described above.
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A useful technical notion.

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Let's suppose the hypothetical MU_{Ξ} exists as described above. For convenience we will work with its BP analog, BP_{Ξ} .

A useful technical notion. Let E be a connective equivariant spectrum with $\pi_*^u E$ (its underlying homotopy groups) free abelian.

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Let's suppose the hypothetical MU_{Ξ} exists as described above. For convenience we will work with its BP analog, BP_{Ξ} .

A useful technical notion. Let E be a connective equivariant spectrum with $\pi_*^U E$ (its underlying homotopy groups) free abelian. A **refinement** of this group is an equivariant map $W \rightarrow E$ where W is underlain by a wedge of spheres mapping to the generators of $\pi_*^U E$.

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Let's suppose the hypothetical MU_{Ξ} exists as described above. For convenience we will work with its BP analog, BP_{Ξ} .

A useful technical notion. Let E be a connective equivariant spectrum with $\pi_*^u E$ (its underlying homotopy groups) free abelian. A **refinement** of this group is an equivariant map $W \rightarrow E$ where W is underlain by a wedge of spheres mapping to the generators of $\pi_*^u E$. The **reduction theorem** for E is the statement that the map

$$E \wedge_W S^0 \rightarrow H\underline{\mathbb{Z}}$$

is an equivariant equivalence.

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If all goes according to plan, $\pi_*^u BP_{\Xi}$ is refined by a map from

$$W = \bigwedge_{n \geq 1} W_n$$

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If all goes according to plan, $\pi_*^u BP_{\Xi}$ is refined by a map from

$$W = \bigwedge_{n \geq 1} W_n$$

with

$$W_n = S^0 \left[\overline{S}^{2 \cdot 3^{n-1} \rho - 1} \right]$$

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where

- ρ denotes the regular representation of C_3 and

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If all goes according to plan, $\pi_*^u BP_{\Xi}$ is refined by a map from

$$W = \bigwedge_{n \geq 1} W_n$$

with

$$W_n = S^0 \left[\overline{S}^{2 \cdot 3^{n-1} \rho - 1} \right]$$

where

- ρ denotes the regular representation of C_3 and
- $\overline{S}^V$ denotes the codimension one skeleton of S^V .

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For $n = 1$ we have

$$W_1 = S^0 \left[\overline{S}^{2\rho-1} \right].$$

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For $n = 1$ we have

$$W_1 = S^0 \left[\overline{S}^{2\rho-1} \right].$$

Here $\overline{S}^{2\rho-1}$ is underlain by $S^4 \vee S^4$,

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For $n = 1$ we have

$$W_1 = S^0 \left[\overline{S}^{2\rho-1} \right].$$

Here $\overline{S}^{2\rho-1}$ is underlain by $S^4 \vee S^4$, and W_1 is underlain by a wedge of spheres with $k + 1$ summands in dimension $4k$ for each $k \geq 0$.

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For $n = 1$ we have

$$W_1 = S^0 \left[\overline{S}^{2\rho-1} \right].$$

Here $\overline{S}^{2\rho-1}$ is underlain by $S^4 \vee S^4$, and W_1 is underlain by a wedge of spheres with $k + 1$ summands in dimension $4k$ for each $k \geq 0$. There is a C_3 -action on the space $X = S^5 \times S^5$ such that $W_1 = \Sigma^\infty \Omega X$.

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Here $\overline{S}^{2\rho-1}$ is underlain by $S^4 \vee S^4$, and W_1 is underlain by a wedge of spheres with $k + 1$ summands in dimension $4k$ for each $k \geq 0$. There is a C_3 -action on the space $X = S^5 \times S^5$ such that $W_1 = \Sigma^\infty \Omega X$.

Equivariantly we have

$$W_1 = S^0 [S^{4\rho}] \wedge \left(S^0 \vee \overline{S}^{2\rho-1} \vee C_{3+} \wedge \left(\bigvee_{i \geq 2} S^{4i} \right) \right).$$

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Equivariantly we have

$$W_1 = S^0 [S^{4\rho}] \wedge \left(S^0 \vee \overline{S}^{2\rho-1} \vee C_{3+} \wedge \left(\bigvee_{i \geq 2} S^{4i} \right) \right).$$

Free summands here contribute torsion free summands to $\pi_* E^{C_3}$,

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For $n = 1$ we have

$$W_1 = S^0 [\overline{S}^{2\rho-1}].$$

Here $\overline{S}^{2\rho-1}$ is underlain by $S^4 \vee S^4$, and W_1 is underlain by a wedge of spheres with $k+1$ summands in dimension $4k$ for each $k \geq 0$. There is a C_3 -action on the space $X = S^5 \times S^5$ such that $W_1 = \Sigma^\infty \Omega X$.

Equivariantly we have

$$W_1 = S^0 [S^{4\rho}] \wedge \left(S^0 \vee \overline{S}^{2\rho-1} \vee C_{3+} \wedge \left(\bigvee_{i \geq 2} S^{4i} \right) \right).$$

Free summands here contribute torsion free summands to $\pi_* E^{C_3}$, so they are irrelevant to the Kervaire invariant problem.

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Hence will ignore the free summands in W_1 and replace it by

$$W'_1 = S^0 [S^{4\rho}] \wedge (S^0 \vee \overline{S}^{2\rho-1}).$$

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Hence will ignore the free summands in W_1 and replace it by

$$W'_1 = S^0 [S^{4\rho}] \wedge (S^0 \vee \overline{S}^{2\rho-1}).$$

We have a map

$$W'_1 = S^0 [S^{4\rho}] \wedge (S^0 \vee \overline{S}^{2\rho-1}) \longrightarrow BP_{\equiv}$$

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Hence will ignore the free summands in W_1 and replace it by

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We have a map

$$W'_1 = S^0 [S^{4\rho}] \wedge (S^0 \vee \overline{S}^{2\rho-1}) \longrightarrow BP_{\Xi}$$

Thus there is an element $Nv_1 \in \pi_{4\rho} BP_{\Xi}$.

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Hence will ignore the free summands in W_1 and replace it by

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$$W'_1 = S^0 [S^{4\rho}] \wedge (S^0 \vee \overline{S}^{2\rho-1}) \longrightarrow BP_{\Xi}$$

Thus there is an element $Nv_1 \in \pi_{4\rho} BP_{\Xi}$. We can invert it and throw away the higher generators.

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Hence will ignore the free summands in W_1 and replace it by

$$W'_1 = S^0 [S^{4\rho}] \wedge (S^0 \vee \overline{S}^{2\rho-1}).$$

We have a map

$$W'_1 = S^0 [S^{4\rho}] \wedge (S^0 \vee \overline{S}^{2\rho-1}) \longrightarrow BP_{\Xi}$$

Thus there is an element $Nv_1 \in \pi_{4\rho} BP_{\Xi}$. We can invert it and throw away the higher generators. The resulting fixed point spectrum looks a lot like tmf , but with periodicity in dimension 36 instead of 72.

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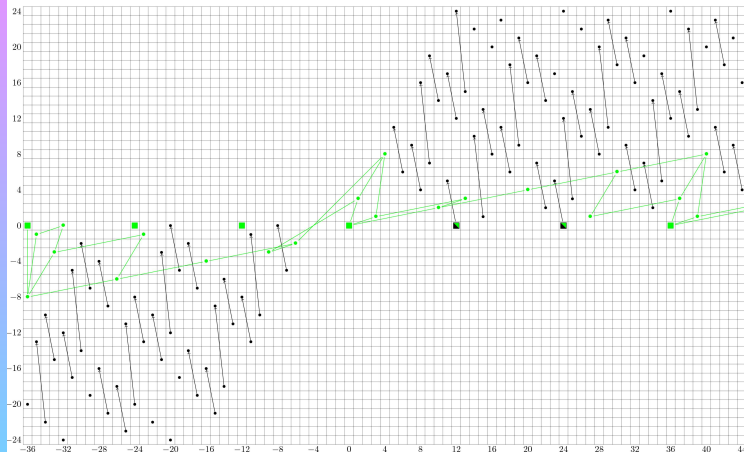
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Here is its slice spectral sequence.



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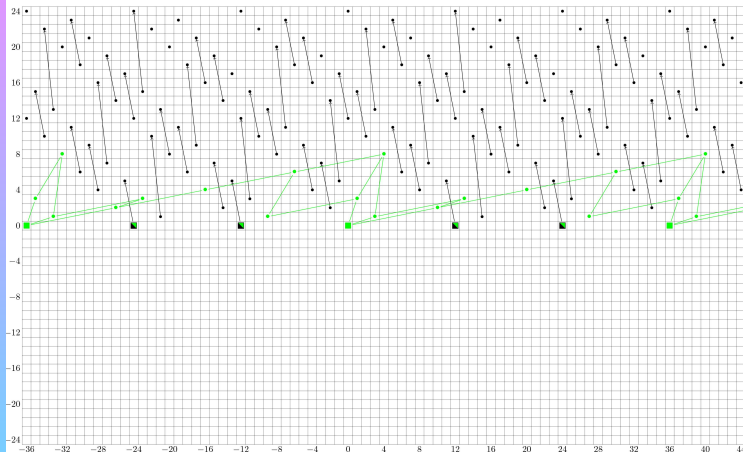
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Here is its homotopy fixed point spectral sequence.



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Now we need to norm up from C_3 to C_9 .

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Now we need to norm up from C_3 to C_9 . Recall that

$$W'_1 = \left(S^0 \vee \overline{S}^{2 \cdot \rho - 1} \right) \wedge S^0 [S^{4\rho}].$$

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$$W'_1 = \left(S^0 \vee \overline{S}^{2 \cdot \rho - 1} \right) \wedge S^0 [S^{4\rho}].$$

The norm functor commutes with smash products.

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$$W'_1 = \left(S^0 \vee \overline{S}^{2 \cdot \rho - 1} \right) \wedge S^0 [S^{4\rho}].$$

The norm functor commutes with smash products. For the first factor we have

$$\begin{aligned} N_3^9 \left(S^0 \vee \overline{S}^{2\rho-1} \right) &= S^0 \vee \left(C_{9+} \wedge_{C_3} \left(\overline{S}^{2\rho-1} \vee S^{3\rho-1} \right) \right) \\ &\quad \vee \left(C_{9+} \wedge S^8 \right) \vee \overline{S}^{\rho_9+2\lambda} \end{aligned}$$

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where λ denotes the 2-dimensional representation of C_9 with a rotation of order 9.

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Now we need to norm up from C_3 to C_9 . Recall that

$$W'_1 = \left(S^0 \vee \overline{S}^{2 \cdot \rho - 1} \right) \wedge S^0 [S^{4\rho}].$$

The norm functor commutes with smash products. For the first factor we have

$$\begin{aligned} N_3^9 \left(S^0 \vee \overline{S}^{2\rho-1} \right) &= S^0 \vee \left(C_{9+} \underset{C_3}{\wedge} \left(\overline{S}^{2\rho-1} \vee S^{3\rho-1} \right) \right) \\ &\quad \vee \left(C_{9+} \wedge S^8 \right) \vee \overline{S}^{\rho_9+2\lambda} \end{aligned}$$

where λ denotes the 2-dimensional representation of C_9 with a rotation of order 9.

For the second factor of W'_1 ,

$$N_3^9 S^0 [S^{4\rho}] = S^0 [S^{4\rho_9}] \wedge \left(S^0 \vee \left(C_{9+} \underset{C_3}{\wedge} \bigvee_{i,j \geq 0} S^{4(i+j+1)\rho} \right) \right).$$

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After inverting the right element in $\pi_{4\rho_9}$, we get a spectrum $\tilde{\Omega}$ whose fixed point set Ω is 972-periodic and detects the θ_j for $j \geq 5$.

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After inverting the right element in $\pi_{4\rho_9}$, we get a spectrum $\tilde{\Omega}$ whose fixed point set Ω is 972-periodic and detects the θ_j for $j \geq 5$. The key question here is

Do we get a Gap Theorem stating that $\pi_{-2}\Omega$ is torsion free?

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Do we get a Gap Theorem stating that $\pi_{-2}\Omega$ is torsion free?

To answer this we need to look at the equivariant homotopy groups of

$$X \wedge \Sigma^{4m\rho_9} H\underline{\mathbb{Z}} \quad \text{and} \quad X \wedge C_{9+} \bigwedge_{C_3} \Sigma^{4n\rho_3} H\underline{\mathbb{Z}}$$

for $m, n \in \mathbb{Z}$, where X is one of the following:

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for $m, n \in \mathbf{Z}$, where X is one of the following:

$$S^0, \quad C_{9+} \bigwedge_{C_3} \overline{S}^{2\rho-1}, \quad C_{9+} \bigwedge_{C_3} S^{3\rho-1} \quad \text{or} \quad \overline{S}^{\rho_9+2\lambda}.$$

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A possible Gap Theorem (continued)

The following table indicates the dimensions in which

$$\pi_i X \wedge \Sigma^{4m\rho_9} H\underline{\mathbb{Z}} \quad \text{and} \quad \pi_i X \wedge C_{9+} \bigwedge_{C_3} \Sigma^{4n\rho_3} H\underline{\mathbb{Z}}$$

can be nontrivial for $m, n \geq 0$, with one caveat as indicated below.

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can be nontrivial for $m, n \geq 0$, with one caveat as indicated below.

X	$\pi_i X \wedge \Sigma^{4m\rho_9} H\underline{\mathbb{Z}}$	$\pi_i X \wedge C_{9+} \wedge_{C_3} \Sigma^{4n\rho_3} H\underline{\mathbb{Z}}$
S^0	$4m \leq i \leq 36m$	$4n \leq i \leq 12n$
$C_{9+} \wedge_{C_3} \overline{S}^{2\rho-1}$	$12m + 1 \leq i \leq 36m + 4$	$4n + 1 \leq i \leq 12n + 4$
$C_{9+} \wedge_{C_3} S^{3\rho-1}$	$12m + 2 \leq i \leq 36m + 8$	$4n + 2 \leq i \leq 12n + 8$
$\overline{S}^{\rho_9+2\lambda}$	$4m + 1 \leq i \leq 36m + 12$	$4n + 3 \leq i \leq 12n + 12$ for $n \geq -1$

A possible Gap Theorem (continued)

Here is a similar table for $m, n \leq -1$.

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Here is a similar table for $m, n \leq -1$.

X	$\pi_j X \wedge \Sigma^{4m\rho_9} H\underline{Z}$	$\pi_j X \wedge C_{9+} \wedge_{C_3} \Sigma^{4n\rho_3} H\underline{Z}$
S^0	$36m \leq i \leq 4m - 3$	$12n \leq i \leq 4n - 3$
$C_{9+} \wedge_{C_3} \overline{S}^{2\rho-1}$	$36m + 4 \leq i \leq 12m - 2$	$12n + 4 \leq i \leq 4n - 2$
$C_{9+} \wedge_{C_3} \overline{S}^{3\rho-1}$	$36m + 8 \leq i \leq 12m - 1$	$12n + 8 \leq i \leq 4n - 1$
$\overline{S}^{\rho_9+2\lambda}$	$36m + 12 \leq i \leq 4m - 2$	$12n + 12 \leq i \leq 4n$ for $n \leq -2$

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Here is a similar table for $m, n \leq -1$.

X	$\pi_j X \wedge \Sigma^{4m\rho_9} H\underline{Z}$	$\pi_j X \wedge C_{9+} \wedge_{C_3} \Sigma^{4n\rho_3} H\underline{Z}$
S^0	$36m \leq i \leq 4m - 3$	$12n \leq i \leq 4n - 3$
$C_{9+} \wedge_{C_3} \overline{S}^{2\rho-1}$	$36m + 4 \leq i \leq 12m - 2$	$12n + 4 \leq i \leq 4n - 2$
$C_{9+} \wedge_{C_3} S^{3\rho-1}$	$36m + 8 \leq i \leq 12m - 1$	$12n + 8 \leq i \leq 4n - 1$
$\overline{S}^{\rho_9+2\lambda}$	$36m + 12 \leq i \leq 4m - 2$	$12n + 12 \leq i \leq 4n$ for $n \leq -2$

In each case the upper bound here is 3 less than the corresponding lower bound in the previous table.

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A possible Gap Theorem (continued)

Here is a similar table for $m, n \leq -1$.

X	$\pi_j X \wedge \Sigma^{4m\rho_9} H\underline{Z}$	$\pi_j X \wedge C_{9+} \wedge_{C_3} \Sigma^{4n\rho_3} H\underline{Z}$
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$C_{9+} \wedge_{C_3} \overline{S}^{3\rho-1}$	$36m + 8 \leq i \leq 12m - 1$	$12n + 8 \leq i \leq 4n - 1$
$\overline{S}^{\rho_9+2\lambda}$	$36m + 12 \leq i \leq 4m - 2$	$12n + 12 \leq i \leq 4n$ for $n \leq -2$

In each case the upper bound here is 3 less than the corresponding lower bound in the previous table. The calculation behind this is the same for $p = 3$ as it was for $p = 2$.

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Here is a similar table for $m, n \leq -1$.

X	$\pi_j X \wedge \Sigma^{4m\rho_9} H\underline{Z}$	$\pi_j X \wedge C_{9+} \wedge_{C_3} \Sigma^{4n\rho_3} H\underline{Z}$
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$C_{9+} \wedge_{C_3} \overline{S}^{2\rho-1}$	$36m + 4 \leq i \leq 12m - 2$	$12n + 4 \leq i \leq 4n - 2$
$C_{9+} \wedge_{C_3} \overline{S}^{3\rho-1}$	$36m + 8 \leq i \leq 12m - 1$	$12n + 8 \leq i \leq 4n - 1$
$\overline{S}^{\rho_9+2\lambda}$	$36m + 12 \leq i \leq 4m - 2$	$12n + 12 \leq i \leq 4n$ for $n \leq -2$

In each case the upper bound here is 3 less than the corresponding lower bound in the previous table. The calculation behind this is the same for $p = 3$ as it was for $p = 2$.

Since $m, n \leq -1$, our upper bound is always ≤ -4 , so we have the desired Gap Theorem.

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Here is a color coded illustration of these fixed point homotopy groups.

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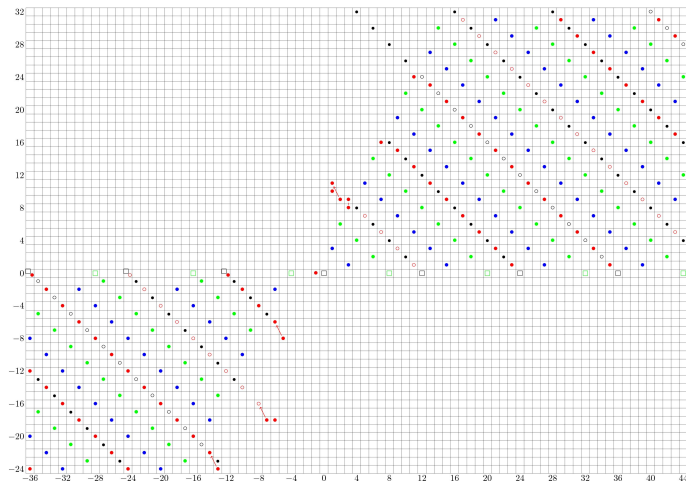
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