Math 443

What is an ∞-category?

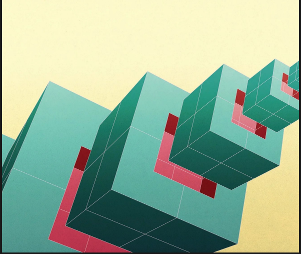

Image appearing in an article by Emily Riehl

in *Scientific American*, October 2021

Doug Ravenel
University of Rochester

December 1, 3 and 8, 2025

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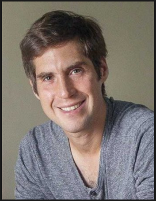

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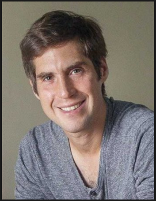

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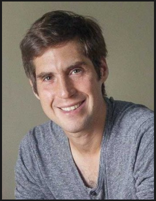

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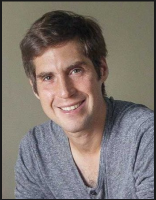

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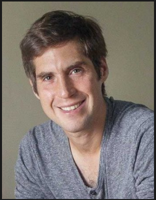

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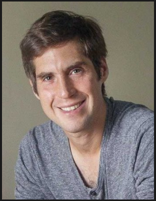

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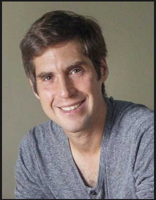

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The inner faces and horns are those for which $0 < i < n$.

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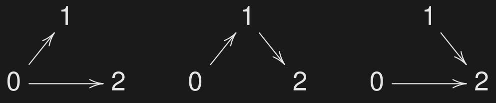

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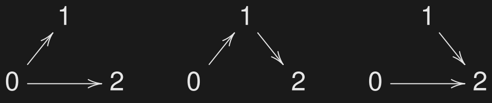

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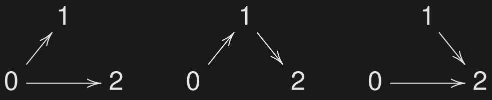

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A Kan complex is a simplicial set X for which every map from a horn $\Lambda_i^n \to X$ extends to Δ^n.

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The topological n-simplex Δ^n_{top} is the space

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Review of simplicial sets (continued)

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Hence $\mathcal{S}\mathrm{et}_\Delta$ is enriched over itself.

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and define NC_n to be the set of functors from $[n]$ to C.

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This simplicial set has the following property: Any simplicial map $\Lambda_i^n \to NC$ for $0 < i < n$ extends uniquely to Δ^n.

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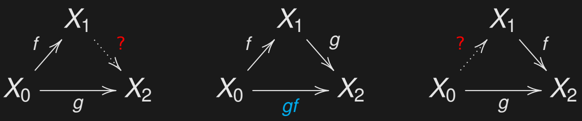

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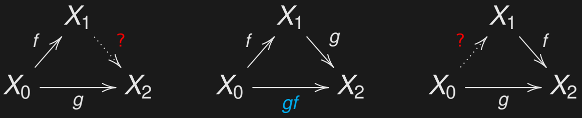

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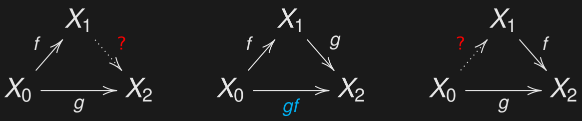

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A small category is thus equivalent to a simplicial set (its nerve) in which each map from an inner horn Λ_i^n extends uniquely to a map from Δ^n.

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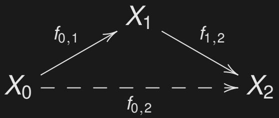

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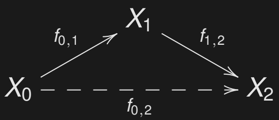

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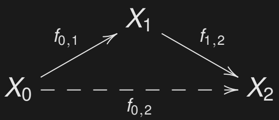

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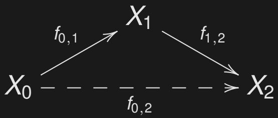

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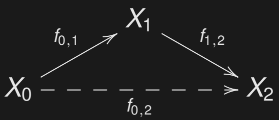

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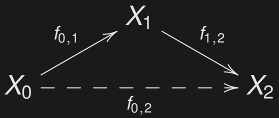

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An ∞*-category (also called a quasicategory)* $\mathcal{C}$ *is a simplicial set in which each simplicial map* $\Lambda_i^n \to \mathcal{C}$ *for* $0 < i < n$ *extends to some map* $\Delta^n \to \mathcal{C}$*. A functor* $F : \mathcal{C} \to \mathcal{C}'$ *from one* ∞*-category to another is a simplicial map.*

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Such an ∞-category is called an ∞-groupoid because all morphisms, i.e., paths in X, are invertible up to homotopy.

The ∞-category of topological spaces

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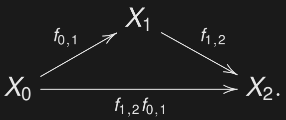

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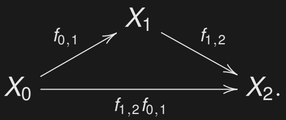

The top two edges can be viewed as a map $\Lambda^1_2 \to N\mathrm{Top}$, with the full diagram being its unique extension to Δ^2.

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$N\mathbf{Top}_2$ is the set of commutative diagrams of the form

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$$\begin{array}{ccc} I & \xrightarrow{\widehat{h}_{0,2}} & \mathbf{Top}(X_0, X_2) \\ 0 & \longmapsto & f_{1,2}f_{0,1} \\ 1 & \longmapsto & f_{0,2} \end{array}$$

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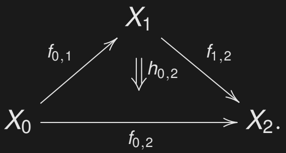

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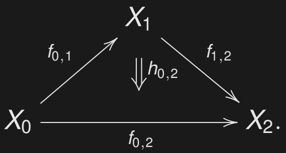

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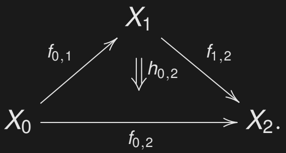

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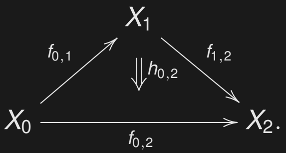

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The set of 3-simplices in $\mathcal{S}$

The following diagram shows four 2-simplices with their homotopies.

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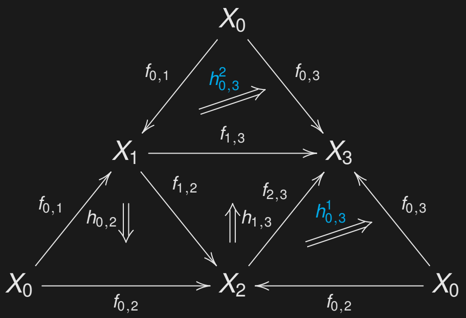

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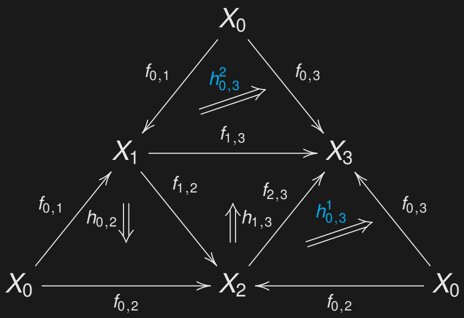

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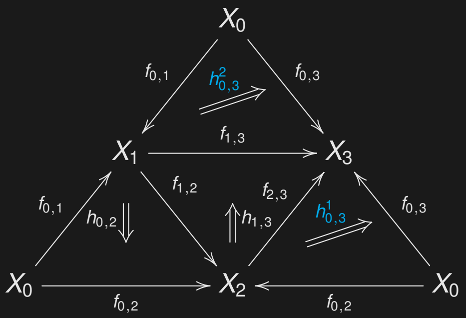
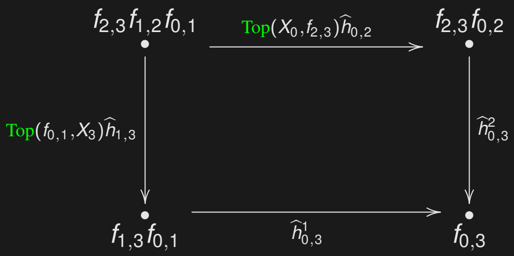


This is the boundary of a 3-simplex in $\mathcal{S}$ iff there is a certain double homotopy adjoint to a map $\widehat{h}_{0,3} : I^2 \to \mathbf{Top}(X_0, X_3)$ shown on the next slide.

The set of 3-simplices in $\mathcal{S}$ (continued)

The diagram on the previous is the boundary of a 3-simplex in $\mathcal{S}$ iff there a map $\widehat{h}_{0,3} : I^2 \to \mathrm{Top}(X_0, X_3)$ of the form



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$$\begin{array}{ccc}
f_{2,3}f_{1,2}f_{0,1} & \xrightarrow{\mathbf{Top}(X_0, f_{2,3})\widehat{h}_{0,2}} & f_{2,3}f_{0,2} \\
{\scriptstyle \mathbf{Top}(f_{0,1}, X_3)\widehat{h}_{1,3}} \downarrow & & \downarrow {\scriptstyle \widehat{h}^2_{0,3}} \\
f_{1,3}f_{0,1} & \xrightarrow[\widehat{h}^1_{0,3}]{} & f_{0,3}
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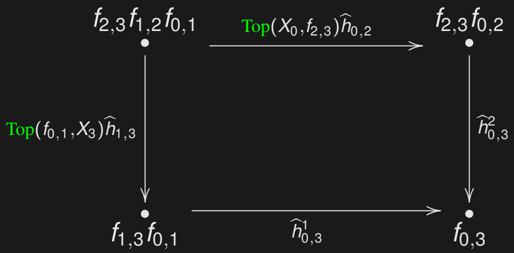

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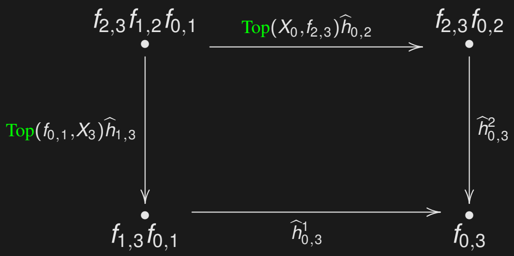

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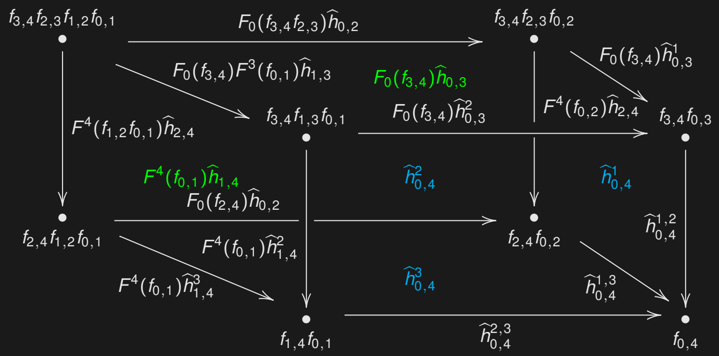

where F_i and F^i denote the endofunctors $\mathbf{Top}(X_i, -)$ and $\mathbf{Top}(-, X_i)$.

What is an ∞-category?
Doug Ravenel
Introduction
Review of simplicial sets
Of all the nerve!
The main definition
The ∞-category of topological spaces
3-simplices in $\mathcal{S}$
4-simplices in $\mathcal{S}$
$\mathcal{S}_{n+1}$ for $n > 3$
A colimit in $\mathcal{S}$
Bousfield localization in ∞-categories
The ∞-category of spectra
References

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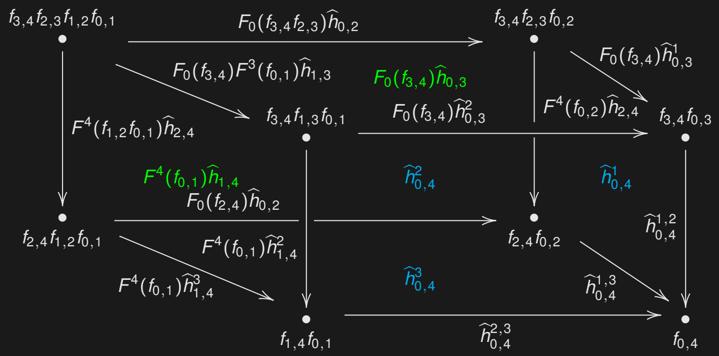

where F_i and F^i denote the endofunctors $\mathbf{Top}(X_i, -)$ and $\mathbf{Top}(-, X_i)$. The restriction of $\widehat{h}_{0,4}$ to the left and top faces are the composite double homotopies indicated in green.

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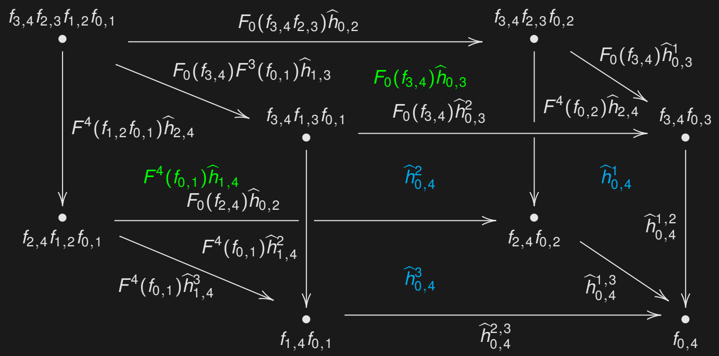

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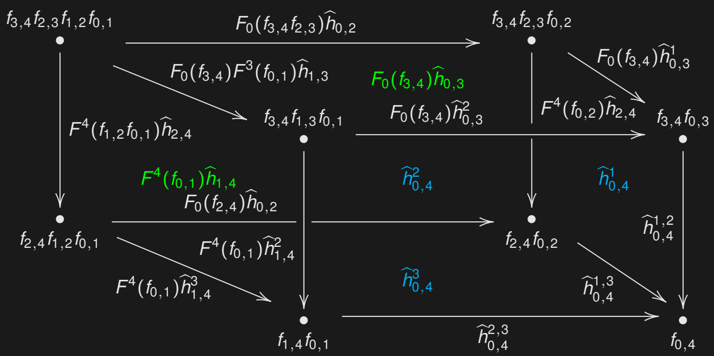
The set of 4-simplices in $\mathcal{S}$ (continued)



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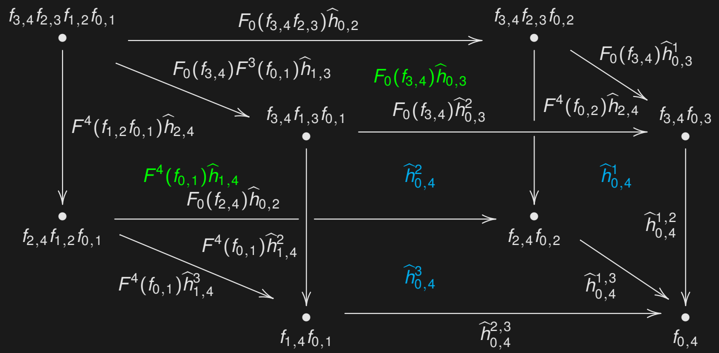
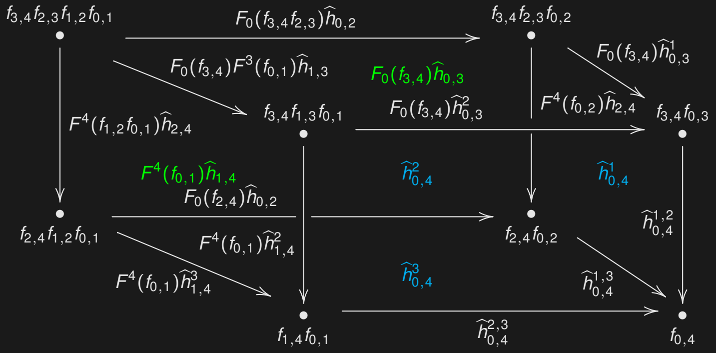


The restriction of $\widehat{h}_{0,4}$ to the back face (not labeled) is the composite

$$\begin{array}{c} I \times I \xrightarrow{\widehat{h}_{2,4}\times\widehat{h}_{0,2}} \mathbf{Top}(X_2, X_4) \times \mathbf{Top}(X_0, X_2) \\ \big\downarrow \text{comp} \\ \mathbf{Top}(X_0, X_4). \end{array}$$

The set of 4-simplices in $\mathcal{S}$ (continued)



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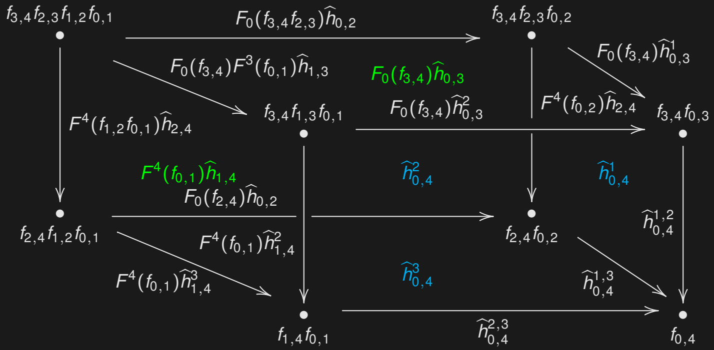

The five labeled faces are associated with the five 3-dimensional faces of the corresponding 4-simplex in $\mathcal{S}$.

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and a map

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$$\{(t_1,\ldots,t_{n-1},0)\} \qquad \text{and} \qquad \{(0,t_2,\ldots,t_n)\}.$$

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To sum up, the ∞-category $\mathcal{S}$ of topological spaces is a simplicial set in which

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More details can be found in [Lur09, 4.2.4].

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When we enlarge the class of weak equivalences (in the category of spaces or spectra) to those maps inducing an isomorphism in Morava E-theory (or Morava K-theory) for a fixed prime p and height n, this fibrant replacement functor is the L_n (or $L_{K(n)}$) of chromatic homotopy theory.

Bousfield localization in ∞-categories (continued)

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3. The inclusion functor $\mathcal{C}' \subseteq \mathcal{C}$ has a left adjoint L. This is the analog of Bousfield's fibrant replacement functor in model category theory.

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