9-8-25

Monday, September 8, 2025 1:58 PM

RECALL THE PULLBACK

PULL

$$\begin{tikzcd} W' \arrow[dr, "\varphi"] \arrow[drr, bend left, "i_2'"] \arrow[ddr, bend right, "i_1'"'] & & \\ & W \arrow[r, dashed, "i_2"] \arrow[d, dashed, "i_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_2 \arrow[d, "f_2"] \\ & X_1 \arrow[r, "f_1"'] & Y \end{tikzcd}$$

PULLBACK SYMBOL

W IS THE PULLBACK IF $f_2 i_2 = f_1 i_1$ AND IT HAS A UNIVERSAL PROPERTY S.T THAT FOR ANY (W', i_1', i_2') WITH $f_2 i_2' = f_1 i_1'$, $\exists!\ \varphi$ WITH $i_2' = i_2 \varphi$ AND $i_1' = i_1 \varphi$

THIS MAY OR MAY NOT EXIST

EXAMPLES.

① IN $\mathbf{Set}$, LET

$$W = \{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\}$$

SUBEXAMPLE: $Y = pt$. THEN $W = X_1 \times X_2$

DUAL DEFINITION: A PUSHOUT IS (AS ABOVE WITH ARROWS ~~REVERSED~~ REVERSED)

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X_1 \arrow[d, dashed] \\ X \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner", very near end] & Z \end{tikzcd}$$

$$X_2 \dashrightarrow Z$$

EXAMPLE: IN $\mathcal{Set}$ $Y = \emptyset$, X_1, X_2 WHATEVER

THEN $Z = X_1 \amalg X_2$ = DISJOINT UNION

EXAMPLE Y, X_1, X_2 ANY SETS

$$Z = X_1 \amalg X_2 / \langle f_1(y) \sim f_2(y) \quad \forall y \in Y \rangle$$

DEFINITION A CW-COMPLEX X IS A SPACE CONSTRUCTED AS FOLLOWS. IT IS THE UNION OF SUBSPACES $X^0 \subset X^1 \subset X^2 \subset \cdots \subset X$ WHERE X^k IS THE <u>k-SKELETON OF X</u>

X^0 IS DISCRETE

$$\begin{tikzcd} J_{k+1} \times S^k \arrow[r, "f_k"] \arrow[d] & X^k \arrow[d] \\ J_{k+1} \times D^{k+1} \arrow[r] \arrow[ur, phantom, "\ulcorner" near end] & X^{k+1} \end{tikzcd}$$

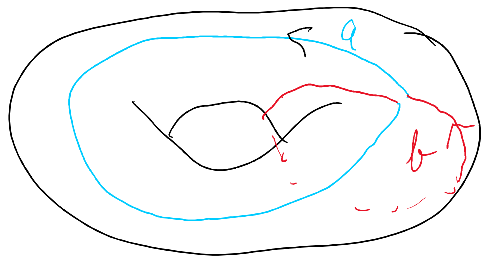
J_{k+1} = DISCRETE SET

f_k IS <u>THE kTH ATTACHING MAP</u>.

X^{k+1} = PUSHOUT

EXAMPLE: THE TORUS

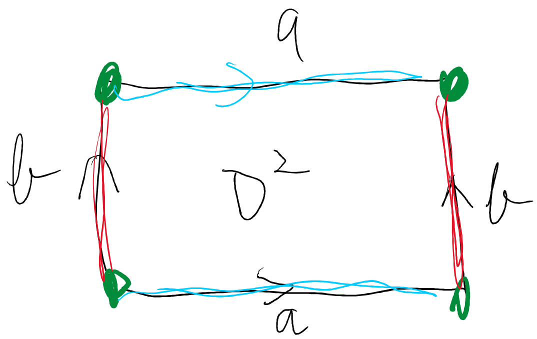
$X^0 = pt.$



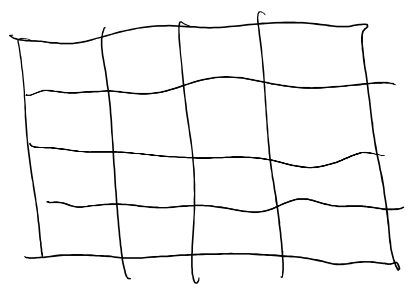
$X^0 = pt$

$J_1 = 2$ pts

$J_2 = 1$ pt



WE GET A CW CX WITH 1 0-CELL, 2 1-CELLS AND 1 2-CELL.



EXAMPLE

$X^0 = pt$
$\|$
X^{n-1}

$X^n = S^n$

$J_0 = 1$ pt

$J_i = \emptyset$ FOR $0 < i < n$

$J_n = 1$ pt

VARIATION LET G BE A DISCRETE (E.G. FINITE) GROUP. SUPPOSE IT ACTS ON EACH SET J_k FOR $k \geq 0$ AND EACH ATTACHING

FOR $k \geq 0$ AND EACH ATTACHING MAP $f_k : S^{k-1} \times J_k \longrightarrow X^{(k-1)}$
$\circlearrowleft G \qquad \circlearrowleft G$

IS EQUIVARIANT.

GENERALIZATION OF PULLBACK

REPLACE PULL BY A SMALL CATEGORY J. ASK FOR A W WITH SIMILAR PROPERTIES,

GIVEN FUNCTOR A $X : J \longrightarrow \mathcal{C}$ (A J-SHAPED DIAGRAM IN $\mathcal{C}$)

LOOK FOR AN OBJECT W WITH: ① FOR EACH $j \in J$ WE HAVE A MORPHISM $W \xrightarrow{i_j} X_j$

X_j = IMAGE OF j UNDER THE FUNCTOR X

② FOR MORPHISM $j \xrightarrow{\beta} j'$ IN J,

THE FOLLOWING $X_j \xrightarrow{X(\beta)} X_{j'}$

THE FOLLOWING COMMUTES

$$\begin{tikzcd} X_j \arrow[rr, "u"] & & X_{j'} \\ & W \arrow[ul, "i_j"] \arrow[ur, "i_{j'}"'] & \end{tikzcd}$$

W SHOULD HAVE A UNIVERSAL PROPERTY AS BEFORE [EXERCISE]

IF WE HAVE THIS, THEN W IS THE <u>LIMIT</u> OF THE FUNCTOR X

A PULLBACK IS A LIMIT.

<u>EXAMPLE</u> LET J BE $\bullet \rightrightarrows \bullet$

A J-SHAPED DIAGRAM IN $\mathcal{C}$ HAS THE FORM $X_0 \rightrightarrows X_1$

THE LIMIT (IF IT EXISTS) IS CALLED THE <u>EQUALIZER</u>

THM IF $\mathcal{C}$ HAS EQUALIZERS

THM IF $\mathcal{C}$ HAS EQUALIZERS, IT HAS ALL SMALL LIMITS

EXAMPLE

$$J = 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow 5 \leftarrow \cdots$$

LET p BE A PRIME

$$\mathbb{Z}/p \leftarrow \mathbb{Z}/p^2 \leftarrow \mathbb{Z}/p^3 \leftarrow \mathbb{Z}/p^4 \leftarrow \cdots$$

THE LIMIT IS $\mathbb{Z}_p$, THE p-ADIC INTEGERS.

EXAMPLE G = GROUP

$\mathcal{B}G$ = ONE OBJECT WITH A MORPHISM FOR EACH $\gamma \in G$.

A FUNCTOR $X : \mathcal{B}G \longrightarrow \mathcal{C}$ IS AN OBJECT IN $\mathcal{C}$ WITH AN ACTION OF G

FOR $\mathcal{C} = \mathcal{S}et$ OR $\mathcal{T}op$, THE LIMIT OF THIS FUNCTOR IS THE

FIXED POINT SET

$$X^G := \{x \in X : \gamma(x) = x \ \ \forall \gamma \in G\}$$

NEXT TIME: COLIMITS