CATEGORIES

DEF A FUNCTOR $F : \mathcal{C} \to \mathcal{D}$ ASSIGNS

1) TO EACH OBJECT X IN $\mathcal{C}$ AN OBJECT $F(X)$ IN $\mathcal{D}$
2) FOR EACH MORPHISM $X \xrightarrow{f} Y$ IN $\mathcal{C}$ WE GET A MORPHISM

$$F(X) \xrightarrow{F(f)} F(Y) \text{ IN } \mathcal{D}$$

FOR $W \xrightarrow{f} X \xrightarrow{g} Y$ IN $\mathcal{C}$

WE GET $F(W) \xrightarrow{F(f)} F(X) \xrightarrow{F(g)} F(Y)$

WITH $F(gf) = F(g)F(f)$.

EXAMPLE ① $\mathcal{C}$ = CATEGORY OF TOP SPACES WITH BASE POINT

$\mathcal{D}$ = CATEGORY OF GROUPS

π_1 IS THE FUNCTOR

EXAMPLE ② LET G BE GROUP AND LET BG BE THE ONE OBJECT CATEGORY WITH A MORPHISM FOR EACH $\gamma \in G$. LET $X : BG \to Top$ BE A FUNCTOR. IT IS A SPACE (ALSO CALLED X) EQUIPPED WITH AN ACTION OF G. FOR EACH $\gamma \in G$ WITH GET A HOMEOMORPHISM $X \circlearrowleft$

③ $Top \xrightarrow{U} Set$ $\quad U$ = FORGETFUL FUNCTOR

Top = CATEGORY OF TOP. SPACES

Set = CATEGORY OF SETS

④ $\mathcal{C}$ =

$$\begin{tikzcd} \text{JOHN} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \text{PAUL} \arrow[d, "\gamma"] \\ \text{GEORGE} \arrow[r, "\delta"] & \text{RINGO} \end{tikzcd}$$

INITIAL (JOHN) TERMINAL (RINGO)

WITH $\gamma\alpha = \delta\beta$

$\mathcal{D} = Sets$ (OR ANY OTHER CATEGORY)

A FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

WITH A COMMUTATIVE DIAGRAM

A FUNCTOR $F : \mathcal{C} \longrightarrow \mathcal{D}$

WITH A COMMUTATIVE DIAGRAM

$$\begin{tikzcd}
F(\text{JOHN}) \arrow[r, "F(\alpha)"] \arrow[d, "F(\beta)"'] & F(\text{PAUL}) \arrow[d, "F(\gamma)"] \\
F(\text{GEORGE}) \arrow[r, "F(\delta)"] & F(\text{RINGO})
\end{tikzcd}$$

WITH $F(\gamma)F(\alpha) = F(\delta)F(\beta)$

DEF AN <u>INITIAL OBJECT</u> I (TERMINAL; J) IN A CATEGORY $\mathcal{C}$ IS ONE WITH A UNIQUE MORPHISM TO (FROM) EVERY OTHER OBJECT IN $\mathcal{C}$

EXAMPLE $\mathcal{C} = A \rightrightarrows B$ NO INITIAL OBJECT

<u>EASY FACT</u> ANY TWO INITIAL (TERMINAL) OBJECTS ARE ISOMORPHIC

<u>DEF</u> A CATEGORY IS <u>POINTED</u> IF EVERY INITIAL OBJECT IS TERMINAL AND VICE VERSA.

<u>ADJOINT FUNCTORS</u> (DAN KAN 1958) $\perp$ = KAN TURNSTILE

$$\begin{tikzcd}
X \in \mathcal{C} \arrow[r, shift left=2, "F \text{ LEFT ADJOINT}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \ni Y \arrow[l, shift left=2, "U \text{ RIGHT ADJOINT}"]
\end{tikzcd} \qquad F \dashv U$$

THESE FUNCTORS ARE <u>ADJOINT</u> IF FOR EACH X, Y AS ABOVE

$$\mathrm{Hom}_{\mathcal{C}}(X, UY) \xrightarrow{\Phi_{X,Y}} \mathrm{Hom}_{\mathcal{D}}(FX, Y)$$

NATURAL BIJECTION

$\parallel$

SET OF MORPHISMS IN $\mathcal{C}$ FROM X TO UY

GIVEN $X \xrightarrow{\alpha} X'$ IN $\mathcal{C}$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X, UY) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathcal{D}}(FX, Y) \\
\mathrm{Hom}_{\mathcal{C}}(X', UY) \arrow[r, "\cong"] \arrow[u, "\alpha^*"] & \mathrm{Hom}_{\mathcal{D}}(FX', Y) \arrow[u, "(F\alpha)^*"']
\end{tikzcd}$$

EXAMPLE $\mathcal{C} = \mathcal{S}et$ = CATEGORY OF SETS

$\mathcal{D} = \mathcal{A}b$ = CAT. OF ABELIAN GPS

$F : \mathcal{C} \to \mathcal{D}$ IS THE FREE ABELIAN GP

$U : \mathcal{D} \to \mathcal{C}$ IS THE FORGETFUL FUNKTOR

LET X BE A SET AND LET A BE AN ABELIAN GP

$$\mathrm{Hom}_{ab}(FX, A) \xrightarrow{\cong} \mathrm{Hom}_{set}(X, UA)$$

$$X \in \mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D} \ni Y \quad (\perp)$$

$$\mathrm{Hom}_{\mathcal{C}}(X, UY) \xrightarrow[\cong]{\Phi_{X,Y}} \mathrm{Hom}_{\mathcal{D}}(FX, Y)$$

SUPPOSE $Y = FX$

$$\mathrm{Hom}_{\mathcal{C}}(X, UFX) \xrightarrow[\cong]{} \mathrm{Hom}_{\mathcal{D}}(FX, FX)$$

THE IDENTITY IN $\mathrm{Hom}_{\mathcal{D}}(FX, FX)$ LEADS TO A MORPHISM $X \xrightarrow{\eta_X} UFX$ IN $\mathcal{C}$

THE UNIT OF THE ADJUNCTION.

SUPPOSE $X = UY \rightsquigarrow FUY \xrightarrow{\varepsilon_Y} Y$

COUNIT OF ADJUNCTION

MORE FUN: PULLBACKS AND PUSHOUTS

DEF SUPPOSE WE A DIAGRAM OF SETS

$$\begin{array}{ccc} X & \xrightarrow{x_2} & A' \\ {\scriptstyle x_1}\downarrow & & \downarrow{\scriptstyle g} \\ A & \longrightarrow & B \end{array}$$

$$A \xrightarrow{f} B$$

THE PULLBACK IS X AS ABOVE WITH A UNIVERSAL PROPERTY

GIVEN ANY OTHER SUCH OBJECT P WITH

$$\begin{tikzcd}
P \arrow[dr, dashed, "\pi"] \arrow[drr, bend left, "\phi_2"] \arrow[ddr, bend right, "\phi_1"'] & & & \\
& X \arrow[r, "x_2"] \arrow[d, "x_1"] & A' \arrow[d, "g"] & a_2 \\
& A \arrow[r, "f"] & B & \\
& a_1 & &
\end{tikzcd}$$

$\exists!\ P \xrightarrow{\pi} X$ WITH $\phi_2 = x_2\pi$ AND $\phi_1 = x_1\pi$

PROP SUCH AN X EXISTS.

PROOF: WE HAVE A MAP $X \to A \times A'$

CONSIDER THE SUBSET

$$X_{f,g} = \{ (a_1, a_2) \in A \times A' : f(a_1) = g(a_2) \} \subset A \times A'$$