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HEADING TOWARD DEFINITION OF HOMOLOGY, WE NEED SOME ALGEBRAIC INFRASTRUCTURE

<u>DEF</u> A CHAIN <u>COMPLEX C OF ABELIAN GROUPS</u> IS A DIAGRAM

$$C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \xleftarrow{d_3} C_3 \leftarrow \cdots\cdots$$

WHERE C_i IS AN ABELIAN GP FOR $i \geq 0$, d_i IS A HOMOMORPHISM AND $d_i d_{i+1} = 0$ FOR $i \geq 0$

$d_i = i$ TH BOUNDARY OPERATOR.

IN EACH C_i $\ker d_i \supset \operatorname{im} d_{i+1}$

$\ker d_i$ = GROUP OF i-CYCLES = Z_i

$\operatorname{im} d_{i+1}$ = GROUP OF i-BOUNDARIES = B_i

$H_i(C) := Z_i / B_i$ = i TH HOMOLOGY GROUP

$$\boxed{\text{STRATEGY: ASSOCIATE TO EACH SPACE } X \text{ A CHAIN COMPLEX } S(X) \text{ [S FOR SINGULAR] AND}}$$

S(X) IS FOR SINGULAR, AND DEFINE $H_i(X) = H_i(S(X))$

EXAMPLE

(1) $C_i = \mathbb{Z}$ FOR EACH $i \geq 0$

$$d_i = \begin{cases} 0 & \text{IF } i \text{ IS ODD} \\ 1 & \text{IF } i \text{ IS EVEN} \end{cases}$$

$$\begin{array}{ccccccccccc} & & 0 & & 1 & & 2 & & 3 & & 4 \\ C: & 0 \leftarrow & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{1} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{1} & \mathbb{Z} \leftarrow \cdots \end{array}$$

$$H_i(C) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ 0 & i = 2 \\ 0 & i > 0 \end{cases}$$

EXAMPLE (2) C_i AS BEFORE

$$d_i = \begin{cases} 0 & i \text{ ODD} \\ 2 & i \text{ EVEN} \end{cases}$$

$$\begin{array}{cccccccccc} 0 & & 1 & & 2 & & 3 & & 4 & \\ \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} \cdots \end{array}$$

$$H_i(C_i) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2 & i \text{ ODD} \\ 0 & i \text{ EVEN}, \ i > 0 \end{cases}$$

DEF A CHAIN MAP $C \xrightarrow{f} C'$

DEF A CHAIN MAP $C \xrightarrow{f} C'$ IS A COLLECTION OF HOMS

$$f_i : C_i \to C_i' \quad \text{SUCH THAT}$$

$x + d_3(u)$

$0 \leftarrow x$

$$\begin{tikzcd}
C_0 \arrow[d, "f_0"'] & C_1 \arrow[l, "d_1"'] \arrow[d, "f_1"] & C_2 \arrow[l, "d_2"'] \arrow[d, "f_2"] & C_3 \arrow[l, "d_3"'] \arrow[d, "f_3"] & \cdots \arrow[l] \\
C_0' & C_1' \arrow[l, "d_1'"'] & C_2' \arrow[l, "d_2'"'] & C_3' \arrow[l, "d_3'"'] & \cdots \arrow[l]
\end{tikzcd}$$

DIAGRAM CHASING

$f_2(x)$

$0 = f_1 d_2(x) = d_2' f_2(x)$

$0 \leftarrow f_2(x + d_3(u)) = f_2(x) + f_2 d_3(u) = f_2(x) + d_3' f_3(u)$

$$f_i d_{i+1} = d'_{i+1} f_{i+1} \quad \forall i \geq 0$$

CLAIM SUCH AN f INDUCES MAP $\quad H_i(C) \xrightarrow{H_i(f)} H_i(C')$

PROOF LET $\alpha \in H_i(C) = Z_i / B_i$

$= \ker d_i / \operatorname{im} d_{i+1}$

CHOOSE $x \in Z_i \subset C_i$ THAT REPRESENTS α

WE FIND THE HOMOLOGY CLASS OF $f_2(x)$ IS INDEPENDENT OF THE CHOICE OF x, SO $f_2(\alpha)$ IS WELL DEFINED.

DEF TWO CHAIN MAPS

DEF TWO CHAIN MAPS $f, g : C \longrightarrow C'$ ARE <u>CHAIN HOMOTOPIC</u> IF THERE ARE HOMS $D_i : C_i \longrightarrow C'_{i+1}$ S.T

$$0 \longleftarrow\!\!| \; x$$

$$\begin{tikzcd}
C_{i-1} \arrow[dr, "D_{i-1}"'] & C_i \arrow[l, "d_i"'] \arrow[d, "f_i - g_i"] \arrow[dr, "D_i"] & \\
& C'_i & C'_{i+1} \arrow[l, "d'_{i+1}"]
\end{tikzcd}$$

$$(f_i - g_i)x$$

$$f_i - g_i = D_{i-1} d_i + d'_{i+1} D_i$$

<u>**PROP**</u> IF f AND g ARE CHAIN HOMOTOPIC, $H_i(f) = H_i(g)$.

PROOF: LET $\alpha \in H_i(C)$ BE REPRESENTED BY x. WE NEED TO SHOW $(f_i - g_i)x \in \mathrm{Im}\, d'_{i+1}$

BY THE CHAIN HOMOTOPY

$$(f_i - g_i)(x) = (D_{i-1} d_i + d'_{i+1} D_i)(x)$$

$$= \cancel{D_{i-1} d_i(x)} + d'_{i+1} D_i(x)$$

AS DESIRED QED

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A SHORT EXACT SEQUENCE OF CHAIN CXS IS A DIAGRAM

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0$$

WITH $\ker g = \operatorname{im} f$.

$$\ker g_i = \operatorname{im} f_i \quad \forall\, i \geq 0$$

<u>THEOREM</u> GIVEN THE ABOVE, WE GET A LONG EXACT SEQUENCE

$$\xrightarrow{\partial_{i+1}} H_i(C') \xrightarrow{f_i} H_i(C) \xrightarrow{g_i} H_i(C'') \xrightarrow{\partial_i} H_{i-1}(C') \longrightarrow \cdots$$

∂_n IS THE <u>CONNECTING HOMOMORPHISM</u>

<u>PROOF</u> CONSIDER THE FOLLOWING DIAGRAM WITH EXACT ROWS

$$\begin{tikzcd}
0 \arrow[r] & C'_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d'_{n+1}"] & C_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d_n"] & C''_{n+1} \arrow[r] \arrow[d, "d''_{n+1}"] & 0 \\
0 & C'_n \arrow[r, "f_n"] & C_n & C''_n & \\
& & y \arrow[r, mapsto, "g_n"] & x &
\end{tikzcd}$$

$$
\begin{tikzcd}
 & {} \arrow[d, "d'_{n+1}"'] & {} \arrow[d, "d_n"'] & {} \arrow[d, "d_{n+1}"] & \\
0 \arrow[r] & C'_n \arrow[r, "f_n"] \arrow[d, "d'_n"'] & C_n \arrow[r, "g_n"] \arrow[d, "d_n"'] & C''_n \arrow[r] \arrow[d, "d''_n"] & 0 \\
0 \arrow[r] & C'_{n-1} \arrow[r, "f_{n-1}"] & C_{n-1} \arrow[r, "g_{n-1}"] & C''_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & y \arrow[r, mapsto] \arrow[d, mapsto] & x \arrow[d, mapsto] \\
u \arrow[r, mapsto] & d_{n-1}(y) \arrow[r, mapsto] & 0
\end{tikzcd}
$$

TO DEFINE $\partial_n : H_n(C'') \longrightarrow H_{n-1}(C')$, LET

$\alpha \in H_n(C'')$ BE REPRESENTED BY x CHOICE

CAN CHOOSE $y \in C_n$ WITH $g_n(y) = x$.

SINCE $g_{n-1}(d_n(y)) = d''_n\, g_n(y) = d''_n(x) = 0$

$\exists!\ u \in C'_{n-1}$ WITH $f_{n-1}(u) = d_{n-1}(y)$

CAN SHOW $d'_{n-1}(u) = 0$ SO

u REPRESENTS A CLASS IN

$\beta \in H_{n-1}(C')$. WE CAN THAT

β IS INDEPENDENT OF THE

CHOICES OF x AND y, SO

WE HAVE OUR CONNECTING

HOMOMORPHISM $\partial_n : H_n(C'') \to H_{n-1}(C')$

WE CAN SHOW TEDIOUSLY EXACTNESS AT

EACH STAGE.

QED

MORE ABOUT STRATEGY

$H_i(X) := H_i(S(X))$ WHERE $S(X)$ IS TO BE NAMED LATER.

FOR A SUBSPACE $A \subset X$,

$S(A)$ IS A SUBCOMPLEX OF $S(X)$.

(1) $$0 \to S(A) \to S(X) \to S(X)/S(A) \to 0$$

WE DEFINE $\underset{\text{RELATIVE HOMOLOGY}}{\underset{\parallel}{H_*(X,A)}} := H_*(S(X)/S(A))$

$S(\emptyset) = 0$ SO $H_*(X,\emptyset) = H_*(X)$

(1) LEADS A LONG EXACT SEQ

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to \cdots$$

THESE GPS SATISFY THE EILENBERG-STEENROD AXIOMS

(1) HOMOTOPY AXIOM IF $f, g : X \to Y$

ARE HOMOTOPIC, $H_*(f) = H_*(g)$

(2) EXACTNESS AXIOM THE

LES ABOVE FOR ANY $A \subset X$

(3) EXCISION AXIOM

GIVEN $B \subset A \subset X$ WITH

CLOSURE $(B) \subseteq$ INTERIOR (A)

THEN

$$H_*(X - B, A - B) \xrightarrow{\cong} H_*(X, A)$$

(4) DIMENSION AXIOM

$$H_i(pt) = \begin{cases} \mathbb{Z} & \text{FOR } i = 0 \\ 0 & \text{FOR } i > 0 \end{cases}$$