9-22-22

Monday, September 22, 2025 1:52 PM

VAN KAMPEN THEOREM

LET $X = A \cup B$ WITH $x_0 \in A \cap B$ WITH $A \cap B$, A, B AND X PATH CONNECTED. DIAGRAM ON RIGHT IS PUSHOUT

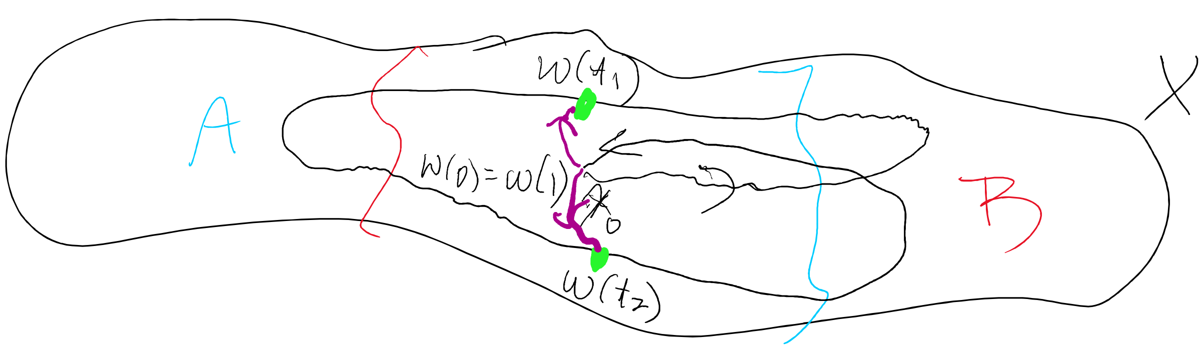
$$\begin{tikzcd} & A \arrow[dr] & \\ A \cap B \arrow[ur] \arrow[dr] & & X \\ & B \arrow[ur] & \end{tikzcd}$$

THEN THE FOLLOWING IS ALSO A PUSHOUT DIAGRAM

$$\begin{tikzcd} & \pi_1(A) \arrow[dr] \arrow[drr, bend left] & & \\ \pi_1(A \cap B) \arrow[ur] \arrow[dr] & & P \arrow[r, "\Phi"] & \pi_1(X) \\ & \pi_1(B) \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd}$$

PROOF. LET P DENOTE THE PUSHOUT OF THE THREE GROUPS. WE HAVE A HOMOMORPHISM $\Phi : P \to \pi_1(X)$.

TO SHOW Φ IS ONTO:



LET $\alpha \in \pi_1 X$ BE REPRESENTED BY A CLOSED PATH $\omega : [0,1] \to X$ AS SHOWN. NOTE

A CLOSED PATH $w:[0,1]\to X$ AS SHOWN. NOTE

$$w(t)\in\begin{cases} B & \text{FOR } 0\le t\le t_1 \\ A & \text{FOR } t_1\le t\le t_2 \\ B & \text{FOR } t_2\le t\le 1\end{cases}$$

$w(0), w(t_1), w(t_2), w(1)\in A\cap B$

IN GENERAL WE CAN FIND

$0=t_0<t_1<t_2<\cdots t_{n-1}<t_n=1$

SUCH THAT i) $w[t_{i-1},t_i]\subset A$ OR B FOR $0\le i<n$

ii) $w(t_i)\in A\cap B$ FOR $0\le i\le n$

iii) WE HAVE A PATH ϕ_i IN $A\cap B$ FROM x_0 TO $w(t_i)$ FOR $0<i<n$

iv) w IS HOMOTOPIC TO THE PRODUCT $w_0 * w_1 * \cdots * w_{n-1}$

WHERE $w_0 = w[0,t_1]*\phi_1^{-1}$

$w_1 = \phi_1 * w[t_1,t_2]*\phi_2^{-1}$

$w_i = \phi_i * w[t_i, t_i\ ] * \phi_i^{-1}$

$$w_i = p_i * w[t_i, t_{i+1}] * p_{i+1}^{-1}$$

EACH w_i IS A CLOSED PATH IN EITHER A OR B.

THIS MEANS Φ IS ONTO.

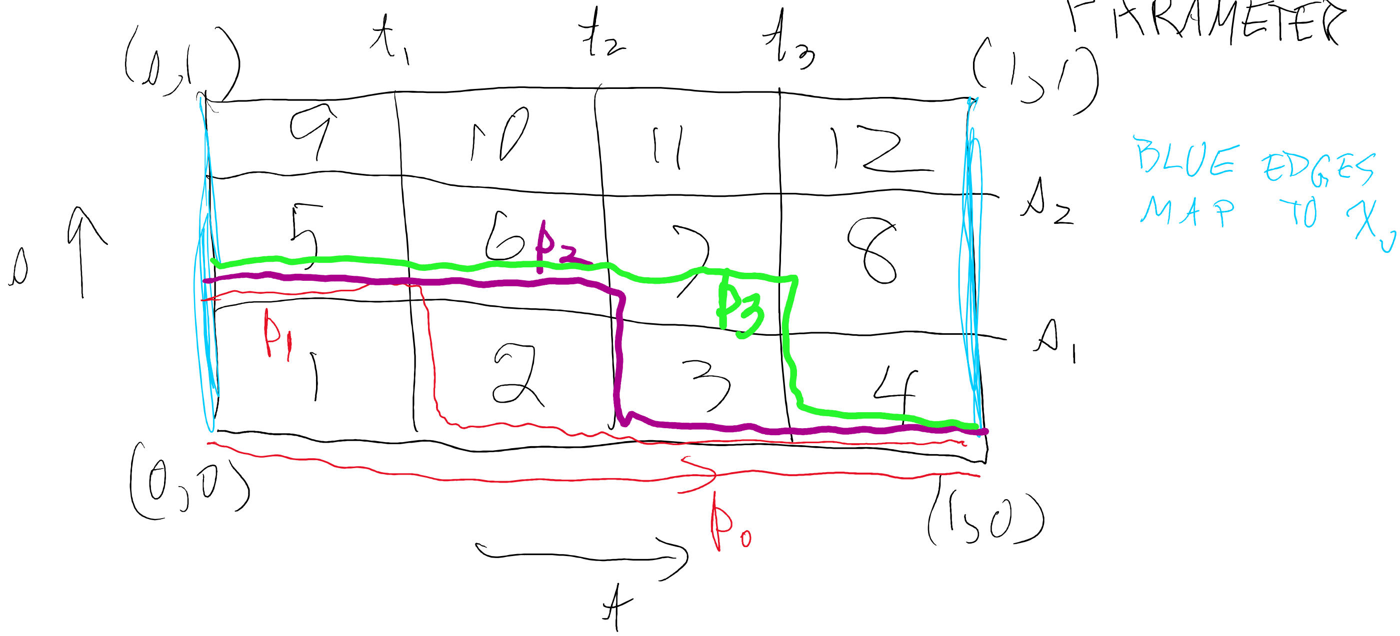
TO SHOW Φ IS 1-1, WE NEED A SIMILAR CONSTRUCTION FOR A HOMOTOPY h BETWEEN PATHS w AND w'

$$h : I^2 \longrightarrow X$$

s = HOMOTOPY PARAMETER

t = PATH PARAMETER

(s,t)



WE CAN DIVIDE I^2 INTO

WE CAN DIVIDE I^2 INTO RECTANGES AS SHOW SUCH THAT $h\left([s_i, s_{i+1}] \times [t_j, t_{j+1}]\right)$

IS CONTAINED IN EITHER A OR B THEN NUMBER THE RECTANGLES AS SHOWN.

h SENDS THE VERTICAL EDGES TO x_0. DEFINE PATH p_i IN I^2 AS SHOWN.

$0 = s_0 < s_1 < s_2 \cdots < s_m = 1$

$0 = t_0 < t_1 < \cdots < t_n = 1$

THE # OF RECTANGLES IS mn.

WE HAVE CLOSED PATHS IN X p_k FOR $0 \leq k \leq mn$.

THE HOMOTOPY BETWEEN p_k AND p_{k+1} COMES FROM A HOMOTOPY IN A OR B.

THIS MEANS Φ IS 1-1.

THIS MEANS Φ IS 1-1.

QED

COVERINGS

<u>DEF</u> A <u>COVERING</u> IS A MAP $\tilde{X} \xrightarrow{p} X$ S.T. EACH $x \in X$ HAS A NBD U S.T. $p^{-1}(U) \cong D \times U$ FOR A DISCRETE D. SUCH A U IS <u>EVENLY COVERED</u>

EXAMPLES

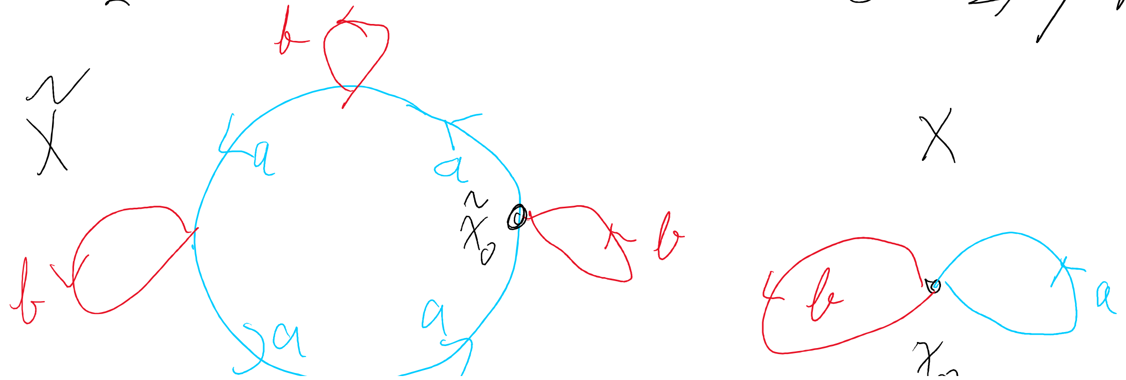
1) $\mathbb{R} \xrightarrow{p} S^1 \subset \mathbb{C}$ $\qquad D = \mathbb{Z}$

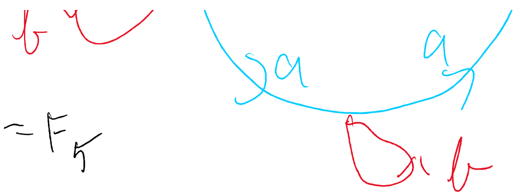
$t \longmapsto e^{2\pi i t}$

2) $S^1 \longrightarrow S^1$ $\qquad 0 \neq n \in \mathbb{Z}$

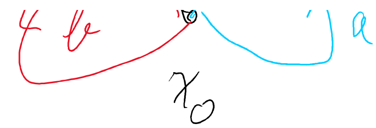
$z \longmapsto z^n$ $\qquad D = \mathbb{Z}/n$

3) $\tilde{X}$ $\qquad\qquad$ X





$\pi_1 \tilde{X} = F_5$



$\pi_1 X = F_2$

THEOREM LET X BE A "NICE" PATH CONNECTED SPACE WITH $\pi_1(X, x_0) = G$. THEN THERE IS A COVERING $\tilde{X} \xrightarrow{p} X$, THE UNIVERSAL COVER OF X, WITH

a) $\tilde{X}$ IS PATH CONNECTED AND $\pi_1(\tilde{X}) = 0$

b) G ACTS FREELY ON $\tilde{X}$ WITH $\tilde{X}/G = X$

c) FOR EACH SUBGP $H \subset G$, THE MAP $\tilde{X}/H \longrightarrow \tilde{X}/G$ IS A COVERING.

d) ANY PATH CONNECTED COVERING OF X IS ISOMORPHIC $\tilde{X}/H$ FOR SOME $H \subset G$.

e) IF $H \subset G$ IS NORMAL, THE ACTION

e) IF $H \subset G$ IS NORMAL, THE ACTION OF G ON $\tilde{X}$ INDUCES AN ACTION OF $G/H =: K$ ON $\tilde{X}/H$ WITH $(\tilde{X}/H)/K = X$

THERE IS A 1-1 CORRESPONDANCE BETWEEN SUBGPS OF $\pi_1(X)$ AND PATH CONNECTED COVERINGS OF X. $\pi_1(\tilde{X}/H) = H$. THE MAP

$$\tilde{X}/H \longrightarrow \tilde{X}/G = X$$

$$\begin{array}{ccc} \pi(\tilde{X}/H) & & \pi_1(\tilde{X}/G) \\ \| & & \| \\ H & \xhookrightarrow{1-1} & G \end{array}$$

ANALOGY WITH GALOIS THEORY

K = FIELD $\hookrightarrow \bar{K}$ = ALGEBRAIC CLOSURE

$G = Gal(\bar{K}/K)$

THERE IS A BIJECTION BETWEEN SUBGPS OF G AND SUBFIELDS

SUBGPS OF G AND SUBFIELDS OF $\bar{K}$ THAT CONTAIN K.

WHAT IS "NICE"?

ADJECTIVE IS "WONDERFUL"

ASSUME WONDERFUL SPACES HAVE BEEN DEFINED

<u>DEF</u> A SPACE X IS <u>LOCALLY WONDERFUL</u> IF EVERY NBD OF EACH $x \in X$ HAS A WONDERFUL SUBNBD.

<u>DEF</u> A SPACE IS <u>SEMI-LOCALLY WONDERFUL</u> IF

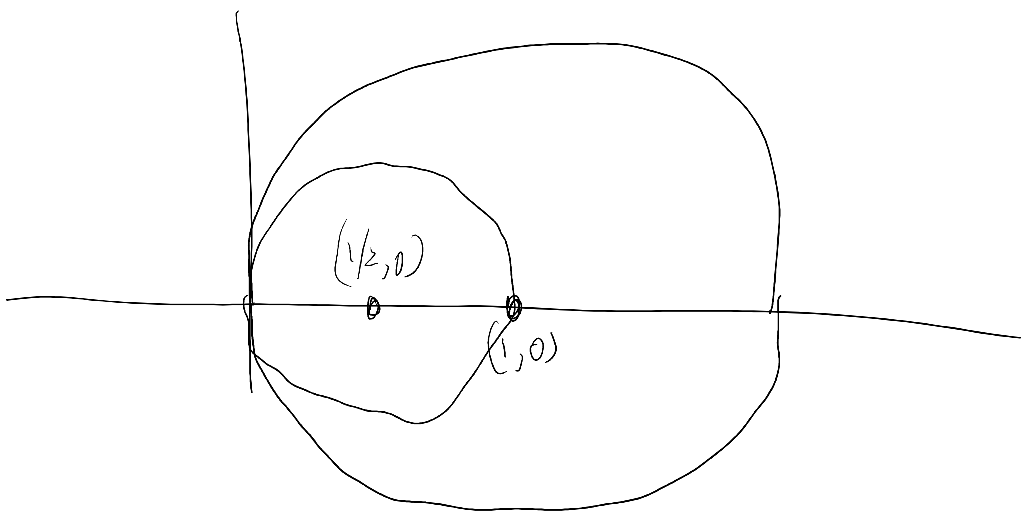
<u>DEF</u> A PATH CONNECTED SPACE X IS <u>SIMPLY CONNECTED</u> IF $\pi_1 X = 0$

DEF A PATH CONN SPACE X IS SEMI-LOCALLY SIMPLY CONNECTED (SLSC) IF EACH NBD W OF EACH

(SLSC) IF EACH NBD W OF EACH $x \in X$ HAS A SUBNBD U SUCH THAT $\pi_1(U) \longrightarrow \pi_1(X)$ IS TRIVIAL.

EXAMPLE <u>HAWAIIAN EARRING</u> H

$H \subset \mathbb{R}^2$.



H = UNION OF ALL CIRCLES THRU $(0,0)$ WITH CENTER $(1/n, 0)$ FOR $n = 1, 2, 3, 4, \ldots$

IT IS <u>NOT</u> SLSC.