9-17-25

Wednesday, September 17, 2025 1:40 PM

THEOREM $\pi_1 S^1 \cong \mathbb{Z}$

WILL CONSTRUCT A HOM $\Phi : \mathbb{Z} \to \pi_1(S^1)$ AND SHOW IT IS AN ISOMORPHISM.

FOR $n \in \mathbb{Z}$

$$I = [0,1] \xrightarrow{w_n} \mathbb{R} \xrightarrow{p} S^1 \subset \mathbb{C}$$

$$t \longmapsto nt$$

$$t \longmapsto e^{2\pi i t}$$

<u>CLAIM A</u>

$$\begin{tikzcd}
\{0\} \arrow[r, hook] \arrow[d] & (\mathbb{R}, 0) \arrow[d, "p"] \\
(I, 0) \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & (S^1, 1)
\end{tikzcd}$$

$\tilde{f}$ IS A <u>LIFTING</u> OF f

THIS IMPLIES THAT Φ IS ONTO

SUPPOSE f IS A CLOSED PATH AND HENCE REPRESENTS SOME $\alpha \in \pi_1 S^1$. THEN $\hat{f}(1) \in p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$

SUPPOSE $\hat{f}(1) = n$. THEN THERE IS AN ENDPOINT PRESERVING HOMOTOPY BETWEEN $\tilde{f}$ AND w_n,

SO $\alpha = \Phi(n)$.

<u>CLAIM B</u>

CONSTANT

CLAIM B

$$\begin{tikzcd}
(I,0) \arrow[r, "\text{CONSTANT}"] \arrow[d, "\text{INCLUSION}"'] & (\mathbb{R},0) \arrow[d, "p"] \\
(I^2, \{0\}\times I) \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & (S^1,1)
\end{tikzcd}$$

$\tilde{h}$ IS A LIFTING OF h

SUPPOSE h IS A HOMOTOPY BETWEEN TWO CLOSED PATHS $h_0(t) := h(0,t)$ AND $h_1(t) := h(1,t)$

THEN $\tilde{h}_0(t) := \tilde{h}(0,t)$ IS A LIFTING OF h_0 AS IN CLAIM A,

$\tilde{h}_1(t) := \tilde{h}(1,t)$ IS A LIFTING OF h_1, AND $\tilde{h}$ IS A HOMOTOPY BETWEEN $\tilde{h}_0$ AND $\tilde{h}_1$. THIS MEANS Φ IS 1-1.

BOTH CLAIMS A AND B ARE SPECIAL CASES OF

CLAIM C

$$\begin{tikzcd}
(X,x_0)\times\{0\} \arrow[r, "f"] \arrow[d] & (\mathbb{R},0) \arrow[d, "p"] \\
(X,x_0)\times(I,0) \arrow[r, "g"] \arrow[ur, dashed, "\tilde{g}"] & (S^1,1)
\end{tikzcd}
\qquad
\begin{tikzcd}
(\tilde{Y},\tilde{y}_0) \arrow[d, "p"] \\
(Y,y_0)
\end{tikzcd}$$

$(X, x_0) \times (I, 0) \xrightarrow{g} (S^1, 1) \qquad (Y, y_0)$

IN CLAIM A, $X = pt$

IN CLAIM B, $X = I$.

WE CAN GENERIZE FURTHER BY REPLACING p BY ANY <u>COVERING</u> $\tilde{Y} \xrightarrow{p} Y$, I.E. A MAP SUCH THAT EACH $y \in Y$ HAS A NBD U S.T. $p^{-1}(U) \approx D \times U$ FOR SOME DISCRETE SET D WHICH IS INDEPENDENT OF y. ASSUME Y IS PATH CONNECTED.

PROOF CLAIM C FOR THE COVERING $\mathbb{R} \to S^1$. ASSUME (WLOG) THAT X IS PATH CONNECTED.

$x \longmapsto f(x)$

$$\begin{tikzcd}
(X, x_0) \times \{0\} \arrow[r, "f"] \arrow[d, "\text{INCLUSION}"'] & (\mathbb{R}, 0) \arrow[d, "p"] \\
(X, x_0) \times (I, 0) \arrow[r, "g"] & (S^1, 1) \supset U
\end{tikzcd}
\qquad p^{-1}(U) = \mathbb{Z} \times U \supset \tilde{U}$$

$(x, 0) \longmapsto$ YES^x

EVENLY

$W := g^{-1}(U)$ $\qquad S^1 \ni y \in U$ EVENLY COVERED

W IS A NBD OF $(x,0)$

LET $\tilde{U} \subset \mathbb{R}$ BE THE COPY OF U CONTAINING $\tilde{g}(x)$. DEFINE THE LIFTING $\tilde{g}$ ON W BY SENDING W TO $\tilde{U}$.

CHOOSE SOME OTHER $x_1 \in W$. LET $y_1 = g(x_1)$ WITH EVENLY COVERED NBD U_1. LET $W_1 = g^{-1}(U_1)$ AND DEFINE $\tilde{g}$ ON W_1 AS BEFORE.

REPEAT UNTIL WE HAVE DEFINED $\tilde{g}$ ON ALL OF $X \times I$. THIS ARGUMENT DOES NOT RELY ON ANY SPECIAL PROPERTY OF $\mathbb{R} \longrightarrow S^1$. IT WORKS FOR ANY COVERING $\tilde{Y} \longrightarrow Y$.

QED

FOR MORE DETAILS SEE HATCHER.

NEXT TOPIC:

VAN KAMPEN THEOREM

LET $X = A \cup B$ WITH BASE POINT $x_0 \in A \cap B$ AND $A \cap B$, A, B AND X ARE ALL PATH CONNECTED. WE HAVE A PUSHOUT DIAGRAM OF SPACE

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X \end{tikzcd}$$

WE ALSO HAVE A DIAGRAM OF GROUPS

$$\begin{tikzcd} \pi_1(A \cap B) \arrow[r] \arrow[d] & \pi_1(A) \arrow[d] \arrow[dr, bend left] & \\ \pi_1(B) \arrow[r] \arrow[rr, bend right] & \text{PUSHOUT GROUP} \arrow[r] & \pi_1(X) \end{tikzcd}$$

VK THEOREM SAYS THIS IS ALSO A PUSHOUT DIAGRAM

EXAMPLES ① $X = S^1 \vee S^1$

$0 \longrightarrow \mathbb{Z}$

ba

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \text{PUSHOUT} \end{array}$$

PUSHOUT = FREE GROUP ON 2 GENERATORS $= F_2 = \mathbb{Z} * \mathbb{Z}$

FREE GROUP = $\{$ WORDS IN $a, a^{-1}, b, b^{-1} \}$

LET F_n BE THE FREE GROUP ON n GENERATORS.

<u>THEOREM</u> $\forall\, n \geq 1$, F_2 HAS A SUBGROUP ISOMORPHIC TO F_n, EVEN FOR $n = \infty$

THIS HAS A TOPOLOGICAL PROOF.

(2) $A = D^2$ $B =$ MOBIUS STRIP

$A \cap B = S^1 =$ BOUNDARY OF EACH

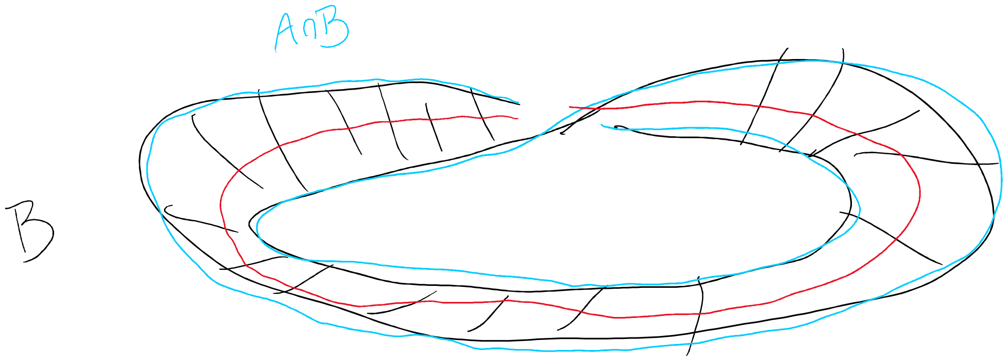
$X = A \cup B = \mathbb{RP}^2$

$\mathbb{Z} = \pi_1(A \cap B) \longrightarrow \pi_1(A) = 0$ VKT SAY

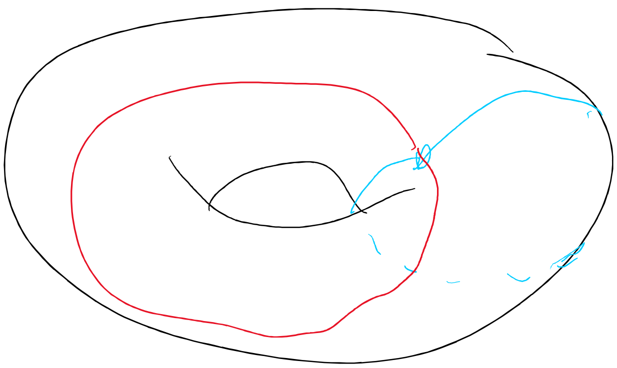
$\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2$

$$\begin{array}{ccc} \mathbb{Z} = \pi_1(A\cap B) & \longrightarrow & \pi_1(A) \\ {\scriptstyle 2}\downarrow & & \downarrow \\ \mathbb{Z} \simeq \pi_1(B) & \longrightarrow & \mathbb{Z}/2 \end{array}$$

$\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$



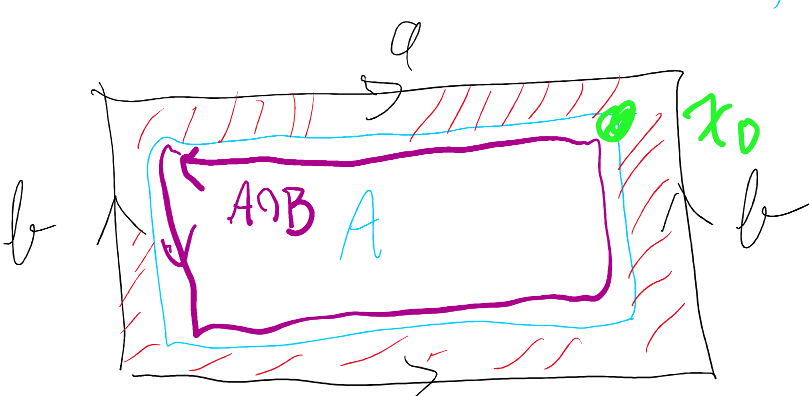
③



$X = $ TORUS

$A \simeq D^2$

$B \simeq S^1 \vee S^1$



$A \cap B = S^1$

$$\text{gen } 1 \longmapsto a^{-1}b^{-1}ab$$

$$\begin{array}{ccc} \mathbb{Z} = \pi_1(A\cap B) & \longrightarrow & \pi_1 B = \mathbb{Z} * \mathbb{Z} \\ \downarrow & & \\ 0 = \pi_1(A) & & \end{array}$$

$$\pi_1(X) = \langle a, b : a^{-1}b^{-1}ab = e \rangle$$

$U = \pi_1(A)$

NOTE $\quad a^{-1}b^{-1}ab = e$

$$b^{-1}ab = a$$

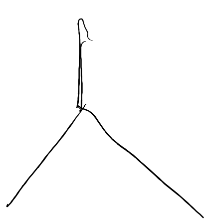
$$ab = ba$$

$$\pi_1(X) = \mathbb{Z} \oplus \mathbb{Z}$$

$$\begin{tikzcd}
1 \in \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "2"'] & 0 \arrow[d] \arrow[dr, bend left] & \\
2 & \mathbb{Z} \arrow[r] \arrow[rr, bend right] & ? & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
G_0 \arrow[r] \arrow[d] & G_1 \arrow[d] \\
G_2 \arrow[r] & G_3
\end{tikzcd}$$

$$? = \mathbb{Z} * 0 / \sim \qquad G_3 = G_1 * G_2 / \sim$$

$$= \mathbb{Z}/\sim$$



$= Y \qquad Y \times I$

$X = D^2 \cup \tilde{Y}$

TRIPLE

$X = D \cup Y$

$D^2 \cap \hat{Y} =$ BLUE CIRCLE

$\overset{3\downarrow}{\tilde{Y}} =$ TRIPLE MOBIUS

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, "3"'] & 0 \arrow[d] \\
\mathbb{Z} \arrow[r] & \mathbb{Z}/3
\end{tikzcd}
$$