9-15-25

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MORE CONSEQUENCES OF $\pi_1 S^1 = \mathbb{Z}$

MOVIE ABOUT ULAM "ADVENTURES OF A MATHEMATICIAN"

BORSUK-ULAM THEOREM

GIVEN A MAP $\mathbb{R}^3 \supset S^2 \xrightarrow{f} \mathbb{R}^2$,

$\exists\, x \in S^2$ WITH $f(x) = f(-x)$

PROOF: ASSUME NO SUCH x EXISTS

DEFINE $g(x) = \dfrac{f(x) - f(-x)}{|f(x) - f(-x)|} \in S^1 \subset \mathbb{R}^2 \cong \mathbb{C}$

NOTE $g(-x) = -g(x)$

$$\begin{tikzcd}
S^2 \arrow[r, "g"] \arrow[d, "-1"'] & S^1 \arrow[d, "-1"] \\
S^2 \arrow[r, "g"] & S^1
\end{tikzcd}$$

WE WILL SEE THAT SUCH A g CANNOT EXIST

$$\begin{tikzcd}
 & & (\mathbb{R}, 0) \arrow[d, "p"'] & t \arrow[d, mapsto] \\
([0,1], 0) \arrow[urr, dashed, "\tilde{h}"] \arrow[r, "h"] & (S^2, *) \arrow[r, "g"] & (S^1, 1) & e^{2\pi i t}
\end{tikzcd}$$

$\mathbb{R}^3 \supset S^2$

$$\begin{tikzcd}
([0,1], 0) \arrow[r, "h"] \arrow[d] & (S, *) \arrow[r] & (S^1, 1) \subset \mathbb{C} \\
S^1 \arrow[ur, "\bar{h}"] & &
\end{tikzcd}$$

$p^{-1}(-1) = \frac{1}{2} + \mathbb{Z}$ $\qquad p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$

$$s \longmapsto (\cos 2\pi s, \sin 2\pi s, 0)$$

CLOSED PATH

LEMMA (TO BE PROVED LATER)

$\exists\, \tilde{h} : [0,1] \to \mathbb{R}$ WITH $p\tilde{h} = gh$

SUPPOSE gh WRAPS AROUND S^1 n TIMES, SO $\tilde{h}(1) = n$.

AND $\tilde{h}(0) = 0$

$h(1/2) = (-1, 0, 0)$

$h(s + 1/2) = -h(s)$ $\quad$ FOR $0 \le s \le 1/2$

$\tilde{h}(s + 1/2) \in p^{-1} h(s + 1/2) = p^{-1}(-h(-s))$

$= 1/2 + p^{-1} h(s)$

CLAIM $\tilde{h}(s + 1/2) = \tilde{h}(s) + q/2$ FOR SOME ODD INTEGER q

FOR $s = 0$, $\tilde{h}(1/2) = \tilde{h}(0) + q/2$

$s = 1/2$ $\quad \tilde{h}(1) = \tilde{h}(1/2) + q/2$

$= \tilde{h}(0) + q$

$= h(0) + g$

AND $g \neq 0$ SINCE g IS ODD.
HENCE gh IS <u>ESSENTIAL</u>, IE
NOT HOMOTOPIC TO A CONSTANT MAP.
THIS IS A CONTRADICTION
BECAUSE g EXTENDS TO S^2

QED

HAM SANDWICH THEOREM

SUPPOSE $K_1, K_2, K_3 \subset \mathbb{R}^3$
ARE DISJOINT COMPACT SUBSETS OF POSITIVE VOLUME
THEN THERE IS A PLANE
WHICH BISECTS ALL THREE.

SKETCH OF PROOF

FOR $x \in S^2 \subset \mathbb{R}^3$, $\exists$ PLANE
$\perp$ TO $\vec{x}$ THAT BISECTS K_1
WE GET A FAMILY OF PLANES
BISECTING K_1 PARAMETRIZED
BY S^2 [illegible]

BISECTING K_1 PARAMETRIZED BY S^2. FOR EACH SUCH PLANE WE TWO NUMBER $f_2(x)$ AND $f_3(x)$ WHERE $f_2(x) = $ (VOLUME OF K_2 LYING ABOVE THE PLANE)

$-$ (VOLUME LYING BELOW)

$f_3(x) = $ SAME FOR K_3

HENCE WE HAVE $f : S^2 \longrightarrow \mathbb{R}^2$

BORSUK-ULAM SAYS

$\exists x$ WITH $f(-x) = f(x)$

BY THE WAY OUR f IS DEFINED

$f(x) = -f(-x)$ FOR ALL x

HENCE AT THE BORSUK-ULAM POINT $f(x) = 0$

THIS IS THE PLANE WE WANT.

QED

FUNDAMENTAL THEOREM OF

FUNDAMENTAL THEOREM OF ALGEBRA (GAUSS ~1800)

LET $f(x) \in \mathbb{C}[x]$ BE NONCONSTANT

THEN $\exists\, \alpha \in \mathbb{C}$ WITH $f(\alpha) = 0$.

THIS IMPLIES $f(x) = (x - \alpha)\, f_1(x)$

PROOF: f DEFINES A MAP $\mathbb{C} \to \mathbb{C}$
$x \mapsto f(x)$

ASSUME NO SUCH α EXISTS,

SO $f(x) \neq 0$ FOR ALL x.

FOR EACH $n \geq 0$

$$\begin{tikzcd}
nz & \mathbb{C} \arrow[r, "f"] & \mathbb{C} - \{0\} \arrow[r] & S^1 \arrow[dll, bend left, "g_n"] \\
z \arrow[u, mapsto] & S^1 \arrow[u] & z \arrow[r, mapsto] & \frac{z}{|z|}
\end{tikzcd}$$

NOTE g_0 IS CONSTANT

. ASSUME FOR SIMPLICITY THAT f IS MONIC AND HAS DEGREE $n > 0$

DEGREE $n > 0$

$$f(re^{it}) = (re^{it})^n + \text{LOWER TERMS}$$

FOR $r \gg 0$, THE LEADING TERM IS BIGGER THAN THE OTHERS.

IT FOLLOWS THAT g_r HAS DEGREE n. IT IS NOT HOMOTOPIC TO g_0. CONTRADICTION

QED

NEXT TIME:

PROOF THAT $\pi_1(S^1) \cong \mathbb{Z}$.