

MORE CONSEQUENCES OF

$$\pi_1 S^1 = \mathbb{Z}$$

MOVIE ABOUT VLAM

"ADVENTURES OF A MATHEMATICIAN"

BORSUK-VLAM THEOREM

GIVEN A MAP $\mathbb{R}^3 \supset S^2 \xrightarrow{f} \mathbb{R}^2$,

$$\exists x \in S^2 \text{ WITH } f(x) = f(-x)$$

PROOF: ASSUME NO SUCH x EXISTS

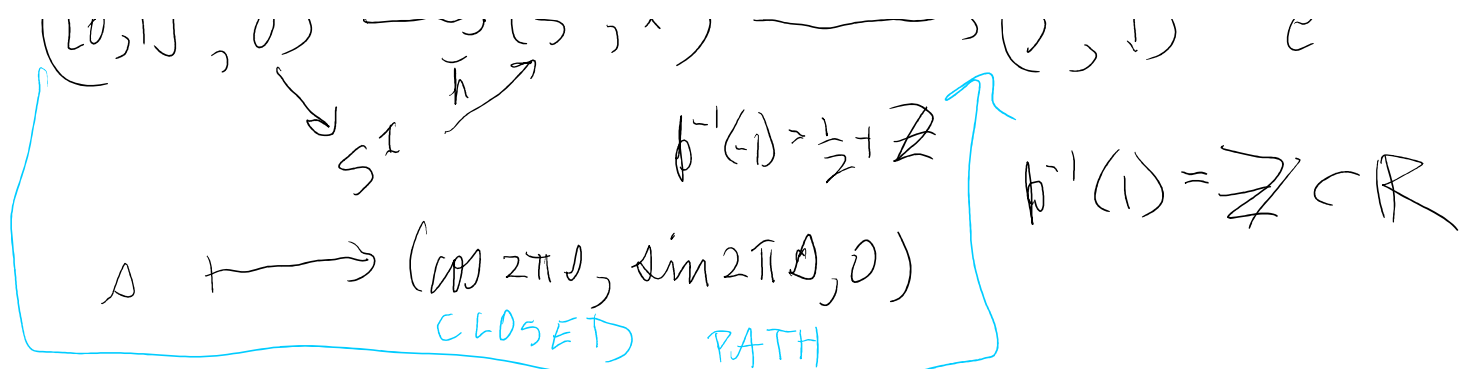
$$\text{DEFINE } g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|} \in S^1 \subset \mathbb{R}^2$$

$$\text{NOTE } g(-x) = -g(x)$$

$$\begin{array}{ccc} S^2 & \xrightarrow{g} & S^1 \\ \downarrow -1 & & \downarrow -1 \\ S^2 & \xrightarrow{g} & S^1 \end{array}$$

WE WILL SEE THAT SUCH A g CANNOT EXIST

$$\begin{array}{ccccc} \uparrow h & \cdots \cdots \cdots & \xrightarrow{\quad} & (\mathbb{R}, 0) & \uparrow t \\ & \mathbb{R}^3 & & \downarrow p & \downarrow \\ ([0, 1], 0) & \xrightarrow{h} (S^2, *) & \xrightarrow{g} & (S^1, 1) & e^{2\pi i t} \end{array}$$



LEMMA (TO BE PROVED LATER)

$$\exists \tilde{h}: [0, 1] \rightarrow \mathbb{R} \text{ WITH } p\tilde{h} = g h$$

SUPPOSE $g h$ WRAPS AROUND S^1 n TIMES, SO $\tilde{h}(1) = n$.

$$\text{AND } \tilde{h}(0) = 0$$

$$h(1/2) = (-1, 0, 0)$$

$$h(s + 1/2) = -h(s) \quad \text{FOR } 0 \leq s \leq 1/2$$

$$\begin{aligned} \tilde{h}(s + 1/2) &\in p^{-1}h(s + 1/2) = p^{-1}(-h(-s)) \\ &= 1/2 + p^{-1}h(s) \end{aligned}$$

CLAIM $\tilde{h}(s + 1/2) = \tilde{h}(s) + 1/2$ FOR

SOME ODD INTEGER q

$$\text{FOR } s = 0, \quad \tilde{h}(1/2) = \tilde{h}(0) + q/2$$

$$\begin{aligned} s = 1/2, \quad \tilde{h}(1) &= \tilde{h}(1/2) + q/2 \\ &= \tilde{h}(0) + q \end{aligned}$$

$$= \tilde{h}(0) + g$$

AND $g \neq 0$ SINCE g IS ODD.
 HENCE gh IS ESSENTIAL, IE
 NOT HOMOTOPIC TO A CONSTANT MAP.
 THIS IS A CONTRADICTION
 BECAUSE g EXTENDS TO S^2
 (QED)

HAM SANDWICH THEOREM

SUPPOSE $K_1, K_2, K_3 \subset \mathbb{R}^3$
 ARE ^{DISJOINT} COMPACT SUBSETS ^{OF POSITIVE VOLUME}
 THEN THERE IS A PLANE
 WHICH BISECTS ALL THREE.

SKETCH OF PROOF

FOR $x \in S^2 \subset \mathbb{R}^3$, $\exists$ PLANE
 $\perp$ TO $\vec{x}$ THAT BISECTS K_1
 WE GET A FAMILY OF PLANES
 BISECTING K_1 PARAMETRIZED
 BY $\sim 2 \dots \dots \dots$

DIRECTING K_1 PARAMETRIZED
 BY S^2 . FOR EACH SUCH PLANE
 WE TWO NUMBER $f_2(x)$ AND $f_3(x)$
 WHERE $f_2(x) = \left(\begin{array}{l} \text{VOLUME OF } K_2 \\ \text{LYING ABOVE THE} \\ \text{PLANE} \end{array} \right) - \left(\begin{array}{l} \text{VOLUME LYING} \\ \text{BELOW} \end{array} \right)$

$f_3(x) = \text{SAME FOR } K_3$

HENCE WE HAVE $f: S^2 \rightarrow \mathbb{R}^2$

BORSUK-ULAM SAYS

$\exists x$ WITH $f(-x) = f(x)$

BY THE WAY OUR f IS DEFINED

$f(x) = -f(-x)$ FOR ALL x

HENCE AT THE BORSUK-ULAM
 POINT $f(x) = 0$

THIS IS THE PLANE WE WANT.

QED

FUNDAMENTAL THEOREM OF

FUNDAMENTAL THEOREM OF ALGEBRA (GAUSS ~ 1800)

LET $f(x) \in \mathbb{C}[x]$ BE NONCONSTANT
 THEN $\exists \alpha \in \mathbb{C}$ WITH $f(\alpha) = 0$.

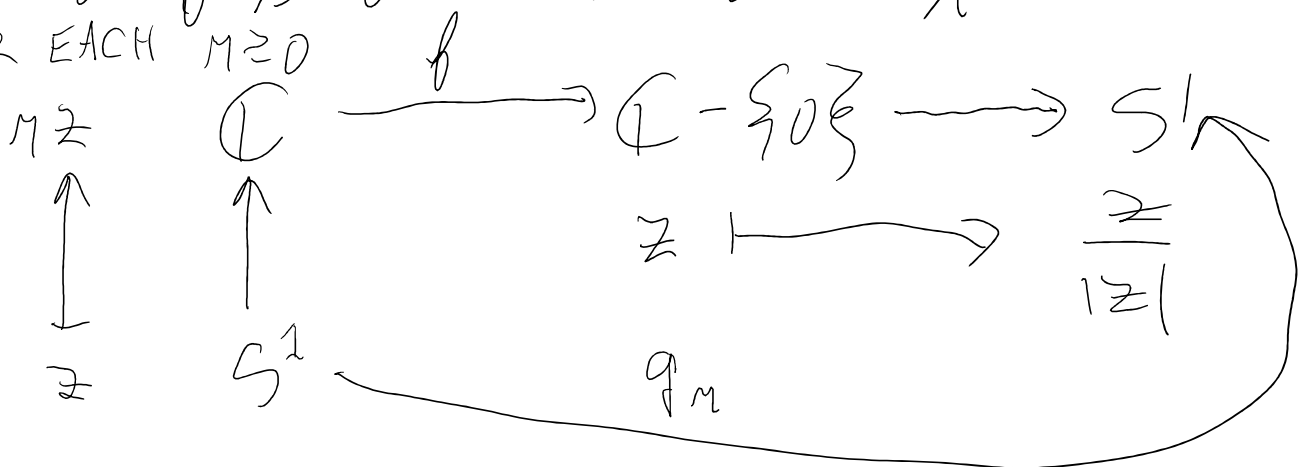
THIS IMPLIES $f(x) = (x - \alpha) f_1(x)$

PROOF: f DEFINES A MAP $\mathbb{C} \rightarrow \mathbb{C}$
 $x \mapsto f(x)$

ASSUME NO SUCH α EXISTS,

SO $f(x) \neq 0$ FOR ALL x .

FOR EACH $m \geq 0$



NOTE g_0 IS CONSTANT

ASSUME FOR SIMPLICITY
 THAT f IS MONIC AND HAS
 DEGREE $n > 0$

DEGREE $n > 0$

$$f(m e^{it}) = (m e^{it})^n + \text{LOWER TERMS}$$

FOR $m \gg 0$, THE LEADING TERM
IS BIGGER THAN THE OTHERS.

IT FOLLOWS THAT g_m
HAS DEGREE n . IT IS NOT
HOMOTOPIC TO g_0 . CONTRADICTION

(QED)

NEXT TIME:

PROOF THAT $\pi_1(S^1) \cong \mathbb{Z}$.