9-10-25

Wednesday, September 10, 2025 1:40 PM

RECALL LIMITS

J = SMALL CATEGORY

$\mathcal{C}$ = ANY CATEGORY

$X : J \longrightarrow \mathcal{C}$ A FUNCTOR, A J-SHAPED

$j \longmapsto X_j$

$\lim_J X$ = AN OBJECT W IN $\mathcal{C}$

WITH A $W \xrightarrow{i_j} X_j$ $\forall j \in J$

S.T.

1) FOR MORPHISM $j \xrightarrow{f} j'$

IN J

$$\begin{tikzcd} & X_j \arrow[dd, "X(f)"] \\ W \arrow[ur, "i_j"] \arrow[dr, "i_{j'}"'] & \\ & X_{j'} \end{tikzcd}$$

COMMUTES

2) W HAS A UNIVERSAL PROPERTY: FOR W' EQUIPPED WITH SIMILAR MAPS $i_j' : W' \longrightarrow X_j$,

$\exists ! \; W' \xrightarrow{\phi} W$ SO THAT

$$\begin{tikzcd} \exists!\, W' \arrow[rr, "\varphi"] \arrow[dr, "i'_j"'] & & W \arrow[dl, "i_j"] \\ & X_j & \end{tikzcd}$$

SO THAT THE TRIANGLE COMMUTES

$\mathrm{COLIM}_J X$ IS AN OBJECT Z IN $\mathcal{C}$ WITH $X_j \xrightarrow{h_j} Z$ WITH SIMILAR PROPERTIES.

WE KNOW THAT LIMITS AND COLIMITS EXIST FOR

$\mathcal{Set}$ SETS

$\mathcal{Ab}$ ABELIAN GROUPS

$\mathcal{Grp}$ GROUPS

$\mathcal{Top}$ TOPOLOGICAL SPACES

$R\text{-}\mathcal{Mod}$ R-MODULES FOR A RING R.

⋮

EXAMPLES

J

1) PUSHOUT $\begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[r] & \bullet \end{tikzcd}$

2) COEQUALIZER $\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}$

3) $J = \mathcal{B}G$ FOR A GROUP G

$\mathcal{C} = \mathcal{S}et$ OR $\mathcal{T}op$

COLIMIT IS ORBIT SPACE

X_G OR X/G.

4) $J = \mathbb{N}$ $\quad 0 \to 1 \to 2 \to 3 \to 4 \to 5 \to \cdots$

$\mathcal{C} = \mathcal{A}b$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "1"] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "3"] & \mathbb{Z} \arrow[r, "4"] & \mathbb{Z} \arrow[r, "5"] & \mathbb{Z} \arrow[r] & \cdots \\
& \mathbb{Z} \arrow[ul, "1"] \arrow[u, "1"] \arrow[ur, "2"] \arrow[urr, "3!"] \arrow[urrr, "4!"] \arrow[urrrr, "5!"] & & & & &
\end{tikzcd}$$

COLIMIT IS $\mathbb{Q}$

WHAT IS THE LIMIT? $\mathbb{Z}$ AS SHOWN.

NOTE $J = \mathbb{N}$ HAS AN INITIAL OBJECT

PROP IF J HAS AN INITIAL OBJECT j_0 AND $J \xrightarrow{X} \mathcal{C}$ IS ANY FUNCTOR, THEN

$$\lim_J X = X_{j_0}$$

EXAMPLE

J = EMPTY CATEGORY

FOR ANY $\mathcal{C}$ THERE IS A UNIQUE FUNCTOR $J \longrightarrow \mathcal{C}$

$\lim_J \phi$ = TERMINAL OBJECT

$\operatorname{colim}_J \phi$ = INITIAL OBJECT

ANOTHER PERSPECTIVE

LET $\mathcal{C}^J$ DENOTE THE CATEGORY OF FUNCTORS $J \xrightarrow{X} \mathcal{C}$

OBJECTS ARE J-SHAPED

DIAGRAMS

A MORPHISM $f : X \longrightarrow Y$ IS A COLLECTION $f_j : X_j \longrightarrow Y_j$ THAT RESPECT MORPHISMS IN J.

$\mathcal{C}^J$ = CATEGORY OF FUNCTORS $J \to \mathcal{C}$

$\parallel$

$\mathrm{Fun}(J, \mathcal{C})$

THERE IS A DIAGONAL FUNCTOR

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Delta} & \mathcal{C}^J \\ \rotatebox{90}{$\in$} & & \\ c & \longmapsto & \text{CONSTANT} \\ & & c\text{-VALUED} \\ & & \text{FUNCTOR } J \to \mathcal{C} \end{array}$$

IF $\mathcal{C}$ HAS LIMITS, WE GET A FUNCTOR

$$\mathcal{C} \underset{\mathrm{colim}}{\overset{\lim}{\leftleftarrows}} \mathcal{C}^J$$

THESE ARE THE LEFT AND RIGHT (IN WHICH ORDER?) ADJOINTS OF Δ.

DEF GIVEN TWO FUNCTORS $F, G : \mathcal{C} \longrightarrow \mathcal{D}$, A **NATURAL TRANSFORMATION** $\theta : F \longrightarrow G$ CONSISTS OF

1) FOR EACH $x \in \mathcal{C}$, A MORPHISM $\theta_x : F(x) \longrightarrow G(x)$ IN $\mathcal{D}$

2) FOR EACH MORPHISM $x \xrightarrow{f} y$ IN $\mathcal{C}$ THE FOLLOWING DIAGRAM COMMUTES

$$\begin{tikzcd}
F(x) \arrow[r, "\theta_x"] \arrow[d, "F(f)"'] & G(x) \arrow[d, "G(f)"] \\
F(y) \arrow[r, "\theta_y"] & G(y)
\end{tikzcd}$$

A NATURAL IS ALSO A MORPHISM IN THE FUNCTOR CATEGORY $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$

BACK DOWN TO EARTH

DEF A MAP OF SPACES $X \xrightarrow{f} Y$ IS A __HOMOTOPY EQUIVALENCE__ IF THERE IS A MAP $X \xleftarrow{g} Y$

SUCH

1) $gf : X \circlearrowleft$ IS HOMOTOPIC TO 1_X, THE IDENTITY MAP ON X
2) SIMILARLY FOR $fg : Y \circlearrowleft$

EXAMPLE

① $X = \mathbb{R}^n$, $Y = pt$

$$\mathbb{R}^n \underset{\text{ORIGIN} \,=\, g}{\overset{f}{\rightleftarrows}} pt$$

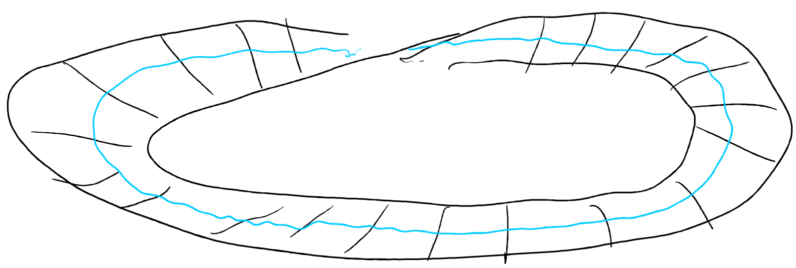
$fg = 1_{pt}$

gf = MAP TO ORIGIN

THE DESIRED HOMOTOPY

$$\mathbb{R}^n \times I \longrightarrow \mathbb{R}^n$$
$$(x, s) \longmapsto sx$$
$$(x, 1) \longmapsto x$$
$$(x, 0) \longmapsto 0$$

② $X = S^1$ $\quad$ Y = MOBIUS STRIP



Y

DEF THE <u>HOMOTOPY CATEGORY</u> $h(\mathcal{T}op)$ OF TOPOLOGICAL SPACES HAS

1) OBJECTS ARE TOP. SPACES
2) MORPHISMS ARE HOMOTOPY CLASSES OF MAPS.

THERE IS A FUNCTOR

$$\mathcal{T}op \longrightarrow h(\mathcal{T}op)$$

$\mathrm{Top} \longrightarrow h(\mathrm{Top})$

WITH NO INVERSE

THEOREM $\pi_1(S^1) \simeq \mathbb{Z}$ WINDING #

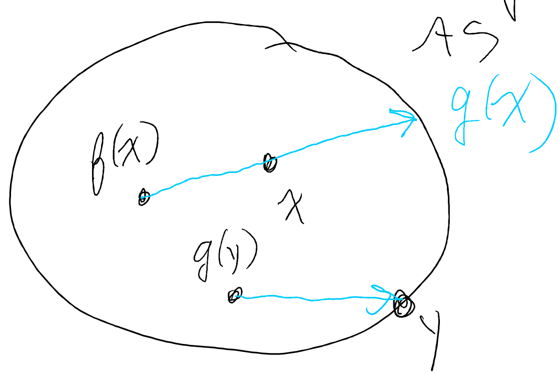
PROOF LATER

COR THE IDENTITY IS NOT HOMOTOPIC TO THE CONSTANT MAP.

BROUWER FIX POINT THEOREM

ANY MAP $D^2 \xrightarrow{f} D^2$ HAS A FIXED POINT, I.E. $\exists x \in D^2$ WITH $f(x) = x$

PROOF: ASSUME NO SUCH POINT EXISTS. DEFINE $g : D^2 \to \partial D^2$ AS SHOWN



NOTE $g|\partial D^2$ IS THE IDENTITY

$$\begin{tikzcd} S^1 \arrow[r, "\text{IDENTITY}"] \arrow[d, "i"] & S^1 \\ {} \arrow[ur] & \end{tikzcd}$$

$$S^1 \times I^1 \longrightarrow D^2 \xrightarrow{g}$$

$$(z, s) \longmapsto s z$$

WE GET A HOMOTOPY BETWEEN 1_{S^1} AND A CONSTANT MAP

CONTRADICTION

QED

SAME HOLDS FOR D^n IF WE KNOW THE IDENTITY ON S^{n-1} IS NOT HOMOTOPIC TO A CONSTANT MAP.