Math443-8-27-25

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RECALL THE MAP $SO(2) \hookrightarrow SO(3)$ INDUCES A HOM

$$\begin{array}{ccc} \pi_1 SO(2) & \longrightarrow & \pi_1 SO(3) \\ \wr\Vert & & \wr\Vert \\ \mathbb{Z} & & \mathbb{Z}/2 \end{array}$$

$SO(2) \approx S^1$ = CIRCLE

$\pi_1 S^1 \longleftrightarrow$ WINDING #

= # OF TIMES A CLOSED PATH GOES AROUND S^1

TOPOLOGY OF $SO(3)$

<u>DEF</u> $\mathbb{R}P^n$ = REAL PROJECTIVE n-SPACE

:= THE SET OF LINES THRU ORIGIN IN $\mathbb{R}^{n+1}$

TO TOPOLOGIZE THIS, LET $U \subset \mathbb{R}^{n+1}$ BE ANY OPEN SET AND LET $\tilde{U} \subset \mathbb{R}P^n$ BE THE SET OF LINES THRU 0 THAT ALSO INTERSECT U.

$\mathbb{R}P^n$ IS AN n-DIMENSIONAL SMOOTH MANIFOLD.

THERE IS A MAP

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$$S^n \xrightarrow{p} \mathbb{RP}^n$$

IT IS A DOUBLE COVERING

S^n = SET OF UNIT VECTORS IN $\mathbb{R}^{n+1}$

$\mathbb{RP}^0$ = POINT

$\mathbb{RP}^1 \cong S^1$ $\qquad S^1 \xrightarrow{p} RP^1$

REMARK: FOR $n > 1$, $\pi_1 S^n = 0$ AND $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2$

IF $S^1 \xrightarrow{f} S^n$ IS NOT ONTO, LET $x \in S^n$ BE A POINT NOT IN THE IMAGE

$$\begin{tikzcd}
S^1 \arrow[r, "f"] \arrow[dr, "f'"'] & S^n \\
 & S^n - \{x\} \cong \mathbb{R}^n \arrow[u]
\end{tikzcd}$$

AND f' IS HOMOTOPIC TO THE CONSTANT MAP.

<u>THEOREM $SO(3) \approx \mathbb{RP}^3$</u>

<u>LEMMA</u> RP^n IS THE QUOTIENT

<u>LEMMA</u> $\mathbb{RP}^n$ IS THE QUOTIENT OF D^n (n-DIMENSIONAL DISK) OBTAINED BY IDENTIFYING ANTIPODAL POINTS ON ITS BOUNDARY $\partial D^n \approx S^{n-1}$

PROOF OF THEOREM. ANY POINT $\neq e$ IN $SO(3)$ CORRESPONDS TO A ROTATION ABOUT SOME AXIS COUNTERCLOCKWISE THRU SOME ANGLE θ WITH $0 < \theta \leq \pi$

WE CAN DEFINE A MAP

$$D^3 \xrightarrow{f} SO(3)$$

$$D^3 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1\}$$

$$(0,0,0) \longmapsto e \in SO(3)$$

OTHERWISE WE CAN WRITE

$$(x,y,z) = r \boxed{\frac{(x,y,z)}{r}} \qquad r = \sqrt{x^2+y^2+z^2}$$

UNIT VECTOR

λ

$f(x,y,z) =$ ROTATION ABOUT λ

$f(x,y,z) =$ ROTATION ABOUT λ WITH ANGLE πr

NOTE THAT ANTIPODAL POINTS ON ∂D^3 MAP TO THE SAME ROTATION IN $SO(3)$

HENCE OUR MAP $D^3 \to SO(3)$ FACTORS THRU $\mathbb{RP}^3$ LEADING TO THE DESIRED HOMEOMORPHISM. QED

WE KNOW $SO(4) \approx \underset{\mathbb{RP}^3}{\overset{}{\underset{\parallel}{SO(3)}}} \times S^3$

$\mathbb{RP}^3 \times S^3$

IN GENERAL THERE IS A MAP $SO(n) \xrightarrow{g} S^{n-1}$

$g :=$ EVALUATION AT SOME PRECHOSEN UNIT VECTOR

THE MAP IS ONTO.

$g^{-1}\left(\substack{\text{any} \\ \text{pt}}\right) \approx SO(n-1)$ DIFFERENT

THESE COPIES OF $SO(n-1)$ IN $SO(n)$ ARE DISJOINT SINCE EACH MAPS TO A POINT IN S^{n-1}!

SPECIAL FEATURE $n=4$: (AND $n=8$ AND $n=2$)

CAN USE QUATERNIONS TO DEFINE A MAP

$$SO(4) \underset{h}{\overset{g}{\rightleftarrows}} S^3 = \text{UNIT QUATERNIONS}$$

$$\exists h: S^3 \to SO(4) \quad \text{s.t.} \quad gh = 1_{S^3}$$

CAN USE THIS TO SHOW

$SO(4) = S^3 \times SO(3)$ QUATERNIONS (NOT COMM)

$SO(8) = S^7 \times SO(7)$ OCTONIONS OR CAYLEY #s (NOT ASSOC)

$SO(2) = S^1 \times SO(1)$ COMPLEX #s.

THERE IS NO 16-DIMENSIONAL DIVISION ALGEBRA

THE COFFEE CUP + OTHERS TRICKS ARE RELATED

THE COFFEE CUP [illegible] TOPICS
ARE RELATED

$$SO(2) \longrightarrow SO(3)$$

$$\mathbb{Z} \cong \pi_1 SO(2) \longrightarrow \pi_1 SO(3) = \mathbb{Z}/2$$

$$2 \longmapsto 0$$

CATEGORY THEORY

DEF A <u>CATEGORY</u> $\mathcal{C}$ CONSISTS OF

1) A COLLECTION OF OBJECTS
2) FOR ANY TWO OBJECTS X, Y A SET OF MORPHISMS

   $X \xrightarrow{f} Y$ $\quad \mathcal{C}(X,Y)$
3) GIVEN $X \xrightarrow{f} Y$ AND $Y \xrightarrow{g} Z$ THERE IS A COMPOSITE MORPHISM

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

COMPOSITION IS ASSOCIATIVE

4) $\mathcal{C}(X,X) \ni$ IDENTITY MORPHISM

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[rr, bend right, "g"'] & X \arrow[r, "1_X"] \arrow[rr, bend left, "f"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

EXAMPLES

1) $\mathbf{Set}$: OBJECTS ARE SETS
MORPHISMS ARE MAPS

2) $\mathbf{Top}$: OBJECTS ARE TOP. SPACES
MORPHISMS ARE CONTINUOUS MAPS

3) $\mathbf{Grp}$: OBJECTS ARE GROUPS
MORPHISMS ARE GROUP HOMOMORPHISM

4) WALKING ARROW $\quad \overset{A}{\bullet} \longrightarrow \overset{B}{\bullet}$

EMPTY CAT
NO OBJECTS

5) ??? $\quad \begin{tikzcd} \overset{A}{\bullet} \arrow[r, shift left] \arrow[r, shift right] & \overset{B}{\bullet} \end{tikzcd}$

TRIVIAL CAT
ONE OBJECT WITH IDENTITY

6) LET G BE A GROUP
$\mathbb{B}G$ = CATEGORY WITH ONE OBJECT
THERE IS ONE MORPHISM FOR EACH $\gamma \in G$ WITH COMPOSITION

EACH $\gamma \in G$ WITH COMPOSITION GIVEN BY GROUP MULTIPLICATION.