8-25-25

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THE COFFEE CUP TRICK + THE GEOMETRY BEHIND IT.

LET $SO(n)$ DENOTE THE SPECIAL n-DIMENSIONAL ORTHOGONAL GROUP

= THE GROUP OF $n \times n$-ORTHOGONAL MATRICES WITH DETERMINANT 1

= GROUP OF ROTATIONS OF $\mathbb{R}^n$!

A MATRIX IS ORTHOGONAL IF IT HAS ORTHONORMAL ROW VECTORS.

AN $n \times n$ MATRIX M OVER $\mathbb{R}$ IS ORTHOGONAL IF

$$M^{-1} = M^{tr} = \text{TRANSPOSE OF } M$$

THIS IMPLIES $\det M = \pm 1$

$SO(n)$ = GP OF $n \times n$ ORTHOGONAL WITH DETERMINANT 1.

$SO(2)$ = GROUP OF ROTATIONS IN $\mathbb{R}^2$

$= \Big(\Big[\cos\theta \quad \sin\theta \Big]$

$$= \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} : 0 \le \theta \le 2\pi \right\}$$

$\approx S^1$ = CIRCLE.

SO(n) HAS A TOPOLOGY THAT IT INHERITS FROM $\mathbb{R}^{(n^2)}$

IT IS KNOWN TO BE A CLOSED SMOOTH MANIFOLD OF DIMENSION $\binom{n}{2}$

WE CAN DESCRIBE IT EXPLICITLY FOR $n \le 4$.

IN THE COFFEE CUP TRICK WE HAVE CLOSED PATH IN SO(3).

WE WANT TO THINK ABOUT CONTINUOUS FAMILES OF CLOSED PATHS IN A SPACE X.

DEF A CLOSED PATH IN X AT $x_0 \in X$ (THE BASE POINT) IS A MAP

$[0,1] \xrightarrow{p} X$

$$I = [0,1] \xrightarrow{p} X$$
$$\{0,1\} \longmapsto x_0$$

DEF TWO MAPS $f_0, f_1 : X \longrightarrow Y$ ARE HOMOTOPIC IF THERE IS A MAP $h : I \times X \longrightarrow Y$ SUCH THAT $h(0,x) = f_0(x)$ AND $h(1,x) = f_1(x)$

h IS A HOMOTOPY. IT CONTINUOUSLY DEFORMS f_0 TO f_1.

$$\partial I = \{0,1\} \subset I$$

EXAMPLE $X = S^1 = I/\partial I \ni x_0$

$Y = Y$ WITH $y_0 \in Y$

WANT BASEPOINT PRESERVING HOMOTOPIES

$$h : I \times S^1 \longrightarrow Y$$

FOR ALL s, $(s, x_0) \longmapsto y_0$

FOR ANY POINTED SPACE (Y, y_0) LET $\pi(Y, y_0)$ BE THE

(Y, y_0), LET $\pi_1(Y, y_0)$ BE THE SET OF HOMOTOPY CLASSES OF CLOSED PATHS IN Y THAT START AND END AT y_0.

<u>THEOREM</u> THIS SET HAS A NATURAL GROUP STRUCTURE.

"NATURAL" MEANS A MAP

$$(Y, y_0) \xrightarrow{\phi} (Y', y_0')$$

INDUCES

$$\pi_1(Y, y_0) \xrightarrow{\pi_1(\phi)} \pi_1(Y', y_0')$$

WHICH IS A GROUP HOMOMORPHISM

HOW TO DEFINE THE BINARY OPERATION IN $\pi_1(Y, y_0)$:

GIVEN CLOSED PATHS ϕ_1, ϕ_2 IN Y

DEFINE A THIRD PATH $\phi_1 * \phi_2$ CONCATENATION

BY. $\phi_1, \phi_2 : [0, 1] \to Y$

$$\phi_1 * \phi_2(t) = \begin{cases} \phi_1(2t) & \text{FOR } 0 \le t \le 1/2 \end{cases}$$

$$p_1 * p_2(t) = \begin{cases} p_1(2t) & \text{FOR } 0 \le t \le 1/2 \\ p_2(2t-1) & \text{FOR } 1/2 \le t \le 1 \end{cases}$$

WE NEED TO VERIFY

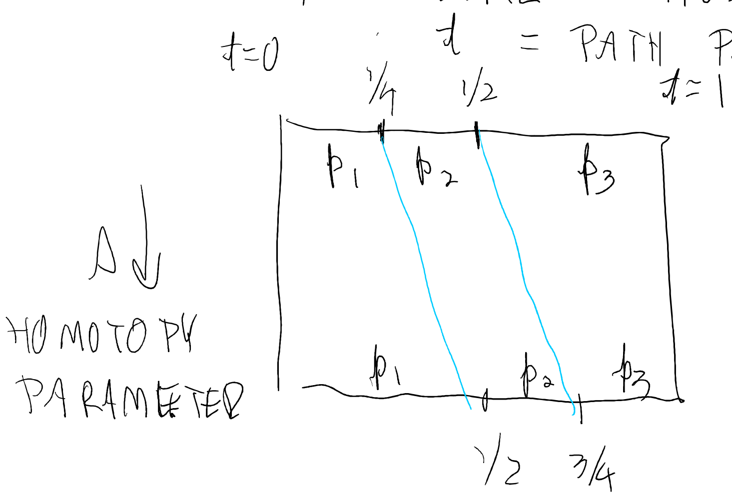
1) IF WE ALTER p_1 AND p_2 BY HOMOTOPIES, WE ARE ALTERING $p_1 * p_2$ BY A HOMOTOPY. HENCE WE GET A WELL DEFINED BINARY OPERATION ON $\pi_1(Y, y_0)$

2) ASSOCIATIVITY: GIVEN THREE CLOSED PATHS p_1, p_2, p_3, WE NEED TO COMPARE

$$(p_1 * p_2) * p_3 \quad \text{AND} \quad p_1 * (p_2 * p_3)$$

THEY ARE HOMOTOPIC.

t = PATH PARAMETER



$(p_1 * p_2) * p_3$

$p_1 * (p_2 * p_3)$

3) IDENTITY ELEMENT IS THE CONSTANT y_0-VALUED PATH p_n

CONSTANT y_0-VALUED PATH p_0

CAN SHOW $p_0 * p \simeq p \simeq p * p_0$

FOR ANY CLOSED PATH p.

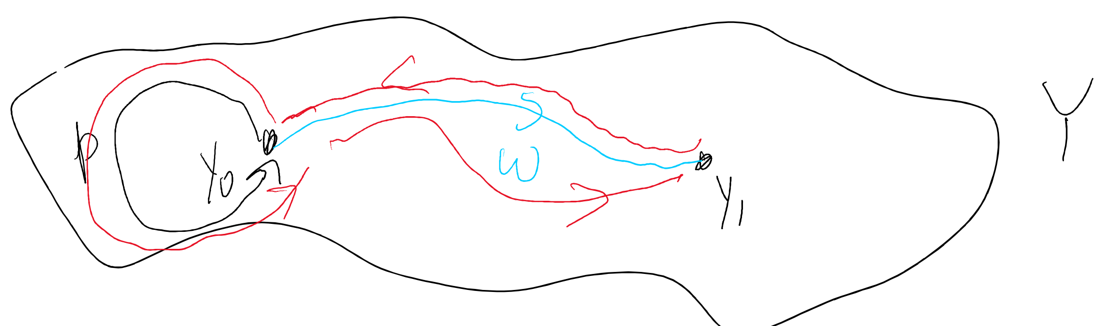
4) INVERSES. THE INVERSE $\bar{p}$ OF A PATH p IS DEFINED BY $\bar{p}(t) = p(1-t)$

CAN SHOW $\bar{p} * p \simeq p_0 \simeq p * \bar{p}$

THIS MAKES $\pi_1(Y, y_0)$ A GROUP, THE <u>FUNDAMENTAL GROUP</u> OF (Y, y_0).

<u>EASY FACTS</u>

IF Y IS PATH CONNECTED, THE CHOICE OF y_0 IS IRRELEVANT



CLAIM $\pi_1(Y, y_0) \cong \pi_1(Y, y_1)$

CLAIM $\pi_1(Y, y_0) \approx \pi_1(Y, y_1)$

CHOOSE A PATH ω FROM y_0 TO y_1

FOR ANY PATH p AT y_0

$\bar{\omega} * p * \omega$ IS A PATH AT y_1

WILL SHOW LATER THAT

1) $\pi_1 SO(2) \approx \mathbb{Z}$

2) $\pi_1 SO(3) \approx \mathbb{Z}/2$

3) THE INCLUSION $SO(2) \to SO(3)$ INDUCES A NONTRIVIAL HOM

$\mathbb{Z} = \pi_1 SO(2) \to \pi_1 SO(3) = \mathbb{Z}/2$

4) ROTATING THE COFFEE CUP GIVES A GENERATOR OF $\pi_1 SO(2) \approx \mathbb{Z}$.

$2 \times \text{GENERATOR} \mapsto 0$

THIS IS WHY THE COFFEE CUP AND BELT TRICKS ARE POSSIBLE.