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INTRODUCTION TO HOMOLOGICAL ALGEBRA

DEF AN R-MODULE P IS PROJECTIVE IF FOR ANY DIAGRAM OF THE FORM

$$\begin{tikzcd} & P \arrow[dl, dashed, "\hat{\alpha}"'] \arrow[d, "\alpha"] & \\ M \arrow[r, "f"'] & N \arrow[r] & 0 \end{tikzcd}$$

WITH f ONTO, $\exists \hat{\alpha} : P \to M$ WITH $f\hat{\alpha} = \alpha$.

SUPPOSE P IS FREE WITH GENERATORS $x_1, x_2, \cdots$. CHOOSE $m_1, m_2 \cdots \in M$ WITH $f(m_i) = \alpha(x_i)$ AND DEFINE $\hat{\alpha}(x_i) = m_i$

HENCE ANY FREE MODULE IS PROJECTIVE.

FOR $R = \mathbb{Z}$, EVERY PROJECTIVE MODULE IS FREE.

EXAMPLE $R = \mathbb{Z}[x]$ $P = (2, x) \subset R$

IT IS PROJECTIVE BUT NOT FREE

$$R^2 \xrightarrow{\varphi} R \qquad \operatorname{Im} \varphi = P$$

$$R^2 = R\{a_1, a_2\}$$

$$a_1 \mapsto 2$$

$$a_2 \mapsto x$$

[illegible]

$a_2 \longmapsto -x$

$$\ker \varphi = \{ x_1 a_1 + x_2 a_2 : 2x_1 + x x_2 = 0 \}$$

CAN SHOW $R^2 \cong \ker \varphi \oplus P$ AND $\ker \varphi$ IS FREE.

PROP ANY DIRECT SUMMAND OF A FREE MODULE IS PROJECTIVE.

NOT NEEDED

PROP ① FOR EACH R-MODULE M, THERE IS A PROJECTIVE P WITH A SURJECTION $P \longrightarrow M$.

PROOF: CHOOSE A SET $\{x_1, x_2 \dots\}$ OF R-MODULE GENERATORS OF M, AND P BE THE FREE R-MODULE ON THIS SET. $\square$

" R HAS ENOUGH PROJECTIVES "

DEF A PROJECTIVE RESOLUTION OF AN R-MODULE M IS A LES

$$0 \leftarrow M \xleftarrow{d_0} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} P_2 \leftarrow \cdots$$

WHERE EACH P_i IS PROJECTIVE

PROP ② EVERY M HAS A PROJ. RES.

PROOF: USE PROP ① TO CONSTRUCT P_0

$$0 \leftarrow M \xleftarrow{d_0} P_0 \xleftarrow{1\text{-}1} \ker d_0 \leftarrow 0$$

USE PROP ① AGAIN TO PRODUCE

$$0 \leftarrow \ker d_0 \underset{d_1}{\xleftarrow{\text{ONTO}}} P_1 \leftarrow \ker d_1 \leftarrow 0$$

REPEAT FOREVER. QED

DEF A RING R IS A PRINCIPAL IDEAL DOMAIN (PID)

IF EACH IDEAL IN R IS PRINCIPAL, I.E. GENERATED BY A SINGLE ELEMENT.

EXAMPLES

$\mathbb{Z}$ IS A PID

ANY FIELD IS A PID

ANY SUBRING OF $\mathbb{Q}$ IS A PID

$\mathbb{Z}[x]$ IS NOT A PID

$(2, x)$ IS NOT PRINCIPAL

PROP IF R IS A PID, THEN EVERY R-MODULE M HAS A PROJECTIVE RES. OF LENGTH 1.

A PROJECTIVE RES. OF LENGTH 1,

$\exists$ PROJECTIVES P_0 AND P_1 WITH.

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow 0$$

$$0 \quad \mathbb{Z}/n \leftarrow \mathbb{Z} \xleftarrow{n} \mathbb{Z} \leftarrow 0$$

EXAMPLE $\quad k =$ FIELD

$$R = k[x]/(x^2)$$

$$M = R/(x) \cong k$$

PROJECTIVE RESOLUTION

$$0 \leftarrow \underset{\substack{\parallel \\ k}}{M} \leftarrow R \xleftarrow{x} R \xleftarrow{x} R \xleftarrow{x} R \xleftarrow{x} \cdots$$

RECALL OUR BASIC DIFFICULTY

GIVEN A SES OF ABELIAN GROUPS $\quad 0 \to A' \to A \to A'' \to 0,$

THE FUNCTORS $- \otimes B$ AND $\mathrm{Hom}(-, B)$ FAIL TO PRESERVE EXACTNESS.

SUPPOSE M AND N ARE R-MODULES AND M HAS A PROJECTIVE RESOLUTION

RESOLUTION

$$0 \leftarrow M \leftarrow \boxed{P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots}$$

CONSIDER THE CHAIN COMPLEX

$$(3) \qquad P_0 \otimes_R N \leftarrow P_1 \otimes_R N \leftarrow P_2 \otimes_R N \leftarrow \cdots$$

THEOREM (FUND THM OF HOMOLOGICAL ALGEBRA) $H_i((3))$ IS INDEPENDENT OF THE CHOICE OF PROJECTIVE RES

$$\boxed{\text{DEF} \quad Tor_i^R(M,N) := H_i((3))}$$

MEMORIZE

REMARKS

1. $Tor_0^R(M,N) = M \otimes_R N$
2. IF R IS A PID THEN $Tor_i^R(M,N) = 0$ FOR $i > 1$ FOR ANY M AND N.
3. IF EITHER M OR N IS PROJ, THEN $Tor_i^R(M,N) = 0$ FOR $i > 0$
4. $Tor_*^R(M,N) \cong Tor_*^R(N,M)$
5. $Tor_1^{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) = \mathbb{Z}/\gcd(m,n)$

DEF $Ext^i_R(M,N)$ IS THE iTH COHOMOLOGY OF THE COCHAIN CX

$$Hom_R(P_0,N) \longrightarrow Hom_R(P_1,N) \longrightarrow Hom_R(P_2,N) \to \cdots$$

$$Hom_R(P_i,N) \longrightarrow Hom_R(P_{i+1},N)$$

$$\begin{tikzcd}
P_i \arrow[d] & P_{i+1} \arrow[l, "d_{i+1}"'] \arrow[dl] \\
N &
\end{tikzcd}$$

SIMILAR REMARKS APPLY

EG $Ext^0_R(M,N) = Hom_R(M,N)$

<u>LEMMA</u> LET P AND P' BE TWO PROJECTIVE RESOLUTIONS OF M. THEY ARE CHAIN HOMOTOPICALLY EQUIVALENT.

<u>PROOF</u>:

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, equal] & P_0 \arrow[l, "d_0"'] \arrow[d, "g_0"] \arrow[dl] & P_1 \arrow[l, "d_1"'] \arrow[d, "g_1"] & P_2 \arrow[l, "d_2"'] \arrow[d, "g_2"] & \cdots \arrow[l, "d_3"'] \\
0 & M \arrow[l] & P_0' \arrow[l, "d_0'"'] & P_1' \arrow[l, "d_1'"'] & P_2' \arrow[l, "d_2'"'] & \cdots \arrow[l, "d_3'"']
\end{tikzcd}$$

$\exists g_0$ BECAUSE d_0' IS ONTO AND P_0 IS PROJECTIVE.

$$\begin{tikzcd}
\ker d_0 \arrow[d] & P_1 \arrow[l] \arrow[dl] \arrow[d, dotted, "g_1"] \\
\ker d_0' & P_1' \arrow[l, "\text{ONTO}"']
\end{tikzcd}$$

$\exists g_1$ FOR SIMILAR REASONS

CAN CONSTRUCT $g_1 : P_1 \longrightarrow P_1'$

THUS WE CAN FIND $g_i : P_i \longrightarrow P_i'$ WITH

$g_i d_{i+1} = d_i' g_{i+1}$ AND g IS A

CHAIN MAP $P \xrightarrow{g} P'$

WE CAN CONSTRUCT $P' \xrightarrow{f} P$

IN THE SAME WAY.

$$P' \xrightarrow{f} P \xrightarrow{g} P'$$

WANT TO SHOW $gf \simeq I_{P'}$

$$\begin{tikzcd}
P_{i-1}' \arrow[dr, "D_{i-1}'"'] & P_i' \arrow[l, "d_i'"'] \arrow[d, "1-g_i f_i"] \arrow[dr, "D_i'"] & \\
 & P_i' & P_{i+1}' \arrow[l, "d_{i+1}'"]
\end{tikzcd}$$

WE NEED MAPS $D_i' : P_i' \longrightarrow P_{i+1}$

SUCH THAT $D_{i-1}' d_i' + d_{i+1}' D_i' = 1 - g_i f_i$

WE CAN CONSTRUCT SUCH D_i

WE CAN CONSTRUCT SUCH D_1 USING THE PROJECTIVITY OF THE P_1'. DETAILS OMITTED

THIS MEANS Tor AND Ext ARE WELL DEFINED.

<u>THEOREM</u> GIVEN SHORT EXACT SEQUENCES

$$0 \to M' \to M \to M'' \to 0$$

AND $$0 \to N' \to N \to N'' \to 0$$

WE GET LONG EXACT SEQUENCES

$$\cdots \to \mathrm{Tor}_1(M', N) \to \mathrm{Tor}_1(M, N) \to \mathrm{Tor}_1(M'', N) \to$$
$$\to \mathrm{Tor}_0(M', N) \to \mathrm{Tor}_0(M, N) \to \mathrm{Tor}_0(M'', N)$$

NOTE $\mathrm{Ext}^*_R(M, N)$ IS CONTRAVARIANT IN M AND COVARIANT IN N. WE HAVE

$$\cdots \leftarrow \mathrm{Ext}^1(M', N) \leftarrow \mathrm{Ext}^1(M, N) \leftarrow \mathrm{Ext}^1(M'', N) \leftarrow$$
$$\leftarrow \mathrm{Ext}^0(M', N) \leftarrow \mathrm{Ext}^0(M, N) \leftarrow \mathrm{Ext}^0(M'', N) \leftarrow 0$$

$$\hookleftarrow \mathrm{Ext}^0(M',N) \leftarrow \mathrm{Ext}^0(M,N) \leftarrow \mathrm{Ext}^0(M'',N) \leftarrow 0$$

AND

$$\cdots \to \mathrm{Ext}^1(M,N') \to \mathrm{Ext}^1(M,N) \to \mathrm{Ext}^1(M,N'') \to$$

$$\hookrightarrow \mathrm{Ext}^0(M,N') \to \mathrm{Ext}^0(M,N) \to \mathrm{Ext}^0(M,N'') \to 0$$

ANOTHER WAY TO DEFINE Ext:

DEF A <u>INJECTIVE</u> R-MODULE I

$$\begin{tikzcd} & I & \\ M \arrow[ur, dashed, "\hat{\alpha}"] & N \arrow[l, "f"] \arrow[u, "\alpha"'] & 0 \arrow[l] \end{tikzcd}$$

IF FOR EACH INJECTION $f: N \to M$ AND $\alpha: N \to I$, $\exists\, \hat{\alpha}: M \to I$ WITH $\hat{\alpha} f = \alpha$.

EXAMPLES $\mathbb{Z}$ IS NOT INJECTIVE $/\mathbb{Z}$ BUT $\mathbb{Q}$ AND $\mathbb{Q}/\mathbb{Z}$ ARE BOTH INJECTIVE

$$\begin{tikzcd} & \mathbb{Q} \\ \mathbb{Z} \arrow[ur, dashed] & \mathbb{Z} \arrow[l, "2"'] \arrow[u] \end{tikzcd}$$

IN GENERAL FREE R-MODULES ARE <u>NOT</u> INJECTIVE

DEF AN INJECTIVE [illegible]

DEF AN INJECTIVE RESOLUTION IS A LONG EXACT SEQUENCE

$$0 \to M \to I_0 \to I_1 \to I_2 \to \cdots$$

WITH EACH I_n INJECTIVE.