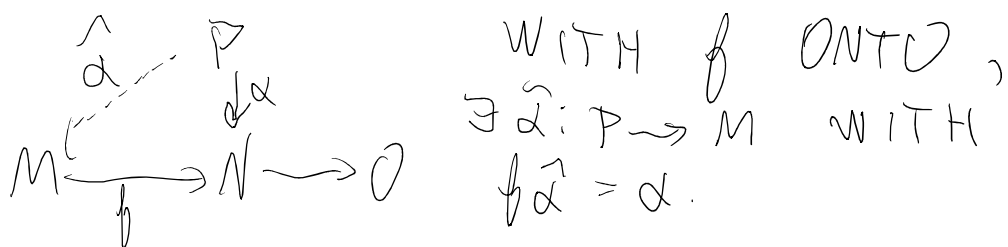


# INTRODUCTION TO HOMOLOGICAL ALGEBRA

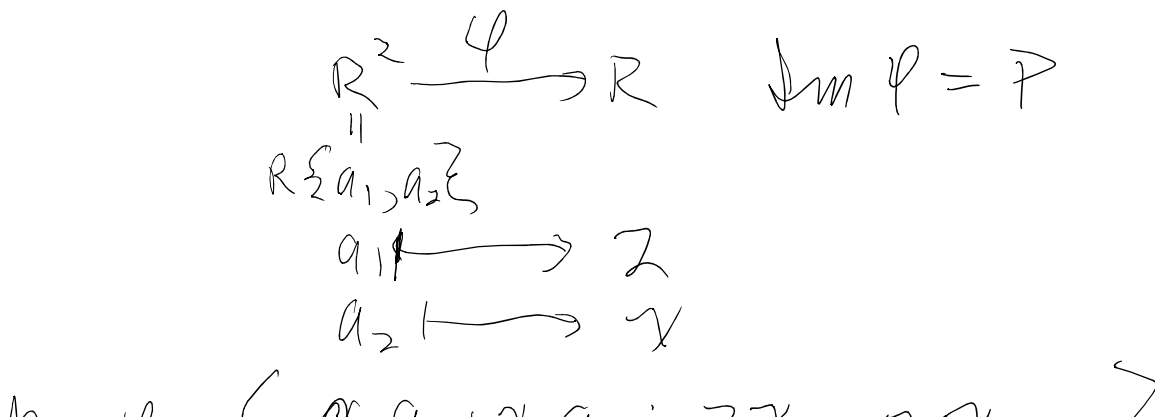
DEF AN  $R$ -MODULE  $P$  IS PROJECTIVE IF FOR ANY DIAGRAM OF THE FORM



SUPPOSE  $P$  IS FREE WITH GENERATORS  $x_1, x_2, \dots$ . CHOOSE  $m_1, m_2, \dots \in M$  WITH  $\beta(m_i) = \alpha(x_i)$  AND DEFINE  $\tilde{\alpha}(x_i) = m_i$ . HENCE ANY FREE MODULE IS PROJECTIVE.

FOR  $R = \mathbb{Z}$ , EVERY PROJECTIVE MODULE IS FREE.

EXAMPLE  $R = \mathbb{Z}[x]$   $P = (2, x) \subset R$   
 IT IS PROJECTIVE BUT NOT FREE



$$\text{Ker } \varphi = \left\{ \sum x_i a_i + x_2 a_2 : 2x_1 + x_2 = 0 \right\}$$

CAN SHOW  $R^2 \cong \text{Ker } \varphi \oplus P$  AND  
 $\text{Ker } \varphi$  IS FREE.

PROP ANY DIRECT SUMMAND OF A  
 NOT NEEDED FREE MODULE IS PROJECTIVE.

PROP FOR EACH R-MODULE  $M$ ,  
 THERE IS A PROJECTIVE  $P$   
 WITH A SURJECTION  $P \rightarrow M$ .

PROOF: CHOOSE A SET  $\{x_1, x_2, \dots\}$   
 OF R-MODULE GENERATORS OF  
 $M$ , AND  $P$  BE THE FREE  
 R-MODULE ON THIS SET.  $\square$

"R HAS ENOUGH PROJECTIVES"

DEF A PROJECTIVE RESOLUTION

OF AN R-MODULE  $M$  IS  
 A LES

$$0 \leftarrow M \xleftarrow{d_0} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} P_2 \leftarrow \dots$$

WHERE EACH  $P_i$  IS PROJECTIVE

PROP ② EVERY  $M$  HAS A PROJ. RES.

PROOF: USE PROP ① TO CONSTRUCT  $\mathcal{P}_0$

$$0 \leftarrow M \xleftarrow{d_0} \mathcal{P}_0 \xleftarrow{d_1} \text{Ker } d_0 \leftarrow 0$$

USE PROP ① AGAIN TO PRODUCE

$$0 \leftarrow \text{Ker } d_0 \xleftarrow[\mathcal{P}_1]{\text{ONTO } d_1} \mathcal{P}_1 \leftarrow \text{Ker } d_1 \leftarrow 0$$

REPEAT FOREVER. QED

DEF A RING  $R$  IS A  
PRINCIPAL IDEAL DOMAIN (PID)

IF EACH IDEAL IN  $R$  IS  
PRINCIPAL, I.E. GENERATED BY

A SINGLE ELEMENT.

EXAMPLES

$\mathbb{Z}$  IS A PID

ANY FIELD IS A PID

ANY SUBRING OF  $\mathbb{Q}$  IS A PID

$\mathbb{Z}[x]$  IS NOT A PID

$(2, x)$  IS NOT PRINCIPAL

PROP IF  $R$  IS A PID, THEN

EVERY  $R$ -MODULE  $M$  HAS

A PROJECTIVE RES. OF LENGTH 1.

A PROJECTIVE RES. OF LENGTH 1,  
 $\exists$  PROJECTIVES  $P_0$  AND  $P_1$  WITH

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow 0$$

$$0 \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} 0$$

EXAMPLE  $R = \text{FIELD}$

$$R = R[x] / (x^2)$$

$$M = R / (x) \cong R$$

PROJECTIVE RESOLUTION

$$0 \leftarrow M \leftarrow R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \dots$$

$\parallel$   
 $R$

RECALL OUR BASIC DIFFICULTY  
 GIVEN A SES OF ABELIAN  
 GROUPS

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

THE FUNCTORS  $- \otimes B$  AND

$\text{Hom}(-, B)$  FAIL TO PRESERVE  
 EXACTNESS.

SUPPOSE  $M$  AND  $N$  ARE  $R$ -MODULES  
 AND  $M$  HAS A PROJECTIVE  
 RESOLUTION

# RESOLUTION

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \dots$$

CONSIDER THE CHAIN COMPLEX

$$(3) \quad P_0 \otimes_R N \leftarrow P_1 \otimes_R N \leftarrow P_2 \otimes_R N \leftarrow \dots$$

THEOREM (FUND THM OF HOMOLOGICAL ALGEBRA)  $H_i(3)$  IS INDEPENDENT OF THE CHOICE OF PROJECTIVE RES

$$\text{DEF } \text{Tor}_i^R(M, N) \stackrel{\text{def}}{=} H_i(3)$$

MEMORIZE

REMARKS

$$(1) \quad \text{Tor}_0^R(M, N) = M \otimes_R N$$

$$(2) \quad \text{IF } R \text{ IS A PID THEN} \\ \text{Tor}_i^R(M, N) = 0 \quad \text{FOR } i > 1 \\ \text{FOR ANY } M \text{ AND } N.$$

$$(3) \quad \text{IF EITHER } M \text{ OR } N \text{ IS} \\ \text{PROJ, THEN } \text{Tor}_i^R(M, N) = 0 \\ \text{FOR } i > 0$$

$$(4) \quad \text{Tor}_x^R(M, N) \cong \text{Tor}_x^R(N, M)$$

$$(5) \quad \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) = \mathbb{Z}/\gcd(m, n)$$

DEF  $\text{Ext}_R^i(M, N)$  IS THE  $i$ 'TH COHOMOLOGY OF THE COCHAIN CX

$$\begin{array}{c} \text{Hom}_R(P_0, N) \longrightarrow \text{Hom}_R(P_1, N) \longrightarrow \text{Hom}_R(P_2, N) \longrightarrow \dots \\ \text{Hom}_R(P_i, N) \longrightarrow \text{Hom}_R(P_{i+1}, N) \\ \begin{array}{ccc} P_i & \xleftarrow{d_{i+1}} & P_{i+1} \\ \downarrow & \swarrow & \\ N & & \end{array} \end{array}$$

SIMILAR REMARKS APPLY

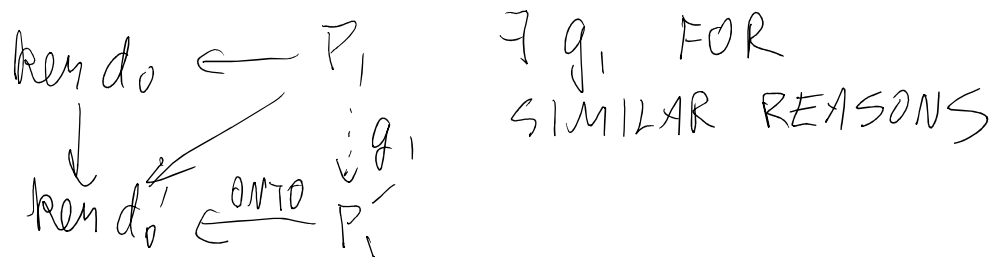
EG  $\text{Ext}_R^0(M, N) = \text{Hom}_R(M, N)$

LEMMA LET  $P$  AND  $P'$  BE TWO PROJECTIVE RESOLUTIONS OF  $M$ . THEY ARE CHAIN HOMOTOPICALLY EQUIVALENT.

PROOF:

$$\begin{array}{ccccccc} 0 \leftarrow M & \xleftarrow{d_0} & P_0 & \xleftarrow{d_1} & P_1 & \xleftarrow{d_2} & P_2 \xleftarrow{d_3} \dots \\ & \parallel & \downarrow g_0 & & \downarrow g_1 & & \downarrow g_2 \\ 0 \leftarrow M & \xleftarrow{d'_0} & P'_0 & \xleftarrow{d'_1} & P'_1 & \xleftarrow{d'_2} & P'_2 \xleftarrow{d'_3} \dots \end{array}$$

$\exists g_0$  BECAUSE  $d'_0$  IS ONTO AND  $P_0$  IS PROJECTIVE.



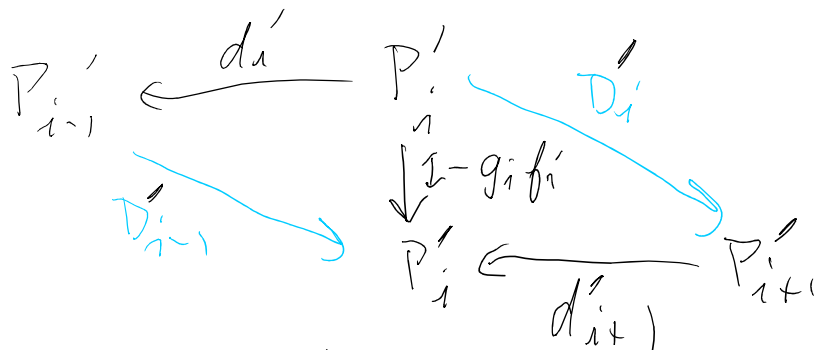
CAN CONSTRUCT  $g_i: P_i \rightarrow P'_i$

THUS WE CAN FIND  $g_i: P_i \rightarrow P'_i$  WITH

$g_i d_{i+1} = d'_i g_{i+1}$  AND  $g$  IS A CHAIN MAP  $P \xrightarrow{g} P'$

WE CAN CONSTRUCT  $P' \xrightarrow{f} P$  IN THE SAME WAY.

$P' \xrightarrow{f} P \xrightarrow{g} P'$   
 WANT TO SHOW  $gf \approx I_{P'}$



WE NEED MAPS  $D'_i: P'_i \rightarrow P'_{i+1}$

SUCH THAT  $D'_{i-1} d'_i + d'_{i+1} D'_i = 1 - g_i f_i$

WE CAN CONSTRUCT SUCH  $D'_i$

WE CAN CONSTRUCT SUCH  $D_i$  USING THE PROJECTIVITY OF THE  $P_i'$ . DETAILS OMITTED

THIS MEANS  $T_{0n}$  AND  $Ext$  ARE WELL DEFINED.

THEOREM GIVEN SHORT EXACT SEQUENCES

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

$$\text{AND } 0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

WE GET LONG EXACT SEQUENCES

$$\dots \rightarrow T_{0n}(M', N) \rightarrow T_{0n}(M, N) \rightarrow T_{0n}(M'', N) \rightarrow$$

$$\rightarrow T_{0n-1}(M', N) \rightarrow T_{0n-1}(M, N) \rightarrow T_{0n-1}(M'', N) \rightarrow \dots$$

NOTE  $Ext_R^*(M, N)$  IS

CONTRAVARIANT IN  $M$  AND

COVARIANT IN  $N$ . WE HAVE

$$\dots \leftarrow Ext^1(M', N) \leftarrow Ext^1(M, N) \leftarrow Ext^1(M'', N) \leftarrow$$

$$\leftarrow Ext^0(M', N) \leftarrow Ext^0(M, N) \leftarrow Ext^0(M'', N) \leftarrow 0$$

$$\hookrightarrow \text{Ext}^p(M', N) \leftarrow \text{Ext}^p(M, N) \leftarrow \text{Ext}^p(M'', N) \leftarrow 0$$

AND

$$\dots \rightarrow \text{Ext}^1(M, N') \rightarrow \text{Ext}^1(M, N) \rightarrow \text{Ext}^1(M, N'') \rightarrow 0$$

$$\hookrightarrow \text{Ext}^0(M, N') \rightarrow \text{Ext}^0(M, N) \rightarrow \text{Ext}^0(M, N'') \rightarrow 0$$

ANOTHER WAY TO DEFINE  $\text{Ext}$ :

DEF A INJECTIVE  $R$ -MODULE  $I$

$$\begin{array}{ccc} & \uparrow & \\ & \alpha & \\ & \dots & \\ & \uparrow & \\ M & \xleftarrow{f} & N \leftarrow 0 \end{array} \quad \begin{array}{l} \text{IF FOR EACH} \\ \text{INJECTION } f: N \rightarrow M \\ \text{AND } \alpha: N \rightarrow I, \\ \exists \hat{\alpha}: M \rightarrow I \text{ WITH} \\ \hat{\alpha} \circ f = \alpha. \end{array}$$

EXAMPLES  $\mathbb{Z}$  IS NOT INJECTIVE /  $\mathbb{Z}$

BUT  $\mathbb{Q}$  AND  $\mathbb{Q}/\mathbb{Z}$  ARE

BOTH INJECTIVE

$$\begin{array}{ccc} & \uparrow & \\ & \mathbb{Q} & \\ & \dots & \\ & \uparrow & \\ \mathbb{Z} & \xleftarrow{f} & \mathbb{Z} \end{array} \quad \begin{array}{l} \text{IN GENERAL FREE} \\ \text{R-MODULES ARE} \\ \text{NOT INJECTIVE} \end{array}$$

DEF  $n$ th INJECTIVE ...

DEF AN INJECTIVE RESOLUTION IS  
A LONG EXACT SEQUENCE

$$0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \dots$$

WITH EACH  $I_n$  INJECTIVE.